Mixed problems for degenerate hyperbolic equations

By

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Recently, Cauchy problems for degenerate hyperbolic equations have been studied in rather general situations (e.g. [3], [6], [7]). What kind of boundary conditions can be combined with such equations as mixed problems? In this paper, we shall define boundary conditions satisfying uniform Lopatinski conditions for a degenerate hyperbolic mixed problem to be a $H^m$-well posed mixed problem. Our method is based on energy inequalities for such mixed problems, which are analogous to those in the non-degenerate case, except for having some degenerate orders. Main difference from the non-degenerate case is that we have to use the generalized pseudo-differential operators ([1], [2]) to get energy inequalities.

§ 1. Problem and assumption.

1.1. Operators $A$ and $B$.

Let us consider a partial differential operator

$$A(t, x; D_t, D_x) = \sum_{j=0}^m A_j(t, x; D_t, D_x),$$

$$A_j(t, x; D_t, D_x) = \sum_{|\nu|=j} a_{j\nu}(t, x) t^{-|\nu|+m_1+\ldots+m_l} D_{i1}^{m_1-j+|\nu|} \ldots D_{i_l}^{m_l-j+|\nu|},$$

where $D_t = \frac{1}{i} \frac{\partial}{\partial t}$, $D_x = \left( \frac{1}{i} \frac{\partial}{\partial x_1}, \ldots, \frac{1}{i} \frac{\partial}{\partial x_n} \right)$, $a_{00} = 1$, $a_{0\nu}(t, x) \in C^0([0, T] \times \mathbb{R}^n)$, $a_{j\nu}(t, x) \in C^0((-\infty, s_0) \times \mathbb{R}^n)$ $(s_0 = \log T)$, and $0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_m$.

We can rewrite

$$t^m A(t, x; D_t, D_x) = \sum_{j=0}^m \sum_{\nu} a_{j\nu}(t, x; D_t, D_x) t^{\nu_1+\ldots+\nu_l} (t D_t)^{m-j},$$

where $a_{j\nu}(t, x; D_t, D_x) = \sum_{|\nu|=j} a_{j\nu}(t, x) t^{\nu_1+\ldots+\nu_l}$, $a_{j\nu}(t, x) \in C^0((-\infty, s_0) \times \mathbb{R}^n)$, and $a_{0\nu} = a_{0\nu}$. We denote

$$\gamma_0 = 0 \leq \gamma_1 = \ldots = \gamma_{m_1} = q_1 < \gamma_{m_1+1} = \ldots = \gamma_{m_1+m_2} = q_2 < \ldots$$

$$\cdots < \gamma_{m_1+\ldots+m_{l-1}+1} = \ldots = \gamma_{m_1+\ldots+m_l} = q_l (m_1+\ldots+m_l=m).$$
and
\[ A^{(0)}_0 = 1, \quad A^{(i)}_0(t, x; \tau, \xi) = \sum_{k=0}^{m_i} a_{0m_1+\ldots+m_{i-1}+k}(t, x; \xi)\tau^{m_i-k} \quad (i=1, \ldots, l). \]

**Assumption(A).**

i) the zeros of \( A_0(t, x; \tau, \xi) \) with respect to \( \tau \) are real distinct for \((t, x, \xi) \in (0, T] \times \mathbb{R}^n \times S^{n-1}\),

ii) \( a_{0m_1+\ldots+m_i}(0, x; \xi) \neq 0 \) for \((x, \xi) \in \mathbb{R}^n \times S^{n-1} \) \((i=1, \ldots, l)\),

iii) the zeros of \( A^{(i)}_0(0, x; \tau, \xi) \) with respect to \( \tau \) are real distinct for \((x, \xi) \in \mathbb{R}^n \times S^{n-1} \) \((i=1, \ldots, l)\).

Let us denote
\[ \frac{\tau}{\xi_i} = \omega_i \quad (i=1, \ldots, l), \]
then
\[ \tau-(\tau_1+\cdots+\tau_m)_{i=1}^m A_j(t, x; \tau, \xi) = \sum_{k=0}^{m} a_{jk}(t, x; \xi)\left( \frac{\tau}{\xi_{k+1}} \right)^{\cdots} \left( \frac{\tau}{\xi_{m_k}} \right) P_j(t, x; \omega_1, \ldots, \omega_l, \xi), \]
where \( P_j(t, x; \omega, \xi) \) is a homogeneous polynomial of order \( m \) with respect to \((\omega, \xi)\).

