J. Math. Kyoto Univ. (JMKYAZ)
22-4 (1983) 619-642

On local solvability of some non-kowalewskian partial differential operators

By

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(Received July 17, 1981)

1. Introduction

We are concerned with local solvability of the partial differential operators. The notion of local solvability in the distribution's sense was introduced by L. Hörmander. Let Ω be a domain of $\boldsymbol{R}^n$ and P be a partial differential operator with smooth coefficients in Ω.

Definition 1. We say that P is locally solvable at the point $x \in \Omega$ if and only if there exists a neighborhood U of x such that for every $f \in C_0^\infty(U)$, there exists $u \in \mathscr{D}'(U)$ which satisfies $Pu=f$ in $\mathscr{D}'(U)$.

Let I be a interval $[-T, T]$, $D_t = \frac{1}{i}\frac{\partial}{\partial t}$, and $D_x^\alpha = \frac{1}{i}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, where $\alpha = (\alpha_1,\dots,\alpha_n) \in N^n$, and $N=(0, 1, 2,\dots)$. In this paper we shall consider the local solvability of the operator

$$(1) \qquad L = D_t + P(x, t, D_x) \qquad (x, t) \in \Omega \times I$$

, where $P(x, t, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x, t) D_x^\alpha$, and $a_\alpha(x, t) \in C^\infty(\Omega \times I)$. When $m=1$, local solvability of L is almost completely decided. (L. Nirenberg and F. Treves [17]). So we consider the case $m \geqslant 2$. In this case, L becomes non-kowalewskian operator. In non-degenerate case, hypoellipticity of parabolic system has been proved by S. Mizohata. In degenerate case, hypoellipticity and well-posedness for Cauchy problem is considered by many people. Some of their works give us some information for L to be locally solvable. But we have little knowledge of necessary condition for L to be locally solvable. For example, Y. Kannai has showed that

$$L_1 = D_t + itD_x^2$$

is hypoelliptic but not locally solvable at the origin, and R. Rubinstein has showed that

$$L_2 = D_t - it^n D_x^2 + it^m D_x \qquad (n;\ \text{even})$$

is not locally solvable at the origin if $\frac{2}{n+1}<\frac{1}{m+1}$.

The purpose of this paper is to seek necessary or sufficient conditions for L to be locally solvable. As a corollary we can show that the sufficient condition of theorem 2 in [7] is also necessary condition for hypoellipticity without an additional assumption that the coefficients of the operator depend only on t. Moreover, we can decide completely local solvability of the operator

$$L_3=D_t+at^lD_x^m+bt^kD_x^n \qquad (m>n,\ x\in R, \text{ and } a,\ b\in C)$$

if $a,\ b\in iR$.

The outline of this paper is as follows. In §2, we shall state the main results. In §3～§6, we shall give their proofs. In §7, we shall give some results about an influence of real part of $P(x, t, D_x)$ on the solvability of L. In §8, we shall investigate the solvability of L_3 as example. In the last section, we shall give some remarks on semi-local solvability.

2. Statement of the results

Let L be an operator given by

$$L=D_t+\sum_{j=1}^{m} a_j(x, t, D_x)$$

, where $a_j(x, t, D_x)=\sum_{|\alpha|=j} a_\alpha(x, t)D_x^\alpha$. We assume that Ω contains the origin and

$$a_j(x, t, D_x)=t^{k_j}\mathring{a}_j(x, t, D_x)+it^{l_j}\mathring{b}_j(x, t, D_x)$$

, where $\mathring{a}_j(x, t, D_x)=\sum_{|\alpha|=j}\mathring{a}_\alpha(x, t)D_x^\alpha$ and $\mathring{b}_j(x, t)D_x=\sum_{|\alpha|=j}\mathring{b}_\alpha(x, t)D_x^\alpha$. Here $\mathring{a}_\alpha(x, t)$ and $\mathring{b}_\alpha(x, t)$ are real-valued smooth functions in $\overline{\Omega\times I}$ such that either $\mathring{a}_\alpha(x, 0)$ $[\mathring{b}_\alpha(x, 0)]$ is not identically zero in any neighborhood of the origin or $\mathring{a}_\alpha(x, t)\,[\mathring{b}_\alpha(x, t)]$ is identically zero in $\Omega\times I$.

Now we introduce an important quantity which is effective when we treat a degenerate operator whether it is kowalewskian or non-kowalewskian operator. (See [1], [12], [18],...) Let us define

$$\nu=\max_{j\in\sigma}\frac{j}{l_j+1}$$

, where $\sigma=\{j;{}^{\exists}\alpha$ such that $|\alpha|=j,\ \mathring{b}_\alpha(x, 0)$ is not identically zero$\}$. We also define σ' by

$$\sigma'=\{j;{}^{\exists}\alpha \text{ such that } |\alpha|=j,\ \mathring{a}_\alpha(x, 0) \text{ is not identically zero}\}.$$

Let $\sigma_0=\left\{j\in\sigma,\ \frac{j}{l_j+1}=\nu\right\}$, and $j_0=\max_{j\in\sigma_0} j$. Then we have

Theorem 1. *Suppose that* $\max_{j\in\sigma'}\frac{j}{k_j+1}\leqslant\nu$. *If* l_{j_0} *is odd and there exists*

$\eta_0 \in S^{n-1}$ such that $\mathring{b}_{j_0}(0, 0, \eta_0)<0$, then L^* is not locally solvable at the origin. (Here L^* is a formal adjoint of L.)

Corollary 1. *Under the same assumption as theorem 1, L is not hypoelliptic.*

This corollary follows from the fact that if $L$is hypoelliptic in $\tilde{\Omega}$, then L^* is locally solvable at every point of $\tilde{\Omega}$. (See [24].)

Remark 1. By this corolllary, we can remove the additional assumption that the coefficients depend only on t of the necessary part of theorem 2 in [7].

On the other hand, if $m \in \sigma_0$ and $\mathring{b}_m(0, 0, \eta)\neq 0$, then the sufficient part is also obtained. In fact, we have

Theorem 2. *Suppose that $\max\limits_{j\in\sigma'} \frac{j}{k_j+1} \leqslant \nu$, $m \in \sigma_0$, and $\mathring{b}_m(0, 0, \xi)\neq 0$ if $\xi \neq 0$. Then, L is locally solvable at the origin if one of the following two conditions holds;*

a) *l_m is even,*

b) *l_m is odd and for every $\eta \in S^{n-1}$, $\mathring{b}_m(0, 0, \eta)<0$.*

Remark 2. For the case that m is even, this theorem is obtained as corollary of the theorems for hypoellipticity of L. (See [7], [8], [11],...)

If we drop the hypothesis $\mathring{b}_m(0, 0, \xi)\neq 0$ in theorem 2, some conditions ensure the hypoellipticity of L. (See [2], [10]) But, in this paper we do not enter in this direction. Instead of it, we look at the hypothesis $m \in \sigma_0$ in theorem 2. If this assumption is dropped, the situation becomes more complicated.

Hereafter we assume that $m \notin \sigma_0$. First we note that theorem 1 contains also some result in this case but does not cover completely it. To give light on this case, we must introduce another scale ρ instead of ν. Let

$$m' = \max_{j\in\sigma_0} j, \quad \text{and} \quad \rho = \max_{\substack{j\in\sigma\setminus\sigma_0 \\ j>m'}} \frac{j-m'}{l_j - l_{m'}} \ (>0).$$

We denote the set $\left\{j \in \sigma\setminus\sigma_0, j>m'\,;\ \rho = \frac{j-m'}{l_j-l_{m'}}\right\} \cup \{m'\}$ by $\tilde{\sigma}$. Let

$$H_0'(x, t, \xi) = i \sum_{j\in\tilde{\sigma}} t^{l_j} \mathring{b}_j(x, 0, \xi).$$

Then $H_0'(x, t, \xi)=|\xi|^{\rho+d} H_0'(x, \tau, \eta)$, where $\tau=|\xi|^\rho$, $\eta=\xi/|\xi| \in S^{n-1}$ and $d=-\rho(l_j+1)+j$ (>0) for $^\forall j \in \tilde{\sigma}$. If $H_0'(x, \tau, \eta)\neq 0$ for any $(x, \tau, \eta)\in \Omega\times(R\setminus 0)\times S^{n-1}$, then the situation is essentially same as the case that $m \in \sigma_0$. But if $H_0'(x, \tau, \eta)$ has a null point τ_0 which differs from 0, then the situation is quite different from the case $m \in \sigma_0$.

For simplicity we assume that for every j, $\mathring{a}_j(x, t, \xi)$ and $\mathring{b}_j(x, t, \xi)$ are independent of t. Then $H_0'(x, \tau, \eta)$ becomes polinomials in τ with smooth coefficients.

