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On the neighborhood of a compact complex curve with topologically trivial normal bundle

By

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Introduction

Let C be a non-singular irreducible compact complex curve imbedded in a complex manifold of dimension 2. As an oriented differentiable manifold, the structure of the neighborhood of the curve C is completely characterized by the Chern class of the normal bundle of C, in other words by the self-intersection number (C^2) of C. This topological structure imposes restrictions on the complex analytic properties of the neighborhood of C. Specifically the curve C has a strongly pseudoconvex neighborhood if and only if (C^2) is negative (see Grauert [3]); on the other hand C has a fundamental system of strongly pseudoconcave neighborhoods if (C^2) is positive (see Suzuki [11]).

The purpose of the present paper is to investigate such complex analytic properties of the neighborhood of the curve C when the self-intersection number (C^2) vanishes. We shall see that, if the complex normal bundle N of C is a general element (in the sense of Lebesgue measure) of the Picard variety $\mathfrak{P}(C)$, then C has either a fundamental system of strongly pseudoconcave neighborhoods or that of pseudoflat neighborhoods. We shall find moreover, in the former case, a restriction on the behavior of plurisubharmonic functions and holomorphic functions having singularities along C. This restriction may be regarded as an expression of the weekness of pseudoconcavity of the neighborhood of C.

In §1, we make some preliminary observations concerning flat line bundles, i.e., complex line bundles whose transition functions are constants of modulus 1. In §2, we define the type (1, 2,..., or infinite) for a curve C whose complex normal bundle N is topologically trivial. This type can be described as follows: A unique structure of flat line bundle is introduced on N, and N is extended uniquely to a flat line bundle F over a neighborhood of C; then the type represents the order of coincidence of F and the complex line bundle $[C]$ corresponding to the divisor C. The curve C is of infinite type if F and $[C]$ coincide *formally*. In §3, the case of finite type is treated. We construct a strongly plurisubharmonic function $\Phi(p)$ defined on a neighborhood of C except on C which tends to $+\infty$ as p approaches C. Letting n be the type of C, we can construct, for any real number $n'>n$, such a function $\Phi(p)$ of order $1/r(p)^{n'}$, where $r(p)$ is the distance of p from C (Theorem 1). But there exists no non-constant plurisubharmonic function which increases slower than $1/r(p)^{n''}$ for $n''<n$ (Theorem 2). This presents a contrast to the case $(C^2)>0$, where we have such a function of order $-\log r(p)$ (see Suzuki [11]). In §4, the case of infinite type is considered. We show that F and $[C]$ coincide on a neighborhood of C, if the complex normal bundle N of C is contained in a subset $\mathfrak{E}$ of $\mathfrak{P}(C)$ (Theorem 3). Here the set $\mathfrak{E}$ consists of the elements of finite order and the elements which are not "well approximated" by those of finite order. Thus Theorem 3 generalizes the result of Arnol'd [1] for elliptic curves. In §5, summarizing the results, we classify the curves C into four classes, and make some supplementary remarks. Finally we give an example, suggested by Arnol'd [1], of a curve of infinite type for which F and $[C]$ do not coincide.

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§1. Preliminaries

1. Flat line bundles

Let $E\xrightarrow{\pi}M$ be a complex line bundle over a complex manifold M. We call E a *flat line bundle* if an open covering $\mathfrak{U}=\{U_i\}_{i\in I}$ of M and a collection of fiber coordinates $\{\zeta_i\}$ of E over U_i are so chosen that the transition functions $t_{ik}=\zeta_i/\zeta_k$ on $U_i\cap U_k$ are constants of modulus 1. Then the system $\{t_{ik}\}$ is a 1-cocycle with coefficients in the multiplicative group $\boldsymbol{T}$ of all complex numbers of modulus 1. Two flat line bundles E and E' with systems of transition functions $\{t_{ik}\}$ and $\{t'_{ik}\}$, respectively, are equivalent, if and only if there exist constants $t_i\in\boldsymbol{T}$, $i\in I$, such that $t'_{ik}=t_{ik}t_i^{-1}t_k$; then they are considered as different expressions of one and the same flat line bundle E. The set of all (equivalence classes of) flat line bundles over M is identified with the first cohomology group $H^1(M, \boldsymbol{T})$ in an obvious manner.

We introduce, on a flat line bundle E, a fiber metric of curvature zero by $|\zeta_i|$ over each U_i. We note that the structure of flat line bundle on a complex line bundle is determined by such a fiber metric.

For a complex line bundle E over M, we denote by $c(E)$ the Chern class of E, and by $c_{\boldsymbol{R}}(E)$ the element of $H^2(M, \boldsymbol{R})$ corresponding to $c(E)$ by the map $H^2(M, \boldsymbol{Z})\to H^2(M, \boldsymbol{R})$.

Proposition 1. (1) *If E is a flat line bundle, then $c_{\boldsymbol{R}}(E)=0$.* (2) *When M is compact, two flat line bundles over M are equivalent if and only if they are equivalent as complex line bundles.* (3) *When M is compact, the necessary and sufficient condition for any complex line bundle with $c_{\boldsymbol{R}}(E)=0$ to admit a structure of flat line bundle is that* $\dim H^1(M, \boldsymbol{C})=2\dim H^1(M, \mathcal{O})$. (a theorem of Kashiwara, see Kodaira [7], pp. 124–126).

Proof. Consider the following commutative diagram of sheaves of abelian groups over M:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \boldsymbol{R} & \longrightarrow & \boldsymbol{T} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \boldsymbol{Z} & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}^* & \longrightarrow & 1\,, \end{array}$$

with exact rows, where $\mathcal{O}$ denotes the sheaf over M of germs of holomorphic functions and $\mathcal{O}^*$ denotes the sheaf over M of germs of non-vanishing holomorphic functions. From this we obtain the commutative diagram

$$\begin{array}{ccccccccc} H^1(M, \boldsymbol{Z}) & \longrightarrow & H^1(M, \boldsymbol{R}) & \longrightarrow & H^1(M, \boldsymbol{T}) & \longrightarrow & H^2(M, \boldsymbol{Z}) & \longrightarrow & H^2(M, \boldsymbol{R}) \\ \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ H^1(M, \boldsymbol{Z}) & \longrightarrow & H^1(M, \mathcal{O}) & \longrightarrow & H^1(M, \mathcal{O}^*) & \longrightarrow & H^2(M, \boldsymbol{Z}) & & \end{array}$$

with exact rows. The assertion (1) follows from this immediately. When M is compact, the vertical maps α and β are injective. In particular the injectivity of β implies the assertion (2). From the diagram we infer that the following two conditions (i) and (ii) are equivalent:

(i) Any $E \in H^1(M, \mathcal{O}^*)$ with $c_{\boldsymbol{R}}(E)=0$ is in the image of the map β.

(ii) The map α is surjective.

If M is compact, the condition (ii) is equivalent to

(iii) $\dim_{\boldsymbol{R}} H^1(M, \boldsymbol{R})=\dim_{\boldsymbol{R}} H^1(M, \mathcal{O})$ (real dimensions),

or

(iv) $\dim H^1(M, \boldsymbol{C})=2 \dim H^1(M, \mathcal{O})$.

The assertion (3) is thus proved. q. e. d.

We note that the condition (iv) holds if M is a compact Kähler manifold, in particular, if M is a compact Riemann surface.

The set consisting of all topologically trivial complex line bundles E ($c(E)=0$) is called the Picard variety of M and denoted by $\mathfrak{P}(M)$. If M is a compact Riemann surface, we can identify $\mathfrak{P}(M)$ with $H^1(M, \boldsymbol{T})$.

2. Holomorphic sections and pluriharmonic sections

A complex valued function h defined on a complex manifold is called pluriharmonic, if h is locally expressed as a sum $f+\bar{g}$ of a holomorphic function f and an anti-holomorphic function $\bar{g}$. If a pluriharmonic function h is represented by two such sums: $h=f+\bar{g}=f'+\bar{g}'$, then we have $f'=f+c$ and $\bar{g}'=\bar{g}-c$, where c is a constant. Indeed, $f'-f=\bar{g}-\bar{g}'$ is holomorphic and anti-holomorphic therefore it is a constant. A differentiable function h is pluriharmonic if and only if the (1, 1)-form $\partial\bar{\partial}h$ vanishes identically, as is well known. We note that, for a pluriharmonic function h, the modulus $|h|$ is a plurisubharmonic function, so that the principle of maximum modulus holds.

A section of a flat line bundle E is called constant (resp. holomorphic, anti-holomorphic, or pluriharmonic), if its expressions with respect to the fiber coordinates are constant (resp. holomorphic, anti-holomorphic, or pluriharmonic) functions. The sheaves of germs of such sections are denoted by $\boldsymbol{C}(E)$, $\mathcal{O}(E)$, $\bar{\mathcal{O}}(E)$ and $\mathcal{H}(E)$, respectively. Denoting by $E^{-1}=\{t_{ik}^{-1}\}=\{\bar{t}_{ik}\}$ the dual of the flat line bundle $E=\{t_{ik}\}$, we have anti-$\boldsymbol{C}$-linear isomorphisms $\boldsymbol{C}(E)\cong\boldsymbol{C}(E^{-1})$, $\mathcal{O}(E)\cong\bar{\mathcal{O}}(E^{-1})$, $\bar{\mathcal{O}}(E)\cong\mathcal{O}(E^{-1})$ and $\mathcal{H}(E)\cong\mathcal{H}(E^{-1})$, by complex conjugation.

Let us consider, following Kashiwara (see Kodaira [7]), the exact sequence of sheaves over M

$$0 \longrightarrow \boldsymbol{C}(E) \xrightarrow{\varphi} \mathcal{O}(E)\oplus\bar{\mathcal{O}}(E) \xrightarrow{\psi} \mathcal{H}(E) \longrightarrow 0,$$

where the map φ is defined by $c\mapsto\varphi(c)=c\oplus(-c)$, and the map ψ is defined by $f\oplus\bar{g}\to\psi(f\oplus\bar{g})=f+\bar{g}$. From this we obtain the exact cohomology sequence

$$\begin{aligned}0 &\longrightarrow H^0(M, \boldsymbol{C}(E)) \longrightarrow H^0(M, \mathcal{O}(E))\oplus H^0(M, \bar{\mathcal{O}}(E)) \longrightarrow H^0(M, \mathcal{H}(E)) \\ &\xrightarrow{\delta} H^1(M, \boldsymbol{C}(E)) \xrightarrow{\varphi^1} H^1(M, \mathcal{O}(E))\oplus H^1(M, \bar{\mathcal{O}}(E)) \xrightarrow{\psi^1} H^1(M, \mathcal{H}(E)).\end{aligned}$$

Let us assume M to be compact. We note first that

$$\begin{aligned}H^0(M, \mathcal{H}(E)) &= H^0(M, \mathcal{O}(E)) = H^0(M, \bar{\mathcal{O}}(E)) = H^0(M, \boldsymbol{C}(E)) \\ &= \begin{cases} \boldsymbol{C}, & \text{if } E=\mathbf{1}, \\ 0, & \text{if } E\neq\mathbf{1}, \end{cases}\end{aligned}$$

where $\mathbf{1}$ denotes the analytically trivial line bundle over M. In fact, for any global section $\{h_i\}\in H^0(M, \mathcal{H}(E))$, $|h_i|$ is constant by the principle of maximum; hence $\{h_i\}$ is a constant section, which can be non-zero only if $E=\mathbf{1}$. Therefore the map δ is a zero-map and the map φ^1 is injective. Thus the sequence

$$0 \longrightarrow H^1(M, \boldsymbol{C}(E)) \xrightarrow{\varphi^1} H^1(M, \mathcal{O}(E))\oplus H^1(M, \bar{\mathcal{O}}(E)) \xrightarrow{\psi^1} H^1(M, \mathcal{H}(E))$$

is exact. Clearly the following three conditions are equivalent: (i) ψ^1 is a zero-map; (ii) φ^1 is surjective; (iii) $\dim H^1(M, \boldsymbol{C}(E)) = \dim H^1(M, \mathcal{O}(E)) + \dim H^1(M, \bar{\mathcal{O}}(E))$. We have thus the following

Proposition 2. *Let E be a flat line bundle over a compact complex manifold M. If*

$$\dim H^1(M, \boldsymbol{C}(E))=\dim H^1(M, \mathcal{O}(E))+\dim H^1(M, \bar{\mathcal{O}}(E)),$$

then the homomorphism $H^1(M, \mathcal{O}(E))\to H^1(M, \mathcal{H}(E))$ is a zero-map.