Moreover we have
\[ \omega_i^{-m_i+1} \omega_j^{-m_j} P_j(t, x; \omega, \xi) \]
\[ = \sum_{k=0}^{m} a_{jk}(t, x; \xi)\omega_i^{-m_i-k} \omega_{j+k}^{m_j+k} \]
\[ + \sum_{k=0}^{m} a_{jk}(t, x; \xi)\omega_i^{m_i-k} \]
\[ + \sum_{k=0}^{m} a_{jk}(t, x; \xi)\omega_i^{m_i-k} \omega_{j+k}^{m_j+k} \]
\[ = P_j^{(0)}(t, x; \omega, \xi) \quad (i=0, \ldots, l, j=0, \ldots, m). \]

Let us consider \( \omega=(\omega_1, \ldots, \omega_l) \) as independent variables in
\[ \Omega_T = \{ \omega \in C^l; |\omega_i| < T^I|\omega_{i+1}| \quad (i=1, \ldots, l-1) \}, \]
where
\[ \delta = \min_{1 \leq i \leq l-1} \{ q_{i+1}-q_i \} > 0. \]

Let us denote for \( a>0, \beta>0 \)
\[ D_{i(a, \beta)} = \{(t, x, \omega, \xi) \in (0, T) \times \mathbb{R}^n \times \Omega_T \times C^*; \frac{|\omega_{i-1}|}{|\xi|} \leq a, \frac{|\omega_{i+1}|}{|\xi|} \geq \beta \} \quad (i=0, 1, \ldots, l) \]
then we have
Lemma 1.1. It holds that
\[ |P_i^{(\ell)}(t, x; \omega, \xi) - A_i^{(\ell)}(t, x; \omega_i, \xi)| \leq C \left\{ \left( \frac{|\omega_i|}{|\xi|} \right)^{\beta - 1} |\xi|^{m_1 + \ldots + m_i} \right\} \]
in \( D_{i(a, b)} \) \((i = 0, 1, \ldots, l)\).

Now we consider a boundary operator of order \( r \) combined to \( A \):
\[
B(t, x; D_t, D_x) = \sum_{j=0}^{r} B_j(t, x; D_t, D_x),
\]
\[
B_j(t, x; D_t, D_x) = \sum_{|\nu| \leq r-j} b_{j\nu}(t, x) t^{-j-|\nu|} \hat{v}_0 D_t^r \hat{v}_0^r D_x^r,
\]
where \( \hat{v}_j = \frac{1}{2}(\gamma_j + \gamma_{j+1}) \), \( b_{0\nu}(t, x) \in C^0([0, T] \times \mathbb{R}^n) \) and \( b_{j\nu}(x, t) \in \mathcal{B}^m((\infty, s_0) \times \mathbb{R}^n) \).

We can rewrite
\[
t^r B(t, x; D_t, D_x) = \sum_{j=0}^{r} \sum_{k=0}^{r} b_{j\nu}(t, x; D_t, D_x) t^{r-j-|\nu|} D_t^r \hat{v}_0 D_x^r.
\]
where
\[
b_{j\nu}(t, x; \xi) = \sum_{|\nu|=k} b_{j\nu}(t, x) \xi^k \quad (k \leq r - j), \quad b_{j\nu} = 0 \quad (k > r - j),
\]
\[
b_{j\nu}(x, t) \in \mathcal{B}^m((\infty, s_0) \times \mathbb{R}^n), \quad b_{0\nu} = b_{0\nu}.
\]
Let us denote
\[
Q(t, x; \omega, \xi) = \sum_{i=1}^{l} \sum_{k=0}^{m_i} Q_i(t, x; \omega, \xi)
\]
\[
= \sum_{i=1}^{l} \sum_{k=0}^{m_i} b_{j_{i+1} \ldots + m_i-1+k}(t, x; \xi) \omega_{i}^{m_i-1/2-k} \omega_{i+1}^{m_i+1} \ldots \omega_{l}^{m_l}.
\]
Moreover we have
\[
Q(t, x; \omega, \xi) = \sum_{i=1}^{l} \sum_{k=0}^{m_i} Q_i(t, x; \omega, \xi) \omega_{i}^{m_i-1/2-k} \omega_{i+1}^{m_i+1} \ldots \omega_{l}^{m_l}.
\]
Let us denote
\[
B_0^{(l)} = 0, \quad B_0^{(i)}(t, x; \tau, \xi) = \sum_{k=0}^{m_i} b_{0 \ldots m_i-1+k}(t, x; \xi) \tau^{m_i-1/2} \quad (i = 1, \ldots, l),
\]
then we have