Let $D=\{-\rho(k_j+1)+j,\ -\rho(l_j+1)+j \quad \text{for} \quad j\in\sigma', \sigma, \text{respectively}\}$

Then D consists of a finite number of elements which we denote by d_l ($l=0, 1, 2,\ldots, q$). Especially, we denote d by d_0. Here we remark that if for every $j\in\sigma'$,

$-l(k_j+1)+j<d$, then for every $l=1, 2,\dots, q$, we have $d_l<d$. Let

$$H_l(x, t, \xi)=\sum_{j\in\sigma'_l} t^{k_j}\mathring{a}_j(x, \xi)+i\sum_{j\in\sigma_l} t^{l_j}b_j(x, \xi)$$

, where $\sigma_l=\{j\in\sigma;\ -\rho(l_j+1)+j=d_l\}$, and $\sigma'_l=\{j\in\sigma';\ -\rho(k_j+1)+j=d_l\}$. Then by definition we have

$$H_l(x, t, \xi)=|\xi|^{\rho+d_l}H_l(x, \tau, \eta)$$

, where $\tau=t|\xi|^\rho$, and $\eta=\xi/|\xi|\in S^{n-1}$.
Here we assume that for each l,

(A–1) $H_l(x, \tau, \eta)=\{\tau-\tau_0(x, \eta)\}^{m_l}g_l(x, \tau, \eta),\ g_l(x, \tau, \eta)\neq 0$ if $(x, \tau, \eta)\in\Omega\times J\times V$

, where $J=\{\tau\in R;\ |\tau-\tau_0(x, \eta)|\leqslant\delta<1)\}$, V is an open subset of S^{n-1}, $g_l(x, \tau, \eta)\in C^\infty(\overline{\Omega\times J\times V})$, and $\tau_0(x, \eta)\in C^\infty(\overline{\Omega\times V})$. Then we have

Theorem 3. *Under the assumption* (A–1), *if m_0 is odd,* $\operatorname{Im} g(x, \tau, \eta)<0$ *for* $(x, \tau, \eta)\in\Omega\times J\times V$, *and for* $l=1, 2,\dots,q$,

$$\frac{d_l}{m_l+1}<\frac{d_0}{m_0+1}<\frac{1}{2},$$

then L^* is not locally solvable at the origin. Moreover, if $\tau_0(x, \eta)$ does not depend on x, then we can replace the condition $\frac{d_0}{m_0+1}<\frac{1}{2}$ *by* $\frac{d_0}{m_0+1}<1$, *and if $H_l(x, \eta)$ does not depend on both x and η, then we can drop the condition* $\frac{d_0}{m_0+1}<\frac{1}{2}$.

Remark 3. If $\tau_0=0$, then this theorem has intersection with theorem 1.

Next, we consider the sufficient condition. We assume that for each l,

(A–2) $$H_l(x, \tau, \eta)=\prod_{j=1}^{r}(\tau-\tau_j(x, \eta))^{m_j^l}h_l(x, \tau, \eta)$$

, where $\tau_j(x, \eta)$and $h_l(x, \tau, \eta)$ are smooth in $\Omega\times R\times S^{n-1}$,
$\tau_1(x, \eta)<\tau_2(x, \eta)<\cdots<\tau_r(x, \eta)$ and $\operatorname{Im} h_0(x, \tau, \eta)\neq 0$,
$h_l(x, \tau, \eta)\neq 0$ if $(x, \tau, \eta)\in\Omega\times R\times S^{n-1}$.
Then we have

Theorem 4. *Under the assumption* (A–2) *and that for j,* $-\rho(k_j+1)+j<d_0$, $\frac{d_l}{m_j^l+1}<\frac{d_0}{m_j^0+1}<\frac{1}{2}$ *for every l and j, and* $\min_{j\in\tilde\sigma} j>\max_{j\in\sigma'\cup\sigma\setminus\tilde\sigma} j$, *if either* i) *for every j, m_j^0 is even or* ii) *for some j_0, $m_{j_0}^0$ is odd, for $j\neq j_0$, m_j^0 is even and* $\operatorname{Im} h_0(x, \tau, \eta)<0$, *then L is locally solvable at the origin. Moreover, if for any j, $\tau_j(x, \eta)$ does not depend on x, then we can replace the assumption* $\frac{d_0}{m_j^0+1}<\frac{1}{2}$ *by* $\frac{d_0}{m_j^0+1}<1$, *and if for any j, $H_j(x, \eta)$ is constant, then we can drop the assumption* $\frac{d_0}{m_j^0+1}<\frac{1}{2}$.

Remark 4. If $r=1$ and $\tau_1(x, \eta)=0$, then this theorem has intersection with theorem A in [14].

3. Proof of theorem 1.

First we state a fundamental lemma which is given in [3].

Lemma 3.1. *If L is locally solvable at the origin, then there exists a neighborhood of the origin U such that for some constants C and positive integers M, N*

$$\left|\int f(x, t)\bar{v}(x, t)dxdt\right| \leqslant C|f|_M|L^*v|_N \tag{3.1}$$

for all $f, v \in C_0^\infty(U)$. *Here* $|u|_M = \sum_{|\alpha|+j<M} \sup |D_x^\alpha D_t^j u(x, t)|$.

We shall prove theorem 1 by contradiction. Namely, under the assumption of the theorem we shall construct functions $f, v \in C_0^\infty(U)$ which never satisfy (3.1) for any U. Before doing so, we begin with some definitions. Let $W \subset \Omega \times I \times (R^n \backslash 0)$ be a open conic set.

Definition 3.2. $f(x, t, \xi) \in C^\infty(W)$ belongs to $S_{\rho,\delta}^{M,\nu}(W)$ if and only if for any j, α, β,

$$|D_x^\alpha D_t^j D_\xi^\beta f(x, t, \xi)| \leqslant C_{\alpha,\beta,j}|\xi|^{M+j\nu-\rho|\beta|+\delta|\alpha|}$$

Definition 3.3. For $u(x, t, \xi) \in S_{\rho,\delta}^{M,\nu}(w)$ and $u_j(x, t, \xi) \in S_{\rho,\delta}^{M_j,\nu}(w)$

$$u(x, t, \xi) \sim \sum_{j=1}^{\infty} u_j(x, t, \xi) \qquad \text{if and only if}$$

$$u(x, t, \xi) - \sum_{j=0}^{N} u_j(x, t, \xi) \in S_{\rho,\delta}^{M_{N+1},\nu}(W) \qquad \text{for any } N \in \boldsymbol{N}.$$

Lemma 3.4. *Suppose that* $0 \leqslant \rho < \delta \leqslant 1$. *Let* $\{M_j\}_{j=0}^\infty \in R$ *be a sequence such that* $M_j \to -\infty$ *as* $j \to \infty$. *If* $u_j(x, t, \xi) \in S_{\rho,\delta}^{M_j,\nu}(W)$, *then there exists* $u(x, t, \xi) \in S_{\rho,\delta}^{M_0,\nu}(W)$ *such that*

$$u(x, t, \xi) \sim \sum_{j=0}^{\infty} u_j(x, t, \xi).$$

This lemma is proved by a standard method as the symbol class of the pseudo-differential operator. So we omit its proof.

First we want to seek an approximate null solution $u(x, t, \xi)$ of the equation

$$L[u(x, t, \xi)e^{ix\xi}]=0 \tag{3.2}$$

in the form $u(x, t, \xi) \sim \sum_{j=0}^{\infty} u_j(x, t, \xi)$, where u_j belongs to $S_{\rho,\delta}^{M_j,\nu}(W)$ ($M_j \to -\infty$ as $j \to \infty$.)

If $f(x, t, \xi) \in C^\infty(\Omega \times I \times (R \backslash 0))$, we have

$$P(fe^{ix\xi}) = e^{ix\xi} \sum_{|\alpha|<m} \frac{1}{\alpha!} P^{(\alpha)}(x, t, \xi) D_x^\alpha f(x, t, \xi) \tag{3.3}$$

, where $P^{(\alpha)}(x, t, \xi)=\partial_\xi^\alpha P(x, t, \xi)=\partial_\xi^\alpha\{\sum_{|\alpha|<m} a_\alpha(x, t)\xi^\nu\}$. In (3.3), we expand $P^{(\alpha)}(x, t, \xi))$ as Taylor series with respect to t. Then we have

$$e^{-ix\xi} L(fe^{ix\xi})=\{(D_t+L_0)+L_1+\cdots+L_j+\cdots\}f$$

, where $L_j(x, t, \xi, D_x)$ whose coefficients are polinomials in t with smooth coefficients satisfies

$$L_j(x, t, \xi, D_x)=|\xi|^{m_j}L_j(x, s, \eta, D_x)(m_0>m_1>\cdots>m_r>\cdots).$$

Here we recall that $s=t|\xi|^\nu$, $\eta=\xi/|\xi|\in S^{n-1}$. Especially, $m_0=\nu$, and

$$L_0(x, t, \xi)=A_0(x, t, \xi)+iB_0(x, t, \xi)$$

$$A_0(x, t, \xi)=\sum_{j\in\sigma_0'} t^{k_j}\mathring{a}_j(x, 0, \xi)$$

$$B_0(x, t, \xi)=\sum_{j\in\sigma_0} t^{l_j}\mathring{b}_j(x, 0, \xi)$$

, where $\sigma_0'=\left\{j;\ \frac{j}{k_j+1}=\nu\right\}$, and $\sigma_0=\left\{j;\ \frac{j}{l_j+1}=\nu\right\}$. In s-variable,

$$D_t+L_0(x, t, \xi)=|\xi|^\nu\{D_s+L_0(x, s, \eta)\}.$$

Let us define $u_0(x, t, \xi)$ by

$$(D_t+L_0)u_0=0,\ u_0(x, 0, \xi)=0;\text{ i.e.,}$$

$$u_0(x, t, \xi)=\exp\left[-i\int_0^s L_0(x, s', \eta)ds'\right]\Big|_{s=t|\xi|^\nu}.$$

Then by virtue of the hypothesis of theorem 1, we have for any N, $j\in\boldsymbol{N}$ and α, $\beta\in\boldsymbol{N}^n$, there exists constant $C_{N,j,\alpha,\beta}$ such that

(3.4) $$|(1+|t|\xi|^\nu|)^N D_t^j D_x^\alpha D_\xi^\beta u_0(x, t, \xi_0)|\leqslant C_{N,j,\alpha,\beta}|\xi|^{-|\beta|+j\nu}$$

for $(x, t)\in\Omega\times I$ and $|\xi|\geqslant 1$

, where $\xi_0=\eta_0|\xi|$, and η_0 is given in the condition of theorem 1. Here we have used the fact that s^Ne^{-s} is bounded if $s>0$. ($^\forall N>0$). We remark that from (3.4) it follows that $u_0(x, t, \xi)\in S^{0}_{1,0}{}^{;\nu}_{;0}(W)$, where $W=\{(x, t, \xi);\ (x, t)\in\Omega\times I,\ |\xi/|\xi|-\eta_0|\leqslant\varepsilon$ (ε is sufficiently small), $|\xi|\geqslant 1\}$.