Now let us assume the conclusion of Proposition 2. Then, for any holomorphic 1-cocycle $\{f_{ik}\}\in Z^1(\mathfrak{U}, \mathcal{O}(E))$, there exists a 0-cochain $\{h_i\}\in C^0(\mathfrak{U}, \mathcal{H}(E))$ such that $\{f_{ik}\}$ is the coboundary of $\{h_i\}$, i.e., $f_{ik}=t_{ik}h_k-h_i$ on $U_i\cap U_k$. The 0-cochain $\{h_i\}$ is uniquely determined if $E\neq\mathbf{1}$, and unique up to an additive constant if $E=\mathbf{1}$. Indeed, if $\{f_{ik}\}$ is the coboundary of two such 0-cochains $\{h_i\}$ and $\{h'_i\}$, then $\{h'_i-h_i\}$ is a pluriharmonic global section of E, which is zero or a constant according as $E\neq\mathbf{1}$ or $E=\mathbf{1}$. We can define an anti-holomorphic (0, 1)-form $\{\omega_i\}$ on M with coefficients in E by $\omega_i=\bar{\partial}h_i$ on each U_i. Obviously, the correspondence $\{f_{ik}\}\mapsto\{\omega_i\}$ gives the Dolbeault isomorphism

$$H^1(M, \mathcal{O}(E))\cong\frac{\{\bar{\partial}\text{-closed } (0, 1)\text{-forms with coefficients in } E\}}{\{\bar{\partial}\text{-exact } (0, 1)\text{-forms with coefficients in } E\}}.$$

The condition of Proposition 2 is satisfied if M is a compact Kähler manifold. In fact, denoting by $\boldsymbol{H}^1(E)$, $\boldsymbol{H}^{0,1}(E)$, and $\boldsymbol{H}^{1,0}(E)$, respectively, the space of all harmonic 1-forms, (0, 1)-forms and (1, 0)-forms on M with coefficients in E, we have $H^1(M, \boldsymbol{C}(E))\cong\boldsymbol{H}^1(E)$, $H^1(M, \mathcal{O}(E))\cong\boldsymbol{H}^{0,1}(E)$, $H^1(M, \bar{\mathcal{O}}(E))\cong\boldsymbol{H}^{1,0}(E)$ and $\boldsymbol{H}^1(E)=\boldsymbol{H}^{0,1}(E)\oplus\boldsymbol{H}^{1,0}(E)$. (see Kodaira [6], [7])

If M is a compact Riemann surface of genus g, we have, by Riemann-Roch theorem,

$$\dim H^1(M, \mathcal{O}(E))=\begin{cases} g & \text{for } E=\mathbf{1}, \\ g-1 & \text{for } E\neq\mathbf{1}. \end{cases}$$

§2. Type of curves

1. Let C be a non-singular irreducible compact complex curve imbedded in a complex manifold S of dimension 2. We assume in all what follows that the normal bundle N of the curve C is topologically trivial.

We choose and fix a finite open covering $\mathfrak{U}=\{U_i\}_{i\in I}$ of C consisting of small disks U_i: $|z_i|<1$, where z_i is a local coordinate of C which covers the closure $\bar{U}_i$ of U_i. Further we choose, for each U_i, a sufficiently small neighborhood V_i of U_i in S in such a way that $V_i\cap C=U_i$ $(i\in I)$, and that $U_i\cap U_k=\emptyset$ implies $V_i\cap V_k=\emptyset$ $(i, k\in I)$. Then $\mathfrak{B}=\{V_i\}_{i\in I}$ is a finite open covering of the neighborhood $V=\bigcup_{i\in I} V_i$ of C. In the course of the following considerations we shall replace, if it is necessary, the neighborhoods V_i by smaller ones satisfying the above conditions. Such smaller neighborhoods will be again denoted by V_i. We are thus concerned with the germs of the neighborhoods of U_i in S. We extend each local coordinate z_i on U_i to a holomorphic function on V_i and denote the extended function also by z_i.

Let $\{w_i\}_{i\in I}$ be a system of holomorphic functions w_i on V_i such that (z_i, w_i) is a local coordinate system on V_i and that $V_i\cap C=U_i$ is defined in V_i by the equation $w_i=0$. The complex line bundle $[C]$ over V corresponding to the divisor C is defined by the multiplicative 1-cocycle $\{w_i/w_k\}$ composed of the non-vanishing holomorphic functions w_i/w_k on $V_i\cap V_k$. The complex normal bundle N of the curve C is identical to the restriction $[C]|C$ of $[C]$ to C. Since $N=[C]|C$ is topologically trivial, it is expressed by a multiplicative 1-cocycle $\{t_{ik}\}\in Z^1(\mathfrak{U}, \boldsymbol{T})$. This implies that there exist non-vanishing holomorphic functions e_i on U_i $(i\in I)$ such that $t_{ik}=e_ie_k^{-1}w_i/w_k$ on $U_i\cap U_k$. We extend e_i to V_i and put $\tilde{w}_i=e_iw_i$. Then $\{\tilde{w}_i\}$ is a system of holomorphic functions satisfying the above conditions for $\{w_i\}$ and further $\tilde{w}_i/\tilde{w}_k|U_i\cap U_k=t_{ik}$ on $U_i\cap U_k$ $(i, k\in I)$.

Let us fix a multiplicative 1-cocycle $\{t_{ik}\}$ representing N and consider the systems $\{w_i\}$ such that $w_i/w_k|U_i\cap U_k=t_{ik}$ on $U_i\cap U_k$. *A system* $\{w_i\}$ *will be called of type* ν *if each* $t_{ik}w_k-w_i$ *vanishes on* $U_i\cap U_k=V_i\cap V_k\cap C$ *with order (at least)* $\nu+1$. If $\{w_i\}$ is a system of type ν, then we can put

$$t_{ik}w_k-w_i=f_{ik}(z_i)w_i^{\nu+1}+\cdots \qquad \text{on} \quad V_i\cap V_k.$$

We regard f_{ik} as a holomorphic function on $U_i\cap U_k$, and further as a holomorphic section over $U_i\cap U_k$ of the flat line bundle $N^{-\nu}$ represented by the fiber coordinate over U_i.

First we assert that $\{f_{ik}\}$ is a 1-cocycle composed of holomorphic sections of $N^{-\nu}$ over $U_i\cap U_k$, i.e., $\{f_{ik}\}\in Z^1(\mathfrak{U}, \mathcal{O}(N^{-\nu}))$. Indeed, we have

$$\begin{aligned}0&=(t_{ij}w_j-w_i)+t_{ij}(t_{jk}w_k-w_j)+t_{ik}(t_{ki}w_i-w_k)\\&=(f_{ij}(z_i)w_i^{\nu+1}+\cdots)+t_{ij}(f_{jk}(z_j)w_j^{\nu+1}+\cdots)+t_{ik}(f_{ki}(z_k)w_k^{\nu+1}+\cdots)\\&=(f_{ij}(z_i)+t_{ij}^{-\nu}f_{jk}(z_j)+t_{ik}^{-\nu}f_{ki}(z_k))w_i^{\nu+1}+\cdots\end{aligned}$$

on $V_i\cap V_j\cap V_k$;

and hence

$$f_{ij}+t_{ij}^{-\nu}f_{jk}+t_{ik}^{-\nu}f_{ki}=0 \qquad \text{on} \quad U_i\cap U_j\cap U_k,$$

which implies the assertion. The 1-cocycle $\{f_{ik}\}$ will be called the ν-th obstruction associated with the system $\{w_i\}$ of type ν.

Now suppose that the ν-th obstruction $\{f_{ik}\}$ is the coboundary of a 0-cochain $\{f_i\}\in C^0(\mathfrak{U}, \mathcal{O}(N^{-\nu}))$, namely, $f_{ik}=t_{ik}^{-\nu}f_k-f_i$ on $U_i\cap U_k$ $(i, k\in I)$. Then putting

$$\tilde{w}_i=w_i-f_i(z_i)w_i^{\nu+1} \qquad \text{on} \quad V_i,$$

we can obtain a system $\{\tilde{w}_i\}$ of type $\nu+1$. Indeed,

$$\begin{aligned}t_{ik}\tilde{w}_k-\tilde{w}_i&=t_{ik}(w_k-f_k(z_k)w_k^{\nu+1})-(w_i-f_i(z_i)w_i^{\nu+1})\\&=(f_{ik}(z_i)w_i^{\nu+1}+\cdots)-(t_{ik}^{-\nu}f_k(z_k)-f_i(z_i))w_i^{\nu+1}+\cdots\end{aligned}$$

is of order at least $\nu+2$.

Next let us consider two systems $\{w_i\}$ and $\{w_i'\}$ of type ν with ν-th obstructions $\{f_{ik}\}$ and $\{f'_{ik}\}$ respectively. We assert that $\{f_{ik}\}$ and $\{f'_{ik}\}$ are cohomologous up to a constant factor. To see this, we put

$$w_i'=ew_i+g_{i|2}(z_i)w_i^2+\cdots+g_{i|\mu}(z_i)w_i^{\mu}+\cdots \qquad \text{on} \quad V_i,$$

where e is a constant different from zero and independent of the index i. We have

$$\begin{aligned}f'_{ik}(z_i)w_i'^{\nu+1}+\cdots&=t_{ik}w_k'-w_i'\\&=t_{ik}(ew_k+g_{k|2}(z_k)w_k^2+\cdots)-(ew_i+g_{i|2}(z_i)w_i^2+\cdots)\\&=ef_{ik}(z_i)w_i^{\nu+1}+\cdots+t_{ik}(g_{k|2}(z_k)w_k^2+\cdots)-(g_{i|2}(z_i)w_i^2+\cdots)\end{aligned}$$

on $V_i\cap V_k$. Comparing the terms of order 2, we see that $t_{ik}^{-1}g_{k|2}-g_{i|2}=0$ on $U_i\cap U_k$. Therefore $\{g_{i|2}\}$ consitutes a global holomorphic section of N^{-1}, and $g_{i|2}$ are all constant. Hence $t_{ik}g_{k|2}w_k^2 - g_{i|2}w_i^2 = g_{i|2}(t_{ik}^2 w_k^2 - w_i^2)$ is of order $\nu+2$ $(>\nu+1)$. Next comparing the terms of order 3, we have $t_{ik}^{-2}g_{k|3} - g_{i|3} = 0$ on $U_i \cap U_k$. We proceed in this manner and finally, comparing the terms of order $\nu+1$, we have

$$e^{\nu+1}f'_{ik} = ef_{ik} + t_{ik}^{-\nu}g_{k|\nu+1} - g_{i|\nu+1} \qquad \text{on} \quad U_i \cap U_k.$$

This shows that $\{e^\nu f'_{ik}\}$ and $\{f_{ik}\}$ are cohomologous.

Definition. (i) The curve C is called of finite type n if there exists a system $\{w_i\}_{i\in I}$ of type n such that the n-th obstruction associated with it is not cohomologous to zero. (ii) The curve C is called of infinite type if, for any system $\{w_i\}_{i\in I}$, the obstruction associated with it is cohomologous to zero.