\[ Q_0^{(l)}(t, x; \omega, \xi) = e^{\omega x}(t, x; \omega, \xi) \quad \text{in} \quad D_i (a, b), \quad (i = 0, 1, \ldots, l). \]

**Lemma 1.2.**

\[ |Q_0^{(l)}(t, x; \omega, \xi) - e^{\omega x}(t, x; \omega, \xi)| \leq C \left( \frac{1}{4} \left( \frac{|\omega|}{|\xi|} \right)^{m_i} + \beta^{i-1/2} \right) |\xi|^{m_i} \text{ in } D_i (a, b), \quad (i = 0, 1, \ldots, l). \]

**1.2. Problem and result.**

Let us use the following notations of norms:

\[ ||u(t)||_{\bar{H}_{-\ell}} = \int_0^t \left( \sum_{|\alpha| + |\beta| + h = k} |\partial_{\alpha} v(t)\partial_{\beta} u(t)|^2 \right)^{1/2} \, dt, \]

\[ ||u(t)||_{\bar{H}_{-\ell}} = ||u(t)||_{\bar{H}_{-\ell}}, \quad \ell = \ldots, m, \]

\[ \langle u(t) \rangle_{\bar{H}_{-\ell}}, \quad \ell = \ldots, m, \]

\[ \left\langle u(t) \right\rangle_{\bar{H}_{-\ell}} = \int_0^t \left( \sum_{|\alpha| + |\beta| + h = k} |\partial_{\alpha} v(t)\partial_{\beta} u(t)|^2 \right)^{1/2} \, dt. \]

for \( h, r = 0, 1, \ldots \) \((h \geq r)\) and \( \kappa > 0 \), where \( \gamma_j = \gamma \) for \( j \geq m + 1, \ \Omega \subset \mathbb{R}^n \) and \( D^\alpha \) means tangential differentiation to \( \partial \Omega \). Moreover, we use the following functional spaces:

\[ \bar{H}_{-\ell}((0, T) \times \Omega) = \{ u; u = 0 \text{ for } t < 0, ||u(T)||_{\bar{H}_{-\ell}} < \infty \}, \]

\[ \bar{H}_{-\ell}((0, T) \times \partial \Omega) = \{ u; u = 0 \text{ for } t < 0, \langle u(T) \rangle_{\bar{H}_{-\ell}} < \infty \}. \]

Now we consider a system of \( 1/2 \) \( m \) boundary operators \( \{ B(t, x; D_t, D_x; j) \} \)

\[ j = 1, \ldots, 1/2 \]

of orders \( \{ r_j \} = 1/2 \) \( 0 \leq r_j \leq m - 1 \), then we also have \( \{ B(t, x; tD_t, D_x; j) \} \) and \( \{ B_0^{(2)}(t, x; \tau, \xi; j) \} \), which we denote

\[ B(t, x; D_t, D_x) = \left( \begin{array}{c} B(t, x; D_t, D_x; 1) \\ \vdots \\ B(t, x; D_t, D_x; 1) \end{array} \right), \]

\[ B(t, x; D_t, D_x) = \left( \begin{array}{c} B(t, x; tD_t, D_x; 1) \\ \vdots \\ B(t, x; tD_t, D_x; 1) \end{array} \right), \]