As for $j\geqslant 1$, we define u_j by inductively

$$(D_t+L_0)u_j=-(L_1u_{j-1}+\cdots+L_ju_0)$$

$$u_j(x, 0, \xi)=0\ ;\text{i.e.,}$$

$$u_j(x, t, \xi)=-|\xi|^{-\nu}\left[\int_0^s iu_0(x, s|\xi|^{-\nu}, \eta)u_0^{-1}(x, s'|\xi|^{-\nu}, \eta)\right.$$

$$\times\{|\xi|^{m_1}L_1(x, s', \eta)u_{j-1}(x, s'|\xi|^{-\nu}, \eta)+\cdots$$

$$\left.+|\xi|^{m_j}L_j(x, s', \eta)u_0(x, s'|\xi|^{-\nu}, \eta)\}ds'\right]\Big|_{s=t|\xi|^\nu}.$$

Since $\delta=\min_j (m_j-m_{j+1})>0$, by induction we can show that

(3.5) $$|(1+|t|^{\nu}|)^N D_t^j D_x^{\alpha} D_{\xi}^{\beta} u_j(x, t, \xi)| \leqslant C_{N,j,\alpha,\beta}|\xi|^{-j\delta-|\beta|+j'\nu} \quad \text{if } (x, t, \xi)\in W.$$

Therefore we have $u_j(x, t, \xi)\in S_{1,0}^{-\delta j,\nu}(W)$. By definition,

(3.6) $$L[\sum_{j=0}^{N} u_j]=L_1 u_N+L_2(u_{N-1}+u_N)+\cdots+L_N(\sum_{j=1}^{N} u_j)+(L-\sum_{j=0}^{N} L_j)(\sum_{j=0}^{N} u_j).$$

Since $m_j=m_j-m_0+\nu\leqslant -j\delta+\nu$, the first N-terms of the right hand side of (3.6) belongs to $S_{1,0}^{-(N+1)\delta+\nu,\nu}(W)$. On the other hand,

$$L-\sum_{j=0}^{N} L_j=\sum_{|\alpha|<m} c_{\alpha}(x, t, \xi)D_x^{\alpha}$$

$$|D_t^j D_x^{\alpha} D_{\xi}^{\beta} c_{\alpha}(x, t, \xi)|\leqslant C_{j,\alpha,\beta}|\xi|^{-(N+1)\delta+j\nu-|\beta|}$$

$$\text{if } (x, t, \xi)\in \Omega\times I\times\{|\xi|\geqslant 1\}.$$

Therefore the last term of the right hand side of (3.6) also belongs to $S_{1,0}^{-(N+1)\delta+\nu,\nu}(W)$. In conclusion, we have $L[\sum_{j=0}^{N} u_j]\in S_{1,0}^{-(N+1)\delta+\nu,\nu}(W)$.

By lemma 3.4, there exists $u(x, t, \xi)\in S_{1,0}^{0,\nu}(W)$ such that

$$u(x, t, \xi)\sim \sum_{j=0}^{\infty} u_j(x, t, \xi).$$

Then $Lu\in \bigcap_{r=0}^{\infty} S_{1,0}^{r,\nu}(W)=S^{-\infty}(W)$, where $S^{-\infty}(W)=\{f(x, t, \xi)\in C^{\infty}(W)$; for any j, N, α, $|D_t^j D_{x,\xi}^{\alpha} f(x, t, \xi)|\leqslant C_{j,\alpha,N}|\xi|^{-N}$ if $(x, t, \xi)\in W\}$. In fact, we have

$$Lu=L[\sum_{j=0}^{N} u_j]+L[u-\sum_{j=0}^{N} u_j].$$

The former term belongs to $S_{1,0}^{-(N+1)\delta+\nu,\nu}$ as we mentioned before. The latter term also belongs to $S_{1,0}^{-(N+1)\delta+\nu,\nu}$ since $u-\sum_{j=0}^{N} u_j\in S_{1,0}^{-(N+1)\delta,\nu}(W)$.

Now let us define $v(x, t)$ and $f(x, t)$ by

$$f(x, t)=F(|\xi|^2 x, |\xi|^{1+\nu}t)$$

$$v(x, t)=\chi(x, t)\int e^{ix\tau\xi}u(x, t, \tau\xi)g(\tau)d\tau$$

, where $F(y, s)\in C_0^{\infty}(R^{n+1})$ is determined later, $\chi(x, t)\in C_0^{\infty}(\Omega\times I)$ with sufficiently small compact support has value 1 identically in some neighborhood of the origin and $g(\tau)\in C_0^{\infty}(R^+)$. Then

(3.7) $$\begin{aligned} Lv=&\chi(x, t)\int e^{ix\tau\xi}u_{\infty}(x, t, \tau\xi)g(\tau)d\tau \\ &+D_t\chi(x, t)\int e^{ix\tau\xi}u(x, t, \tau\xi)g(\tau)d\tau \\ &+\tilde{\chi}(x, t)Q(x, t, D_x)\int e^{ix\tau\xi}u(x, t, \tau\xi)g(\tau)d\tau \end{aligned}$$

, where $u_\infty(x, t, \xi)\in S^{-\infty}(W)$, $\tilde{\chi}(x, t)=0$ if $|x|<\varepsilon$ (ε is sufficiently small, and $Q(x, t, D_x$ is a partial differential operator with degree at most m − 1 with smooth coefficient. Let us denote each term in the right hand side of (3.7) by I_1, I_2, I_3, respectively. Then it is obvious that for any N there exists a constant C such that

$$|I_j|_M\leqslant C|\xi|^{-N} \qquad (j=1, 2) \qquad \text{if} \quad \xi=\eta_0|\xi|,\ |\xi|\geqslant 1.$$

In fact for $j=1$, this follows from the fact $Lu\in S^{-\infty}(W)$, and for $j=2$, this follows from the fact that for sufficiently small t, $D_t\chi(x, t)=0$ and (3.4), (3.5). Let us look at I_3. We integrate by part to obtain

$$I_3=\tilde{\chi}(x, t)\int e^{ix\tau\xi}\sum_{|\alpha|\leqslant m-1} Q^{(\alpha)}(x, t, \tau\xi)D_x^\alpha u(x, t, \tau\xi)g(\tau)d\tau$$

$$=\tilde{\chi}(x, t)\int \frac{1}{(ix\tau\xi)}{}^N e^{ix\tau\xi}(-1)^N\partial_\tau^N\sum_{|\alpha|\leqslant m-1}(Q^{(\alpha)}D_x^\alpha u)(x, t, \tau\xi)g(\tau)d\tau.$$

Then we have

$$|I_3|_M\leqslant C_{M,N}|\xi|^{M+m-1-N} \quad \text{if} \quad \xi=\eta_0|\xi|,\ |\xi|\geqslant 1.$$

since $u\in S^{0,\nu}_{1,0}(W)$. In conclusion, we have

$$|Lv|_M\leqslant C_{M,N}|\xi|^{-N} \qquad \text{for any} \quad N \quad \text{if} \quad \xi=\eta_0|\xi|,\ |\xi|\geqslant 1. \tag{3.8}$$

On the other hand,

$$\int f(x, t)\bar{v}(x, t)dxdt$$

$$=|\xi|^{-2n-1-\nu}\int F(y, s')\bar{v}(|\xi|^{-2}y, |\xi|^{-1-\nu}s')dyds',$$

$$v(|\xi|^{-2}y, |\xi|^{-1-\nu}s')=\chi(|\xi|^{-2}y, |\xi|^{-1-\nu}s')$$

$$\times\int e^{-iy\cdot\tau\xi/|\xi|^2}u(|\xi|^{-2}y, |\xi|^{-1-\nu}s', \xi)g(\tau)d\tau.$$

Then, on the support of $F(y, s')$ and $g(\tau)$, $u_j(|\xi|^{-2}y, |\xi|^{-1-\nu}s', \xi)$ uniformly tends to 1 if $j=0$ and to 0 if $j\geqslant 1$, respectively. Therefore if we choose $F(y, s')$ and $g(\tau)$ such that

$$\int F(y, s')dyds'\neq 0 \qquad \text{and} \quad \int g(\tau)d\tau\neq 0,$$

then for some positive constant C,

$$\left|\int f(x, t)\bar{v}(x, t)dxdt\right|\geqslant C|\xi|^{2n+1+\nu}$$

if $\xi=\eta_0|\xi|$ and $|\xi|$ is sufficiently large. This is in contradiction with (3.1) since (3.8) and

$$|f|_M\leqslant C|\xi|^{(3+\nu)M}.$$

This completes the proof of theorem 1.

