J. Math. Kyoto Univ. (JMKYAZ)
22-4 (1983) 685-699

Lie algebra of foliation preserving vector fields

By

Kazuhiko FUKUI and Naoto TOMITA

(Communicated by Prof. H. Toda Aug. 19, 1981)

Introduction

Let M and M' be connected paracompact C^∞-manifolds and $\mathfrak{X}(M)$ and $\mathfrak{X}(M')$ the Lie algebras of all C^∞-vector fields with compact support on M and M' respectively. A well-known theorem of Pursell-Shanks [10] may be stated as follows.

Theorem. *There exists a Lie algebra isomorphism Φ of $\mathfrak{X}(M)$ onto $\mathfrak{X}(M')$ if and only if there exists a C^∞-diffeomorphism φ of M onto M' such that $d\varphi=\Phi$.*

The above result still holds for Lie algebras of all infinitesimal automorphisms of several geometric structures on manifolds. Indeed, Omori [9] proved the corresponding result in case of volume structures, symplectic structures, contact structures and fibering structures with compact fibers. The case of complex structures was proved by Amemiya [1]. Koriyama [8] proved that this is still true for submanifolds regarding a submanifold as a geometric structure. Furthermore the first author [6] has proved the corresponding result in case of Lie algebras of G-invariant C^∞-vector fields with compact support on paracompact, connected, free G-manifolds when G is a compact connected semi-simple Lie group such that the automorphism group of its Lie algebra is connected. The corresponding result is no longer true when the automorphism group of its Lie algebra is not connected, G is not semi-simple or G does not act freely.

Let $(M, \mathscr{F})$ be a foliated manifold and $\mathfrak{X}(M, \mathscr{F})$ (resp. $\mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})$) be the Lie algebra of all foliation preserving (resp. leaf preserving) C^∞-vector fields with compact support on M. Then we have the following theorem, due to Amemiya [1], which can be also proved by using the methods of Pursell-Shanks [10] and Omori [9].

Theorem A. *There exists a Lie algebra isomorphism Φ of $\mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})$ onto $\mathfrak{X}_{\mathscr{L}}(M', \mathscr{F}')$ if and only if there exists a foliation preserving diffeomorphism φ of M onto M' such that $d\varphi=\Phi$.*

Theorem A implies that if $\mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})$ is algebraically isomorphic to $\mathfrak{X}_{\mathscr{L}}(M', \mathscr{F}')$, then $\mathfrak{X}(M, \mathscr{F})$ is algebraically isomorphic to $\mathfrak{X}(M', \mathscr{F}')$. Conversely, does $\mathfrak{X}(M, \mathscr{F})$ characterize $\mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})$?

The purpose of this paper is to prove Pursell-Shanks type theorem for certain foliated manifolds.

We call $\mathscr{F}$ compact Hausdorff if all leaves of $\mathscr{F}$ are compact and its leaf space is Hausdorff. Then we have the following theorem.

Theorem B. *Let $\mathscr{F}$ and $\mathscr{F}'$ be compact Hausdorff foliations on M and M' respectively. Then there exists a Lie algebra isomorphism Φ of $\mathfrak{x}(M, \mathscr{F})$ onto $\mathfrak{x}(M', \mathscr{F}')$ if and only if there exists a C^∞-foliation preserving diffeomorphism φ of $(M, \mathscr{F})$ onto $(M', \mathscr{F}')$ such that $d\varphi = \Phi$.*

This result is an extension of the corresponding result in case of fibering structures with compact fibers due to Omori [9].

Next we consider codimension one foliations.

Theorem C (Theorem 5.5). *Let $(M, \mathscr{F})$ and $(M', \mathscr{F}')$ be generalized Reeb foliated manifolds (see §5 for definition). If $\Phi: \mathfrak{x}(M, \mathscr{F}) \to \mathfrak{x}(M', \mathscr{F}')$ is a Lie algebra isomorphism, then there exists a foliation preserving diffeomorphism $\varphi: (M, \mathscr{F}) \to (M', \mathscr{F}')$.*

Furthermore we have the following theorem.

Theorem D (Theorem 6.4). *Let $\mathscr{F}$ and $\mathscr{F}'$ be foliations without holonomy on closed manifolds M and M' respectively. If $\Phi: \mathfrak{x}(M, \mathscr{F}) \to \mathfrak{x}(M', \mathscr{F}')$ is a Lie algebra isomorphism, then there exists a foliation preserving diffeomorphism $\varphi: (M, \mathscr{F}) \longrightarrow (M', \mathscr{F}')$.*

The key to the proofs of our theorems is to find and characterize maximal ideals of $\mathfrak{x}(M, \mathscr{F})$. In §2, we find and characterize maximal ideals of $\mathfrak{x}(M/\mathscr{F})$ which is the quotient Lie algebra $\mathfrak{x}(M, \mathscr{F})/\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$. This section is an equivariant version of §2 in Koriyama [8].

In §3, for $\mathscr{F}$ compact Hausdorff foliation, we find and characterize maximal ideals of $\mathfrak{x}(M, \mathscr{F})$ using the facts given in §2. In §4, we prove Theorem B. In §5, §6, we prove Theorems C, D respectively.

§1. Preliminaries

Let M be a paracompact connected C^∞-manifold without boundary of dimension n and $\mathscr{F}$ a C^∞-foliation of M of codimension q.

Definition 1.1. A vector field X on M is called a foliation preserving (resp. leaf preserving) vector field if transformations $\{\phi_t\}$ generated by X are foliation preserving (resp. leaf preserving) diffeomorphisms (cf. Fukui [5]).

We denote by $\mathfrak{x}(M, \mathscr{F})$ (resp. $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$) the Lie algebra of all C^∞-foliation preserving (resp. leaf preserving) vector fields on M with compact support.

Remark 1.2. $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ is an ideal of $\mathfrak{x}(M, \mathscr{F})$.

We denote by $\mathfrak{x}(M/\mathscr{F})$ the quotient Lie algebra $\mathfrak{x}(M, \mathscr{F})/\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$. Then we have an exact sequence of Lie algebras;

$$0 \longrightarrow \mathfrak{x}_{\mathscr{L}}(M, \mathscr{F}) \longrightarrow \mathfrak{x}(M, \mathscr{F}) \xrightarrow{d\pi} \mathfrak{x}(M/\mathscr{F}) \longrightarrow 0$$

where each map is a Lie algebra homomorphism.

Definition 1.3. $\mathscr{F}$ is called a compact foliation if all leaves of $\mathscr{F}$ are compact.

Let π: $M \to M/\mathscr{F}$ be the map which identifies each leaf to a point and let $M/\mathscr{F}$ have the quotient topology.

Definition 1.4. A compact foliation $\mathscr{F}$ is called Hausdorff if $M/\mathscr{F}$ is a Hausdorff space.

Remark 1.5. The example due to Sullivan [12] says that all compact foliations are not Hausdorff.

Remark 1.6. For $q=1$, every compact foliation is Hausdorff. Indeed, $M/\mathscr{F}$ is a manifold with boundary or without boundary. For $q=2$, every compact foliation on a compact manifold is also Hausdorff. (see Epstein [3], Edwards, Millett and Sullivan [2]). Furthermore Edwards, Millett and Sullivan in [2] showed that in the presence of a certain homological assumption, every compact foliation on a compact manifold is Hausdorff for $q \geqq 3$.

For compact Hausdorff foliations, Epstein [3] proved the following theorem.