Our problem is to seek a solution \( u \in \bar{H}_{-\ell}((0, T) \times \Omega) \) satisfying

\[ (P) \]

\[ A(t, x; D_t, D_x) u = f \text{ in } (0, T) \times \Omega, \]

\[ B(t, x; D_t, D_x; j) u = g_j \text{ on } (0, T) \times \partial \Omega \quad (j = 1, \ldots, 1/2), \]

for given \( f, g_j \in \bar{H}_{-\ell}((0, T) \times \Omega) \) and \( t^m f, t\nu g_j \in \bar{H}_{-\ell}((0, T) \times \partial \Omega) \). By the definition of \( A \) and \( B \), \((P)\) is equivalent to

\[ (P) \]

\[ A(t, x; tD_t, D_x) u = f \text{ in } (0, T) \times \Omega, \]

\[ B(t, x; tD_t, D_x; j) u = g_j \text{ on } (0, T) \times \partial \Omega \quad (j = 1, \ldots, 1/2), \]

where \( f = t^m f \) and \( g_j = t^\nu g_j \).

Beside the assumption \((A)\), we assume the uniform Lopatinski condition for \( 0 < t \leq T \) (Assumption \((B)\)), which will be explained in the following §1.3, then our
main result in this paper is stated as follows:

**Theorem.** Under the assumptions (A) and (B), there exists a solution of (P) and it holds the energy inequalities: there exists positive constant $C_0$ such that

$$
(\mathcal{E})_h \kappa ||u(t)||_{m-1+h,e}^2 + \langle u(t) \rangle_{m-1+h,e}^2 
\leq C_h \left\{ \frac{1}{\kappa} ||u_{m-1+h,e}^m||_{m-1+h,e}^2 + \sum_{j=1}^{1/2m} \langle \tau^j (B(;j)u(t)) \rangle_{m-1+h-[r_j,e]}^2 \right\}
$$

for $\kappa \geq C_h$ and $u \in H_{m+h,e}((0,T) \times \Omega)$ ($h=0, 1, \ldots$).

We remark that (E)$_h$ is equivalent to

$$
(\mathcal{E})_h \kappa ||u(t)||_{m-1+h,e}^2 + \langle u(t) \rangle_{m-1+h,e}^2 
\leq C_h \left\{ \frac{1}{\kappa} ||u_{m-1+h,e}^m||_{m-1+h,e}^2 + \sum_{j=1}^{1/2m} \langle (B(;j)u(t)) \rangle_{m-1+h-[r_j,e]}^2 \right\}
$$

The proof of the theorem will be given in the following sections only for the special case of $Q=R^+_n$. But when $\partial Q$ is smooth and compact, the theorem is also shown by the usual techniques.

If we assume $\gamma_1 > 0$, then we can show that the problem (P) has finite dependence domain in the same way as in [6]. Hence we have

**Corollary.** Under the assumptions (A), (B) and $\gamma_1 > 0$, the problem (P) is $C^m$-well posed.

**1.3. Uniform Lopatinski conditions for $0 < t \leq T$.**

Hereafter we consider only the case of $Q=R^+_n$. So we shall use slightly different notations from the above:

$$
Q=\{(x, y) ; x>0, y=(y_1, \ldots, y_{n-1}) \in R^{n-1} \},
$$

$$
C^n=\{(\xi, \eta) ; \xi \in C^1, \eta=(\eta_1, \ldots, \eta_{n-1}) \in C^{n-1} \}.
$$

Let

$$
A_0(t, x, y ; \tau, \xi, 0) = a_0 m_{1+\ldots+m_{n-1}} (t, x, y ; \xi),
$$

where

$$
k_{ij} < 0 \text{ if } 1 \leq j \leq \frac{1}{2} m_i, \quad k_{ij} > 0 \text{ if } \frac{1}{2} m_i + 1 \leq j \leq m_i.
$$

Let us modify $\{B(;j)\}$ into $\{\mathcal{B}(;j)\}$:

$$
\mathcal{B}(t, x, y ; tD_t, D_x, D_y ; j) = (tD_t - i\tau^{r+1} \sqrt{|D_t|^2+1}) \ldots (tD_t - i\tau^{m-1} \sqrt{|D_t|^2+1}) B(t, x, y ; tD_t, D_x, D_y ; j).
$$

