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A remark on the Hölder continuity of the solution for a certain elliptic equation with irregular coefficients

By

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§0. A solution for an elliptic equation, even it may satisfy the equation in a weak sense, is expected to have certain regularity properties in the interior of the domain in question. In this paper we treat the elliptic equation of the following form

$$-\triangle u+\sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j}+c(x)u=f(x)\,, \tag{0.1}$$

where $\triangle$ denotes the Laplace operator in R^n. We denote by $B(x, r)$ the ball in R^n with the center x and radius r, and $B_R=B(0, R)$ in abbreviation. Since we are interested in the interior regularity properties of the solution u, we may confine our considerations within a neighborhood B_{R_0} of the origin. We impose the following conditions on b_j, c and f.

c and f belong to $L^1(B_{R_0})$, b_j's belong to $L^2(B_{R_0})$ and there exist constants B, C, F and θ $(0<\theta\leqslant 1)$ such that the following inequalities hold

$$\begin{cases} \sum_{j=1}^{n}\int_{B(x,r)\cap B_{R_0}}|b_j(y)|^2\,dy\leqslant B^2r^{n-2+\theta}, & \int_{B(x,r)\cap B_{R_0}}|c(y)|\,dy\leqslant Cr^{n-2+\theta} \\ \int_{B(x,r)\cap B_{R_0}}|f(y)|dy\leqslant Fr^{n-2+\theta} & \text{for every ball } B(x, r) \text{ in } R^n. \end{cases} \tag{0.2}$$

We say that $u\in H^1(B_{R_0})$ (the L^2-Sobolev space of order 1) is a weak solution of (0, 1) in B_{R_0}, when u satisfies

$$\int_{B_{R_0}}\nabla u\cdot\nabla\varphi+\sum_{j=1}^{n} b_j\frac{\partial u}{\partial x_j}\varphi+cu\varphi dx=\int_{B_{R_0}}f\varphi dx$$

for all $\varphi\in C_0^1(B_{R_0})$. Since $b_j(\partial u/\partial x_j)$ and cu^2 are integrable under the conditions (see Lemma 1.2), the above definition makes sense.

Now we state our main result in this paper.

Theorem. *Under the condition* (0.2), *if a function* $u\in H^1(B_{R_0})$ *is a weak solution of* (0.1), *then* u *is equivalent to a Hölder continuous function with* *exponent* θ $(0<\theta\leqslant 1)$ *in the interior.*

We should mention some other conditions which guarantee the Hölder continuity of the solution. If $b=0$ and c and f satisfy the Stummel condition

$$(0.2)' \quad \sup_{x\in B_{R_0}}\int_{B_{R_0}}\frac{(|c|^2+|f|^2)}{|x-y|^{n-4+2\theta}}dy\leqslant L^2 \quad \text{for some constants } L \text{ and } \theta\ (0<\theta<1/2),$$

the theorem is contained in Lemma 5.1 of S. Agmon [1]. We can see easily that if c and f satisfy (0.2)′ for some θ, then they satisfy (0.2) with the same θ. If b belongs to $L^p(B_{R_0})$ and f to $L^{p/2}(B_{R_0})$ for some $p>n$, the theorem is essentially included in the works of C. B. Morrey and G. Stampacchia (see [2] and [3]). It can be easily seen that (0.2) holds with $\theta=2(1-n/p)$ in this case. When $n=2$, the theorem is included in §5.4 of C. B. Morrey [2]. So we may confine ourselves to the case $n\geqslant 3$ in the following considerations.

We remark that the considerations in this paper carry over without any essential changes to the system of equations of the following form,

$$-\Delta\vec{u}+B\nabla\vec{u}+C\vec{u}=\vec{f},$$

where $\vec{u}={}^t(u_1, u_2,\dots, u_N)$, $\vec{f}={}^t(f_1, f_2,\dots,f_N)$, $\nabla\vec{u}=\left(\frac{\partial u_i}{\partial x_j}\right)$, $B=(b_{ij}(x))$ and $C=(c_{ij}(x))$.

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§1. To begin with, we prepare some lemmas which will be used in the proof of the theorem. We denote the partial derivative of u in x_j by $\nabla_j u$, $|\nabla u|^2=\sum_{j=1}^n|\nabla_j u|^2$, the volume of the unit sphere in R^n by γ_n and $\Gamma_n=(n-2)\gamma_n$. We abuse K to denote various constants which do not depend on the coefficients or solutions.