Theorem 1.7. *Let $\mathscr{F}$ be a compact Hausdorff C^∞-foliation on M. Then there is a "generic leaf" L_0 with the property that there is an open dense subset of M, where all the leaves have trivial holonomy and are all diffeomorphic to L_0. Given a leaf L, we can describe a neighborhood $U(L)$ of L, together with the foliation on the neighborhood as follows. There is a finite group G_L of $O(q)$. G_L acts freely on L_0 on the right and $L_0/G_L \cong L$. Let D^q be the unit q-disk. We foliate $L_0 \times D^q$ with leaves of the form $L_0 \times \{pt\}$. This foliation is preserved by the diagonal action of G_L, defined by $g(x, y)=(x \cdot g^{-1}, g \cdot y)$ for $g \in G_L$, $x \in L_0$ and $y \in D^q$. So we have a foliation induced on $U = L_0 \underset{G_L}{\times} D^q$. The leaf corresponding to $y=0 \in D^q$ is L_0/G_L. Then there is a C^∞-imbedding ϕ: $U \to M$ with $\phi(U)=U(L)$, which preserves leaves and $\phi(L_0/G_L)=L$.*

Definition 1.8. A leaf L is called singular if G_L is not trivial.

Definition 1.9. A singular leaf L is called isolated if the action of G_L has only the origin of D^q as fixed points.

Remark 1.10. Let $\mathrm{Fix}(G_L)$ denote the fixed point set of the action of G_L. If G_L is a cyclic group, the action is semifree.

Now, consider what is a standard coordinate of such compact Hausdorff foliation $\mathscr{F}$. This is a local coordinate $(U; x^1,\dots, x^q, y^1,\dots, y^r)$ $(q+r=n)$ such that for any fixed $x^1,\dots, x^q$, a local coordinate of the leaf is given by $y^1,\dots, y^r$. For such local coordinate, every $X \in \mathfrak{x}(M, \mathscr{F})$ is described as follows;

$$X \equiv \sum_{i=1}^{q} a_i(x^1,\dots, x^q) \frac{\partial}{\partial x^i} + \sum_{j=1}^{r} b_j(x^1,\dots, x^q, y^1,\dots, y^r) \frac{\partial}{\partial y^j},$$

where a_i and b_j $(1 \leqq i \leqq q,\ 1 \leqq j \leqq r)$ are C^∞-functions on U.

Lemma 1.11. *If $X \in \mathfrak{X}(M, \mathscr{F})$ satisfies $X(p) \neq 0$ $(p \in M)$, then there is a local coordinate $(U;\ x^1,\dots, x^q, y^1,\dots, y^r)$ at p such that*

1) $X \equiv \dfrac{\partial}{\partial x^1}$ on a neighborhood of p, or

2) $X \equiv \dfrac{\partial}{\partial y^1} + \sum_{i=1}^{q} f_i(x^1,\dots, x^q) \dfrac{\partial}{\partial x^i}$ with $f_i(0,\dots, 0) = 0$

where the origin of the coordinate corresponds to the point p. Especially, when p is contained in some isolated singular leaf, 1) *does not occur.*

Proof. Easy computations.

Let G be a finite subgroup of the orthogonal group $O(q)$. G acts linearly on R^q. We denote by $\mathfrak{X}(R^q)$ the Lie algebra of all C^∞-vector fields on R^q with compact support.

Definition 1.12. A vector field $X \in \mathfrak{X}(R^q)$ is called G-invariant if $Tg \circ X = X \circ g$ for all $g \in G$, where Tg is the tangent of $g\colon R^q \to R^q$. We call X a G-vector field.

Remark 1.13. Since the map g is linear, the tangent map Tg is equal to the map g.

The set $\mathfrak{X}_G(R^q) = \{X \in \mathfrak{X}(R^q) \mid X$ is G-invariant$\}$ is a Lie subalgebra of $\mathfrak{X}(R^q)$.

Definition 1.14. Two (G-)vector fields X, Y are (G-)equivalent at the origin if there exist an open neighborhood $U(\ni 0)$ and a C^∞- (G-) diffeomorphism h of R^q such that $Yh^{-1}(p) = Dh^{-1}(p) \cdot X(p)$ for $p \in U$.

Theorem 1.15 (*Equivariant linearization theorem*).
If a G-vector field X is equivalent at the origin to a G-linear vector field Y by a C^∞-diffeomorphism h such that $Dh(0)$ is equal to the unit matrix, then X and Y are also G-equivalent at the origin.

Proof. From the assumption, there exist an open neighborhood $U(\ni 0)$ and a C^∞-diffeomorphism h such that $Yh^{-1}(p) = Dh^{-1}(p) \cdot X(p)$. Put $\tilde{h}^{-1} = \frac{1}{|G|} \sum_{g \in G} g \cdot h^{-1} \cdot g^{-1}$, where $|G|$ denotes the order of G. Since $Dh(0)$ is equal to the unit matrix, it is easy to see that $\tilde{h}$ is a G-diffeomorphism in some neighborhood $V(\subset U)$ of the origin. Since Y is G-linear, for $p \in V$

$$\begin{aligned} Y(\tilde{h}^{-1}(p)) &= Y\Big(\frac{1}{|G|} \sum_{g \in G} g \cdot h^{-1} \cdot g^{-1}(p)\Big) \\ &= \frac{1}{|G|} \sum_{g \in G} Y(g \cdot h^{-1} \cdot g^{-1}(p)) \end{aligned}$$

$$= \frac{1}{|G|} \sum_{g \in G} g Y(h^{-1} \cdot g^{-1}(p))$$

$$= \frac{1}{|G|} \sum_{g \in G} g Dh^{-1}(g^{-1}(p)) X(g^{-1}(p)) \qquad \text{(because } X \text{ and } Y \text{ are equivalent)}$$

$$= \frac{1}{|G|} \sum_{g \in G} g Dh^{-1}(g^{-1}(p)) g^{-1} X(p) \qquad (X \text{ is } G\text{-invariant})$$

$$= \frac{1}{|G|} \sum_{g \in G} D(g \cdot h^{-1} \cdot g^{-1})(p) X(p)$$

$$= D\tilde{h}^{-1}(p) X(p).$$

This completes the proof.

Remark 1.16. For $X \in \mathfrak{X}_G(R^q)$, $\bar{X} = \frac{1}{|G|} \sum gXg^{-1}$ is equal to X.

Remark 1.17. For $X \in \mathfrak{X}(R^q)$, $\bar{X} = \frac{1}{|G|} \sum_{g \in G} gXg^{-1}$ is G-invariant. Furthermore if $j^k(X)(0) = 0$ for all $k \geqq 1$, then $j^k(\bar{X})(0) = 0$ for all $k \geqq 1$, where $j^k(X)(0)$ (resp. $j^k(\bar{X})(0)$) denotes the k-jet of X (resp. $\bar{X}$) at 0.

§2. Characterization of maximal ideals of $\mathfrak{X}(M/\mathscr{F})$

Let G be a finite subgroup of the orthogonal group $O(q)$. G acts linearly on R^q. We denote by $\mathfrak{X}(R^q)$ the Lie algebra of all C^∞-vector fields on R^q with compact support and $\mathfrak{X}_G(R^q) = \{X \in \mathfrak{X}(R^q) \mid X \text{ is } G\text{-invariant}\}$ is a Lie subalgebra of $\mathfrak{X}(R^q)$. We assume that the action of G is semifree.