Then we have

$$
\mathcal{B}_0(t, x, y ; \tau, \xi, \eta ; j) = (tD_t - i\tau^{r+1} |\eta|) \ldots (tD_t - i\tau^{m-1} |\eta|) B_0(t, x, y ; \tau, \xi, \eta ; j),
$$

and we have, denoting $m_{1+\ldots+m_{i-1}} \leq r_j < m_{1+\ldots+m_i}$,
Now we shall define the basic Lopatinski determinant by

$$R(t, x, y; \omega, \eta) = \det \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_0(t, x, y; \omega, \xi, \eta; f) \xi^{1/2m - k}}{P_0(t, x, y; \omega, \xi, \eta)} d\xi \right)_{j=1, \ldots, 1/2m}$$

where $P_0(t, x, y; \omega, \xi, \eta) = \prod_{j=1}^{1/2m} (\xi - \xi_j^+(t, x, y; \omega, \eta))$ and $\{\xi_j^+(t, x, y; \omega, \eta)\}_{j=1, \ldots, 1/2m}$ are zeros of $P_0(t, x, y; \omega, \xi, \eta)$ with respect to $\xi$ satisfying $\text{Im} \xi_j^+ > 0$ when $\text{Im} \omega_i < 0 (i = 1, \ldots, l)$. Moreover, denoting

$$\begin{align*}
B_{0}^{(i)}(t, x, y; \tau, \eta) &= \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_0(t, x, y; \tau, \xi, \eta) \xi^{1/2m + \cdots + 1/2m - k}}{A_0^{(i)}(t, x, y; \tau, \xi, \eta)} d\xi \right)_{j=1, \ldots, 1/2m_i} \\
\delta_{0}^{(i)}(t, x, y) &= (\delta_{0}^{(i)}(t, x, y; 1, \kappa_i)(t, x, y, 0))_{k=1, \ldots, 1/2m_i},
\end{align*}$$

where $A_0^{(i)}(t, x, y; \tau, \xi, \eta) = \prod_{j=1}^{1/2m_i}(\xi - \xi_j^{(i)}(t, x, y; \tau, \eta))$ and $\{\xi_j^{(i)}(t, x, y; \tau, \eta)\}_{j=1, \ldots, 1/2m_i + \cdots + 1/2m}$ are zeros of $A_0^{(i)}(t, x, y; \tau, \xi, \eta)$ with respect to $\xi$ satisfying $\text{Im} \xi_j^{(i)} > 0$ when $\text{Im} \omega_i < 0 (i = 1, \ldots, l)$, we shall define a system of $l+1$ Lopatinski determinants by

$$R(t, x, y; \tau, \eta) = \det \left( B_{0}^{(i)}(t, x, y; \tau, \eta) \delta_{0}^{(i)}(t, x, y) \right)_{i=0, 1, \ldots, l},$$

where $R(t, x, y; \tau, \eta)$ is positively homogeneous of order $\epsilon_i = \frac{3}{2} \left( \frac{1}{2} m_1 + \cdots + \frac{1}{2} m_l \right)$ with respect to $(\tau, \eta)$. Then, as shown in §2, we have

**Lemma 1.3.**

$$R(t, x, y; \omega, \eta) \omega_i^{-(i+1-i)} \cdots \omega_i^{-(i-1-i)}$$

$$= c_i(t, x, y) R^{(i)}(t, x, y; \omega_i, \eta) + \epsilon_i(t, x, y; \omega_i, \eta) |\eta|^{k}(i = 0, 1, \ldots, l),$$

where $|c_i(t, x, y)| > c > 0$ and

$$\epsilon_i(t, x, y; \omega_i, \eta) \to 0 \quad \text{as} \quad \beta T^s + \beta^{-1} \to 0,$$

uniformly in $(t, x, y, \omega, \eta) \in (0, T) \times R^1 \times R^{n-1} \times \Omega \times R^{n-1}; |\omega_i| \leq |\eta| \leq |\omega_{i+1}|$.

Here we assume

**Assumption(B).** $(A, B)$ satisfies the following two conditions, which we call uniform Lopatinski conditions for $0 < t \leq T$:

i) $\inf_{\tau \in R^{n-1}} |R(t, 0, y; \tau, \eta)| = 0$ for $0 < t \leq T$, $\text{Im} \tau = 0, \eta \in R^{n-1}, (\tau, \eta) \neq (0, 0)$,
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\( \inf_{y \in \mathbb{R}^{n-1}} |R(i)(0, 0, y; \tau, \eta)| \neq 0 \) for \( \text{Im} \tau \leq 0, \eta \in S^{n-2} \) (\( i = 0, 1, \ldots, l \)).