Lemma 1.1. *Any* $u\in H_0^1(B_R)$ *is represented as*

$$(1.1) \qquad u(x)=\frac{1}{\gamma_n}\sum_{j=1}^n\int_{B_R}\frac{(x_j-y_j)}{|x-y|^n}\nabla_j u(y)dy$$

almost everywhere on B_R.

Proof. Since $\frac{1}{\Gamma_n|x|^{n-2}}$ is a fundamental solution for the Laplace operator, (1.1) holds obviously for a smooth function with compact support. The result follows by approximating to u in $H_0^1(B_R)$ by such smooth functions.

Lemma 1.2. *Let* c *belong to* $L^1(B_R)$ *and* u *to* $H^1(B_{R+a})$ $(a>0)$. *Suppose that there exist constants* C, L *and* λ $(0\leqslant\lambda\leqslant n-2+\theta)$ *such that*

$$(1.2) \quad \int_{B(x,r)\cap B_R}|c|dy\leqslant Cr^{n-2+\theta} \quad (0<\theta\leqslant 1)$$

$$(1.3) \quad \int_{B(x,r)\cap B_{R+a}}|\nabla u|^2+|u|^2dy\leqslant L^2r^\lambda \quad (0\leqslant\lambda\leqslant n-2+\theta) \text{ for each } B(x,r), \text{ then}$$

cu^2 *belongs to* $L^1(B_R)$ *and satisfies*

(1.4) $\int_{B(x_0,r)\cap B_R} |cu^2|dx \leqslant KCL^2 r^{\theta-\varepsilon+\lambda} (0<\varepsilon<\theta)$, *where* K *depends only on* n, ε, R *and* a.

Proof. When we choose a smooth cut-off function φ which is equal to 1 on B_R and whose support is contained in B_{R+a}, we have by Lemma 1.1

$$\begin{aligned}|u(x)| &\leqslant \frac{1}{\gamma_n}\int_{B_{R+a}} \frac{|\nabla(\varphi u)(y)|}{|x-y|^{n-1}} dy\\ &\leqslant K\int_{B_{R+a}} \frac{1}{|x-y|^{n-1}}(|\nabla u(y)|+|u(y)|)dy\\ &\leqslant K\Big(\int_{B_{R+a}} \frac{1}{|x-y|^{n-\varepsilon}}dy\Big)^{1/2}\Big(\int_{B_{R+a}} \frac{1}{|x-y|^{n-2+\varepsilon}}(|\nabla u(y)|^2+|u(y)|^2)dy\Big)^{1/2}\\ &\leqslant K(R+a)^{\varepsilon/2}\Big(\int_{B_{R+a}} \frac{1}{|x-y|^{n-2+\varepsilon}}(|\nabla u(y)|^2+|u(y)|^2)dy\Big)^{1/2}.\end{aligned}$$

Then we have

$$\int_{B(x_0,r)\cap B_R} |cu^2|dx \leqslant K(R+a)^{\varepsilon}\int_{B(x_0,r)\cap B_R}\int_{B_{R+a}} \frac{|c(x)|(|\nabla u(y)|^2+|u(y)|^2)}{|x-y|^{n-2+\varepsilon}} dy\,dx.$$

We divide the integral into two parts I_1+I_2 where

$$I_1=\int_{B(x_0,r)\cap B_R}\int_{B_{R+a}\cap B(x_0,2r)} \quad \text{and} \quad I_2=\int_{B(x_0,r)\cap B_R}\int_{B_{R+a}\setminus B(x_0,2r)}.$$

i) Estimate of I_1. In order to evaluate the integral, we set

$$\varphi(\rho, y)=\int_{B(x_0,r)\cap B(y,\rho)\cap B_R} |c(x)|dx.$$

Then we can see that $\varphi(\rho, y)$ satisfies

$$\varphi(\rho, y)\leqslant \begin{cases} C\rho^{n-2+\theta} & \text{if } 0\leqslant\rho\leqslant r\\ Cr^{n-2+\theta} & \text{if } \rho\geqslant r,\end{cases} \tag{1.5}$$

and that the integral with respect to x can be written as

$$\int_{B(x_0,r)\cap B_R} \frac{|c(x)|}{|x-y|^{n-2+\varepsilon}}dx=\int_0^{3r} \rho^{-n+2-\varepsilon}\, d\varphi(\rho, y) \quad \text{for} \quad |x-y|\leqslant 3r$$

where the integral of the right hand side is taken to be the Lebesgue-Stieltjes integral with respect to $\rho=|x-y|$. Using (1.5), we find