Lemma 2.1. *If $X \in \mathfrak{X}_G(R^q)$ does not vanish at p ($\notin Fix(G)$) in R^q, then for any $Z \in \mathfrak{X}_G(R^q)$ there exist a neighborhood U of p in R^q and a vector field $Y \in \mathfrak{X}_G(R^q)$ such that $[X, Y] = Z$ on U.*

Proof. We consider the orbit map $h: R^q \to R^q/G$. Then $h|_{R^q - Fix(G)}$ is a finite covering. The differential dh maps any element Z of $\mathfrak{X}_G(R^q)$ to some element $\bar{Z}$ in $\mathfrak{X}(R^q - \mathrm{Fix}(G)/G)$. Let V be a neighborhood of p in R^q such that $h|_V$ is hoemomorphic. Put $\bar{p} = h(p)$, $\bar{V} = h(V)$. Since $\bar{X}(\bar{p}) \neq 0$, it is easy to see that there exists a local coordinate $(\bar{U}; x^1, \ldots, x^q)$ $(\bar{U} \subset \bar{V})$ at $\bar{p}$ such that $\bar{X} = \frac{\partial}{\partial x^1}$ on $\bar{V}$. By the usual argument (see Koriyama [8] Lemma 2.1.) there exist a local coordinate $(\bar{U}; x^1, \ldots, x^q)$ $(\bar{U} \subset \bar{V})$ at $\bar{p}$ and a vector field $\bar{Y}$ on $\bar{U}$ such that $[\bar{X}, \bar{Y}] = \bar{Z}$ on $\bar{U}$. Let U be a component of $h^{-1}(\bar{U})$ which contains p. We can easily lift the vector field $\bar{Y}$ to a G-invariant vector field Y on R^q. Then $[X, Y] = Z$ on U. This completes the proof.

Lemma 2.2. *For each point $p \in R^q$ such that $p \notin Fix(G)$, we set $\mathscr{I}_p = \{X \in \mathfrak{X}_G(R^q) \mid X(p) = 0, j^k(X)(p) = 0$ for all $k \geqq 1\}$. Then for each point $p \in R^q$, $\mathscr{I}_p$ is an ideal of $\mathfrak{X}_G(R^q)$.*

Proof. Easy computation.

Lemma 2.3. *Let $p(\notin Fix(G))$ in R^q be a given point. If $\mathscr{I}$ is an ideal of $\mathfrak{x}_G(R^q)$ and $X(p)=0$ for all $X\in\mathscr{I}$, then $\mathscr{I}\subset\mathscr{I}_p$.*

Proof. Since $p\notin Fix(G)$, there exists a local coordinate $(\bar{U}; x^1,\dots, x^q)$ at p such that $\bar{U}\cap Fix(G)=\emptyset$ and $g_1\bar{U}\cap g_2\bar{U}=\emptyset$ for any distinct $g_1, g_2\in G$. Hence appropriate extensions of $\frac{\partial}{\partial x^i}$ $(i=1,\dots, q)$ are contained in $\mathfrak{x}_G(R^q)$. We also denote the extended vector fields by the same letters. For any $X=\sum_{i=1}^{q} a^i\frac{\partial}{\partial x^i}\in\mathscr{I}$, $\left[\frac{\partial}{\partial x^j}, X\right]=\sum_{i=1}^{q}\frac{\partial a^i}{\partial x^j}\frac{\partial}{\partial x^i}$ for all $j=1,\dots, q$. As $\mathscr{I}$ is an ideal, $\left[\frac{\partial}{\partial x^j}, X\right]\in\mathscr{I}$. From the assumption for $\mathscr{I}$, $\left(\frac{\partial a^i}{\partial x^j}\right)(p)=0$ for all $i, j=1,\dots, q$. By induction on k, we have $j^k(X)(p)=0$ for all $k\geqq 1$. Therefore $\mathscr{I}\subset\mathscr{I}_p$. This completes the proof.

The next lemma is well known.

Lemma 2.4. *Let A be an arbitrary Lie algebra. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A such that $\mathfrak{a}\supset\mathfrak{b}$. Then $(A/\mathfrak{b})/(\mathfrak{a}/\mathfrak{b})\cong A/\mathfrak{a}$.*

Now from Lemma 2.5 till Theorem 2.12, we assume that the action of G has only the origin of R^q as fixed points.

Lemma 2.5. *The subset $\mathfrak{x}_G^1(R^q)=\{X\in\mathfrak{x}_G(R^q)\,|\,j^1(X)(0)=0\}$ is an ideal of $\mathfrak{x}_G(R^q)$.*

Proof. Easy computations.

Let π: $\mathfrak{x}_G(R^q)\to\mathfrak{x}_G(R^q)/\mathfrak{x}_G^1(R^q)\cong gl_G(q, R)$ be the natural projection which is a Lie algebra homomorphism, where $gl_G(q, R)$ denotes the set of G-invariant endmorphisms of R^q. We denote by $\{g_\lambda\}_{\lambda\in\Lambda}$ the set of maximal ideals of $gl_G(q, R)$. Then for any $\lambda\in\Lambda$, $\pi^{-1}(g_\lambda)$ is a maximal ideal of $\mathfrak{x}_G(R^q)$.

Proposition 2.6. *If $\mathfrak{m}$ is a maximal ideal of $\mathfrak{x}_G(R^q)$ such that $\mathfrak{m}\supset\mathfrak{x}_G^1(R^q)$, then there is a $\lambda_0(\in\Lambda)$ such that $\mathfrak{m}=\pi^{-1}(g_{\lambda_0})$.*

Proof. Let $\mathfrak{m}\subsetneqq\mathfrak{x}_G(R^q)$ be a maximal ideal such that $\mathfrak{m}\supset\mathfrak{x}_G^1(R^q)$. By Lemma 2.4, $\mathfrak{m}/\mathfrak{x}_G^1(R^q)$ is a proper ideal of $\mathfrak{x}_G(R^q)/\mathfrak{x}_G^1(R^q)\cong gl_G(q, R)$. From the maximality of $\mathfrak{m}$, $\mathfrak{m}/\mathfrak{x}_G^1(R^q)$ is a maximal ideal of $gl_G(q, R)$. Hence $\mathfrak{m}/\mathfrak{x}_G^1(R^q)$ should be equal to g_{λ_0} for some $\lambda_0\in\Lambda$. This completes the proof.

We set $\mathfrak{x}_0(R^q)=\{X\in\mathfrak{x}(R^q)\,|\,X(0)=0\}$ and

$$\mathfrak{x}_G^\infty(R^q)=\{X\in\mathfrak{x}_G(R^q)\,|\,j^k(X)(0)=0\quad\text{for all}\quad k\geqq 1\}.$$

Lemma 2.7. *If $\mathfrak{m}$ is a maximal ideal of $\mathfrak{x}_G(R^q)$ such that $\mathfrak{m}\supset\mathfrak{x}_G^\infty(R^q)$, then $j^1(\mathfrak{m})(0)$ is a proper ideal of $gl_G(q, R)$, where $j^1(\mathfrak{m})(0)$ is the image of $\mathfrak{m}$ under the natural projection;*

$$\pi: \mathfrak{X}_G(R^q) \longrightarrow \mathfrak{X}_G(R^q)/\mathfrak{X}_G^1(R^q) \cong gl_G(q, R).$$

Proof. Assume $j^1(\mathfrak{m})(0)=gl_G(q, R)$. Take a vector field $X \in \mathfrak{m}$ such that $j^1(X)(0)$ is the unit matrix. Then by Sternberg's linearization theorem [11], there exists a local coordinate $(U; x^1,\dots, x^q)$ at 0 such that X is equivalent at the origin to $\sum_{i=1}^{q} x^i \frac{\partial}{\partial x^i}$, via a C^∞-diffeomorphism h such that $Dh(0)$ is the unit matrix. Then by Theorem 1.15, the G-vector field $X \in \mathfrak{X}_G(R^q)$ is G-equivalent at the origin to the G-vector field $\sum_{i=1}^{q} x^i \frac{\partial}{\partial x^i}$.

On the other hand, by the same argument as in Lemma 2.10 of Koriyama [6], we have that for any $Z \in \mathfrak{X}_G^1(R^q)$, there exists a vector field $Y \in \mathfrak{X}_0(R^q)$ such that $Z-[X, Y] \in \mathfrak{X}_0^\infty(R^q)$.