By the above observations we infer that, if the curve C is of finite type n, then there exists no system of type $\nu > n$; for any system of type $\nu < n$, the ν-th obstruction is cohomologous to zero; and that, for any system of type n, the n-th obstruction is not cohomologous to zero. On the other hand, if C is of infinite type, then there exists a system of type ν for any arbitrarily large ν.

So far we have fixed the open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ and the multiplicative 1-cocycle $\{t_{ik}\}$ defining the complex normal bundle N. But it is easy to see that the definition of the type of the curve C is independent of the choice of $\mathfrak{U}$ and $\{t_{ik}\}$.

2. It is necessary for the later purposes to represent the obstructions in a different way. Let $n(\leqq +\infty)$ be the type of the curve C and let $\{w_i\}$ be a system of type ν $(\nu \leqq n)$ such that $t_{ik}w_k - w_i = f_{ik}(z_i)w_i^{\nu+1} + \cdots$ on $V_i \cap V_k$.

We can regard the multiplicative 1-cocycle $\{t_{ik}\} \in Z^1(\mathfrak{U}, \boldsymbol{T})$ as a multiplicative 1-cocycle on the nerve of the covering $\mathfrak{V} = \{V_i\}$ of V. Then $\{t_{ik}\} \in Z^1(\mathfrak{V}, \boldsymbol{T})$ defines a flat line bundle F over V. The restriction $F|C$ of F to the curve C is identical to the complex normal bundle $N = [C]\,|\,C$ of C. But generally F and $[C]$ do not coincide on any small neighborhood of C (see also §4, 1).

Now let us consider the system $\{w_i^{-\nu}\}$ of meromorphic functions $w_i^{-\nu}$ on V_i. The system $\{w_i^{-\nu}\}$ is regarded as an additive Cousin data composed of meromorphic sections $w_i^{-\nu}$ of $F^{-\nu}$. Indeed,

$$\begin{aligned} t_{ik}^{-\nu}w_k^{-\nu} - w_i^{-\nu} &= w_i^{-\nu}(1 + f_{ik}(z_i)w_i^\nu + \cdots)^{-\nu} - w_i^{-\nu} \\ &= -\nu f_{ik}(z_i) + \cdots \end{aligned}$$

is holomorphic on $V_i \cap V_k$. This shows also that the ν-th obstruction is identical, up to the constant factor $-\nu$, to the restriction $\{-\nu f_{ik}\}$ to C of the 1-cocycle $\{t_{ik}^{-\nu}w_k^{-\nu} - w_i^{-\nu}\} \in Z^1(\mathfrak{V}, \mathcal{O}(F^{-\nu}))$ corresponding to the Cousin data $\{w_i^{-\nu}\}$. We shall sometimes call $\{-\nu f_{ik}\}$ the ν-th obstruction.

3. Suppose that the curve C is rational. Then the complex normal bundle N is analytically trivial. Since $H^1(C, \mathcal{O}) = 0$, all the obstructions are cohomologous to zero. Therefore C is a priori of infinite type. Suppose next that C is elliptic. We have $H^1(C, \mathcal{O}(N^{-\nu})) \neq 0$ if and only if $N^{-\nu}$ is analytically trivial. Therefore, if N is of infinite order, then C is of infinite type; and if N is of finite order m, then the type of C is either infinite or finite n, n being a multiple of m.

When a compact Riemann surface C, a topologically trivial complex line bundle N, and a positive integer n (or infinity) are preassigned, we can easily construct an example of imbedding of C in a complex manifold S of dimension 2 in such a way that the complex normal bundle of C is N and that C is of type n, as long as the above conditions posed a priori are satisfied (see also Miyajima [9]). But this is not always possible if S is required to be compact, as we shall see in the forthcoming paper.

§ 3. The case of finite type

1. In this section we assume the curve C to be of finite type n. Let us take open coverings $\mathfrak{U}=\{U_i\}_{i\in I}$ and $\mathfrak{V}=\{V_i\}_{i\in I}$ and a system $\{w_i\}_{i\in I}$ of type n as in the preceding section. To represent the distance from C of a point p in the neighborhood V of C, we take a non-negative continuous function $r(p)$ on V which has the form $r(p)=\rho_i(p)|w_i(p)|$ on V_i, where ρ_i is a positive smooth function such that $\rho_i=1$ on $U_i=V_i\cap C$. The first purpose of this section is to prove the following

Theorem 1. *In the above situation there exist, for any real number n' greater than n, a neighborhood V_0 of C and a strongly plurisubharmonic function $\Phi(p)$ on V_0-C which increases with the same order as $1/r(p)^{n'}$ when p approaches C.*

Corollary. *The curve C has a fundamental system of strongly pseudoconcave neighborhoods.*

2. Let us begin with some preliminaries. Let E be a flat line bundle over the neighborhood $V=\cup V_i$ of C defined by a multiplicative 1-cocycle $\{\tau_{ik}\}\in Z^1(\mathfrak{V}, \boldsymbol{T})$. We denote by $\mathscr{D}(E)$ the sheaf over V of all germs of differentiable sections of E, and by $\mathscr{J}^\nu(E)$ the subsheaf of $\mathscr{D}(E)$ consisting of germs of differentiable sections of E which vanish on C with order ν. A differentiable 1-cochain $\{\varphi_{ik}\}\in C^1(\mathfrak{V}, \mathscr{D}(E))$ is called a 1-*cocycle modulo* $\mathscr{J}^\nu(E)$ if we have

$$\varphi_{ij}+\tau_{ij}\varphi_{jk}+\tau_{ik}\varphi_{ki}\in\Gamma(V_i\cap V_j\cap V_k, \mathscr{J}^\nu(E)),\ i, j, k\in I,$$

where $\Gamma(X, \mathscr{S})$ denotes as usual the set of all sections over X of a sheaf $\mathscr{S}$. The set of all differentiable 1-cocycles modulo $\mathscr{J}^\nu(E)$ is denoted by $Z^1(\mathfrak{V}, \mathscr{D}(E), \mathrm{mod}\,\mathscr{J}^\nu(E))$. A 1-cochain $\{\varphi_{ik}\}\in C^1(\mathfrak{V}, \mathscr{D}(E))$ is called the *coboundary modulo* $\mathscr{J}^\nu(E)$ of a 0-cochain $\{\varphi_i\}\in C^0(\mathfrak{V}, \mathscr{D}(E))$ if we have

$$\varphi_{ik}-\tau_{ik}\varphi_k+\varphi_i\in\Gamma(V_i\cap V_k, \mathscr{J}^\nu(E)),\ i, k\in I.$$

We denote by $\mathscr{I}^\nu(E)$ the subsheaf of $\mathscr{O}(E)$ consisting of germs of holomorphic sections which vanish on C with order ν, i.e., $\mathscr{I}^\nu(E)=\mathscr{O}(E)\cap\mathscr{J}^\nu(E)$. We set $Z^1(\mathfrak{V}, \mathscr{O}(E), \mathrm{mod}\,\mathscr{I}^\nu(E))=C^1(\mathfrak{V}, \mathscr{O}(E))\cap Z^1(\mathfrak{V}, \mathscr{D}(E), \mathrm{mod}\,\mathscr{J}^\nu(E))$, whose elements are called *holomorphic* 1-*cocycles modulo* $\mathscr{I}^\nu(E)$.

Lemma 1. *For any* $\{\varphi_{ik}\}\in Z^1(\mathfrak{B}, \mathcal{O}(E), \bmod \mathcal{I}^{\nu+1}(E))$, $\nu=0, 1,\dots, n$, *there exists a differentiable* 0-*cochain* $\{\varphi_i\}\in C^0(\mathfrak{B}, \mathcal{D}(E))$ *such that*

(i) $\{\varphi_{ik}\}$ *is the coboundary of* $\{\varphi_i\}$ *modulo* $\mathcal{I}^{\nu+1}(E)$,

(ii) *each* φ_i *is of the form*

$$\varphi_i(p)=\sum_{\lambda,\mu\geq 0,\lambda+\mu\leq\nu}\varphi_{i|\lambda\mu}(z_i(p))w_i(p)^\lambda\overline{w_i(p)}^\mu,$$

where $\varphi_{i|\lambda\mu}(z_i)$ *are harmonic functions of the variable* z_i.

Proof. Since the restriction $\{\varphi_{ik}|U_i\cap U_k\}$ of $\{\varphi_{ik}\}$ to C is in $Z^1(\mathfrak{U}, \mathcal{O}(E|C))$, we have a 0-cochain $\{\varphi_{i|00}\}$ consisting of harmonic sections on U_i such that $\{\varphi_{ik}|U_i\cap U_k\}$ is the coboundary of $\{\varphi_{i|00}\}$ (see § 1, 2). We extend each $\{\varphi_{i|00}\}$ to a pluriharmonic function on V_i depending only on the variable z_i, which we denote again by $\varphi_{i|00}$. Then we obtain a 0-cochain $\{\varphi_{i|00}\}\in C^0(\mathfrak{B}, \mathcal{D}(E))$ such that $\{\varphi_{ik}\}$ is the coboundary of $\{\varphi_{i|00}\}$ modulo $\mathcal{I}^1(E)$. This shows the lemma for $\nu=0$. We proceed by induction for $\nu\geq 1$. Assume the lemma for any flat line bundle E with $\nu-1$ in the place of ν. Since $\varphi_{ik}-\tau_{ik}\varphi_{k|00}+\varphi_{i|00}$ is pluriharmonic and vanishes on $U_i\cap U_k$, we have the decomposition

$$\varphi_{ik}-\tau_{ik}\varphi_{k|00}+\varphi_{i|00}=\xi_{ik}+\bar{\eta}_{ik} \qquad \text{on} \quad V_i\cap V_k,$$

where ξ_{ik} is holomorphic, $\bar{\eta}_{ik}$ is anti-holomorphic, and they vanish on $U_i\cap U_k$. Such a decomposition is obviously unique. Since $\{\varphi_{ik}-\tau_{ik}\varphi_{k|00}+\varphi_{i|00}\}$ is in $Z^1(\mathfrak{B}, \mathcal{D}(E), \bmod \mathcal{I}^{\nu+1}(E))$, we see easily that

$$\{\xi_{ik}\}\in Z^1(\mathfrak{B}, \mathcal{O}(E), \bmod \mathcal{I}^{\nu+1}(E))$$

and

$$\{\eta_{ik}\}\in Z^1(\mathfrak{B}, \mathcal{O}(E^{-1}), \bmod \mathcal{I}^{\nu+1}(E^{-1})).$$

Now, setting $\varphi'_{ik}=\xi_{ik}w_i^{-1}$, we have

$$\{\varphi'_{ik}\}\in Z^1(\mathfrak{B}, \mathcal{O}(E\otimes F^{-1}), \bmod \mathcal{I}^{\nu}(E\otimes F^{-1})),$$

where $F=\{t_{ik}\}$ is the flat line bundle defined in §2, 2. Indeed, from $t_{ik}w_k-w_i=O(w_i^{n+1})$ it follows that $t_{ik}^{-1}w_k^{-1}=w_i^{-1}+O(w_i^{n-1})$; hence

$$\begin{aligned}\varphi'_{ij}&+\tau_{ij}t_{ij}^{-1}\varphi'_{jk}+\tau_{ik}t_{ik}^{-1}\varphi'_{ki}\\&=\xi_{ij}w_i^{-1}+\tau_{ij}t_{ij}^{-1}\xi_{jk}w_j^{-1}+\tau_{ik}t_{ik}^{-1}\xi_{ki}w_k^{-1}\\&=(\xi_{ij}+\tau_{ij}\xi_{jk}+\tau_{ik}\xi_{ki})w_i^{-1}+O(w_i^n)=O(w_i^\nu).\end{aligned}$$

Similarly, setting $\varphi''_{ik}=\eta_{ik}w_i^{-1}$, we have

$$\{\varphi''_{ik}\}\in Z^1(\mathfrak{B}, \mathcal{O}(E^{-1}\otimes F^{-1}), \bmod \mathcal{I}^{\nu}(E^{-1}\otimes F^{-1})).$$

Now, by the hypothesis of induction, we have a 0-cochain $\{\varphi'_i\}\in C^0(\mathfrak{B}, \mathcal{D}(E\otimes F^{-1}))$ of the form

$$\varphi'_i=\sum_{\lambda,\mu\geq 0,\lambda+\mu\leq\nu-1}\varphi'_{i|\lambda\mu}(z_i)w_i^\lambda\bar{w}_i^\mu,$$

such that $\{\varphi'_{ik}\}$ is the coboundary of $\{\varphi'_i\}$ modulo $\mathscr{I}^\nu(F\otimes F^{-1})$; and a 0-cocain $\{\varphi''_i\}\in C^0(\mathfrak{V}, \mathscr{D}(E^{-1}\otimes F^{-1}))$ of the form

$$\varphi''_i = \sum_{\lambda,\mu\geqq 0,\lambda+\mu\leqq \nu-1} \varphi''_{i|\lambda\mu}(z_i)w_i^\lambda\bar{w}_i^\mu,$$

such that $\{\varphi''_{ik}\}$ is the cobundary of $\{\varphi''_i\}$ modulo $\mathscr{I}^\nu(E^{-1}\otimes F^{-1})$. Let $\varphi_i=\varphi_{i|00}+\varphi'_iw_i+\bar{\varphi}''_i\bar{w}_i$. Then $\{\varphi_i\}$ is a 0-cochain of the desired properties. q.e.d.