(1.6)

$$\begin{aligned}\int_{B(x_0,r)\cap B_R} \frac{|c(x)|}{|x-y|^{n-2+\varepsilon}} dx &= \rho^{-n+2-\varepsilon}\varphi(\rho, y)\Big|_0^{3r} + (n-2+\varepsilon)\int_0^{3r} \rho^{-n+1+\varepsilon}\varphi(\rho, y)d\rho\\ &\leqslant KCr^{\theta-\varepsilon}(\theta>\varepsilon).\end{aligned}$$

Inserting the inequality (1.6) into the integral I_1, we have

$$I_1=\int_{B_{R+a}\cap B(x_0,2r)}(|\nabla u(y)|^2+|u(y)|^2)\left(\int_{B_R\cap B(x_0,r)}\frac{|c(x)|}{|x-y|^{n-2+\varepsilon}}dx\right)dy$$

$$\leqslant KCr^{\theta-\varepsilon}\int_{B_{R+a}\cap B(x_0,2r)}|\nabla u(y)|^2+|u(y)|^2dy\leqslant KCL^2r^{\theta-\varepsilon+\lambda}.$$

ii) Estimate of I_2. We set, as above

$$\mathscr{U}(\rho,x)=\int_{[B_{R+a}\backslash B(x_0,2r)]\cap B(x,\rho)}|\nabla u(y)|^2+|u(y)|^2dy.$$

Then $\mathscr{U}$ satisfies

$$\mathscr{U}(\rho,x)\leqslant\begin{cases}L^2\rho^\lambda & \text{if } 0\leqslant\rho\leqslant R\\ L^2R^\lambda & \text{if } \quad\rho\geqslant R.\end{cases}\tag{1.7}$$

Since $x\in B(x_0,r)$ and $y\in B_R\backslash B(x_0,2r)$ in this case, $|x-y|\geqslant r$ holds for these x and y. Then we have

$$I_2=\int_{B(x_0,r)\cap B_R}|c(x)|\left(\int_{B_{R+a}\backslash B(x_0,2r)}\frac{|\nabla u(y)|^2+|u(y)|^2}{|x-y|^{n-2+\varepsilon}}dy\right)dx$$

$$=\int_{B(x_0,r)\cap B_R}|c(x)|\left(\int_r^{2R+2a}\rho^{-n+2-\varepsilon}\,d\mathscr{U}(\rho,x)\right)dx\qquad(\rho=|x-y|)$$

$$=\int_{B(x_0,r)\cap B_R}|c(x)|\left[\rho^{-n+2-\varepsilon}\mathscr{U}(\rho,x)\Big|_r^{2R+2a}+(n-2+\varepsilon)\int_r^{2R+2a}\rho^{-n+1-\varepsilon}\mathscr{U}(\rho,x)d\rho\right]dx$$

$$\leqslant KL^2(1+r^{-n+2-\varepsilon+\lambda})\int_{B(x_0,r)\cap B_R}|c(x)|dx$$

$$\leqslant KCL^2r^{\theta-\varepsilon+\lambda}.$$

Thus we find

$$\int_{B(x_0,r)\cap B_R}|cu^2|dx\leqslant K(R+a)^\varepsilon CL^2r^{\theta-\varepsilon+\lambda}$$

and obtain the lemma.

Remark. Since the constant $KCr^{\theta-\varepsilon}$ can be made arbitrary small as $r\to0$, we observe that the form $J[u]=\int_{R^n}|\nabla u|^2+c|u|^2dx$, $u\in H^1(R^n)$ is bounded below under the conditions. However we shall not use this fact in the following arguments.