**Remark 1.** Assuming \( R(0)(0, 0, y) \neq 0 \), let us denote

\( \mathcal{B}(t, x, y; \tau, \xi, \eta) \)

\[ = (\delta_{\mathcal{B}}^{(1)}(t, x, y) \ldots \delta_{\mathcal{B}}^{(1)}(t, x, y))^{-1} \mathcal{B}_o(t, x, y; \tau, \xi, \eta), \]

and

\( R(i)(t, x, y; \tau, \eta) \)

\[ = \det \left( \frac{1}{2\pi i} \int_{\tau^{1/2}} \frac{B(i)(t, x, y; \tau, \xi, \eta; f)^{\xi_{1/2m_1+\ldots+1/2m_l-1}}}{A^0(i)(t, x, y; \tau, \xi, \eta)} \, d\xi \right)_{j, k = 1, \ldots, 1/2m_1+\ldots+1/2m_l} \]

(\( i = 1, \ldots, l \)), then we have

\( R(i)(t, x, y; \tau, \eta) = \frac{1}{r^0} R(0)(t, x, y)^{-1} R(i)(t, x, y; \tau, \eta). \)

**Remark 2.** Let \( \beta|\eta| \leq \omega_1 \), then we have

\( R(1)(\omega_1, \eta) \omega_1^{-s_1} = e^{R(0)} + o(1) \) \((e \neq 0)\),

and

\[ R(1)(\omega_1, \eta) \omega_1^{-s_1} = \left( \frac{\omega_1}{|\omega_1|} \right)^{-s_1} R^{(1)} \left( \frac{\omega_1}{|\omega_1|}, 0 \right) + o(1) \]

as \( \beta T^s + \beta^{-1} \to 0 \), therefore we have

\( R(1)(\tau, 0) = e^{R(0)} \tau^{-s_1}. \)

Here we have that (ii) of Assumption (B) is equivalent to

\( \inf_{y \in \mathbb{R}^{n-1}} |R(i)(0, 0, y; \tau, \eta)| \neq 0 \) for \( \text{Im} \tau \leq 0, \eta \in \mathbb{R}^{n-1}, (\tau, \eta) \neq (0, 0), \)

\( \inf_{y \in \mathbb{R}^{n-1}} |R(i)(0, 0, y; \tau, \eta)| \neq 0 \) for \( \text{Im} \tau \leq 0, \eta \in S^{n-2} \) (\( i = 2, \ldots, l \)).

**Example 1.** Let

\[ D_1^2 - t(D_x^2 + D_y^2) \]

\[ = t^{-3}[t(D_1^2 - t^2(D_x^2 + D_y^2) + \ldots) = t^{-3}A(t, tD_t, D_x, D_y), \]

then

\[ A_0(t; \tau, \xi, \eta) = \tau^2 - \tau^0(\xi^2 + \eta^2) = \tau^3[t^{-3}\tau^2 - (\xi^2 + \eta^2)], \]

\[ A^{(1)}_0(\omega_1, \xi, \eta) = \omega_1^2(\xi^2 + \eta^2), \omega_1 = t^{-3/2}\tau. \]

Let

\[ aD_1 + t^{1/4}(-D_x + bD_y) \]

\[ = t^{-7/4}[at^{3/4}(tD_t) + t^{1/4}(-D_x + bD_y)] \quad (a, b \in \mathbb{R}^1, a > |b|) \]
\[ B_0(t; \tau, \xi, \eta) = t^{3/4} + t^{3/4}(-\xi + b\eta) = t^{3/4}(at^{-3/2} + (-\xi + b\eta)), \]
\[ B_0^{(1)}(\omega_1, \xi, \eta) = \omega_1 + (-\xi + b\eta). \]

Since \( \{A_0^{(1)}(\tau, \xi, \eta), B_0^{(1)}(\tau, \xi, \eta)\} \) satisfies uniform Lopatinski condition, \( \{A, B\} \) satisfies uniform Lopatinski conditions for \( 0 < t \leq T \).