Now consider the vector field $\frac{1}{|G|}\sum_{g\in G} g(Z-[X, Y])g^{-1}$ which is G-invariant and is contained in $\mathfrak{X}_G^\infty(R^q)$ from Remark 1.17. Furthermore since X and Z are G-invariant, it is equal to $Z-\left[X, \frac{1}{|G|}\sum g Y g^{-1}\right]$. Since $X \in \mathfrak{m}$ and $\mathfrak{m}$ is an ideal of $\mathfrak{X}_G(R^q)$, we obtain $\left[X, \frac{1}{|G|}\sum g Y g^{-1}\right] \in \mathfrak{m}$. So $Z \in \mathfrak{m}$, hence $\mathfrak{X}_G^1(R^q) \subset \mathfrak{m}$. By Proposition 2.6, $\mathfrak{m}=\pi^{-1}(G_\lambda)$ for some $\lambda \in \Lambda$. Then we have $j^1(\mathfrak{m})(0) \subsetneqq gl_G(q, R)$, contradicting the assumption. This completes the proof.

Proposition 2.8. *If $\mathfrak{m}$ is a maximal ideal of $\mathfrak{X}_G(R^q)$ and $\mathfrak{m} \supset \mathfrak{X}_G^\infty(R^q)$, then $\mathfrak{m}=\pi^{-1}(g_{\lambda_0})$ for some $\lambda_0 \in \Lambda$.*

Proof. From Lemma 2.7, $j^1(\mathfrak{m})(0)$ is a proper maximal ideal of $gl_G(q, R)$. Thus $j^1(\mathfrak{m})(0)=g_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Then we have $\mathfrak{m} \subset \pi^{-1}(g_{\lambda_0})$. By the maximality of $\mathfrak{m}$, $\mathfrak{m}=\pi^{-1}(g_{\lambda_0})$. This completes the proof.

Lemma 2.9. *If $\mathfrak{m}$ is a maximal ideal of $\mathfrak{X}_G(R^q)$ and $\mathfrak{m} \not\supset \mathfrak{X}_G^\infty(R^q)$, then $j^1(\mathfrak{m})(0)=gl_G(q, R)$.*

Proof. Assume that $j^1(\mathfrak{m})(0)$ is a proper ideal of $gl_G(q, R)$. Since $j^1(\mathfrak{m})(0)$ is a maximal ideal of $gl_G(q, R)$, $j^1(\mathfrak{m})(0)=g_\lambda$ for some $\lambda \in \Lambda$. Then $\mathfrak{m}=\pi^{-1}(g_\lambda) \supset \mathfrak{X}_G^\infty(R^q)$, contradicting the assumption. Hence $j^1(\mathfrak{m})(0)=gl_G(q, R)$. This completes the proof.

Lemma 2.10. *Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{X}_G(R^q)$ such that $j^1(\mathfrak{m})(0)=gl_G(q, R)$. If for any point $p(\neq 0) \in R^q$, there exists a vector field $Y \in \mathfrak{m}$ such that $Y(p) \neq 0$, then $\mathfrak{m} \supset \mathfrak{X}_G^\infty(R^q)$.*

Proof. We set $\mathscr{S}=\{X \in \mathfrak{X}_G^\infty(R^q) \mid \operatorname{supp} X \not\ni 0\}$.

Firstly we prove that $\mathscr{S} \supset \mathfrak{m}$. Let X be an arbitrary element of $\mathscr{S}$. Since supp X is compact, there are $X_i \in \mathscr{S}$ and local coordinates $(V_i; x_i^1,\dots, x_i^q)$ $(i=1, 2,\dots, r)$ such that $X=X_1+\cdots+X_r$, $\operatorname{supp} X_i \subset V_i$ and $g_1 V_i \cap g_2 V_i=\emptyset$ $(i=1, 2,\dots, r)$ for any distinct $g_1, g_2 \in G$.

If we want to prove that $X \in \mathfrak{m}$, it suffices to prove that $X_i \in \mathfrak{m}$ for each i. From the assumption, for each $V_i (i=1, 2,\dots, r)$, there is a vector field $Y_i \in \mathfrak{m}$ such that $Y_i= \frac{\partial}{\partial x_i^1}$ on V_i. Because the argument is local, we may delete the indices. By the same argument as in Lemma 2.1, we can prove that there exists a vector field $Z\in \mathfrak{x}_G(R^q)$ such that supp $Z\subset V$ and $[X, Z]=X$ on V. Therefore we have $X\in \mathfrak{m}$, hence $\mathscr{I}\subset \mathfrak{m}$.

Now we continue the proof of Lemma 2.10. Since $j^1(\mathfrak{m})(0)=gl_G(q, R)$ by Theorem 1.15, there are a vector field $X\in m$ and a local coordinate $(U; x^1,\dots, x^q)$ at 0 such that $X=\sum_{i=1}^{q} x^i \frac{\partial}{\partial x^i}$ on U. By the same argument as in the last part of the proof of Lemma 1.13 in [8], we see that for any $Z\in \mathfrak{x}_G^\infty(R^q)$, there exist a vector field $Y\in \mathfrak{x}_0(R^q)$ and an open neighborhood $W(\subset U)$ of 0 such that $[X, Y]=Z$ on W. Since X and Z are G-invariant, for $\bar{Y}=\frac{1}{|G|}\sum_{g\in G} gYg^{-1}\in \mathfrak{x}_G(R^q)$, we have $[X, \bar{Y}]=\frac{1}{|G|}\sum_{g\in G} g[X, Y]g^{-1}=\frac{1}{|G|}\sum_{g\in G} gZg^{-1}=Z$ on W.

Since $[X, \bar{Y}]\in \mathfrak{m}$ and supp$(Z-[X, \bar{Y}])\not\ni 0$, $Z-[X, \bar{Y}]\in \mathscr{I}\subset \mathfrak{m}$, hence $Z\in \mathfrak{m}$. Therefore we have $\mathfrak{x}_G^\infty(R^q)\subset \mathfrak{m}$. This completes the proof.

Proposition 2.11. *Let m be a maximal ideal of $\mathfrak{x}_G(R^q)$ such that $\mathfrak{m}\not\supset \mathfrak{x}_G^\infty(R^q)$. Then there exists a point $p\in R^q$ such that $p\neq 0$, $\mathfrak{m}=\mathscr{I}_p$, and the orbit of p by G is unique.*

Proof. By Lemma 2.9, $j^1(\mathfrak{m})(0)=gl_G(q, R)$. By Lemma 2.10, there exists a point $p(\neq 0)\in R^q$ such that $X(p)=0$ for all $X\in \mathfrak{m}$. By Lemma 2.3, $\mathfrak{m}\subset \mathscr{I}_p$. Since $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}=\mathscr{I}_p$. From the maximality of $\mathfrak{m}$, $G\cdot p$ is uniquely determined. This completes the proof.

Theorem 2.12. *Any maximal ideal of $\mathfrak{x}_G(R^q)$ should be equal to one of the following ideals;*

(i) $\mathscr{I}_{g_\lambda}=\pi^{-1}(g_\lambda)$: ideal with finite codimension and corresponding to $0\in R^q$,

(ii) $\mathscr{I}_p$: ideal with infinite codimension and corresponding to $G\cdot p$, $p\neq 0$.

Proof. The result is an immediate consequence of Propositions 2.8 and 2.11.

Now we consider the case of dim Fix$(G)\geqq 1$.

Lemma 2.13. *For any ideal $\mathfrak{m}$ of $\mathfrak{x}_G(R^q)$, there exists a point $p\in R^q$ such that $X(p)=0$ for all $X\in \mathfrak{m}$.*

Proof. We assume that for any point $p\in R^q$, there is a vector field $X\in \mathfrak{m}$ such that $X(p)\neq 0$. Then applying Remark 1.17 to Lemma 3.1 of [8], we easily prove that for any $Z\in \mathfrak{x}_G(R^q)$, there exist a vector field $Y\in \mathfrak{x}_G(R^q)$ and a nieghborhood U of p such that $[X, Y]=Z$ on U. Hence by the method which was used to prove $\mathscr{I}\supset \mathfrak{m}$ in Lemma 2.10, we have $\mathfrak{m}=\mathfrak{x}_G(R^q)$. This completes the proof.