Lemma 1.3. *Let g and h belong to $L^2(B_R)$ and satisfy*

$$\int_{B(x,r)\cap B_R}|g|^2dy\leqslant G^2r^{n-2+\theta}\qquad(0<\theta\leqslant1)\tag{1.8}$$

$$\int_{B(x,r)\cap B_R}|h|^2dy\leqslant H^2r^\lambda\qquad(0\leqslant\lambda\leqslant n-2+\theta)\qquad \textit{for every } B(x,r).\tag{1.9}$$

Then $V(x)=\frac{1}{\Gamma_n}\int_{B_R}\frac{g(y)h(y)|}{|x-y|^{n-2}}dy$ is defined for almost all x and belongs to $H^1(D)$ for any bounded domain D in R^n. Furthermore $V(x)$ satisfies

(1.10) $$\int_{B(x_0,r)} |\nabla V(x)|^2+|V(x)|^2 dx \leqslant K(GH)^2 r^{\theta+\lambda} \qquad \textit{for every } B(x_0, r).$$

Proof. Since $g(y)h(y)\in L^1(B_R)$, we find $V(x)\in L^1(D)$ for any bounded domain D. By approximating to gh in $L^1(B_R)$ by smooth functions, we can see easily that

(1.11) $$\nabla_j V(x)=\frac{1}{\gamma_n}\int_{B_R}\frac{x_j-y_j}{|x-y|^n}g(y)h(y)dy$$

holds for almost all x, where the derivatives are taken in a weak sense.

We devide the integral in the right hand side of (1.11) into two parts as $I_1(x)+I_2(x)$ where

$$I_1(x)=\int_{B_R\cap B(x_0,2r)} \quad\text{and}\quad I_2(x)=\int_{B_R\setminus B(x_0,2r)}.$$

i) Estimate of $I_1(x)$. We have, by Schwarz' inequality,

$$|I_1(x)|\leqslant\frac{1}{\gamma_n}\left(\int_{B_R\cap B(x_0,2r)}\frac{|g(y)|^2}{|x-y|^{n-2+\varepsilon}}dy\right)^{1/2}\left(\int_{B_R\cap B(x_0,2r)}\frac{|h(y)|^2}{|x-y|^{n-\varepsilon}}dy\right)^{1/2} \quad (0<\varepsilon<\theta).$$

By the same argument as in obtaining (1.6), we have

(1.12) $$\int_{B_R\cap B(x_0,2r)}\frac{|g(y)|^2}{|x-y|^{n-2+\varepsilon}}dy\leqslant KG^2r^{\theta-\varepsilon} \quad (0<\varepsilon<\theta).$$

Then it follows

$$\begin{aligned}\int_{B(x_0,r)}|I_1(x)|^2dx &\leqslant KG^2r^{\theta-\varepsilon}\int_{B_R\cap B(x_0,2r)}|h(y)|^2\left(\int_{B(x_0,r)}\frac{1}{|x-y|^{n-\varepsilon}}dx\right)dy\\ &\leqslant KG^2r^{\theta}\int_{B_R\cap B(x_0,2r)}|h(y)|^2dy\\ &\leqslant K(GH)^2r^{\theta+\lambda}.\end{aligned}$$

ii) Estimate of $I_2(x)$. When we set

$$\varphi(\rho, x)=\int_{[B_R\setminus B(x_0,2r)]\cap B(x,\rho)}|g(y)h(y)|dy,$$

we can easily verify

$$\varphi(\rho, x)\leqslant\begin{cases} GH\rho^{(n+\theta+\lambda)/2-1} & \text{if } 0\leqslant\rho\leqslant R\\ GHR^{(n+\theta+\lambda)/2-1} & \text{if } \quad \rho\geqslant R.\end{cases}$$

Since $|x-y|\geqslant r$ in this case, we have, as in the proof of Lemma 1.2,

$$\begin{aligned}|I_2(y)| &\leqslant\frac{1}{\gamma_n}\int_r^{3R}\rho^{-n+1}d\varphi(\rho, x) \qquad (\rho=|x-y|)\\ &\leqslant\frac{1}{\gamma_n}\left[\rho^{-n+1}\varphi\Big|_r^{3R}+(n-1)\int_r^{3R}\rho^{-n}\varphi(\rho, x)d\rho\right]\\ &\leqslant KGH(1+r^{(-n+\theta+\lambda)/2}).\end{aligned}$$

Then it follows

$$\int_{B(x_0,r)} |I_2(x)|^2 dx \leqslant K(GH)^2 r^{\theta+\lambda}.$$

Thus we obtain

$$\int_{B(x_0,r)} |\nabla u(x)|^2 dx \leqslant K(GH)^2 r^{\theta+\lambda}.$$

We observe that the estimate of $V(x)$ proceeds just as above, and the lemma follows.