**Example 2.** Let
\[ t^2D_1^4 - (D_x^2 + D_y^2)D_2^2 + (D_x^2 + D_y^2)^2 \]
\[ = t^{-2}[(tD_1)^2 - (D_x^2 + D_y^2)](tD_1)^2 - t^2(D_x^2 + D_y^2)] + \ldots \]
\[ = t^{-2}A(t; tD_1, D_x, D_y), \]
then we have
\[ A_0(t; \tau, \xi, \eta) = \tau^4 - (\xi^2 + \eta^2)\tau^2 - t^2(\xi^2 + \eta^2)^2, \]
and \( A \) satisfies Assumption (A) with \( m_1 = m_2 = 2, q_1 = 0, q_2 = 1, \) where
\[ A_0^{(1)}(\omega_1, \xi, \eta) = \omega_1 - (\xi^2 + \eta^2), \omega_1 = \tau, \]
\[ A_0^{(2)}(\omega_2, \xi, \eta) = (\xi^2 + \eta^2) - (\omega_2 - (\xi^2 + \eta^2)). \]
\( \omega_2 = t^{-1}\tau. \)

Now let us consider boundary operators
\[ 1 = B(1; 1), D_x^2 = t^{1/2}(\ell^{1/2}D_x^2) = t^{-1/2}B(1; 2). \]

Setting
\[ \begin{cases} 
\bar{B}(t; tD_1, D_x, D_y; 1) = (tD_1 - itD_y^2 + 1)(tD_1 - it^{1/2}D_y^2 + 1), \\
\bar{B}(t; tD_1, D_x, D_y; 2) = (tD_1 - itD_y^2 + 1)t^{1/2}D_x^2, 
\end{cases} \]
we have
\[ \begin{aligned}
\bar{B}_0(t; \tau, \xi, \eta; 1) &= \tau^3 - i|\eta|\tau^2 - t^{1/2}|\eta|^2\tau + it^{3/2}|\eta|^3, \\
\bar{B}_0(t; \tau, \xi, \eta; 2) &= (\tau - it|\eta|)t^{1/2}\xi^2, \\
\bar{B}^{(1)}(\omega_1, \xi, \eta; 1) &= \omega_1 - i|\eta|, \quad \bar{B}^{(1)}(\omega_2, \xi, \eta; 1) = -|\eta|^2(\omega_2 - i|\eta|), \\
\bar{B}^{(1)}(\omega_1, \xi, \eta; 2) = 0, \quad \bar{B}^{(2)}(\omega_2, \xi, \eta; 2) = (\omega_2 - i|\eta|)\xi^2. 
\end{aligned} \]

Since
\[ R^{(0)} = 1, \quad R^{(1)}(\omega_1, \eta) = \omega_1 - i|\eta|, \quad R^{(2)}(\omega_2, \eta) = |\eta|^2(\omega_2 - i|\eta|)\xi^2(\eta^2 + \sqrt{\omega_2^2 - |\eta|^2}), \]
we have
\[ |R^{(0)}| = 1 \]
\[ |R^{(1)}(\omega_1, \eta)| \geq (|\omega_1|^2 + |\eta|^2)^{1/2} \text{ if } \text{Im } \omega_1 \leq 0, \]
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\[ |R^2(\omega_1, \eta)| \geq |\eta|^4(\omega_2)^2 + |\eta|^2 \] if \( \text{Im} \omega_2 \leq 0, \)
hence \( (A, B; 1), B; (1) \) satisfies uniform Lopatinski conditions for \( 0 < t \leq T. \)

\section{2. Basic Properties.}

\subsection{2.1. Decomposition of \( A_0. \)}

Since (ii) of the assumption (A) means

\[ \varepsilon_0 = \min_{1 \leq i \leq k} \inf_{(t, x, y) \in \mathbb{R}^{n+1}} |a_{m_1+\ldots+m_i}(t, x, y; \xi, \eta)| > 0, \]
there exist \( \varepsilon_0 > 0 \) and \( C_0 > 0 \) such that

\[ |a_{m_1+\ldots+m_i}(t, x, y; \xi, \eta)| \geq \frac{1}{2} \varepsilon_0 (|\xi| + |\eta|)^{m_1+\ldots+m_i} \]
in \( (t, x, y, \xi, \eta) \in (0, T) \times \mathbb{R}^1 \times \mathbb{R}^{n-1} \times C^1 \times \mathbb{R}^{n-1}; \xi \in G = G_+ \cup G_- \), where

\[ G_z = \{ \xi \in C^1; \text{Im} \xi = \pm \varepsilon_0|\eta|, |\xi| > C_0|\eta| \}. \]