3. Proof of Theorem 1. Consider the additive Cousin data $\{w_i^{-n}\}$ and the corresponding holomorphic 1-cocycle $\{t_{ik}^{-n}w_k^{-n}-w_i^{-n}\}\in Z^1(\mathfrak{V}, \mathscr{O}(F^{-n}))$. Applying Lemma 1, for $\nu=n$, to this 1-cocycle, we obtain a differentiable 0-cochain $\{\varphi_i\}\in C^0(\mathfrak{V}, \mathscr{D}(F^{-n}))$ of the form

$$\varphi_i = \sum_{\lambda,\mu\geqq 0,\lambda+\mu\leqq n} \varphi_{i|\lambda\mu}(z_i)w_i^\lambda\bar{w}_i^\mu,$$

where $\varphi_{i|\lambda\mu}$ are harmonic, such that

$$t_{ik}^{-n}(w_k^{-n}-\varphi_k)-(w_i^{-n}-\varphi_i)\in\Gamma(V_i\cap V_k, \mathscr{I}^{n+1}(F^{-n})),\ i,\ k\in I.$$

By the assumption that the curve C is of type n, none of $\varphi_{i|00}$ is holomorphic. We can assume here, choosing the system $\{w_i\}$ suitably, that $\varphi_{i|00}$ are all anti-holomorphic. To see this we put

$$t_{ik}^{-n}w_k^{-n}-w_i^{-n}=-nf_{ik}(z_i)+\cdots \qquad \text{on} \quad V_i\cap V_k.$$

The n-th obstruction $\{-nf_{ik}\}\in Z^1(\mathfrak{U}, \mathscr{O}(N^{-n}))$ is the coboundary of the harmonic 0-cochain $\{\varphi_{i|00}\}$. We decompose each $\varphi_{i|00}$ into a sum $f_i+\bar{g}_i$ of a holomorphic function f_i and an anti-holomorphic function $\bar{g}_i$, and define a new system $\{\tilde{w}_i\}$ by

$$\tilde{w}_i^{-n}=w_i^{-n}-f_i(z_i) \qquad \text{on} \quad V_i.$$

Then we have

$$\begin{aligned} t_{ik}^{-n}\tilde{w}_k^{-n}-\tilde{w}_i^{-n} &= -nf_{ik}(z_i)-t_{ik}^{-n}f_k(z_k)+f_i(z_i)+\cdots \\ &= t_{ik}^{-n}\bar{g}_k(z_k)-\bar{g}_i(z_i)+\cdots \qquad \text{on} \quad V_i\cap V_k. \end{aligned}$$

This implies that the n-th obstruction associated with $\{\tilde{w}_i\}$ is the coboundary of the anti-holomorphic 0-cochain $\{\bar{g}_i\}$.

Thus we may assume that the 0-cochain $\{\varphi_i\}$ is of the form

$$\varphi_i=\bar{g}_i(z_i)+\sum_{\lambda,\mu\geqq 0,1\leqq\lambda+\mu\leqq n}\varphi_{i|\lambda\mu}(z_i)w_i^\lambda\bar{w}_i^\mu \qquad \text{on} \quad V_i.$$

Adding to each $w_i^{-n}-\varphi_i$ a correction term $\alpha_i\in\Gamma(V_i, \mathscr{I}^{n+1}(F^{-n}))$, we obtain a global differentiable section σ of F^{-n} over V with "pole" of order n on C:

$$\sigma=w_i^{-n}-\bar{g}_i(z_i)-\sum_{\lambda,\mu\geqq 0,1\leqq\lambda+\mu\leqq n}\varphi_{i|\lambda\mu}(z_i)w_i^\lambda\bar{w}_i^\mu+\alpha_i \qquad \text{on} \quad V_i.$$

Let us calculate the complex Hessian $H(|\sigma|^a)$ of the function $|\sigma|^a$ for $a>0$. We restrict our consideration in V_i and omit the index i. We write $s=|\sigma|^2$. Since

$$s=|w|^{-2n}-w^{-n}g-\bar{w}^{-n}\bar{g}-w^{-n}\overline{\sum\varphi_{\lambda\mu}w^{\lambda}\bar{w}^{\mu}}-\bar{w}^{-n}\sum\varphi_{\lambda\mu}w^{\lambda}\bar{w}^{\mu}+|g|^2+O(|w|)\,,$$

we have

$$s_w=-nw^{-n-1}\bar{w}^{-n}+\cdots,\ s_z=-w^{-n}g_z+\cdots,$$

and

$$H(s)=\begin{pmatrix} s_{w\bar{w}} & s_{w\bar{z}} \\ s_{z\bar{w}} & s_{z\bar{z}} \end{pmatrix}=\begin{pmatrix} n^2|w|^{-2n-2}+\cdots & O(|w|^{-n}) \\ O(|w|^{-n}) & |g_z|^2+\cdots \end{pmatrix}$$

Here we have $s_{z\bar{z}}=|g_z|^2+\cdots$, because $\varphi_{\lambda\mu}(z)$ are harmonic. From

$$(|\sigma|^a)_w=(s^{\frac{a}{2}})_w=\frac{a}{2}s^{\frac{a}{2}-1}s_w,$$

$$(|\sigma|^a)_{w\bar{z}}=\frac{a}{2}s^{\frac{a}{2}-1}s_{w\bar{z}}+\frac{a}{2}\left(\frac{a}{2}-1\right)s^{\frac{a}{2}-2}s_w s_{\bar{z}},\ \text{etc.},$$

it follows that

$$\frac{2}{a}|\sigma|^{2-a}H(|\sigma|^a)=\frac{2}{a}s^{1-\frac{a}{2}}H(s^{\frac{a}{2}})=H(s)+\left(\frac{a}{2}-1\right)s^{-1}\begin{pmatrix} |s_w|^2 & s_w s_{\bar{z}} \\ s_z s_{\bar{w}} & |s_z|^2 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{a}{2}n^2|w|^{-2n-2}+\cdots & \left(\frac{a}{2}-1\right)nw^{-1}\bar{w}^{-n}\bar{g}_z+\cdots \\ \left(\frac{a}{2}-1\right)n\bar{w}^{-1}w^{-n}g_z+\cdots & \frac{a}{2}|g_z|^2+\cdots \end{pmatrix},$$

since $|\sigma|\sim|w|^{-n}$, $s\sim|w|^{-2n}$. Therefore

$$\det\left(\frac{2}{a}|\sigma|^{2-a}H(|\sigma|^a)\right)=(a-1)n^2|w|^{-2n-2}|g_z|^2+\cdots.$$

Now we put $Z=\bigcup_{i\in I}\{p\in U_i|(g_i)_{z_i}(z_i(p))=0\}$. Since Z is the set of the zeros of the holomorphic 1-form ∂g_i on C with coefficients in N^{-n} which does not vanish identically, Z consists of a finite number of points. By the above calculation, we infer that, for $a>1$, there is a neighborhood V' of $C-Z$ such that $|\sigma|^a$ is strongly plurisubharmonic on $V'-(C-Z)$.

Our intention is to modify $|\sigma|^a$ to obtain a strongly plurisubharmonic function on V_0-C, where V_0 is a sufficiently small neighborhood of C. Let q be a point of Z and assume that q is in U_i. We define a function $\beta_q(p)$ on V_i by

$$\beta_q(p)=\rho(|z_i(p)-z_i(q)|)|z_i(p)-z_i(q)|^2|w_i(p)|^{(2-a)n},$$

where $\rho(x)$ is a non-negative smooth function of the variable x, $0\leqq x<+\infty$, such that

$$\rho(x)=\begin{cases} 1 & \text{for}\quad 0\leqq x\leqq x_0 \\ 0 & \text{for}\quad x\geqq 2x_0 \end{cases}$$

x_0 being a sufficiently small number. Then we have

$$H(\beta_q)=|w_i|^{(2-a)n}\begin{pmatrix} O(|w_i|^{-2}) & O(|w_i|^{-1}) \\ O(|w_i|^{-1}) & O(1) \end{pmatrix},$$

and

$$(\beta_q)_{z\bar{z}}=|w_i|^{(2-a)n} \qquad \text{if} \quad |z_i(p)-z_i(q)|<x_0.$$

We define $\Phi_a=|\sigma|^a+\varepsilon\sum_{q\in z}\beta_q$, where $\varepsilon>0$. If ε is sufficiently small, we can find a neighborhood V_0 of C such that Φ_a is strongly plurisubharmonic on V_0-C. Since $n'>n$, the function $\Phi_{n'/n}$ has the desired property. Thus Theorem 1 is proved.

4. Let us consider the function $|\sigma|^a$ with $0<a<1$. There is a neighborhood V' of $C-Z$ such that the complex Hessian $H(|\sigma|^a)$ of $|\sigma|^a$ has one positive and one negative eigenvalues at every point in $V'-(C-Z)$. We define $\Phi_a=|\sigma|^a-\varepsilon\sum_{q\in z}\beta_q$, where ε is a sufficiently small positive number. Then there is a neighborhood V_0 of C such that Φ_a has the above property at every point in V_0-C. With the aid of this function, we prove the following

Theorem 2. *Let V be a neighborhood of the curve C and let Ψ be a plurisubharmonic function on $V-C$. If $\Psi(p)=o(1/r(p)^{n''})$ as p approaches C, where n'' is a positive real number smaller than n, then there exists a neighborhood V_0 of C such that Ψ is constant on V_0-C.*

Corollary. *Let f be a holomorphic function on $V-C$, where V is a neighborhood of C. If $\log^+|f(p)|=o(1/r(p)^{n''})$ as p approaches C, then f is constant.*

To prove Theorem 2, we show first the following

Lemma 2. *In the situation of Theorem 2, if the plurisubharmonic function Ψ is bounded from above, then there exists a neighborhood V_0 of C such that Ψ is constant on V_0-C.*

Proof. By a theorem of Grauert and Remmert [5], Ψ is extended to a plurisubharmonic function on V, which we denote also by Ψ. By Theorem 1, we can take in V a relatively compact strongly pseudoconcave neighborhood V_0 of C. We will show that Ψ is constant on V_0. Let the maximum of Ψ on $\bar{V}_0$ be attained at a boundary point p_0 of V_0. Since V_0 is strongly pseudoconcave, there exist a neighborhood W of p_0 and an analytic set X in W such that $X\subset\bar{V}_0\cap W$ and such that $X\cap\partial V_0=\{p_0\}$, where ∂V_0 denotes the boundary of V_0. Since Ψ is plurisubharmonic, there exists a point p_1 in $X-\{p_0\}$ such that $\Psi(p_1)\geqq\Psi(p_0)$. This shows that the maximum of Ψ is attained at an interior point of V_0. Hence Ψ is constant on V_0. q.e.d.