The following propositions are direct consequences of the preceding lemmas.

Proposition 1.4. *(see Theorem 3.7.5. of C. B. Morrey [2]). Let f belong to $L^1(B_R)$ and satisfy*

$$\int_{B(x_0,r)\cap B_R} |f| dy \leqslant F r^{n-2+\theta} \quad (0<\theta\leqslant 1) \qquad \text{for every ball } B(x_0, r). \quad \text{Then}$$

$$W_R(x) = \frac{1}{\Gamma_n} \int_{B_R} \frac{f(y)}{|x-y|^{n-2}} dy$$

belongs to $H^1(D)$ for any bounded domain D in R^n and satisfies

$$\int_{B(x_0,r)} |\nabla W_R(x)|^2 + |W_R(x)|^2 dx \leqslant K F^2 r^{n-2+2\theta}. \tag{1.13}$$

Proof. The proposition follows from Lemma 1.3 by putting $g(y) = \text{sign}(f(y)) \times |f(y)|^{1/2}$, $h(y) = |f(y)|^{1/2}$ and $\lambda = n-2+\theta$.

Proposition 1.5. *Suppose b_j belongs to $L^2(B_R)$ and satisfies*

$$\int_{B(x,r)\cap B_R} |b|^2 dy \leqslant B^2 r^{n-2+\theta} \quad \text{where} \quad |b|^2 = \textstyle\sum_{j=1}^n |b_j|^2.$$

Suppose also u belongs to $H^1(B_R)$ and satisfies

$$\int_{B(x,r)\cap B_R} |\nabla u|^2 dy \leqslant L^2 r^{\lambda} \quad (0\leqslant\lambda\leqslant n-2+\theta). \quad \text{Then}$$

$$U_R(x) = \frac{1}{\Gamma_n} \int_{B_R} \frac{b(y)\cdot\nabla u(y)}{|x-y|^{n-2}} dy$$

belongs to $H^1(D)$ for any bounded domain D in R^n, and satisfies

$$\int_{B(x_0,r)} |\nabla U_R(x)|^2 + |U_R(x)|^2 dx \leqslant K B^2 L^2 r^{\theta+\lambda}. \tag{1.14}$$

Proof. To verify this, we have only to substitute b for g and ∇u for h in Lemma 1.3.

Proposition 1.6. *Suppose c belongs to $L^1(B_{R+a})$ and satisfies*

$\int_{B(x,r)\cap B_R} |c| dy \leqslant C r^{n-2+\theta}$. *Suppose also u belongs to $H^1(B_{R+a})$ and satisfies*

$$\int_{B(x,r)\cap B_{R+a}} |\nabla u|^2+|u|^2\, dy \leqslant L^2 r^\lambda \quad (0\leqslant\lambda\leqslant n-2+\theta) \text{ for every } B(x, r).$$

Then

$$V_R(x)=\frac{1}{\Gamma_n}\int_{B_R}\frac{c(y)u(y)}{|x-y|^{n-2}}dy$$

belongs to $H^1(D)$ *for any bounded domain D and satisfies*

(1.15) $\int_{B(x_0,r)} |\nabla V_R(x)|^2+|V_R(x)|^2 dx \leqslant KC^2L^2 r^{\theta+\lambda+(\theta-\varepsilon)}$, *where K depends on* n, ε, R *and* a $(0<\varepsilon<\theta)$.

Proof. By Lemma 1.3, it follows

$$\int_{B(x,r)\cap B_R} |cu^2| dy \leqslant KCL^2 r^{\theta-\varepsilon+\lambda} \quad \text{for every } B(x, r).$$

The proposition follows from Lemma 1.3 by putting $g=\operatorname{sign}(c(y))|c(y)|^{1/2}$ and $h=|c|^{1/2}u$.