The following two lemmas are easily proved.

Lemma 2.14. *Let $\mathfrak{m}$ be an ideal of $\mathfrak{x}_G(R^q)$ and $p\in R^q$ be a point such that $X(p)=0$ for all $X\in \mathfrak{m}$. Then for $p\notin Fix(G)$ we have $j^k(X)(p)=0$ for all $k\geqq 1$. For $p\in Fix(G)$, let $(U; x^1,\dots, x^s, x^{s+1},\dots, x^q)$ be a local coordinate at p such that $U\cap Fix(G)=\{x^{s+1}=\cdots=x^q=0\}$. Then for any $X=\sum_{i=1}^{q} f_i(x)\frac{\partial}{\partial x_i}\in \mathfrak{m}$, we have $\frac{\partial^r f_i}{x^{i_1}\cdots\partial x^{i_r}}(p)=0$ $(i=1, 2,\dots, q;\ r\geqq 1;\ 1\leqq i_j\leqq s)$.*

Lemma 2.15. *Let $\mathfrak{m}$ be an ideal of $\mathfrak{x}_G(R^q)$ such that $X(p)=0$ for all $X\in\mathfrak{m}$ at a point $p\in R^q$. Then $p\notin Fix(G)$ if and only if $\mathfrak{m}$ does not contain* $\mathrm{Ker}\, r_*$, *where r_*: $\mathfrak{x}_G(R^q)\to\mathfrak{x}(Fix(G))$ is the Lie algebra homomorphism induced from the restriction map r: $R^q\to Fix(G)$.*

Theorem 2.16. *Any maximal ideal of $\mathfrak{x}_G(R^q)$ should be equal to one of the following ideals;*
(i) $\mathscr{I}_p$: ideal corresponding to $G\cdot p$ $(p\notin Fix(G))$,
(ii) $r_*^{-1}\bar{\mathscr{I}}_p$: *ideal corresponding to p $(p\in Fix(G))$,*
where $\bar{\mathscr{I}}_p$ is the maximal ideal of $\mathfrak{x}(Fix(G))$ corresponding to p.

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{x}_G(R^q)$. By Lemma 2.13 there is a point $p\in R^q$ such that $X(p)=0$ for all $X\in\mathfrak{m}$. If p is not contained in Fix(G), then by Lemma 2.14 we have $\mathfrak{m}\subset\mathscr{I}_p$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m}=\mathscr{I}_p$ and the orbit $G\cdot p$ is uniquely determined. If p is contained in Fix(G), then by Lemma 2.14 we have $r^*(\mathfrak{m})\subset\bar{\mathscr{I}}_p$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m}=r_*^{-1}\bar{\mathscr{I}}_p$ and the point p is uniquely determined. This completes the proof.

Corollary 2.17. *Let $\mathscr{F}$ be a compact Hausdorff foliation on M. Then any maximal ideal of $\mathfrak{x}(M/\mathscr{F})$ should be equal to one of the following ideals;*
(i) *ideal with finite codimension and corresponding to an isolated singular leaf,*
(ii) *ideal with infinite codimension and corresponding to a non-isolated singular leaf,*
(iii) *ideal with infinite codimension and correspondeing to a non-singular leaf.*

Proof. The result is an immediate consequence of Theorems 2.12 and 2.16 since it is sufficient to prove the cases that the actions of the groups in Theorem 1.7 are semifree.

§3. Characterization of maximal ideals of $\mathfrak{x}(M, \mathscr{F})$

In this section we assume that $\mathscr{F}$ is a compact Hausdorff foliation.

Let $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ be a Lie subalgebra of $\mathfrak{x}(M, \mathscr{F})$ whose elements are tangent to leaves of $\mathscr{F}$. Then we know that $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ is an ideal of $\mathfrak{x}(M, \mathscr{F})$ and we have the following exact sequence:

$$0 \longrightarrow \mathfrak{x}_{\mathscr{L}}(M) \longrightarrow \mathfrak{x}(M, \mathscr{F}) \xrightarrow{d\pi} \mathfrak{x}(M/\mathscr{F}) \longrightarrow 0.$$

Then it is easy to see the following.

Lemma 3.1. *Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{x}(M, \mathscr{F})$. If $d\pi(\mathfrak{m})\neq\mathfrak{x}(M/\mathscr{F})$, then $d\pi(\mathfrak{m})$ is a maximal ideal of $\mathfrak{x}(M/\mathscr{F})$.*

Lemma 3.2. *Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{x}(M, \mathscr{F})$. If $d\pi(\mathfrak{m})=\mathfrak{x}(M/\mathscr{F})$, then there are an isolated singular leaf L and a point $p\in L$ such that for all $X\in\mathfrak{m}$, $X(p)=0$.*

Proof. Assume that for any point p in isolated singular leaves, there exists a vector field $X \in \mathfrak{m}$ with $X(p) \neq 0$.

First we prove that $\mathfrak{x}_{\mathscr{L}}(M) \subset \mathfrak{m}$. Let $\{L_i\}_{i \in N}$ be the set of all isolated singular leaves of $\mathscr{F}$. From Theorem 1.7, N is a countable set. Thus we may consider N as the set of natural numbers or its finite subset. Take neighborhoods $U(L_i)$ $(\cong L_i \underset{G_i}{\times} D^q)$, $U_{1/3}(L_i)$ $(\cong L_i \underset{G_i}{\times} D^q(1/3))$ of an isolated singular leaf L_i for each $i \in N$ such that $U(L_i) \cap U(L_j) = \emptyset$ for any i, j $(i \neq j)$, where $D^q(1/3)$ is the disk of radius 1/3. Set $V = \bigcap_{i \in N} \{M - U_{1/3}(L_i)\}$, which is an open set of M. Let h: $[0, 1] \to [0, 1]$ be a C^∞-function such that

$$h(t) = \begin{cases} 1 & \left(0 \leqq t \leqq \frac{1}{2}\right) \\ 0 & \left(\frac{3}{4} \leqq t \leqq 1\right) \end{cases}$$

and μ: $D^q \to R$ be a C^∞-function defined by $\mu(x^1, \ldots, x^q) = h((x^1)^2 + \cdots + (x^q)^2)$. We define a C^∞-function λ_i: $U(L_i) \cong L_i \underset{G_i}{\times} D^q \to R$ by $\lambda_i(p, x^1, \ldots, x^q) = \mu(x^1, \ldots, x^q)$ for $p \in L_i$, $(x^1, \ldots, x^q) \in D^q$. For any $Y \in \mathfrak{x}_{\mathscr{L}}(M)$, we set $Y_i = \lambda_i Y$ $(i \in N)$. Since supp Y is compact, we may assume that there is $i_0 \in N$ such that $Y_i = 0$ for $i > i_0$. If we set $Y_0 = Y - Y_1 - Y_2 - \cdots - Y_{i_0}$, then we have that $Y = Y_0 + Y_1 + \cdots + Y_{i_0}$, supp $Y_i \subset U(L_i)$ for each $i \in N$ and supp $Y_0 \subset V$.

Hence if we want to prove that $Y \in \mathfrak{m}$, it suffices to prove that (i) $Y_0 \in \mathfrak{m}$ and (ii) $Y_i \in \mathfrak{m}$ for $i = 1, 2, \ldots, i_0$.

(i) From $d\pi(\mathfrak{m}) = \mathfrak{x}(M/\mathscr{F})$, for each point p in V, there is a vector field $X \in \mathfrak{m}$ with $X(p) \neq 0$. So by Lemma 1.11, there is a local coordinate $(U; x^1, \ldots, x^q, y^1, \ldots, y^r)$ at p such that $X = \frac{\partial}{\partial x^1}$ on U. Thus by the usual argument (cf. Lemma 2.1), we can easily prove that $Y_0 \in \mathfrak{m}$.