Now let us prove Theorem 2. In view of Lemma 2, it suffices to derive a contradiction from the assumption that Ψ is unbounded from above. We take the function Φ_a with $a=\frac{n''}{n}$. Since $\Phi_a\sim 1/r^{n''}$ and $\limsup_{p\to C} r(p)^{n''}\Psi(p)=0$, we have

$\limsup_{p \to C} \Psi(p)/\Phi_a(p) = 0$. We choose in V a relatively compact strongly pseudoconcave neighborhood V_0 of C such that, at every point in $V_0 - C$, the complex Hessian of Φ_a has one positive and one negative eigenvalues. We may assume that Ψ is non-positive on the boundary ∂V_0 of V_0, since it suffices to prove the theorem for $\Psi - A$ in the place of Ψ, A being a sufficiently large positive number. The function Ψ/Φ_a is upper semi-continuous and takes a positive value at some point in $V_0 - C$, since Ψ is unbounded. Therefore Ψ/Φ_a attains its maximum B at an interor point p_0 of $V_0 - C$. We have $\Psi(p) \leqq B\Phi_a(p)$, $p \in V_0 - C$ and $\Psi(p_0) = B\Phi_a(p_0)$. Since the complex Hessian of Φ_a at p_0 has a negative eigenvalue, there exists a holomorphic map f of the unit disk $\{\zeta \in \boldsymbol{C} \mid |\zeta| < 1\}$ to $V_0 - C$ such that $f(0) = p_0$ and that $\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}(\Phi_a \circ f)(0) < 0$. We have

$$\frac{1}{2\pi}\int_0^{2\pi}(\Phi_a \circ f)(\rho e^{i\theta})d\theta < \Phi_a(f(0)) = \Phi_a(p_0),$$

if ρ is sufficiently small. From this and the fact that $\Psi \circ f$ is subharmonic, we have

$$\Psi(p_0) \leqq \frac{1}{2\pi}\int_0^{2\pi}(\Psi \circ f)(\rho e^{i\theta})\,d\theta \leqq \frac{B}{2\pi}\int_0^{2\pi}(\Phi_a \circ f)(\rho e^{i\theta})d\theta$$
$$< B\Phi_a(p_0) = \Psi(p_0),$$

which is a contradiction. Thus Theorem 2 is proved.

§4. The case of infinite type

1. We consider in this section the case where the curve C is of infinite type.

Some definitions are necessary to state the result. As was mentioned in §1, the Picard variety $\mathfrak{P}(C)$ of the compact Riemann surface C can be identified with the multiplicative group $H^1(C, \boldsymbol{T})$ consisting of all flat line bundles over C. As a real Lie group, $\mathfrak{P}(C) = H^1(C, \boldsymbol{T})$ is a torus of dimension $2g$, where g is the genus of C. We introduce on $\mathfrak{P}(C)$ an invariant distance d, i.e., $d(E_1^{-1}, E_2^{-1}) = d(E_1, E_2)$ and $d(E_1 \otimes G, E_2 \otimes G) = d(E_1, E_2)$ for any $E_1, E_2, G \in \mathfrak{P}(C)$. There exist on $\mathfrak{P}(C)$ infinitely many such distances, but obviously they are all equivalent to one another. Now we denote by $\mathfrak{E}_0$ the subset of $\mathfrak{P}(C)$ consisting of all elements of finite order. We denote by $\mathfrak{E}_1$ the subset of $\mathfrak{P}(C) - \mathfrak{E}_0$ consisting of all elements E such that

$$-\log d(\mathbf{1}, E^\nu) = O(\log \nu) \quad \text{as} \quad \nu \longrightarrow +\infty.$$

This condition is equivalent to the condition: There exists a positive number α such that

$$d(\mathbf{1}, E^\nu) \geqq (2\nu)^{-\alpha} \quad \text{for} \quad \nu = 1, 2, \dots.$$

Clearly the set $\mathfrak{E}_1$ is determined independently of the choice of the invariant distance d. It is easy to see that $\mathfrak{P}(C) - \mathfrak{E}_1$ is of Lebesgue measure zero. In this sense the elements of $\mathfrak{E}_0 \cup \mathfrak{E}_1$ are general. But we note that $\mathfrak{E}_1$ is the union of a countable number of nowhere dense closed sets; it is therefore a set of the first category.

The purpose of this section is to prove the following

Theorem 3. *Suppose that the curve C is of infinite type and that the complex normal bundle N of C is contained in $\mathfrak{E}_0 \cup \mathfrak{E}_1$. Then there exist an open covering $\{V'_i\}$ of a neighborhood V' of C and a system $\{u_i\}$ of holomorphic functions u_i on V'_i such that $u_i=0$ is a local equation of $C \cap V'_i$ on each V'_i and that u_i/u_k is a constant of modulus 1 on each $V'_i \cap V'_k$.*

We note that the conclusion of Theorem 3 is equivalent to either of the following statements (i), (ii):

(i) *There exists a multiplicative holomorphic function u with divisor C on V'. Here, by a multiplicative function on V' we mean a function u defined on a covering manifold of V' whose modulus $|u|$ is a (single-valued) function on V'.*

(ii) *The restriction $[C] \mid V'$ of the complex line bundle $[C]$ to V' admits a structure of flat line bundle, i.e., $[C] \mid V' = F \mid V'$. (See §2, 2.)*

Corollary. *If the curve C is of infinite type and N is of finite order m, then there exists an m-valued multiplicative holomorphic function u with divisor C on a neighborhood V' of C, such that u^m is a (single-valued) holomorphic function with divisor mC.*

2. Construction of formal power series. We choose and fix a finite open covering $\mathfrak{B} = \{V_i\}$ of a neighborhood of C consisting of small dicylinders V_i of the form $|w_i|<1$, $|z_i|<1$, where (w_i, z_i) is a local coordinate system of the manifold S which covers the closure $\bar{V}_i$ of V_i. We assume that the functions w_i satisfy the conditions: (i) $w_i=0$ is a local equation of $C \cap V_i$ in each V_i, and (ii) the system $\{w_i\}$ is of type (at least) 1, i.e., the restriction of w_i/w_k to $C \cap V_i \cap V_k$ is a constant of modulus 1. Then the transformation of local coordinates on $V_i \cap V_k$ is of the form

$$(1) \qquad \begin{cases} w_k(p) = \varphi_{ki}(w_i(p), z_i(p)) = t_{ki} w_i(p) + \sum_{\nu=2}^{\infty} \varphi_{ki|\nu}(z_i(p)) w_i(p)^\nu, \\ z_k(p) = \psi_{ki}(w_i(p), z_i(p)). \end{cases}$$

We set $U_i = C \cap V_i$. Then $\mathfrak{U} = \{U_i\}$ is a finite open covering of C by disks U_i: $|z_i|<1$, and the transformation of local coordinates on $U_i \cap U_k$ is of the form

$$z_k(p) = \psi_{ki}(0, z_i(p)).$$

To prove Theorem 3, it suffices to construct a system $\{u_i\}$ of holomorphic functions u_i defined respectively on a neighborhood V'_i ($\subseteq V_i$) of U_i satisfying the conditions: (i) each u_i is of the form

$$u_i(p) = g_i(w_i(p), z_i(p)) = w_i(p) + (\text{terms of order} \geqq 2),$$

and (ii) $u_i = t_{ik} u_k$ on $V'_i \cap V'_k$. By (1), the condition (ii) is equivalent to

$$(2) \qquad g_i(w_i, z_i) = t_{ik} g_k(\varphi_{ki}(w_i, z_i), \psi_{ki}(w_i, z_i)),$$

where $(w_i, z_i) = (w_i(p), z_i(p))$, $p \in V'_i \cap V'_k$.

We will determine each $u_i=g_i(w_i, z_i)$ as an implicit function defined by the equation

$$w_i=f_i(u_i, z_i)=u_i+\sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i^\nu,$$

where $f_i(u_i, z_i)$ is a power series in u_i whose coefficients $f_{i|\nu}(z_i)$ are holomorphic functions of the variable z_i, $|z_i|<1$. The condition (2) is equivalent to

$$\varphi_{ki}(f_i(u_i, z_i), z_i)=f_k(t_{ki}u_i, \psi_{ki}(f_i(u_i, z_i), z_i)). \tag{3}$$

We expand the left-hand side of (3) into the power series

$$\varphi_{ki}(f_i(u_i, z_i), z_i)=t_{ki}(u_i+\sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i^\nu)+t_{ki}\sum_{\nu=2}^{\infty} h'_{ik|\nu}(z_i)u_i^\nu, \tag{4}$$

where

$$t_{ki}\sum_{\nu=2}^{\infty} h'_{ik|\nu}(z_i)u_i^\nu=\sum_{\nu=2}^{\infty} \varphi_{ki|\nu}(z_i)\Big(u_i+\sum_{\mu=2}^{\infty} f_{i|\mu}(z_i)u_i^\mu\Big)^\nu. \tag{5}$$

The right-hand side of (3) is expanded into the form

$$t_{ki}u_i+\sum_{\nu=2}^{\infty} f_{k|\nu}(\psi_{ki}(f_i(u_i, z_i), z_i))(t_{ki}u_i)^\nu.$$

Letting

$$f_{k|\nu}(\psi_{ki}(w_i, z_i))=f_{k|\nu}(\psi_{ki}(0, z_i))+\sum_{\mu=1}^{\infty} f_{ki|\nu\mu}(z_i)w_i^\mu,$$

we have

$$\begin{aligned}(6)\quad & f_k(t_{ki}u_i, \psi_{ki}(f_i(u_i, z_i), z_i))\\ &=t_{ki}u_i+\sum_{\nu=2}^{\infty} f_{k|\nu}(\psi_{ki}(0, z_i))(t_{ki}u_i)^\nu+t_{ki}\sum_{\nu=2}^{\infty} h''_{ik|\nu}(z_i)u_i^\nu,\end{aligned}$$

where

$$t_{ki}\sum_{\nu=2}^{\infty} h''_{ik|\nu}(z_i)u_i^\nu=\sum_{\nu=2}^{\infty}\Big[\sum_{\mu=1}^{\infty} f_{ki|\nu\mu}(z_i)(u_i+\sum_{\lambda=2}^{\infty} f_{i|\lambda}(z_i)u_i^\lambda)^\mu\Big](t_{ki}u_i)^\nu. \tag{7}$$

We infer from (5) and (7) that, if $f_{i|2},\dots,f_{i|\nu}$, $i\in I$, are determined, then $h'_{ik|\nu+1}$ and $h''_{ik|\nu+1}$ are determined independently of $f_{i|\nu+1}, f_{i|\nu+2},\dots$.

To obtain $f_i(u_i, z_i)$, $i\in I$, satisfying (3) as formal power series in u_i, it suffices to determine successively the systems $\{f_{i|\nu+1}\}_{i\in I}$, $\nu=1, 2,\dots$, in such a way that

$$t_{ik}^{-\nu}f_{k|\nu+1}(\psi_{ki}(0, z_i))-f_{i|\nu+1}(z_i)=h_{ik|\nu+1}(z_i), \tag{8, ν}$$

for $z_i=z_i(p)$, $p\in U_i\cap U_k$, are satisfied, where we have set

$$h_{ik|\nu+1}=h'_{ik|\nu+1}-h''_{ik|\nu+1}.$$

Suppose that $\{f_{i|2}\},\dots,\{f_{i|\nu}\}$ satisfying (8, 1),…, (8, $\nu-1$), respectively, are already determined. We shall show that $\{-h_{ik|\nu+1}\}$ is the ν-th obstruction ($\in Z^1(\mathfrak{U}, \mathcal{O}(N^{-\nu}))$) associated with a system of functions of type ν (see §2, 1). Then by the assumption that the curve C is of infinite type, $\{f_{i|\nu+1}\}$ satisfying (8, ν) will be obtained.