4. Proof of theorem 3.

In theorem 1, the most dominant terms in essentially one. But in theorem 3, the most dominant terms are more than or equal to 2. In other words, the operator L has terms which have interaction with each other to lead L to no-solvability, in this case. Therefore a slight different treatment is necessary for the proof of theorem 3.

First we have

Lemma 4.1. *For any $j \in \sigma \backslash \tilde{\sigma}$,*

$$d_j = -\rho(l_j+1)+j < d = -\rho(l_k+1)+k$$

, *where $^\forall k \in \tilde{\sigma}$.*

Proof. Let $j \in \sigma \backslash \tilde{\sigma}$. Then we have three possibility;

a) $j \in \sigma \backslash \sigma_0$ and $j < m'$,

b) $j \in \sigma \backslash (\sigma_0 \cup \tilde{\sigma})$, $j > m'$, and $\dfrac{j-m'}{l_j - l_{m'}} < \rho$,

c) $j \in \sigma_0$, and $j < m'$.

At first we note that $\rho < \nu$. In fact,

$$\nu - \rho = \frac{m'}{l_{m'}+1} - \frac{k-m'}{l_k - l_{m'}} = \frac{m'(l_k+1)-k(l_{m'}+1)}{(l_{m'}+1)(l_k - l_{m'})} > 0$$

since $l_k - l_{m'} > 0$ if $k \in \tilde{\sigma}$.

When a) holds,

$$d - d_j = \left(\frac{m'-j}{l_{m'}-l_j} - \rho\right)(l_{m'}-l_j) > 0.$$

For, if $l_{m'} < l_j$, $\dfrac{m'-j}{l_{m'}-l_j} < 0$, and if $l_{m'} > l_j$,

$$\frac{m'-j}{l_{m'}-l_j} - \nu = \frac{m'(l_j+1)-j(l_{m'}+1)}{(l_{m'}-l_j)(l_{m'}+1)} > 0.$$

When b) holds,

$$d - d_j = \left(\frac{j-m'}{l_j - l_{m'}} - \rho\right)(l_j - l_{m'}) > 0 \qquad \text{since} \quad l_j > l_{m'}.$$

When c) holds,

$$d - d_j = (-\rho+\nu)(l_{m'}-l_j) > 0 \qquad \text{since} \quad l_{m'} - l_j > 0. \qquad \text{Q.E.D.}$$

In theorem 3, we shall also construct an approximate null solution $u(x, t, \xi)$ of the equation $L[ue^{ix\cdot\xi}]=0$. In §3, we localized $u(x, t, \xi)$ in (x, t) sapce, but in this section we shall localize it in (x, τ) space, where $\tau = t|\xi|^\rho$ because of $d>0$.

First we recall that

$$L = D_t + \sum_{l=0}^{q} H_l(x, t, D_x)$$

$$H_l(x, t, \xi) = |\xi|^{d_l+\rho} H_l(x, \tau, \eta).$$

Taking account of (3.3), we define $u_0(x, t, \xi)$ by

$$\{D_t + H_0(0, t, \xi)\} u_0(x, t, \xi) = 0$$

$$u_0(x, \tau_0(0, \eta)|\xi|^{-\rho}, \xi) = e^{-(|x|\cdot|\xi|^{\delta})^2}.$$

Namely,

$$u_0(x, t, \xi) = \exp\left[-i\int_{\tau_0(0,\eta)}^{\tau} H_0(0, \tau', \eta)|\xi|^d d\tau' - (|x|\cdot|\xi|^{\delta})^2\right]\Big|_{\tau=t|\xi|^{\rho}}$$

, where δ is a positive number determined later. Hereafter, we denote $\tau_0(0, \eta)$ by τ_0 for simplicity.

Let

$$W = \{(x, t, \xi) \in \Omega \times I \times (R^n \backslash O);\ |t|\xi|^{\rho} - \tau_0| < c_0,\ \xi/|\xi| \in V \text{ and } |\xi| \geqslant 1\}$$

where c_0 is sufficiently small number such that $|\tau - \tau_0| \leqslant c_0$ is contained in J.

Then we have

Proposition 4.2. *If $(x, t, \xi) \in W$, then*

(4.3) $$|D_x^{\alpha} D_{\xi}^{\beta} D_t^j u_0(x, t, \xi)| \leqslant C_{\alpha,\beta,j} |\xi|^{\delta|\alpha| - (1-\frac{d}{k+1})|\beta| + (\rho + \frac{d}{k+1})j}$$

for all α, β, j,

where $k = m_0$.

Proof. By the hypothesis of theorem 3.

$$\mathrm{Im}\int_{\tau_0(0,\eta)}^{\tau} H_0(0, \tau', \eta) d\tau' = \mathrm{Im}\int_{\tau_0}^{\tau} (\tau' - \tau_0)^k g(0, \tau', \eta) d\tau'$$

$$< -\frac{c}{k+1}(\tau - \tau_0)^{k+1} (c > 0).$$

If $(x, \tau, \eta) \in \Omega \times J \times V$, then for $\tau = t|\xi|^{\rho}, \eta = \xi/|\xi|$

$$\left| D_x^{\alpha} D_{\xi}^{\beta} D_{\tau}^{j} \left[\int_{\tau_0}^{\tau} H_0(x, \tau', \eta) d\tau'\right]\right| \leqslant c'(\tau - \tau_0)^{(k+1-|\alpha|-|\beta|-j)_+} \cdot |\xi|^{-|\beta|}$$

, where $(l)_+ = \max(l, 0)$. It is easily seen that these two inequalities and Leipniz' rule yield to (4.3) since $X^N e^X$ is bounded if $X > 0$ for any $N > 0$. Q. E. D.

Let us define $u_j(x, t, \xi)$ by

(4.4) $$\{D_t + H_0(0, t, \xi)\} u_j(x, t, \xi) = -\sum_{j'=1}^{j} P_{j'}(x, t, \xi, D_x) u_{j-j'}(x, t, \xi),$$

$u_j(x, \tau_0|\xi|^{-\rho}, \xi)=0$

, where we denote by $P_{j'}(x, t, \xi, D_x)$

$$P_{j'}(x, t, \xi, D_x)=\sum_{|\alpha|=j'-1}\frac{1}{\alpha!}\{\sum_{l=1}^{r} H_l^{(\alpha)}(x, t, \xi)D_x^\alpha+\tilde{H}^{(\alpha)}(x, t, \xi)D_x^\alpha\}$$

Here $\tilde{H}(x, t, \xi)=H_0(x, t, \xi)-H_0(0, t, \xi)$. The solution of (4.4) is given by

$$u_j(x, t, \xi)=-i|\xi|^{-\rho}\int_{\tau_0}^{\tau} u_0(x, \tau|\xi|^{-\rho}, \xi)u_0^{-1}(x, \tau'|\xi|^{-\rho}, \xi) \times(-\sum_{j'=1}^{j} P_{j'}u_{j-j'})(x, \tau'|\xi|^{-\rho}, \xi)d\tau'|_{\tau=t|\xi|^\rho}. \tag{4.5}$$

Then we have

Proposition 4.3. *Let $\delta=\dfrac{d}{k+1}+\delta'$. If $(x, t, \xi)\in W$ and δ' is sufficiently small positive number, then there exist $\varepsilon>0$ such that*

$$|D_x^\alpha D_\xi^\beta D_t^l u_j(x, t, \xi)|\leqslant C_{\alpha,\beta,l,j}|\xi|^{-j\varepsilon+\delta|\alpha|-(1-\delta)|\beta|+(\rho+\frac{d}{k+1})l}. \tag{4.6}$$

Proof. By (A-1), if we expand $\tau_0(x, \eta)$ as Taylor series with respect to x, we have

$$\tilde{H}(x, \tau, \eta)=\sum_{j=0}^{k}(\tau-\tau_0(0, \eta))^{k-j}h_j(x, \tau, \eta)$$

$$H_l(x, \tau, \eta)=\sum_{j=0}^{m}(\tau-\tau_0(0, \eta))^{m_l-j}h_j^l(x, \tau, \eta)$$

, where h_j and h_j^l are smooth in $\Omega\times J\times V$ and satisfy

$$h_0=O(|x|),\ h_j=O(|x|^j) \qquad (j=1,\dots, k)$$

$$h_j^l=O(|x|^j) \qquad (l=1,\dots, q, \text{ and } j=1,\dots, m)$$

Let $\varepsilon_1=\max_l\left(\dfrac{d}{k+1}-\dfrac{d}{m_l+1}, \delta'\right)>0$. Then if $(x, t, \xi)\in W$,

$$\left|D_x^\alpha D_\xi^\beta D_\tau^j\left[\int_{\tau_0}^{\tau}\tilde{H}(x, \tau', \eta)|\xi|^d\, d\tau'\cdot u_0(x, \tau, \eta)\right]\right| \leqslant C_{\alpha,\beta,j}|\xi|^{-\varepsilon_1+\delta|\alpha|-(1-\delta)|\beta|+\frac{d}{k+1}j+\rho},$$

$$\left|D_x^\alpha D_\xi^\beta D_\tau^j\left[\int_{\tau_0}^{\tau}H_l(x, \tau', \eta)|\xi|^d d\tau'\cdot u_0(x, \tau, \eta)\right]\right| \leqslant C_{\alpha,\beta,j,l}|\xi|^{-\varepsilon_1+\delta|\alpha|-(1-\delta)|\beta|+\frac{d}{k+1}j+\rho}$$

, where $\tau=t|\xi|^\rho$ and $\eta=\xi/|\xi|$. Here we have used the fact that X^Ne^{-X} is bounded if $X>0$ for any $N>0$.