(ii) We shall prove that $Y_i \in \mathfrak{m}$. Take a point p in the isolated singular leaf L_i and a vector field X with $X(p) \neq 0$. By Lemma 1.11, there is a local coordinate $(U; x^1, \ldots, x^q, y^1, \ldots, y^r)$ at p such that $X = \frac{\partial}{\partial y^1} + \sum_{i=1}^{q} f_j(x^1, \ldots, x^q) \frac{\partial}{\partial x^j}$ with $f_j(0, \ldots, 0) = 0$ $(j = 1, 2, \ldots, q)$ on U. With respect to this coordinate, Y_i is expressed as $Y_i = \sum_{j=1}^{r} g_j(x^1, \ldots, x^q, y^1, \ldots, y^r) \frac{\partial}{\partial y^j}$ on U. We consider the following system of differential equations on U:

$$\frac{\partial h_j}{\partial y^1} + \sum_{k=1}^{q} f_k \frac{\partial h_j}{\partial x^k} = g_j \qquad (j = 1, 2, \ldots, r).$$

These can be solved on some neighborhood $V(\subset U)$ for given g_j $(j = 1, 2, \ldots, r)$. If we set $Z = \sum_{j=1}^{r} h_j \frac{\partial}{\partial y^j}$ then $[X, Z] = Y_i$ on V. Performing the same argument for local coordinates at other points in L_i, we can prove that $Y_i \in \mathfrak{m}$.

Next, we prove that $\mathfrak{m} = \mathfrak{x}(M, \mathscr{F})$. From $d\pi(\mathfrak{m}) = \mathfrak{x}(M/\mathscr{F})$, for any $X \in \mathfrak{x}(M, \mathscr{F})$, there exists a vector field $Z \in \mathfrak{m}$ such that $d\pi(X) = d\pi(Z)$. Therefore $X - Z \in \mathfrak{x}_{\mathscr{L}}(M)$. Since $\mathfrak{x}_{\mathscr{L}}(M) \subset \mathfrak{m}$ and $Z \in \mathfrak{m}$, we have that $X \in \mathfrak{m}$, hence $\mathfrak{m} = \mathfrak{x}(M, \mathscr{F})$. This is a contradiction. This completes the proof.

From Lemmas 3.1 and 3.2, we have the following.

Theorem 3.3. *Any maximal ideal of $\mathfrak{x}(M, \mathscr{F})$ should be equal to one of the following ideals;*

(i) $\mathfrak{m}_{\bar{p}}=d\pi^{-1}\mathscr{I}_{\bar{p}}$: *ideal corresponding to Lemma 3.1, where $\mathscr{I}_{\bar{p}}$ is the maximal ideal of $\mathfrak{x}(M/\mathscr{F})$ corresponding to a point $\bar{p}\in M/\mathscr{F}$.*

(ii) $\mathfrak{A}_p$: *ideal corresponding to Lemma 3.2, where p is a point of an isolated singular leaf.*

Remark 3.4. We can characterize the ideal $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$. This is given by the intersection of all maximal ideals of type (i).

Remark 3.5. Let $(U: x^1,\dots, x^q, y^1,\dots, y^r)$ be a local coordinate at p in an isolated singular leaf. Then by the similar way as in the proof of Lemma 2.3, we can prove that any element of $\mathfrak{A}_p$ as above vanishes at p with all of its derivatives with respect to $y^1,\dots, y^r$.

Proposition 3.6. *Let $(M, \mathscr{F})$ and $(M', \mathscr{F}')$ be C^∞ manifolds with compact Hausdorff foliation. If Φ: $\mathfrak{x}(M, \mathscr{F})\to\mathfrak{x}(M', \mathscr{F}')$ is a Lie algebra isomorphism, then $\Phi(m_{\bar{p}})=m_{\bar{p}'}$ and $\Phi(\mathfrak{A}_p)=\mathfrak{A}_{p'}$, where $\bar{p}$, $\bar{p}'$ are points of $M/\mathscr{F}$, $M'/\mathscr{F}'$ respectively and p, p' are points of isolated singular leaves of $\mathscr{F}$, $\mathscr{F}'$ respectively.*

Proof. We consider the following exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{x}_{\mathscr{L}}(M) & \longrightarrow & \mathfrak{x}(M, \mathscr{F}) & \longrightarrow & \mathfrak{x}(M/\mathscr{F}) & \longrightarrow & 0 \\ & & \cup & & \cup\!\!\!\!\!\! \shortmid\,\shortmid & & \| & & \\ 0 & \longrightarrow & \mathfrak{x}_{\mathscr{L}}(M)\cap\mathfrak{A}_p & \longrightarrow & \mathfrak{A}_p & \longrightarrow & d\pi(\mathfrak{A}_p) & \longrightarrow & 0. \end{array}$$

Then there is a Lie algebra homomorphism

$$\iota\colon \mathfrak{x}_{\mathscr{L}}(M)/\mathfrak{x}_{\mathscr{L}}(M)\cap\mathfrak{A}_p \longrightarrow \mathfrak{x}(M, \mathscr{F})/\mathfrak{A}_p.$$

Clearly ι is injective. Furthermore, since $\mathfrak{x}(M/\mathscr{F})=d\pi(\mathfrak{A}_p)$ we can prove that ι is surjective. Hence $\mathfrak{x}(M, \mathscr{F})/\mathfrak{A}_p\cong\mathfrak{x}_{\mathscr{L}}(M)/\mathfrak{x}_{\mathscr{L}}(M)\cap\mathfrak{A}_p$. Let $(U; x^1,\dots, x^q, y^1,\dots, y^r)$ be a local coordinate at p. Then the formal Taylar expansion of $X\in\mathfrak{x}_{\mathscr{L}}(M)$ at p with respect to $y^1,\dots, y^r$ is a homomorphism of $\mathfrak{x}_{\mathscr{L}}(M)$ onto the product of the rings of formal power series and its kernel is exactly $\mathfrak{x}_{\mathscr{L}}(M)\cap\mathfrak{A}_p$. Therefore

$$\mathfrak{x}(M, \mathscr{F})/\mathfrak{A}_p\cong\mathfrak{x}_{\mathscr{L}}(M)/\mathfrak{x}_{\mathscr{L}}(M)\cap\mathfrak{A}_p$$

$$\cong \overbrace{C^\infty(x^1,\dots, x^q)[[y^1,\dots, y^r]]\times\cdots\times C^\infty(x^1,\dots, x^q)[[y^1,\dots, y^r]]}^{r},$$

where $C^\infty(x^1,\dots, x^q)$ is the ring of the germs of C^∞-functions at p and the algebraic structure on the right hand is induced from the algebraic structure on the left hand. On the other hand, we can easily prove that

$$\mathfrak{x}(M, \mathscr{F})/\mathfrak{m}_{\bar{p}}\cong\mathfrak{x}(M/\mathscr{F})/\overline{m}_{\bar{p}}$$

$$\cong \begin{cases} \overbrace{R[[x^1,\dots,x^q]]\times\cdots\times R[[x^1,\dots,x^q]]}^{q} \\ \quad \text{if } p \text{ is contained in non-singular leaves,} \\ \overbrace{R[[x^1,\dots,x^s]]\times\cdots\times R[[x^1,\dots,x^s]]}^{s}\ (s<q) \\ \quad \text{if } p \text{ is contained in non-isolated singular leaves (see Lemma 2.14),} \\ gl_G(q, R)/g_\lambda \text{ (for some } \lambda) \\ \quad \text{if } p \text{ is contained in isolated singular leaves.} \end{cases}$$

Comparing their commutative Lie subalgebras, we have that any $\mathfrak{A}_p$, $\mathfrak{m}_{\bar{p}}$ are not isomorphic to $\mathfrak{m}'_{\bar{p}'}$, $\mathfrak{A}_{p'}$ respectively. This completes the proof.