Consider the functions $v_i=\tilde{g}_i(w_i, z_i)$, $i\in I$, defined implicitly by the equations

$$w_i=f_i^\nu(v_i, z_i)=v_i+\sum_{\mu=2}^{\nu} f_{i|\mu}(z_i)v_i^\mu,$$

respectively; and let $\{v_i(p)\}$ be the system of functions $v_i(p)=\tilde{g}_i(w_i(p), z_i(p))$ on V_i. It follows from (4) and (6) that

$$\begin{aligned} w_k-f_k^\nu(t_{ki}v_i, z_k) &= \varphi_{ki}(f_i^\nu(v_i, z_i), z_i)-f_k^\nu(t_{ki}v_i, \psi_{ki}(f_i(v_i, z_i), z_i)) \\ &= t_{ki}h_{ik|\nu+1}(z_i)v_i^{\nu+1}+\cdots \end{aligned}$$

where $w_i=w_i(p)$, $z_i=z_i(p)$, $v_i=v_i(p)$, $w_k=w_k(p)$, $z_k=z_k(p)$, $p\in V_i\cap V_k$, and where $\cdots$ denotes the term which vanishes on $C\cap V_i\cap V_k$ with order $\geqq \nu+2$. Therefore

$$\tilde{g}_k(w_k-t_{ki}h_{ik|\nu+1}(z_i)v_i^{\nu+1}+\cdots, z_k)=t_{ki}v_i.$$

Hence

$$v_k+t_{ki}h_{ik|\nu+1}(z_i)v_i^{\nu+1}+\cdots=t_{ki}v_i,$$

or, multiplying by t_{ik},

$$t_{ik}v_k-v_i=-h_{ik|\nu+1}(z_i)v_i^{\nu+1}+\cdots.$$

This implies that $\{-h_{ik|\nu+1}\}$ is the ν-th obstruction asscoiated with $\{v_i\}$.

Thus we can obtain $f_i(u_i, z_i)=u_i+\sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i^\nu$, $i\in I$, as formal power series.

3. Estimate of obstructions. For two power series $a(u)=\sum_{\nu=0}^{\infty} a_\nu u^\nu$ and $A(u)=\sum_{\nu=0}^{\infty} A_\nu u^\nu$, $A_\nu\geqq 0$, we write $a(u)\ll A(u)$, when $|a_\nu|\leqq A_\nu$ for $\nu=0, 1, 2,\ldots$. We shall show that the power series $f_i(u_i, z_i)=u_i+\sum_{\nu=2}^{\infty} f_{i|\nu}(z_i)u_i^\nu$, $i\in I$, of the preceding paragraph can be constructed in such a way that there is a power series $A(u)=u+\sum_{\nu=2}^{\infty} A_\nu u^\nu$ with constant coefficients and with positive radius of convergence satisfying

$$f_i(u_i, z_i)\ll A(u_i), \qquad i\in I. \tag{9}$$

If we write $f_i^\nu(u_i, z_i)=u_i+\sum_{\mu=2}^{\nu} f_{i|\mu}(z_i)u_i^\mu$ and $A^\nu(u)=u+\sum_{\mu=2}^{\nu} A_\mu u^\mu$, then (9) is equivalent to the conditions for $\nu=1, 2,\ldots$

$$f_i^\nu(u_i, z_i)\ll A^\nu(u_i), \qquad i\in I. \tag{10, ν}$$

Suppose that $f_i^\nu(u_i, z_i)$, and $A^\nu(u)$ satisfying (10, ν) are already determined. We shall first make an estimate of $h'_{ik|\nu+1}$ and $h''_{ik|\nu+1}$, in terms of $A_2,\ldots, A_\nu$. Let R be a sufficiently large number such that $|\varphi_{ki|\mu}(z_i)|\leqq R^\mu$, $\mu=2, 3,\ldots$. Then from

(5) it follows that

$$\sum_{\mu=2}^{\nu+1} h'_{ik}(z_i)u_i^\mu \ll \sum_{\mu=2}^{\infty} R^\mu(A^\nu(u_i))^\mu = \frac{R^2(A^\nu(u_i))^2}{1-RA^\nu(u_i)}.$$

Let $\mathfrak{U}^*=\{U_i^*\}$ be an open covering of C such that each U_i^* is relatively compact in U_i. We choose a sufficiently large number Q such that, for every point p in $U_i \cap U_k^*$, the closed disk

$$\Delta_p=\{q \in V_i | z_i(q)=z_i(p),\ |w_i(q)|=1/Q\}$$

is contained in V_k. Since

$$|f_{k|\mu}(\psi_{ki}(w_i(q),\ z_i(q)))| \leqq A_\mu, \qquad \text{on} \quad \Delta_p \subseteq V_i \cap V_k,\ \mu=2,\dots,\nu,$$

we have

$$|f_{ki|\mu\lambda}(z_i(p))| \leqq A_\mu Q^\lambda, \qquad \text{on} \quad U_i \cap U_k^*,\ \mu=2,\dots,\nu,\ \lambda=1, 2,\dots.$$

Therefore, by (7), we have

$$\sum_{\mu=2}^{\nu+1} h''_{ik|\mu}(z_i)u_i^\mu \ll \sum_{\mu=2}^{\nu} [\sum_{\lambda=1}^{\infty} A_\mu Q^\lambda (A^\nu(u_i))^\lambda] u_i^\mu$$

$$=(A^\nu(u_i)-u_i)\frac{QA^\nu(u_i)}{1-QA^\nu(u_i)} \ll \frac{Q(A^\nu(u_i))^2}{1-QA^\nu(u_i)},$$

for $z_i=z_i(p)$, $p \in U_i \subset U_k^*$. Thus we have

$$\sum_{\mu=2}^{\nu+1} h_{ik|\mu}(z_i(p))u_i^\mu \ll \frac{M'(A^\nu(u_i))^2}{1-M'A^\nu(u_i)} \qquad \text{for} \quad p \in U_i \cap U_k^*,$$

where $M'=2\max\{R, R^2, Q\}$.
This implies that

$$|h_{ik|\nu+1}(z_i(p))| \leqq \left(\text{the coefficient of } u^{\nu+1} \text{ in } \frac{M'(A^\nu(u))^2}{1-M'A^\nu(u)}\right),$$

for $p \in U_i \cap U_k^*$.

To make an estimate of $h_{ik|\nu+1}$ on $U_i \cap U_k$, we use the fact that $\{h_{ik|\nu+1}\}$ is a 1-cocycle. For any point p in $U_i \cap U_k$, there exists a $j \in I$ such that $p \in U_j^*$. From

$$h_{ij|\nu+1}(z_i(p))+t_{ij}^{-\nu}h_{jk|\nu+1}(z_j(p))+t_{ik}^{-\nu}h_{ki|\nu+1}(z_k(p))=0$$

it follows that

$$|h_{ik|\nu+1}(z_i(p))| \leqq |h_{ij|\nu+1}(z_i(p))|+|h_{jk|\nu+1}(z_j(p))|.$$

Hence, setting $M=2M'$, we have

$$(11) \qquad |h_{ik|\nu+1}(z_i(p))| \leqq \left(\text{the coefficient of } u^{\nu+1} \text{ in } \frac{M(A^\nu(u))^2}{1-MA^\nu(u)}\right),$$

for $p \in U_i \cap U_k$.

4. Proof of convergence for the case $N \in \mathfrak{E}_0$ Let E be a complex line bundle over a compact Riemann surface C defined by a multiplicative 1-cocycle on the nerve of a finite open covering $\mathfrak{U}=\{U_i\}$ of C. For a 0-cochain $\mathfrak{f}^0=\{f_i\}\in C^0(\mathfrak{U}, \mathcal{O}(E))$ and a 1-cochain $\mathfrak{f}^1=\{f_{ik}\}\in C^1(\mathfrak{U}, \mathcal{O}(E))$, we define, respectively,

$$\|\mathfrak{f}^0\|=\max_i \sup_{p\in U_i} |f_i(p)|,$$

and

$$\|\mathfrak{f}^1\|=\max_{i,k} \sup_{p\in U_i\cap U_k} |f_{ik}(p)|.$$

Lemma 3. (*Kodaira-Spencer* [8]) *There exists a constant* $K=K(E)$ *such that, for any* 1*-cocycle* $\mathfrak{f}^1\in Z^1(\mathfrak{U}, \mathcal{O}(E))$ *with* $\|\mathfrak{f}^1\|<+\infty$ *which is cohomologous to zero, there exists a* 0*-cochain* $\mathfrak{f}^0\in C^0(\mathfrak{U}, \mathcal{O}(E))$ *such that* $\mathfrak{f}^1$ *is the coboundary of* $\mathfrak{f}^0$ *and that* $\|\mathfrak{f}^0\|\leqq K\|\mathfrak{f}^1\|$.

Now returning to the proof of Theorem 3, let us consider the case $N\in\mathfrak{E}_0$, i.e., the case where N is of finite order, say of order m. We put $K=\max_{\nu=1,2,\dots} K(N^{-\nu})=\max_{\nu=1,2,\dots,m} K(N^{-\nu})$. Since the obstructions $\{h_{ik|\nu+1}\}$ are in $Z^1(\mathfrak{U}, \mathcal{O}(N^{-\nu}))$, we can determine $\{f_{i|\nu+1}\}$ satisfying (8, ν) in such a way that

$$\|\{f_{i|\nu+1}\}\|\leqq K\|\{h_{ik|\nu+1}\}\|. \tag{12}$$

We define the power series $A(u)=u+\sum_{\nu=2}^{\infty} A_\nu u^\nu$ as the solution of the functional equation

$$A(u)-u=K\frac{M(A(u))^2}{1-MA(u)},$$

where M is as in (11). Clearly $A(u)$ has a positive radius of convergence. Suppose that $f_i^\nu(u_i, z_i)=u_i+\sum_{\mu=2}^{\nu} f_{i|\mu}(z_i)u_i^\mu$, $i\in I$, satisfying (8, 1),..., (8, $\nu-1$) and (10, ν) are already determined. Then we can obtain $f_{i|\nu+1}(z_i)$, $i\in I$, satisfying (8, ν) and (12). By (11) we have

$$\begin{aligned}\|\{f_{i|\nu+1}\}\|\leqq K\|\{h_{ik|\nu+1}\}\|&\leqq K\left(\text{the coefficient of } u^{\nu+1} \text{ in } \frac{M(A(u))^2}{1-MA(u)}\right)\\ &=A_{\nu+1}.\end{aligned}$$

Hence $f_i^{\nu+1}(u_i, z_i)=f_i^\nu(u_i, z_i)+f_{i|\nu+1}(z_i)u_i^{\nu+1}$, $i\in I$, satisfy (10, $\nu+1$). Thus we can obtain $f_i(u_i, z_i)$, $i\in I$, satisfying (9). Theorem 3 is thereby proved for the case $N\in\mathfrak{E}_0$.

5. Let C be a compact Riemann surface and let $\mathfrak{U}=\{U_i\}_{i\in I}$ be a *fixed* finite open covering of C by disks U_i: $|z_i|<1$. For two flat line bundles E_1 and E_2 over C, we define a distance $d(E_1, E_2)$ by

$$d(E_1, E_2)=\inf \max_{i,k\in I} |t_{ik}^{(1)}-t_{ik}^{(2)}|,$$

where the infimum is taken over the sets of all multiplicative 1-cocycles $\{t_{ik}^{(1)}\}$ and $\{t_{ik}^{(2)}\}$ representing E_1 and E_2 respectively. Clearly the distance thus defined is an invariant distance on $\mathfrak{P}(C)=H^1(C, \boldsymbol{T})$ (see no. 1 of this section). We have

$$d(\mathbf{1}, E)=\inf \max_{i,k\in I} |1-t_{ik}|,$$

where the infimum is taken over the set of all multiplicative 1-cocycles $\{t_{ik}\}$ representing the flat line bundle E; or equivalently

$$d(\mathbf{1}, E)=\inf \max_{i,k\in I} |t_i-t_{ik}t_k|,$$

where $\{t_{ik}\}$ is a fixed multiplicative 1-cocycle representing E and the infimum is taken over the set $C^0(\mathfrak{U}, \boldsymbol{T})$ of all multiplicative 0-cochains $\{t_i\}$.