Let $\varepsilon_2=1-2\delta$. Then if δ' is sufficiently small positive number, $\varepsilon_2>0$. When $|\alpha|>0$, we also have the following estimates.

$$\left|D_x^\gamma D_\xi^\beta D_\tau^l\left[\int_{\tau_0}^{\tau} u_0(x,\tau,\eta)u_0^{-1}(x,\tau',\eta)H^{(\alpha)}(x,\tau',\eta)|\xi|^{d-|\alpha|}D_x^\alpha u_0(x,\tau',\eta)d\tau'\right]\right|$$

$$\leqslant C|\xi|^{-\varepsilon_2|\alpha|+\delta|\gamma|-(1-\delta)|\beta|+\frac{d}{k+1}l+\rho},$$

$$\left|D_x^\gamma D_\xi^\beta D_\tau^j\left[\int_{\tau_0}^{\tau} u_0(x,\tau,\eta)u_0^{-1}(x,\tau',\eta)H_l^{(\alpha)}(x,\tau',\xi)|\xi|^{d'-|\alpha|}D_x^\alpha u_0(x,\tau',\eta)d\tau'\right]\right|$$

$$\leqslant C|\xi|^{-\varepsilon_1-\varepsilon_2|\alpha|+\delta|\gamma|-(1-\delta)|\beta|+\frac{d}{k+1}j+\rho}.$$

Therefore, let $\varepsilon=\min(\varepsilon_1, \varepsilon_2)$, then (4.6) follows from these estimates and Leipniz' rule. Q.E.D.

This proposition means that $u_j(x, t, \xi)\in S^{-j,(\rho+\frac{d}{k+1})}_{1-\delta,\delta}(W)$. Therefore by Lemma 3.4, there eixsts $u(x, t, \xi)\in S^{0,(\rho+\frac{d}{k+1})}_{1-\delta,\delta}(W)$ such that

$$u(x, t, \xi)\sim\sum_{j=0}^{\infty}u_j(x, t, \xi).$$

By the same way as §3, we have $Lu\in S^{-\infty}(W)$.

Now let us define $v(x, t)$, $f(x, t)$ by

$$f(x, t)=F(|\xi|x, |\xi|^{1+\rho+d}(t-|\xi|^{-\rho}\tau_0))$$

$$v(x, t)=\chi_1(x)\chi_2(|\xi|^\rho t-\tau_0)e^{ix\xi}u(x, t, \xi)$$

, where $F(y, s)\in C_0^\infty(R^{n+1})$ is determined later, $\chi_1(x)\in C_0^\infty(\Omega')$ equals to 1 identically in some neighborhood of the origin, where Ω' is any small neighborhood of the origin in Ω, and $\chi_2(s)\in C_0^\infty(R)$ has a support contained in $\left\{|s|<\frac{1}{2}c\right\}$ and equals to 1 identically in a neighborhood of o; $\left\{|s|<\frac{1}{4}c\right\}$. Here we recall that c is a constant which appears in the definition of W. We note that if $|\xi|\to\infty$, the support of $\chi_2(|\xi|^\rho t-\tau_0))$ is contained in any neighborhood of the origin.

$$\begin{aligned}Lv=&\chi_1(x)\chi_2(|\xi|^\rho t-\tau_0)e^{ix\xi}u_\infty(x, t, \xi)\\&+\chi_1(x)D_t\chi_2(|\xi|^\rho t-\tau_0)e^{ix\xi}u(x, t, \xi)\\&+\tilde{\chi}_1(x)\chi_2(|\xi|^\rho t-\tau_0)Q(x, t, D_x)e^{ix\xi}u(x, t, \xi)\end{aligned}$$

, where $u_\infty(x, t, \xi)\in S^{-\infty}(W)$, $\tilde{\chi}_1(x)=0$ if x is sufficiently small, and $Q(x, t, D_x)$ is a partial differential operator with degree at most $m-1$ with smooth coefficients. On the support of $D_t\chi_2(|\xi|^\rho t-\tau_0)$ or of $\tilde{\chi}_1(x)$, by definition

$$|D_x^\alpha D_t^j u_0(x, t, \xi)|\leqslant C_{\alpha,j,N}|\xi|^{-N}\qquad\text{for any }\alpha, j\text{ and }N>0.$$

Therefore, we have

$$(4.7)\qquad |Lv|_M<C_{M,N}|\xi|^{-N}\qquad\text{for any}\quad N>0\text{ if }\xi/|\xi|\in V\text{ and }|\xi|\to\infty.$$

On the other hand,

$$\int f(x, t)\bar{v}(x, t)dxdt$$

$$=|\xi|^{-n-1-\rho-d}\int F(y, s)\bar{v}(|\xi|^{-1}y, |\xi|^{-\rho}(|\xi|^{-1-d}s+\tau_0))dyds,$$

$$v(|\xi|^{-1}y, |\xi|^{-\rho}(|\xi|^{-1-d}s+\tau_0))$$

$$=\chi_1(|\xi|^{-1}y)\chi_2(|\xi|^{-1-d}s)e^{iy\eta}u(|\xi|^{-1}y, |\xi|^{-\rho}(|\xi|^{-1-d}s+\tau_0), \xi)\,.$$

Since $|\tau-\tau_0|<c|\xi|^{-1-d}$ if $(y, s, \eta)\in \operatorname{supp} F(y, s)\times V$, $u_0(|\xi|^{-1}y, |\xi|^{-\rho}(|\xi|^{-1-d}s+\tau_0), \xi)$ tends to 1 uniformly as $|\xi|\to\infty$, $\eta\in V$ on the support of F, and $u_j(|\xi|^{-1}y, |\xi|^{-\rho}(|\xi|^{-1-d}s+\tau_0), \xi)$ tends to 0 uniformly as $|\xi|\to\infty$, $\eta\in V$ on $\operatorname{supp} F$ $(j\geqslant 1)$. Therefore if we choose F such that

$$\int F(y, s)e^{-iy\eta}dyds \neq 0, \text{ then}$$

$$\left|\int f(x, t)\bar{v}(x, t)dxdt\right| \geqslant C|\xi|^{-n-1-\rho-d} \quad (C>0)$$

for sufficiently large $|\xi|$ such that $\eta=\xi/|\xi|\in V$. This is in contradiction with (3.1) since $|f|_M\leqslant C_M|\xi|^{(1+\rho+d)M}$ and (4.7).

This completes the proof of the first part of theorem 3. If $\tau_0(x, \eta)$ is independent of x, then let $\delta=\delta'$ such that $\varepsilon_3=1-\dfrac{d}{k+1}-\delta'>0$. Then proposition 4.3 is valid for $\varepsilon=\min(\varepsilon_1, \varepsilon_3)$. If $H_l(x, \eta)$ is independent of both x and η, then let $\delta=\delta'$ such that $\varepsilon_4=1-\delta'>0$. Then proposition 4.3 is also valid for $\varepsilon=\min(\varepsilon_1, \varepsilon_4)$. In the above both case, the subsequent reasoning is also valid. So we have finished the proof of theorem 3.

5. The proof of theorem 2

As we have mentioned before, the essential part of theorem 2 is obtained from the results of hypoellipticity of L^* ([7]), [8], [11]). But to make this paper self-contained as much as possible, we give its proof. (See also [23]).

By hypothesis, on each connected component of S^{n-1}, $\mathring{b}_m(0, 0, \xi)$ has the same sign. So let us define $T(\xi)\in C^\infty(R^n\backslash 0)$ by

$$\begin{aligned} T(\xi)&=-T && \text{if a) is valid and } \mathring{b}_m(0, 0, \xi)<0,\\ &=+T && \text{if a) is valid and } \mathring{b}_m(0, 0, \xi)>0,\\ &=0 && \text{if b) is valid.}\end{aligned} \tag{5.1}$$

We define a sequence $K_j(x, \xi, t, s)$ by

$$K_0(x, \xi, t, s)=\exp\left[-i\int_s^t P(x, \tau, \xi)d\tau\right], \tag{5.2}$$

$$K_j(x, \xi, t, s)=-i\int_s^t K_0(x, \xi, t, \tau)\sum_{k=1}^{j}\sum_{|\alpha|=k}\frac{1}{\alpha!}P^{(\alpha)}(x, \tau, \xi)D_x^\alpha K_{j-k}(x, \xi, \tau, s)d\tau.$$

, where $P(x, t, \xi)=\sum_{j<m} a_j(x, t, \xi)$.