Corollary 3.7. *Under the assumption of Proposition* 3.6, *we have* $\Phi(\mathfrak{X}_{\mathscr{L}}(M))=\mathfrak{X}_{\mathscr{L}}(M')$.

Proof. This is an immediate consequence of Remark 3.4 and Proposition 3.6.

§4. Proof of Theorem B

To begin with, we state the following theorem due to Amemiya [1].

Theorem 4.1. *Let* $(M, \mathscr{F})$ *and* $(M', \mathscr{F}')$ *be foliated manifolds which are not necessarily compact Hausdorff. If* $\Phi: \mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})\to\mathfrak{X}_{\mathscr{L}}(M', \mathscr{F}')$ *is a Lie algebra isomorphism, then there is a foliation preserving diffeomorphism* $\varphi: (M, \mathscr{F}) \longrightarrow (M', \mathscr{F}')$ *such that* $d\varphi=\Phi$.

Remark 4.2. Amemiya proved this theorem by characterizing maximal subalgebras in $\mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})$ of finite codimension. We can also prove this theorem by characterizing maximal ideals of $\mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})$.

Proof of Theorem B. From Corollary 3.7 and Theorem 4.1, there is a foliation preserving diffeomorphism $\varphi: (M, \mathscr{F})\to(M', \mathscr{F}')$ such that $d\varphi=\Phi$ on $\mathfrak{X}_{\mathscr{L}}(M, \mathscr{F})$. Therefore we prove that $d\varphi=\Phi$ on $\mathfrak{X}(M, \mathscr{F})$. Let S, S' be the sets of all singular leaves of $\mathscr{F}$, $\mathscr{F}'$ respectively. Then the map $\pi|_{M-S}$: $M-S\to M-S/\mathscr{F}$ is a projection of a fiber bundle with compact fiber. By the same argument as in [9, §X.7], we see that for any point $p\in M-S$, there are local coordinates$(U; x^1,\dots, x^q, y^1,\dots, y^r)$ at p and $(V; \bar{x}^1,\dots, \bar{x}^q, \bar{y}^1,\dots, \bar{y}^r)$ at $\varphi(p)=p'\in M'$ such that for any

$$X=\sum_{i=1}^{q} f_i(x)\frac{\partial}{\partial x^i}+\sum_{j=1}^{r} g_j(x, y)\frac{\partial}{\partial y^j}\in\mathfrak{X}(M, \mathscr{F}),$$

$$\Phi(X)=\sum_{i=1}^{q}(f_i\circ\varphi^{-1})\frac{\partial}{\partial \bar{x}^i}+\sum_{j=1}^{r}(g_j\circ\varphi^{-1})\frac{\partial}{\partial \bar{y}^j}\quad\text{on } V,$$

and moreover $\bar{x}^i\circ\varphi(x^1,\dots, x^q, y^1,\dots, y^r)=x^i$ for $i=1, 2,\dots, q$ and $\bar{y}^j\circ\varphi(x^1,\dots, x^q,y^1,\dots, y^r)=y^j$ for $j=1, 2,\dots, r$. Hence for any $X=\sum_{i=1}^{q} f_i\frac{\partial}{\partial x^i}+\sum_{j=1}^{r} g_j\frac{\partial}{\partial y^j}$ on U, we have $d\varphi(X)=\sum_{i=1}^{r}(f_i\circ\varphi^{-1})\frac{\partial}{\partial\bar{x}^i}+\sum_{j=1}^{r}(g_j\circ\varphi^{-1})\frac{\partial}{\partial\bar{y}^j}$ on V. Since p is an arbitrary point in $M-S$, for any $X \in \mathfrak{x}(M, \mathscr{F})$, $d\varphi(X)=\Phi(X)$ on $M'-S'$. From the continuity of vector fields, $d\varphi(X)(p')=\Phi(X)(p')$ for any $p' \in S'$. Hence $d\varphi=\Phi$. This completes the proof.

§5. Generalized Reeb foliations

In §5 and §6, we consider codimension one foliations.

Definition 5.1. A compact foliated manifold $(M, \mathscr{F})$ $(\partial M \neq \emptyset)$ is called a generalized Reeb component if the following three conditions are satisfied;
(1) all leaves in Int M are non-compact and proper,
(2) the holonomy groups of all leaves in Int M are trivial and
(3) each of the elements of the holonomy group of each compact leaf of $\mathscr{F}$ can be represented by a local diffeomorphism of $R_+=[0, \infty)$, leaving fixed 0, which is C^∞-tangent to identity at 0 and whose second derived function is non-negative or non-positive in some neighborhood of 0.

Definition 5.2. $\mathscr{F}$ is called a generalized Reeb foliation on a closed oriented manifold M if there is a decomposition of $(M, \mathscr{F})$ such that $(M, \mathscr{F})=\bigcup_{i=1}^{k}(M_i, \mathscr{F}_i)$, where each $(M_i, \mathscr{F}_i)$ denotes a generalized Reeb component.

Then applying Lemma 1.9 of [5] to transformations $\{\phi_t\}$ generated by any $X \in \mathfrak{x}(M, \mathscr{F})$, we have the following.

Proposition 5.3. *Let $(M, \mathscr{F})$ be a generalized Reeb foliated manifold with k generalized Reeb components. Then $\mathfrak{x}(M/\mathscr{F})$ is a k-dimensional trivial Lie algebra.*

Proposition 5.4. *Let $(M, \mathscr{F})$ be as above. Then any maximal ideal of $\mathfrak{x}(M, \mathscr{F})$ should be equal to one of the following ideals;*
(i) $\mathfrak{m}_\lambda=d\pi^{-1}(g_\lambda)$: *ideal with codimension one, where g_λ is a maximal ideal of $\mathfrak{x}(M/\mathscr{F})\cong R^k$.*
(ii) $\mathfrak{A}_p$: *ideal with infinite codimension and $d\pi(\mathfrak{A}_p)\cong R^k$, where p is a point of a compact leaf.*

Proof. The proof is similar to those of Lemmas 3.1 and 3.2 and omitted.

Theorem 5.5. *Let $(M, \mathscr{F})$ and $(M', \mathscr{F}')$ be generalized Reeb foliated manifolds. If Φ: $\mathfrak{x}(M, \mathscr{F})\to\mathfrak{x}(M', \mathscr{F}')$ is a Lie algebra isomorphism, then there is a foliation preserving diffeomorphism φ: $(M, \mathscr{F})\to(M', \mathscr{F}')$.*

Proof. From Proposition 5.4, the ideal $\mathfrak{x}_{\mathscr{S}}(M, \mathscr{F})$ is given by the intersection of all maximal ideals of type (i). Thus this isomorphism of $\mathfrak{x}(M, \mathscr{F})$ to $\mathfrak{x}(M', \mathscr{F}')$ induces an isomorphism of $\mathfrak{x}_{\mathscr{S}}(M, \mathscr{F})$ to $\mathfrak{x}_{\mathscr{S}}(M', \mathscr{F}')$. Hence by Theorem 4.1, we have the result.

§6. Foliations without holonomy

Proposition 6.1. (*see Theorem* 1.2 *and Proposition* 5.3 *of Imanishi* [7]). *Let* M *be a compact* C^{∞}*-manifold and* $\mathscr{F}$ *a transversely orientable codimension one foliation without holonomy of class* C^{∞}. *Then one of the followings occurs*:
(i) *the leaves of* $\mathscr{F}$ *are fibers of a fibration of* M *onto* S^1,
(ii) *all leaves of* $\mathscr{F}$ *are everywhere dense in* M.