We denote by δ the coboundary map $C^0(\mathfrak{U}, \mathcal{O}(E))\to C^1(\mathfrak{U}, \mathcal{O}(E))$. If E is a flat line bundle and $E\neq\mathbf{1}$, then δ is injective.

Lemma 4. *There exists a positive constant K such that, for any flat line bundle E over C and for any 0-cochain $\mathfrak{f}\in C^0(\mathfrak{U}, \mathcal{O}(E))$, the inequality*

$$d(\mathbf{1}, E)\|\mathfrak{f}\|\leqq K\|\delta\mathfrak{f}\|$$

holds.

Proof. We observe first that, there exists a positive constant ε_0 such that, for any flat line bundle E and for any $\mathfrak{f}=\{f_i\}\in C^0(\mathfrak{U}, \mathcal{O}(E))$ with $\|\mathfrak{f}\|=1$ and $\|\delta\mathfrak{f}\|\leqq\varepsilon_0$, we have $\min_i \inf_{p\in U_i} |f_i(p)|\geqq 1/2$. To see this let us assume the contrary. Then we can find sequences of flat line bundles $E_\nu=\{t_{ik}^{(\nu)}\}$ and of 0-cochains $\mathfrak{f}_\nu=\{f_i^{(\nu)}\}\in C^0(\mathfrak{U}, \mathcal{O}(E_\nu))$, $\nu=1, 2,\dots,$ such that $\|\mathfrak{f}_\nu\|=1$, $\min_i \inf_{p\in U_i} |f_i^{(\nu)}(p)|<1/2$, and that $\|\delta\mathfrak{f}_\nu\|\to 0$ as $\nu\to\infty$. We can find subsequences E_{ν_κ} and $\mathfrak{f}_{\nu_\kappa}$, $\kappa=1, 2,\dots,$ such that, for each i, $k\in I$, the sequence $t_{ik}^{(\nu_\kappa)}$ tends to a limit $t_{ik}^{(0)}$, and that, for each $i\in I$, the sequence $f_i^{(\nu_\kappa)}$ converges uniformly on every compact set in U_i to a holomorphic function $f_i^{(0)}$. Then $\{t_{ik}^{(0)}\}$ is a multiplicative 1-cocycle and defines a flat line bundle E_0 over C. Moreover, since $t_{ik}^{(\nu_\kappa)}f_k^{(\nu_\kappa)}-f_i^{(\nu_\kappa)}$ converges to 0, we get $t_{ik}^{(0)}f_k^{(0)}-f_i^{(0)}=0$ on $U_i\cap U_k$; which implies that $\{f_i^{(0)}\}$ is a global section of E_0. Therefore each $f_i^{(0)}$ is a constant and clearly we have $|f_i^{(0)}|=1$. Let $\mathfrak{U}^*=\{U_i^*\}_{i\in I}$ be an open covering of C such that each U_i^* is relatively compact in U_i and such that $U_i\cap U_k\neq\emptyset$ implies $U_i^*\cap U_k^*\neq\emptyset$. Let p be any point in U_i and assume that p is in U_k^*. From

$$\begin{aligned}|f_i^{(\nu_\kappa)}(p)-f_i^{(0)}(p)|&\leqq|f_i^{(\nu_\kappa)}(p)-t_{ik}^{(\nu_\kappa)}f_k^{(\nu_\kappa)}(p)|+|t_{ik}^{(\nu_\kappa)}f_k^{(\nu_\kappa)}(p)-t_{ik}^{(0)}f_k^{(0)}(p)|\\ &\leqq\|\delta\mathfrak{f}_{\nu_\kappa}\|+|t_{ik}^{(\nu_\kappa)}f_k^{(\nu_\kappa)}(p)-t_{ik}^{(0)}f_k^{(0)}(p)|\end{aligned}$$

it follows that $f_i^{(\nu_\kappa)}$ converges uniformly on U_i. This contradicts that $\min_{i\in I} \inf_{p\in U_i} |f_i^{(\nu_\kappa)}(p)|<1/2$, and the statement is proved.

In view of the fact that the distance $d(\mathbf{1}, E)$ is bounded, it suffices to prove the following asssertion: (*) There exists a positive constant K_0 such that, for any flat line bundle E and for any $\mathfrak{f}\in C^0(\mathfrak{U}, \mathcal{O}(E))$ with $\|\mathfrak{f}\|=1$ and $\|\delta\mathfrak{f}\|\leqq\varepsilon_0$, the inequality $d(\mathbf{1}, E)\leqq K_0\|\delta\mathfrak{f}\|$ holds.

We note that, for $\mathfrak{f}=\{f_i\}$ satisfying the above conditions, we have $\min \inf|f_i(p)|\geqq$

1/2; and hence

$$\left|\log \frac{1}{|f_k(p)|} - \log \frac{1}{|f_i(p)|}\right| \leqq 2||f_k(p)| - |f_i(p)||$$

$$\leqq 2|t_{ik} f_k(p) - f_i(p)| \leqq 2\|\delta\mathfrak{f}\|,$$

for $p \in U_i \cap U_k$.

Now we need the following

Sublemma. *There exists a positive constant K_1 such that, if $\{h_i\}$ is a system of bounded non-negative (real-valued) harmonic functions h_i on U_i satisfing the conditions:*

(i) $\min_{i \in I} \inf_{p \in U_i} h_i(p) = 0,$

(ii) $|h_k(p) - h_i(p)| \leqq \varepsilon$, *for* $p \in U_i \cap U_k$, $i, k \in I$,

then we have $\max_{i \in I} \sup_{p \in U_i} h_i(p) \leqq K_1 \varepsilon$.

Proof of the sublemma. By Harnack theorem, there exists a positive constant L such that, for every non-negative harmonic function h on U_i and for every pair of points p and p' in U_i^*, the inequality $h(p) \leqq Lh(p')$ holds. By the condition (i), there exist an $i_0 \in I$ and a point p_0 in $\overline{U}_{i_0}$ such that $\liminf_{p \in U_i, p \to p_0} h_{i_0}(p) = 0$. For any $k \in I$ and a point q in U_k, we can choose a sequence $i_1, \ldots, i_l \in I$ of length l such that $p_0 \in U_{i_1}^*$, $q \in U_{i_l}^*$ and that $U_{i_\nu}^* \cap U_{i_{\nu+1}}^* \neq \emptyset$ $(\nu = 1, \cdots, l-1)$. We take points p_ν $(\nu = 1, \cdots, l-1)$ respectively in $U_{i_\nu}^* \cap U_{i_{\nu+1}}^*$. We have $h_{i_\nu}(p_\nu) \leqq Lh_{i_\nu}(p_{\nu-1})$; and by the condition (ii), $h_{i_{\nu+1}}(p_\nu) \leqq h_{i_\nu}(p_\nu) + \varepsilon$. Hence

$$h_k(q) \leqq h_{i_l}(q) + \varepsilon \leqq Lh_{i_l}(p_{l-1}) + \varepsilon \leqq \cdots \leqq (L^l + L^{l-1} + \cdots + L + 1)\varepsilon.$$

The sequence $i_1, \ldots, i_l$ can be always so chosen that the length l does not exceed a fixed number l_0. Thus, $K_1 = L^{l_0} + \cdots + 1$ is a constant of the desired property. q.e.d.

If we apply the sublemma for $h_i(p) = \log \frac{1}{|f_i(p)|}$ and $\varepsilon = 2\|\delta\mathfrak{f}\|$, we have $\max_i \sup_{p \in U_i} \log \frac{1}{|f_i(p)|} \leqq 2K_1 \|\delta\mathfrak{f}\|$.

The following fact is easily proved: There exists a positive constant K_2 such that, if f is a holomorphic function on U_i such that $1 - \varepsilon \leqq |f| \leqq 1$, then it holds $|f(p) - f(p')| \leqq K_2 \varepsilon$ for any points p and p' in U_i^*.

To prove the assertion (*), we choose a point p_i in each U_i^* and put $t_i = f_i(p_i)/|f_i(p_i)|$. When $U_i \cap U_k \neq \emptyset$, we take a point p in $U_i^* \cap U_k^*$. Then

$$|t_{ik} t_k - t_i| \leqq |t_k - f_k(p)| + |t_{ik} f_k(p) - f_i(p)| + |f_i(p) - t_i|.$$

We have $|t_{ik} f_k(p) - f_i(p)| \leqq \|\delta\mathfrak{f}\|$;

and

$$
\begin{aligned}
|t_k - f_k(p)| &= \left| \frac{f_k(p_k)}{|f_k(p_k)|} - f_k(p) \right| \\
&\leqq \left| \frac{f_k(p_k)}{|f_k(p_k)|} - f_k(p_k) \right| + |f_k(p_k) - f_k(p)| \\
&= (1 - |f_k(p_k)|) + |f_k(p_k) - f_k(p)| \\
&\leqq (2K_1 + K_2 \cdot 2K_1) \|\delta \mathfrak{f}\|,
\end{aligned}
$$

since $1 - |f_k(p_k)| \leqq \log \frac{1}{|f_k(p_k)|} \leqq 2K_1 \|\delta\mathfrak{f}\|$. Similarly we have

$$|f_i(p) - t_i| \leqq (2K_1 + K_2 2K_1) \|\delta\mathfrak{f}\|.$$

Therefore, putting $K_0 = 1 + 2(2K_1 + K_2 2K_1)$, we have $|t_{ik}t_k - t_i| \leqq K_0 \|\delta\mathfrak{f}\|$, which proves the assertion (*). Thus Lemma 4 is proved.

6. Proof of convergence for the case $N \in \mathfrak{E}_1$. A proof of the following lemma is found in Siegel [10], though it is not stated explicitly.

Lemma 5. *Let ε_ν, $\nu = 1, 2, \dots$, be a sequence of positive numbers satisfying the conditions:*

(i) *There exists a positive number α such that*

$$\varepsilon_\nu < (2\nu)^\alpha, \qquad \text{for} \quad \nu = 1, 2, \dots.$$

(ii) $\varepsilon_{\nu-\mu}^{-1} \leqq \varepsilon_\nu^{-1} + \varepsilon_\mu^{-1}$ for $\nu > \mu$.

Then the formal power series $A(u) = u + \sum_{\nu=2}^{\infty} A_\nu u^\nu$ satisfying the functional equation

$$\sum_{\nu=2}^{\infty} \varepsilon_{\nu-1}^{-1} A_\nu u^\nu = \frac{M(A(u))^2}{1 - MA(u)}, \qquad M > 0, \tag{13}$$

has a positive radius of convergence.

Now we return to the proof of Theorem 3 for the case $N \in \mathfrak{E}_1$. We put $\varepsilon_\nu^{-1} = \frac{1}{K} d(\mathbf{1}, N^{-\nu}) = \frac{1}{K} d(\mathbf{1}, N^\nu)$, where K has the same meaning as in Lemma 4. Then the condition (i) is satisfied by the assumption $N \in \mathfrak{E}_1$. The condition (ii) is satisfied because

$$d(\mathbf{1}, N^{\nu-\mu}) = d(N^\mu, N^\nu) \leqq d(\mathbf{1}, N^\mu) + d(\mathbf{1}, N^\nu).$$

Let M have the same meaning as the end of no. 3, and let $A(u)$ be the solution of the equation (13). Then $A(u)$ has a positive radius of convergence and we have

$$f_i(u_i, z_i) \ll A(u_i), \; i \in I,$$

in the same manner as in no. 4. Thus the series $f_i(u_i, z_i)$ are convergent. Theorem 3 for the case $N \in \mathfrak{E}_1$ is thereby proved.