Lemma 5.1. *If the diameters of Ω and I are sufficiently small, we have*

$$\left|\int_s^t \tau^{l_m}\tilde{b}_m(x, \tau, \xi)d\tau\right| \geqslant C|t^{l_m+1}-s^{l_m+1}|\,|\xi|^m \qquad (C>0) \quad \text{if } (x, t, s)\in\Omega\times I\times I.$$

Lemma 5.2. ([12]) *If* $\dfrac{M}{k+1} \geqslant \dfrac{N}{l+1}$ *and* $M\geqslant N$, *then*

$$|t-s| \max(|t|^l, |s|^l)|\xi|^N \leqslant C\{|t^{k+1}-s^{k+1}|\,|\xi|^M\}^{(N/M)} \qquad \textit{for} \quad |t|, |s|\leqslant 1.$$

Lemma 5.3. *Let $k(t, s)$ be a measurable function defined on a measurable set E such that*

$$\left|\int_E k(t, s)dt\right|, \left|\int_E k(t, s)ds\right| \leqslant C.$$

Then the operator K which is defined by

$$Kf=\int_E k(t, s)f(s)ds$$

is a bounded operator on $L^2(E)$ with norm $\leqslant C$.

Hereafter we assume that Ω and I are sufficiently small such that lemma 5.1 holds.

Let us define the operator $K_j(x, \xi)$ by

$$K_j(x, \xi)f=\int_{T(\xi)}^t k_j(x, \xi, t, s)f(s)ds \qquad \text{for} \quad f\in C_0^\infty(I).$$

Then

Proposition 5.4. *For each (x, ξ), $K_j(x, \xi)$ is a bounded opeator on $L^2(I)$ such that*

$$(5.3) \quad \|D_x^\alpha D_\xi^\beta K_j(x, \xi)\| \leqslant C_{\alpha,\beta}|\xi|^{-j-|\beta|} \qquad \textit{for} \quad (x, \xi)\in\Omega\times(R^n\backslash 0)\cap\{|\xi|\geqslant 1\}.$$

, where we denote the norm of the bounded operator on $L^2(I)$ by $\|\cdot\|$.

Proof. First by virtue of lemma 5.2, we have

$$\left|\int_s^t |\operatorname{Im} a_j(x, \tau, \xi)|d\tau\right| \leqslant cX^{(j/m)} \qquad j=1,\dots, m-1,$$

, where $X=|t^{l_m+1}-s^{l_m+1}|\,|\xi|^m$. Moreover,

$$\left|\int_s^t |D_x^\alpha D_\xi^\beta a_j(x, \tau, \xi)|d\tau\right| \leqslant cX^{(j/m)}|\xi|^{-|\beta|} \qquad j=1,\dots, m.$$

Then (5.1), lemma 5.1, and lemma 5.3 yield (5.3) since $X^N \exp(-C_0X+\sum_{j=0}^{m-1} c_jX^{\delta_j})$ $(0<\delta_j<1)$ is bounded for any $N>0$. $(C_0>0)$ Q.E.D.

By this proposition, $K_j(x, \xi)$ is a vector-valued pseudo-differential operator which has been used in [23], etc.....

Let us define the operator Q by

$$Qf = \frac{1}{(2\pi)^n} \psi_0(x) \int e^{ix\xi} \varphi(\xi) \sum_{j=0}^{N} K_j(x, \xi) \hat{f}(\xi, t) d\xi$$

$$\text{for} \quad f(x, t) \in C_0^\infty(\Omega; L^2(I))$$

, where N is sufficiently large positive integer, $\psi_0(x) \in C_0^\infty(\Omega)$ identically equal to 1 in some neighborhood Ω' of the origin, and $\varphi(\xi) \in C^\infty(R^n)$ vanish on $\{|\xi| < 1\}$.

Let $\psi_1(x) \in C_0^\infty(\Omega)$ identically equals to 1 on the support of $\psi_0(x)$. Then

$$L\psi_1 Q = \psi_0(Id + R + (\varphi(D_x) - Id)),$$

$$Rf = \frac{1}{(2\pi)^n} \int e^{ix\xi} \varphi(\xi) \sum_{k=1}^{N+1} \sum_{|\alpha| \leq k} \frac{1}{\alpha!} P^{(\alpha)}(x, \xi) D_x^\alpha K_{N+1-k}(x, \xi) \hat{f}(\xi, t) d\xi.$$

If N is sufficiently large, $R' = R + \varphi(D_x) - Id$ has a continuous kernel;

$$R'f = \int_{T(\xi)}^{t} k(x, y, t, s) f(y, s) ds dy$$

, where $k(x, y, t, s) \in C^0(\Omega \times \Omega \times I \times I)$. Therefore by virtue of lemma 5.3, if Ω and I are sufficiently small, $\psi_0 R'$ becomes a bounded operator on $L^2(\Omega \times I)$ with norm $\leqslant \frac{1}{2}$. So $(Id + \psi_0 R')^{-1}$ exists. Let us define $u(x, t) \in L^2(\Omega \times I)$ by

$$u(x, t) = \psi_1 Q (Id + \psi_0 R')^{-1} f \qquad \text{for} \quad f \in L^2(\Omega \times I).$$

Then we have

$$Lu = f \qquad \text{in} \quad \mathscr{D}'(\Omega' \times I). \qquad \text{Q.E.D.}$$

6. The proof of theorem 4

We assume that ii) hols. In the case i), we can prove theorem 4 essentially in the same way as ii). So we omit it.

Let us define $k_j(x, \xi, t, s)$ by

$$k_0(x, \xi, t, s) = \exp\left[-i \int_s^t H_0(x, s', \xi) ds'\right],$$

$$k_j(x, \xi, t, s) = -i \int_s^t k_0(x, \xi, t, s') \sum_{j'=0}^{j-1} \Big[\sum_{|\alpha| = j-1-j'} \frac{1}{\alpha!} \sum_{l=1}^{q} H_l^{(\alpha)}(x, s', \xi) D_x^\alpha$$

$$+ \sum_{|\beta| = j-j'} \frac{1}{\beta!} H_0^{(\beta)}(x, s', \xi) D_x^\beta \Big] k_{j'}(x, \xi, s', s) ds'.$$

Let us define also the operator $K_j(x, \xi)$ by

$$K_j(x, \xi) f = \int_{\tau_{j0}(x, \eta)|\xi|^{-\rho}}^{t} k_j(x, \xi, t, s) f(s) ds$$

for $f\in C_0^\infty(I)$.
Then we have

Proposition 6.1. *For each $(x, \xi)\in\Omega\times(R^n\backslash 0)$, $K_j(x, \xi)$ is a bounded operator on $L^2(I)$ such that for some $\varepsilon>0$ and $\frac{1}{2}>\delta>0$,*

$$\|D_x^\alpha D_\xi^\beta K_j(x, \xi)\|\leqslant C_{\alpha,\beta,j}|\xi|^{-\varepsilon j+\delta|\alpha|-(1-\delta)|\beta|} \quad \text{if } |\xi| \text{ is large.} \tag{6.1}$$

Proof. Let $\tau=t|\xi|^\rho$, and $\tau'=s|\xi|^\rho$. Then

$$(K_j(x, \xi)f)(\tau|\xi|^{-\rho})=|\xi|^{-\rho}\int_{\tau_{j0}(x,\eta)}^{\tau} k_j(x, \xi, \tau|\xi|^{-\rho}, \tau'|\xi|^{-\rho})f(\tau'|\xi|^{-\rho})d\tau',$$

$$k_0(x, \xi, \tau|\xi|^{-\rho}, \tau'|\xi|^{-\rho})=\exp\left[\int_{\tau'}^{\tau} -iH_0(x, \tau'', \eta)|\xi|^d d\tau''\right].$$

We denote $k_0(x, \xi, \tau|\xi|^{-\rho}, \tau'|\xi|^{-\rho})$ by $\tilde{k}_0(x, \xi, \tau, \tau')$ and $\min_j m_j$ by m_0. Then we obtain

$$|D_x^\alpha D_\xi^\beta \tilde{k}_0(x, \xi, \tau, \tau')|\leqslant C|\xi|^{\frac{d}{m_0+1}|\alpha|-(1-\frac{d}{m_0+1})|\beta|} \tag{6.2}$$

if $\tau>\tau'>\tau_{j_0}(x, \eta)$ or $\tau<\tau'<\tau_{j_0}(x, \eta)$.
To see this, we divided R_τ into $r+1$ intervals:

$$J_1=\{\tau\in R;\ \tau<\tau_1(x, \eta)\},$$

$$J_k=\{\tau\in R;\ \tau_{k-1}(x, \eta)<\tau<\tau_k(x, \eta)\} \qquad (k=2,\dots, r),$$

$$J_{r+1}=\{\tau\in R;\ \tau_r(x, \eta)<\tau\}.$$

Hereafter we always assume that $\tau<\tau'<\tau_{j_0}$ or $\tau_{j_0}<\tau'<\tau$. If $|\tau-\tau'|>c_0$ $(c_0>0)$, then by (A–2) and ii) we have

$$\mathrm{Re}\left\{-i\int_{\tau'}^{\tau} H_0(x, \tau'', \eta)d\tau''\right\}\leqslant -C_1.$$

If $\tau, \tau'\in J_k$ and $|\tau-\tau_l|+|\tau'-\tau_l|$ is sufficiently small, then we have

$$\mathrm{Re}\left\{-i\int_{\tau'}^{\tau} H_0(x, \tau'', \eta)d\tau''\right\}\leqslant -C_2\,\big||\tau'-\tau_l|^{m_l+1}-|\tau-\tau_l|^{m_l+1}\big|$$

for $l=k, k-1$.
If $\tau, \tau'\in J_1$ or J_{r+1}, $|\tau-\tau'|$ is sufficiently small, and $|\tau|$ is sufficiently large, then we have

$$\mathrm{Re}\left\{-i\int_{\tau'}^{\tau} H_0(x, \tau'', \eta)d\tau''\right\}\leqslant -C_3|\tau-\tau'|\,|\tau|^{\tilde{l}},$$

where $\tilde{l}=\max_{j\in\sigma} l_j$. In the above estimates, C_j $(j=1, 2, 3)$ is some positive number.