Proposition 6.2. *Let* $(M, \mathscr{F})$ *be as above, of type* (ii). *Then* $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ *is of finite codimension in* $\mathfrak{x}(M, \mathscr{F})\leqq 1$, *hence* $\dim \mathfrak{x}(M/\mathscr{F})\leqq 1$.

Proof. Let $(U; x, y^1,\dots, y^{n-1})$ be a local coordinate of M for any fixed x, a local coordinate of the leaf is given by $y^1,\dots, y^{n-1}$. For such local coordinate, every $X\in\mathfrak{x}(M, \mathscr{F})$ is described as follows:

$$X=f(x)\frac{\partial}{\partial x}+\sum_{j=1}^{n-1} g_j(x, y)\frac{\partial}{\partial y^j}.$$

Let $\tau\mathscr{F}$ be the subbundle of $\tau M\to M$ determined by the foliation $\mathscr{F}$. $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ is the space of sections of $\tau\mathscr{F}\to M$. By the canonical projection of τM onto $\tau M/\tau\mathscr{F}$, $\mathfrak{x}(M, \mathscr{F})$ defines a subspace S of $\Gamma(\tau M/\tau\mathscr{F})$. Let i_L: $L\to M$ be an inclusion. We see that if $X\in i_L^*(S)$ is such that $X(p)=0$ for some $p\in L$, then there exists a neighborhood V of p in L such that $X|_V\equiv 0$. Since L is connected, $X\equiv 0$. This implies

$$\dim\{i_L^*(S)\}=\dim\{i_L^*(S)_p\}\leqq 1.$$

Since $\bar{L}=M$, X, $Y\in S$ are equal if and only if $i_L^*(X)=i_L^*(Y)$. This completes the proof.

Proposition 6.3. *Let* $(M, \mathscr{F})$ *be as above, of type* (ii). *If* $\dim\mathfrak{x}(M/\mathscr{F})=1$, *then* $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ *is a unique maximal ideal of* $\mathfrak{x}(M, \mathscr{F})$.

Proof. From the assumption, we have a following exact sequence: $0\to\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})\to\mathfrak{x}(M, \mathscr{F})\xrightarrow{d\pi}R\to 0$. Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{x}(M, \mathscr{F})$. Suppose that $d\pi(\mathfrak{m})=R$. Then we prove that $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})\subset\mathfrak{m}$. From the assumption of $d\pi(\mathfrak{m})=R$, there are a local coordinate $(U; x, y^1,\dots, y^{n-1})$ and a vector field $X\in\mathfrak{m}$ such that $X=\frac{\partial}{\partial x}$ on U. Then by the usual argument (cf. Lemma 2.1), we can prove that for any $Y\in\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$, there exists a vector field $Z\in\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ such that $[X, Z]=Y$ on U. Hence $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})\subset\mathfrak{m}$.

Next we prove that $\mathfrak{m}=\mathfrak{x}(M, \mathscr{F})$. Since $d\pi(\mathfrak{m})=R$, for any $X\in\mathfrak{x}(M, \mathscr{F})$, there is a vector field $Y\in m$ such that $d\pi(X)=d\pi(Y)$. Therefore $X-Y\in\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$. Since $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})\subset\mathfrak{m}$ and $Y\in m$, we have $X\in\mathfrak{m}$, hence $\mathfrak{m}=\mathfrak{x}(M, \mathscr{F})$. This is a contradiction. This completes the proof.

Theorem 6.4. *Let* $\mathscr{F}$ *and* $\mathscr{F}'$ *be transversely orientable foliations without holonomy on closed manifolds* M *and* M' *respectively. If* Φ: $\mathfrak{x}(M, \mathscr{F})\to\mathfrak{x}(M', \mathscr{F}')$ *is a Lie algebra isomorphism, then there is a foliation preserving diffeomorphism*

$$\varphi\colon (M, \mathscr{F}) \longrightarrow (M', \mathscr{F}').$$

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{x}(M, \mathscr{F})$. If $\mathscr{F}$ is a foliation of type (i) in Proposition 6.1, then $\mathfrak{m}$ is of infinite codimension in $\mathfrak{x}(M, \mathscr{F})$ and $\mathfrak{x}(M, \mathscr{F})/\mathfrak{m} \cong R[[x]]$ (see Proposition 3.6). If $\mathscr{F}$ is a foliation of type (ii) in Proposition 6.1 and $\dim \mathfrak{x}(M/\mathscr{F})=1$, $\mathfrak{m}$ is equal to $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ and is of finite codimension in $\mathfrak{x}(M, \mathscr{F})$. If $\mathscr{F}$ is a foliation of type (ii) in Proposition 6.1 and $\dim \mathfrak{x}(M/\mathscr{F})=0$, then $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})=\mathfrak{x}(M, \mathscr{F})$, thus $\mathfrak{m}$ is a maximal ideal of $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ and $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})/\mathfrak{m} \cong C^{\infty}(x)[[y^1,\dots, y^{n-1}]]$ (see Proposition 3.6). Therefore if $\mathscr{F}$ is a foliation of type (i), $\mathscr{F}'$ must be of type (i). In this case, Omori [9] proved this theorem. If $\mathscr{F}$ is a foliation of type (ii) and $\dim \mathfrak{x}(M/\mathscr{F})=1$, $\mathscr{F}'$ must be of type (ii) and $\dim \mathfrak{x}(M'/\mathscr{F}')=1$. Hence $\mathfrak{x}_{\mathscr{L}}(M', \mathscr{F}')$ is also a unique maximal ideal of $\mathfrak{x}(M', \mathscr{F}')$, which is isomorphic to $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$. By Theorem 4.1, there exists a foliation preserving diffeomorphism of $(M, \mathscr{F})$ to $(M', \mathscr{F}')$. If $\mathscr{F}$ is a foliation of type (ii) and $\dim \mathfrak{x}(M/\mathscr{F})=0$, $\mathscr{F}'$ must be of type (ii) and $\dim \mathfrak{x}(M'/\mathscr{F}')=0$. Hence $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ is isomorphic to $\mathfrak{x}_{\mathscr{L}}(M', \mathscr{F}')$. This completes the proof.

DEPARTMENT OF MATHEMATICS,
KYOTO SANGYO UNIVERSITY

References

[1] I. Amemiya, Lie algebra of vector fields and complex structure, Jour. Math. Soc. Japan, **27** (1975), 545–549.
[2] R. Edwards, K. Millett and D. Sullivan, Foliations with all leaves compact, Topology, **16** (1977), 13–32.
[3] D. B. A. Epstein, Periodic flows on three manifolds, Ann. of Math., **95**–2 (1972), 66–82.
[4] D. B. A. Epstein, Foliations with all leaves compact, Ann. Inst. Fourier, Grenoble, **26**–1 (1976), 265–282.
[5] K. Fukui, On the homotopy type of some subgroups of Diff (M^3), Japan. J. Math., **2**–2 (1976), 249–267.
[6] K. Fukui, Pursell-Shanks type theorem for free G-manifolds, Publ. R. I. M. S. Kyoto Univ., **17**–1 (1981), 249–265.
[7] H. Imanishi, On the throrem of Denjoy-Sacksteder for codimension one foliations without holonomy, Jour. Math. Kyoto Univ., **14**–3 (1974), 607–634.
[8] A. Koriyama, On Lie algebra of vector fields with invariant submanifolds, Nagoya Math. Jour., **55** (1974), 91–110.
[9] H. Omori, Infinite dimensional Lie transformation group, Lecture Notes in Math., **427** (1976), Springer-Verlag.
[10] L. E. Pursell and M. E. Shanks, The Lie algebra of a smooth manifold, Proc. Amer. Math. Soc., **5** (1954), 468–472.
[11] S. Sternberg, Local contractions and a theorem of Poincaré, Amer. Jour. Math., **79** (1957), 809–824.
[12] D. Sullivan, A counterexample to the periodic orbit conjecture, Publ. Math. I. H. E. S., **46** (1976), 5–14.