§5. Conclusion

1. Classification. We have so far investigated the structure of the neighborhood of a non-singular irreducible compact complex curve C with topologically trivial normal bundle. In view of the obtained results we may classify such curves into four classes as follows. Let n denote the type of C and let m denote the order of the complex normal bundle N of C. Curves with $n<\infty$ constitute class (α). Curves with $n=\infty$, $m<\infty$ constitute class (β'). Curves with $n=\infty$, $m=\infty$ are divided into two classes: A curve C belongs to class (β'') if there is a multiplicative holomorphic function with divisor C on a neighborhood of C; otherwise C belongs to class (γ) (see Theorem 3 and the following remark for equivalent criteria).

We know by Theorem 3 that, if C belongs to class (β'), then there exists an m-valued multiplicative holomorphic function with divisor C on a neighborhood of C; and that if C is of infinite type and N is in $\mathfrak{E}_1$, then C belongs to class (β''). We shall give an example of a curve of class (γ) at the end of this section.

2. Neighborhood of a curve of class (β') or (β''). We have seen in §3 the "strong pseudoconcavity" of the neighborhood of a curve of class (α). The following remarks will make clear the "pseudoflatness" of the neighborhood of a curve of class (β') or (β'').

1° Let C be a curve of class (β') or (β''), and let u be a multiplicative holomorphic function with divisor C on a neighborhood V of C. We take a sufficiently small number $\varepsilon>0$ so that $V_\varepsilon=\{p\in V \mid |u(p)|<\varepsilon\}$ is relatively compact. Then the neighborhoods $V_r=\{p\in V \mid |u(p)|<r\}$, $0<r\leqq\varepsilon$, are pseudoconvex and pseudoconcave (but not strongly). We may call such neighborhoods pseudoflat.

There is no strongly pseudoconcave neighborhood of C which is contained and relatively compact in V. To see this it suffices to consider the plurisubharmonic function $|u|$ (or the pluriharmonic function $\log|u|$, if one prefers) on V; the above fact is known by the same reasoning as in Lemma 2.

2° Suppose that C belongs to class (β'). Then u^m is a (single-valued) holomorphic function on V with divisor mC, m being the order of N. The curves Γ_c, $|c|\leqq\varepsilon^m$, defined by $u^m-c=0$, are irreducible and compact. Hence any plurisubharmonic function or holomorphic function on a neighborhood of Γ_c is dependent on u^m.

It is clear that, if there exists a non-constant holomorphic function on a neighborhood of C, then C belongs to class (β').

3° Suppose that C belongs to class (β''). Let us consider the compact real-analytic hypersurfaces $\Sigma_r=\{p\in V \mid |u(p)|=r\}$, $0<r\leqq\varepsilon$; and the holomorphic foliation $\mathscr{F}$ defined by the multiplicative function u on V (see Suzuki [11]). The leaf L of the foliation $\mathscr{F}$ through a point p_0 in V is defined as follows: We take a small neighborhood W of p_0 and a branch u_* of u on W, and let L' be the curve on W defined by the equation $u_*(p)-u_*(p_0)=0$; the leaf L is the analytic continuation of L'. Every leaf L in V_ε, except for C, is contained in a hypersurface Σ_r; L is non-compact and dense

in Σ_r.

Let Ψ be a plurisubharmonic function on a neighborhood of Σ_r, $0<r\leqq\varepsilon$. Then Ψ is constant on Σ_r. To see this let the maximum of Ψ on Σ_r be attained at a point p_0. Then, by the principle of maximum, Ψ is constant on the leaf L through p_0. Hence Ψ is constant on Σ_r by the upper semi-continuity.

It follows that any holomorphic function on a neighborhood of Σ_r is constant. In particular, there is no non-constant holomorphic function on $V-C$, for any neighborhood V of C.

3. Curves in the neighborhood of C. Let us examine how many compact complex curves are distributed in a small neighborhood V of the curve C. We may assume that V is a tubular neighborhood. If Γ is a 2-cycle in V, then $\Gamma\sim mC$ (homologous), where m is an integer; and hence we have the intersection numbers $(\Gamma^2)=(\Gamma, C)=0$ because $(C^2)=0$. In particular, if Γ is a compact complex curve, then $\Gamma\sim mC$, $m>0$; and if further Γ is irreducible and $\Gamma\neq C$, then Γ and C do not intersect, i.e., $\Gamma\subset V-C$.

Suppose that C belongs to class (α). Then there exists no compact curve other than C in a sufficiently small neighborhood V. To see this, we consider the strongly plurisubharmonic function Φ on $V-C$ constructed in §3. If there were a compact curve Γ in $V-C$, then the restriction of Φ to Γ would be a non-constant subharmonic function; this contradiction proves our assertion.

Suppose that C belongs to class (β'). We have the irreducible compact curves Γ_c: $u^m-c=0$. It is clear that they are the only irreducible compact curves in a small neighborhood of C. We note that $\Gamma_c\sim mC$ ($c\neq 0$), m being the order of the complex normal bundle N of C.

Suppose that C belongs to class (β''). Then there is no compact curve other than C in the neighborhood V_ε. To see this, let Γ be an irreducible compact curve in V_ε. Since the restriction of $|u|$ to Γ is a subharmonic function, it is a constant. We take an arbitrary point p_0 on Γ and a small neighborhood W of p_0. Let u_* be a branch of u on W. Since $|u_*|=|u|$ is constant on $\Gamma\cap W$, u_* is constant there. This implies that Γ coincides with the leaf L through p_0 of the foliation $\mathscr{F}$. But L is compact if and only if $L=C$. The assertion is thus proved.

By the above observations we have the following proposition:

Suppose that there is a sequence of irreducible compact curves Γ_ν, $\nu=1, 2,\ldots$, with the properties:
(i) For any small neighborhood V of C, there is a number ν_0 such that, if $\nu\geqq\nu_0$, then $\Gamma_\nu\subset V$.
(ii) Letting $\Gamma_\nu\sim m_\nu C$, there is no ν_0 such that $m_{\nu_0}=m_{\nu_0+1}=\cdots$.
Then the curve C belongs to class (γ).

The author does not know whether, conversely, there exist infinitely many compact curves in the neighborhood of a curve of class (γ). (See Arnol'd [1] 4, 5.)

4. Example of a curve of class (γ) Let us first make an observation concerning iterations of some holomorphic functions (Cremer [2]).

Let φ: $\boldsymbol{C}\to\boldsymbol{C}$ be an entire function. A system of distinct points $(c_1, c_2,\dots, c_m)$ is called a cycle of order m if we have $\varphi(c_1)=c_2,\dots, \varphi(c_{m-1})=c_m$ and $\varphi(c_m)=c_1$. A point c is called a fixed point of the l-th iterate $\varphi_l=\varphi\circ\cdots\circ\varphi$ (l times) of φ if $\varphi_l(c)=c$. It is clear that the set of all fixed points of φ_l is the union of all cycles whose orders divide l.

Suppose that φ is a polynomial of degree $d\geqq 2$ of the form

$$\varphi(w)=a_1 w+a_2 w^2+\cdots+w^d,$$

where $|a_1|=1$ and $a_1^l\neq 1$ for $l=1, 2,\dots$. Suppose further that a_1 satisfies the condition: There is a number $A>1$ such that $\liminf_{l\to\infty} A^l|1-a_1^l|^{\frac{1}{d^l-1}}=0$. Then there exists a sequence of cycles $(c_{\nu,1}, c_{\nu,2},\dots, c_{\nu,m_\nu})$ of order m_ν, $\nu=1, 2,\dots$, such that $\max_{1\leqq k\leqq m_\nu} |c_{\nu,k}|\to 0$ and $m_\nu\to\infty$ as $\nu\to\infty$.

There exist uncountably many a_1 which satisfy the condition (see [2], p. 155).

To prove the above assertion, consider the fixed points of the l-th iterate $\varphi_l(w)=a_1^l w+\cdots+w^{d^l}$ of φ. They are the roots of the equation.

$$(a_1^l-1)w+\cdots+w^{d^l}=0.$$

Since the product of the roots except for 0 of the equation is $(-1)^{d^l-1}(a_1^l-1)$, there is at least one fixed point c of φ_l such that $0<|c|\leqq|1-a_1^l|^{\frac{1}{d^l-1}}$. Therefore, for any number $r>0$, we can find a sequence of fixed points c_ν of φ_{l_ν}, $\nu=1, 2,\dots$, such that $0<|c_\nu|\leqq A^{-l_\nu}r$.

We choose a sufficiently small r so that

$$K=\sup_{|w|<r}\left|\frac{\varphi(w)}{w}\right|<A, \text{ which is possible because } \left|\frac{\varphi(w)}{w}\right|\to 1$$

as $w\to 0$. Then we have, for $k=1, 2,\dots, l_\nu-1$,

$$|\varphi_k(c_\nu)|<K^k A^{-l_\nu}r<(K/A)^{l_\nu}r<r.$$

Let m_ν be the integer$\geqq 1$ such that $\varphi_k(c_\nu)\neq c_\nu$ for $k=1,\dots, m_\nu-1$, and $\varphi_{m_\nu}(c_\nu)=c_\nu$, and consider the cycles $(c_{\nu,1}, c_{\nu,2},\dots, c_{\nu,m_\nu})=(c_\nu, \varphi(c_\nu),\dots, \varphi_{m_\nu-1}(c_\nu))$ of order m_ν. Since $\max_{1\leqq k\leqq m_\nu}|c_{\nu,k}|<(K/A)^{l_\nu}r$, we have $\max_{1\leqq k\leqq m_\nu}|c_{\nu,k}|\to 0$ as $\nu\to\infty$. We have $m_\nu\to\infty$ because there are only a finite number of cycles of order smaller than a fixed number.

Now let C be a compact Riemann surface of positive genus. We take a closed Jordan curve J on C such that $U_0=C-J$ is connected, and a neighborhood U_1 of J such that $U_0\cap U_1=U_1-J$ consists of two connected components U' and U''. Let Δ_0: $|w_0|<r_0$ be a disk of radius r_0, and let Δ_1: $|w_1|<r_1$ be a disk of radius r_1. We construct a complex manifold S from the disjoint union of $U_0\times\Delta_0$ and $U_1\times\Delta_1$ by the following identification: Identify (p_0, w_0) in $U_0\times\Delta_0$ and (p_1, w_1) in $U_1\times\Delta_1$ if either $p_0=p_1\in U'$, $w_0=w_1$, or $p_0=p_1\in U''$, $w_0=\varphi(w_1)$. Here φ is a polynomial as above, r_1 is sufficiently small so that φ is injective on Δ_1, and r_0 is sufficiently large so that $\sup_{w\in\Delta_1}|\varphi(w_1)|<r_0$.

We identify the curve on S defined by $w_0=0$ on $U_0\times\Delta_0$ and $w_1=0$ on $U_1\times\Delta_1$

with C. Thus C is imbedded in S with topologically trivial normal bundle. Consider the curves Γ_ν, $\nu=1, 2,\dots$, on S defined by $\prod_{k=1}^{m_\nu}(w_0-c_{\nu,k})=0$ on $U_0\times\Delta_0$ and $\prod_{k=1}^{m_\nu}(w_1-c_{\nu,k})=0$ on $U_1\times\Delta_1$. For sufficiently large ν, the curves Γ_ν are irreducible and compact, and $\Gamma_\nu\sim m_\nu C$. Thus the sequence of the curves Γ_ν satisfies the conditions of the proposition in no. 3. The curve C therefore belongs to class (γ).

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