Finally, since $-\Gamma_n^{-1}|x|^{-n+2}$ is a fundamental solution for the Laplace operator, we can see easily the following proposition.

Proposition 1.7. *Let f belong to* $L^1(B_R)$. *If we set*

$$W_R(x)=\frac{1}{\Gamma_n}\int_{B_R}\frac{f(y)}{|x-y|^{n-2}}dy,$$

then we find

$$\int_{B_R}\nabla W_R\cdot\nabla\varphi dx=\int_{B_R} f\varphi dx \qquad \text{for all } \varphi\in C_0^1(B_R)$$

§**2.** Now, we can prove the theorem. Our proof is based on the following Dirichlet growth theorem which is called Morrey's lemma.

Lemma 2.1. (*see Theorem* 3.5.2 *in C. B. Morrey* [2]) *Let u belong to* $H^1(B_{R+a})$ *and satisfy*

(2.1) $$\int_{B(x,r)} |\nabla u|^2 dy \leqslant L^2\left(\frac{r}{a}\right)^{n-2+2\theta} \quad (0<\theta\leqslant 1) \text{ for all } x\in B_R \text{ and } 0\leqslant r\leqslant$$

a with a constant L. Then u is equivalent to a Hölder continuous function in B_R *and satisfies*

$$|u(x)-u(x')|\leqslant KLa^{1-n/2}\left(\frac{|x-x'|}{a}\right)^\theta \qquad \text{for almost all} \quad |x-x'|\leqslant a/2.$$

Owing to the lemma, we can see we have only to show that ∇u satisfies the growth condition (2.1) to prove the theorem.

Proof of the theorem. Let $u \in H^1(B_{R_0})$ be a solution of the equation

$$-\Delta u + \sum_{j=1}^{n} b_j \nabla_j u + cu = f.$$

For any $R'(0<R'<R_0)$ and large integer N, we can choose concentric balls B_{R_j} of radius R_j such that $R' = R_{2N} < R_{2N-1} < \cdots < R_1 < R_0$ and $R_{j-1} - R_j = \text{constant}$, say $2a$. We put

$$
\begin{aligned}
U_{R_j}(x) &= \frac{1}{\Gamma_n} \int_{B_{Rj}} \frac{b(y)\cdot \nabla u(y)}{|x-y|^{n-2}} dy \\
V_{R_j}(x) &= \frac{1}{\Gamma_n} \int_{B_{Rj}} \frac{c(y)u(y)}{|x-y|^{n-2}} dy \qquad (2.2) \\
W_{R_j}(y) &= \frac{1}{\Gamma_n} \int_{B_{Rj}} \frac{f(y)}{|x-y|^{n-2}} dy.
\end{aligned}
$$

Then we can see, by Proposition 1.4, 1.5, 1.5 and 1.7, that the integrals make sense and $v_{R_1} = u + U_{R_1} + V_{R_1} - W_{R_1}$ satisfies $\Delta v_{R_1} = 0$ in a weak sense in the interior of B_{R_1}. Furthermore by Proposition 1.5 and 1.6 with $L = \|u\|_{H^1(B_{R_0})}$, $\lambda = 0$ and $r = R_1$, it follows

$$\|v_{R_1}\|_{L^2(B_{R_1})} \leqslant K_1[(B+C)\|u\|_{H^1(B_{R_0})} + F]. \tag{2.3}$$

Since v_{R_1} is equivalent to a harmonic function B_{R_1}, the magnitude of v_{R_1} and ∇v_{R_1} in B_{R_2} can be estimated by the L^2-norm of v_{R_1} in B_{R_1}. That is:

$$
\begin{aligned}
|v_{R_1}(x)| + |\nabla v_{R_1}(x)| &\leqslant K a^{-n/2} \|v_{R_1}\|_{L^2(B_{R_1})} \\
&\leqslant K a^{-n/2} [(B+C)\|u\|_{H^1(B_{R_0})} + F)]
\end{aligned} \tag{2.4}
$$

holds for almost every x in B_{R_2}.