Now let us investigate each cases more in detial. If the first case holds, it is obvious that (6.2) is valid since for any $N>0$, $|\xi|^N\exp\left\{-i\int_{\tau'}^{\tau} H_0(x, \tau'', \eta)d\tau''|\xi|^d\right\}$

is bounded as $|\xi|\to\infty$. If the second case holds, since $|\tau|$ and $|\tau'|$ are bounded, we have

$$\left|D_x^\alpha D_\xi^\beta \int_{\tau'}^{\tau} H_0(x, \tau'', \eta)|\xi|^d d\tau''\right|$$

$$\leqslant C\int_{\tau'}^{\tau} |\tau''-\tau_l|^M d\tau'' |\xi|^{d-|\beta|}$$

$$\leqslant C|\tau-\tau'| \max(|\tau-\tau_l|^M, |\tau'-\tau_l|^M)|\xi|^{\frac{M+1}{m_l+1}} |\xi|^{\frac{d}{m_l+1}(|\alpha|+|\beta|)-|\beta|}$$

$$\leqslant C'\{||\tau'-\tau_l|^{m_l+1}-|\tau-\tau_l|^{m_l+1}||\xi|^d\}^{\frac{M+1}{m_l+1}} |\xi|^{\frac{d}{m_l+1}(|\alpha|+|\beta|)-|\beta|},$$

where $M=(m_l-|\alpha|-|\beta|)_+$ and $l=k$ or $k+1$. If this combine with above estimate, we have (6.2) since $X^N e^{-X}$ is bounded if $X>0$. If the last case holds,

$$\left|D_x^\alpha D_\xi^\beta \int_{\tau'}^{\tau} H_0(x, \tau'', \eta)\,|\xi|^d d\tau''\right| \leqslant C|\tau-\tau'|\,|\tau|^l|\xi|^{d-|\beta|}.$$

Therefore we have also (6.2).

For $j\geqslant 1$, we have

$$(6.3)\qquad |D_x^\alpha D_\xi^\beta \tilde{k}_j(x, \xi, \tau, \tau')|\leqslant C|\xi|^{-\varepsilon j+\frac{d}{m_0+1}|\alpha|-(1-\frac{d}{m_0+1})|\beta|}$$

, where $\tilde{k}_j(x, \xi, \tau, \tau')=|\xi|^{-\rho}k_j(x, \xi, \tau|\xi|^{-\rho}, \tau'|\xi|^{-\rho})$. In fact, since $\frac{d_l}{m_j^l+1} < \frac{d}{m_j+1} < \frac{1}{2}$, let us define ε by

$$\varepsilon=\min(\varepsilon_1, \varepsilon_2), \text{ where } \varepsilon_1=\min_{j,l}\left(d\frac{m_j^l+1}{m_j+1}-d_l\right) \text{ and } \varepsilon_2=1-\frac{2d}{m_0+1}.$$

Then since $|D_x^\alpha D_\xi^\beta H_l(x, \tau'', \eta)|\leqslant|\tau''-\tau_k|^{M_k^l}|\xi|^{d_l-|\beta|}$, in the second case,

$$|D_x^\alpha D_\xi^\beta \int_{\tau'}^{\tau} \tilde{k}_0(x, \xi, \tau, \tau'')|\xi|^{d_l-|\beta|}H_l^{(\gamma)}(x, \tau'', \eta)D_x^\gamma k_0(x, \xi, \tau'', \tau')d\tau''|$$

$$\leqslant C\{||\tau'-\tau_k|^{m_k+1}-|\tau-\tau_k|^{m_k+1}||\xi|^d\}^{\frac{\tilde{M}_k^l+1}{m_k+1}} |\xi|^{\frac{d}{m_k+1}(|\alpha|+|\beta|)-|\beta|}|\xi|^{-(|\gamma|+1)\varepsilon}$$

, where $M_k^l=(m_k^l-|\alpha|-|\beta|)_+$ and $\tilde{M}_k^l=(m_k^l-|\alpha|-|\beta|-|\gamma|)_+$. This estimate and similar estimate for $H_0^{(\gamma)}(x, \tau'', \eta)$ in place of $H_l^{(\gamma)}(x, \tau'', \eta)$ which has a change such that $|\xi|^{-|\gamma|\varepsilon}$ instead of $|\xi|^{-(|\gamma|+1)\varepsilon}$, combine with Leipniz'rule yield (6.3). In the first and third cases, by the same way as above, we have also (6.3).

If we return to the variable t, s, then (6.1) follows from (6.2), (6.3), and lemma 5.3. Q. E. D.

Remark 6.2. If for any j, $\tau_j(x, \eta)$ is independent of x, then it is easily seen that proposition 6.1 also holds for

$$\varepsilon=\min\left(1-\frac{d}{m_0+1}\ \varepsilon_1\right).$$

If for any j, $H_j(x, \eta)$ is independent of both x and η, then proposition 6.1 is valid for $\varepsilon=\varepsilon_1$ and $\delta=0$.

Let us define the operator Q by

$$Qf=\frac{1}{(2\pi)^n}\int e^{ix\xi}\varphi(\xi)\sum_{j=0}^{N}K_j(x, \xi)\hat{f}(\xi, t)d\xi$$
$$\text{for } f\in C_0^\infty(\Omega; L^2(I))$$

, where $\varphi(\xi)\in C^\infty(R^n)$ identically equals to 0 if $|\xi|\leqslant M$, to 1 if $|\xi|\geqslant 2M$ and N is large positive number determined later. Here M is a sufficiently large positive number such that

$$|\tau_{j_0}(x, \eta)|\,|\xi|^{-\rho}\leqslant\frac{1}{2}T \qquad \text{if} \quad |\xi|>M$$

, where T appear in the definition of $I=[-T, T]$.

Let $\varphi_0(x)\in C_0^\infty(\Omega)$ be a function which identically equals to 1 on the support of $\varphi_1(x)\in C_0^\infty(\Omega)$ which identically equals to 1 in some neighborhood of the origin $V\subset\Omega$. Then we have

$$L\varphi_0Q\varphi_1f=\varphi_1f+\varphi_0Rf+R'f,$$

$$Rf(x, t)=\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\varphi(\xi)\sum_{j=0}^{N}\sum_{l=1}^{q}\sum_{|\alpha|=j}^{m}\frac{1}{\alpha!}H_l^{(\alpha)}(x, t, \xi)D_x^\alpha K_{N-j}(\varphi_1\hat{f})(\xi, t)d\xi,$$

and

$$R'f(x, t)=\int_\Omega K(x, x', t)f(x', t)dx' \qquad \text{for} \quad f\in C_0^\infty(R^n\times I)$$

, where $K(x, x', t)\in C_0^\infty(R^n\times R^n\times I)$, and $\operatorname{supp}_{x,x'} K\subset\Omega\times\Omega$.

Let N be a positive number such that $m-\varepsilon N<-(n+1)$. Then by virtue of proposition 6.1, we have

$$\|\tilde{R}f\|_{L^2(I)}\leqslant C\int[|\varphi(\xi)|+1]\frac{1}{(1+|\xi|)^{n+1}}\|(\varphi_1f)^\wedge(\xi, t)\|_{L^2(I)}d\xi$$
$$\leqslant C'\|f(x, t)\|_{L^2(\Omega\times I)}, \text{ where } \tilde{R}=\varphi_0R+R'.$$

By this inequality, we have

$$\|\tilde{R}f\|_{L^2(\Omega\times I)}\leqslant C'(\text{volume of }\Omega)^{\frac{1}{2}}\|f(x, t)\|_{L^2(\Omega\times I)}\leqslant\frac{1}{2}\|f(x, t)\|_{L^2(\Omega\times I)}$$

if Ω is sufficiently small. Therefore $(Id-\varphi_0R-R')^{-1}$ exists and becomes a bounded operator on $L^2(\Omega\times I)$. Let

$$u=\varphi_0Q\varphi_1(Id-\varphi_0R-R')^{-1}f \qquad \text{for} \quad f\in L^2(\Omega\times I).$$

Then

$$u\in L^2(\Omega\times I), \qquad \text{and} \quad Lu=f \qquad \text{in} \quad \mathscr{D}'(V\times I).$$

This complete the proof of the first part of theorem 4. For the second part of theorem 4, by virtue of remark 6.2, the same reasoning as above is also valid.

Q.E.D.

Remark 6.3. If $H_l(x, t, \xi) \equiv 0$ $(l=1, 2, \ldots, q)$ and $H_0(x, t, \xi)$ is independent of x, it is seen that for any $f \in L^2(R^n \times I)$, there exists $u \in L^2(R^n \times I)$ such that $Lu=f$ since a perturbation terms do not exist.

7. On a perturbation of L by a operator $A(t, D_x)$ with real coefficients.

In this section we shall investigate more carefully the influence of the real part of $P(x, t, \xi)$ on the solvability property of L. We note that its influence on the regularity property is vital. (See [13]).

Let

$$A(t, D_x) = \sum_{|\alpha|<M} A_\alpha(t) D_x^\alpha$$

, where $A_\alpha(t)$ is real-valued function which belongs to $C_0^\infty(I)$. In this section we assume that the coefficients belong to $\mathscr{B}(R^n \times I)$ and have compact support with respect to t.

Let us consider the operator B which is given by

$$Bf = \frac{1}{(2\pi)^n} \int e^{ix\xi} \exp\left\{-i \int_0^t A(\tau, \xi) d\tau\right\} \hat{f}(\xi, t) d\xi \qquad \text{for } f \in C_0^\infty(R^n \times I).$$

Then

$$\{L + A(t, D_x)\} Bf = B\{L\} f. \tag{7.1}$$

Definition 7.1. L is solvable in $L^2(R^n \times I)$ if and only if for any $f \in L^2(R^n \times I)$ there exists $u \in L^2(R^n \times I)$ such that $Lu=f$.

Theorem 7.2. *L is solvable in $L^2(R^n \times I)$ if and only if $L + A(t, D_x)$ is solvable in $L^2(R^n \times I)$.*

Proof. By Lemma 5.3, B is a bounded operator on $L^2(R^n \times I)$. Let us define B^{-1} by

$$B^{-1} f = \frac{1}{(2\pi)^n} \int e^{ix\xi} \exp\left\{i \int_0^t A(\tau, \xi) d\tau\right\} \hat{f}(\xi, t) d\xi.$$

Then we have $B^{-1}B = BB^{-1}$. This equality and (7.1) yield the desired result.

Q.E.D.

Remark 7.3. In theorem 2, if the coefficients of L depend only on t and $\sigma' \cup \sigma \setminus \sigma_0 = \emptyset$, then L is solvable in $L^2(R^n \times I)$.

If the coefficients of L depend only on t, then the more precise result for necessary part than this theorem.

Theorem 7.4. *Suppose that the coefficients of L depend only on t and L satisfies the hypothesis of theorem* 1. *Let*

$$A(t, D_x) = \sum_{\alpha \in \sigma_A} t^{l_\alpha} \mathring{A}_\alpha(t) D_x^\alpha$$

, *where* $\mathring{A}_\alpha(t)$ *is real-valued smooth function in I with* $\mathring{A}_\alpha(0) \neq 0$. *If for every* $\alpha \in \sigma_A$

$$|\alpha| - \frac{l_\alpha + 1}{l_{j_0} + 1} j_0 < 1,$$

then $L^* + A(t, D_x)$ *is not locally solvable at the origin.*

Proof. Let

$$v(x, t) = \chi(x, t) \int e^{ix\tau\xi} u(t, \tau\xi) g(\tau) d\tau,$$

and

$$f(x, t) = F(|\xi|^2 x, |\xi|^{1+\nu} t)$$

, where $\chi(x, t)$, $g(\tau)$, and $F(y, s)$ are the same functions as §3, and

$$u(t, \xi) = \exp\left[\int_0^t |\xi|^\nu \{A(\tau, \xi) + P(\tau, \xi)\} d\tau\right].$$

Then taking account of

$$-\nu(l_\alpha + 1) + |\alpha| < 1$$

for any $\alpha \in \sigma_A$, it is easy to see that the same reasoning as §3 is valid. Q.E.D.

8. Example

In this section, to illustrate the result of the previous section we consider the operator given by

$$L = D_t + a t^k D_x^m + b t^l D_x^n \qquad \text{on} \quad R^2 \qquad (m > n \geqslant 1)$$

, where a, b, C and k, are non-negative integers. We then want to seek a necessary and sufficient condition on a, b, k, l, m and n for L to be locally solvable at the origin. To do so, let us investigate L in various cases.

1) $\dfrac{m}{k+1} \geqslant \dfrac{n}{l+1}$ and $a \notin R$.

In this case, by theorem 1 and 2, the necessary and sufficient condition for L to be locally solvable at the origin is

either k is even or

k is odd, m is even and $\mathrm{Im}(a) < 0$.

2) $\frac{m}{k+1} \geqslant \frac{n}{l+1}$, $a \in R$ and $b \notin R$.

In this case, by theorem 1, 2, 7.2 and remark 7.3, the necessary and sufficient condition for L to be solvable in $L^2(R \times I)$ is

either l is even or

l is odd, n is even and $\operatorname{Im}(b)<0$.

Moreover, if $\frac{m}{k+1} = \frac{n}{l+1}$, then this condition is also the necessary and sufficient condition for L to be locally solvable at the origin.

3) a and $b \in R$.

In this case, by theorem 7.2, L is locally solvable at the origin.

4) $\frac{m}{k+1} \leqslant \frac{n}{l+1}$, $a \notin R$, and $b \in R$.

In this case, the necessary and sufficient condition for L to be solvable in $L^2(R \times I)$ is the one which is obtained by replacing l, n and b by k, m and a, respectively in the condition in the case 2). Morevoer, if $\frac{m}{k+1} = \frac{n}{l+1}$, then this condition is also the necessary and sufficient condition for L to be locally solvable at the origin.

5) $\frac{m}{k+1} < \frac{n}{l+1}$, both a and $b \in iR$.

In this case, L is locally solvable at the origin if and only if the one of the following conditions holds.

i) l is odd, k is odd, m and n are even, and

$$\operatorname{Im}(a) \quad \text{and} \quad \operatorname{Im}(b)<0,$$

ii) l is even, k is odd, m is even, and

$$\operatorname{Im}(a)<0,$$

and

iii) l is even, k is even, $m+n$ is even, and

$$\operatorname{Im}(a)\operatorname{Im}(b)>0.$$

In fact, by theorem 3 and 4 (also theorem 1), the necessary and sufficient condition for L to be locally solvable at the origin is

for $\eta=\pm 1$, $\operatorname{Im}\left(\frac{a}{k+1} t^{k+1}\eta^m + \frac{b}{l+1} t^{l+1}\eta^n\right)$ does not any minimal value in R_t.

Here we remark that if the above function has a minimal value at $t=0$, then the non-solvability result follows from theorem 1 or 3.

6) $\frac{m}{k+1} < \frac{n}{l+1}$, $a \in R$ and $b \notin R$.

In this case, by theorem 1, 2, 7.2, and remark 7.3, the necessary and sufficient condition for L to be locally solvable at the origin is the same one as the case 2).

Remark 8.1. When $a \in R$ or $b \in R$, the above result is improved by theorem 7.4. Especially, if $n=1$, then we can replace 'solvable in $L^2(R \times I)$' by 'locally solvable at the origin' in the case 2) and 4).

9. Final remark

In this section we shall introduce a notion semi-local solvability in order to make a difference between theorem 1 and 3 more clear.

Let us first begin with a definition.

Definition 9.1. L is semi-locally solvable at the origin with respect to $t>0$ ($t<0$) if and only if there exists a neighborhood of the origin V such that for any $f\in C_0^\infty(V_+)(C_0^\infty(V_-))$ there exists $u\in\mathscr{D}'(V_+)(\mathscr{D}'(V_-))$ such that $Lu=f$, where $V_+=V\cap\{t>0\}$ ($V_-=V\cap\{t<0\}$).

Then we have the following theorems.

Theorem 8.1. *Suppose that L satisfies the conditions of theorem 1 and $m\in\sigma_0$. Then L is semi-locally solvable at the origin with respect to both $t>0$ and $t<0$.*

Thjorem 8.2. *Suppose that L satisfies the conditions of theorem 3 and $\tau_0(x, \eta)\neq 0$. Then L^* is not semi-locally solvable at the origin with respect to $(\tau_0/|\tau_0|)t>0$.*

The proofs of these theorem follows from the proofs of theorem 2 and 4 if we replace I by I_+ or $I_-(I_+=I\cap\{t>0\}, I_-=I\cap\{t<0\}.)$ Here we note that for semi-local solvability, the similar lemma as lemma 3.1 is also valid if we replace U by U_+ or U_-.

For theorem 4, we have a variant of this theorem.

Theorem 8.3. *Suppose that (A–2) holds in $\tau\geqslant 0$ ($\tau\leqslant 0$) instead of $\tau\in R$, every τ_j ($j=1,\dots, r$) is non-negative (non-positive) and the other all conditions of theorem 4 are satisfied. Then L is semi-locally solvable at the origin with respect to $t>0$ ($t<0$).*

Example 1. (Lewy's operator).

$$D_t+iD_x-2i(t+ix)D_y \qquad (\text{on} \quad R^3)$$

is not semi-locally solvable at the origin with respect to both $t>0$ and $t<0$.

Example 2 (Mizohata's operator).

$$D_t+it^kD_x \qquad (\text{on} \quad R^2)$$

is semi-locally solvable at the origin with respect to both $t>0$ and $t<0$.

Example 3.

$$D_t-it^nD_x^4+it^mD_x^2 \qquad \left(n;\ \text{even},\ m;\ \text{odd},\ \frac{4}{n+1}<\frac{2}{m+1}\right)(\text{on } R^2)$$

is semi-locally solvable at the origin with respect to $t>0$ but is not semi-locally solvable at the origin with respect to $t<0$.

Acknowledgement. The author wishes to express his sincere gratitude to Professor S. Mizohata for his advices and encouragement.

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