Using Proposition 1.5 and 1.7 with $L = \|u\|_{H^1(B_{R_0})}$, $\lambda = 0$ and $R = R_1$, we have

$$
\begin{aligned}
&\int_{B(x_0,r)} |\nabla U_{R_1}|^2 + |U_{R_1}|^2 dx \leqslant KB^2 \|u\|^2_{H^1(B_{R_0})} r^\theta \\
&\int_{B(x_0,r)} |\nabla V_{R_1}|^2 + |V_{R_1}|^2 dx \leqslant KC^2 \|u\|^2_{H^1(B_{R_0})} r^{\theta+(\theta-\varepsilon)} \quad \text{for every } B(x_0, r).
\end{aligned} \tag{2.5}
$$

Using Proposition 1.4 with $R = R_1$, we have

$$\int_{B(x_0,r)} |\nabla W_{R_1}|^2 + |W_{R_1}|^2 dx \leqslant KF^2 r^{n-2+2\theta}. \tag{2.6}$$

We recall that $u = -U_{R_1} - V_{R_1} + W_{R_1} + v_{R_1}$. Then it follows, by (2.4), (2.5) and (2.6)

(2.7) $\int_{B(x,r)\cap B_{R_2}} |\nabla u|^2 + |u|^2 dy \leqslant L_2^2 r^\theta$ for every $B(x, r)$, where L_2 depends on B, C, $\|u\|_{H^1(B_{R_0})}$, etc.

Next we employ again Proposition 1.5 and 1.6 with $L = L_2$, $R = R_3$ and $\lambda = \theta$. Then we obtain

$$(2.8)\quad \begin{aligned}&\int_{B(x_0,r)} |\nabla U_{R_3}|^2+|U_{R_3}|^2dx\leqslant KB^2L_2^2r^{2\theta}\\ &\int_{B(x_0,r)} |\nabla V_{R_3}|^2+|V_{R_3}|^2dx\leqslant KC^2L_2^2r^{2\theta+\theta-\varepsilon} \qquad \text{for every} \quad B(x, r).\end{aligned}$$

By Proposition 1.4 with $R=R_3$, we have

$$\int_{B(x_0,r)} |\nabla W_{R_3}|^2+|W_{R_3}|^2dx\leqslant KF^2r^{n-2+2\theta} \qquad \text{for every} \quad B(x, r).$$

When we set, as above, $v_{R_3}=u+U_{R_3}+V_{R_3}-W_{R_3}$, v_{R_3} is equivalent to a harmonic function and

$$(2.9)\quad \begin{aligned}|\nabla v_{R_3}(x)|+|v_{R_3}(x)|&\leqslant Ka^{-n/2}\|v_{R_3}\|_{L^2(B_{R_0})}\\ &\leqslant Ka^{-n/2}[(B+C)\|u\|_{H^1(B_{R_0})}+F]\end{aligned}$$

holds almost every x in B_{R_4}. Since $u=-U_{R_3}-V_{R_3}+W_{R_3}+v_{R_3}$, we find, by the same argument as above,

$$(2.10)\quad \int_{B(x,r)\cap B_{R_4}} |\nabla u|^2+|u|^2dy\leqslant L_4^2r^{2\theta} \qquad \text{for every} \quad B(x, r).$$

Repeating these arguments k times, we obtain

$$(2.11)\quad \int_{B(x,r\cap B_{R_{2k}}} |\nabla u|^2+|u|^2dy\leqslant L_{2k}^2r^{k\theta} \qquad \text{for every} \quad B(x, r).$$

Since we can choose beforehand sufficiently large N such that $N\theta$ exceeds $n-2+2\theta$, we conclude

$$\int_{B(x,r)\cap B_{R'}} |\nabla u|^2dy\leqslant L^2r^{n-2+2\theta} \qquad \text{for every} \quad B(x, r) \text{ in } R^n.$$

The theorem then follows from Morrey's lemma.

Added in proof: If $n\geqslant 3$, $u=\log r$ $(r^2=\sum_{j=1}^n x_j^2)$ belongs to $H^1(B_R)$ and is a solution of

$$-\triangle u+c(x)u=0 \quad \text{where} \quad c(x)=\frac{n-2}{r^2\log r}.$$

Then it follows that we can not expect the Hölder continuity (even the boundedness) of the weak solution, when we assume, instead of (0.2),

$$\int_{B(x,r)\cap B_{R_0}} |c(y)|dy\leqslant C\frac{r^{n-2}}{|\log r|}.$$

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