Let us remark that

\[ A_0^{(i)}(t, x, y; \tau, \xi, 0) = a_{m_1+\ldots+m_i}(t, x, y; \xi, 0) \prod_{j=1}^{m_i} (\tau - \kappa_{ij}(t, x, y) \xi), \]
where

\[ d = \inf \left( \min_{i, x, y} \left| \kappa_{ij}(t, x, y) \right|, \min_{i, x, y} \left| \kappa_{ij}(t, x, y) \right|, \right) \]

\[ \min_{i, j} |\kappa_{ij}(t, x, y)| > 0, \]
then we have

\textbf{Lemma 2.1.} There exist \( \varepsilon_1 > 0 \) and \( C_1 > 0 \) such that

\[ |A_0^{(i)}(t, x, y; \omega_i, \xi, \eta)| > \varepsilon_1 |\xi|^{m_1+\ldots+m_i} \]
in \( (t, x, y, \omega_i, \xi, \eta) \in (0, T) \times \mathbb{R}^1 \times \mathbb{R}^{n-1} \times C^1 \times \mathbb{R}^{n-1}; |\xi| > C_1|\eta|, \xi \in \bigcup_{j=1}^{m_i} G_{ij}, \)
where

\[ G_{ij} = \{ \xi \in C^1; \left| \frac{\xi}{\omega_i} - \kappa_{ij}(t, x, y) \right| > \frac{1}{2} d \}. \]

\textbf{Proof.} Owing to the homogeneity of

\[ A_0^{(i)}(t, x, y; \omega_i, \xi, \eta) = a_{m_1+\ldots+m_i}(t, x, y; \xi, \eta) \omega_i^{m_1+\ldots+m_i}(t, x, y; \xi, \eta), \]
the inequality

\[ |A_0^{(i)}(t, x, y; \omega_i, \xi, \eta)| \geq \varepsilon_1 |\xi|^{m_1+\ldots+m_i}; \]
is valid for \( |\xi| > C_0|\eta| \), if \( |\xi| \omega_i < a \) (a: small enough) or \( |\xi| \omega_i > b \) (b: large enough). Hereafter we consider the case when \( a < |\xi| \omega_i < b \) is satisfied. Then we have
for $|\xi|>C_1\eta$. On the other hand, we have

$$
|A_0^{(i)}(t, x, y; \omega, \xi, \eta) - A_0^{(i)}(t, x, y; \omega, \xi, 0)| \leq \varepsilon |\xi|^{m_i+\cdots+m_i}
$$

for $|\xi|>C_1\eta$. Setting $\varepsilon_1 = \frac{1}{2} \varepsilon_0 \left(\frac{d^2}{2d}\right)^{m_i} = \varepsilon$, we have

$$
|A_0^{(i)}(t, x, y; \omega, \xi, \eta)| \geq \varepsilon_1 |\xi|^{m_i+\cdots+m_i}
$$

for $|\xi|>C_1\eta$ and $\xi \not\subset \bigcup_{j=1}^{m_i} G_{ij}$.

Hereafter let us fix

$$
a = \frac{1}{2} d, \quad b = d^{-1} + \frac{1}{2} d,
$$

so that

$$
\mathbb{C} \ni a|\omega| \leq |\xi| \leq b|\omega| \supset \bigcup_{j=1}^{m_i} G_{ij} \quad (i=1, \ldots, l).
$$

Let us denote for a large parameter $\beta \left( \geq \frac{1}{a} \right)$

$$
G_{i0}(\beta) = \{\xi; |\omega| \leq \beta |\xi| \leq |\omega| + 1\} \quad (i=0, \ldots, l),
$$

where

$$
G_{0}(\beta) = \{\xi; \beta |\xi| \leq |\omega|\}, \quad G_{i1}(\beta) = \{\xi; \beta |\xi| \geq |\omega|\},
$$

then the complex $\xi$ plane is covered by $\bigcup_{i=0}^{l} G_{i0}(\beta)$, and we may assume

$$
G_{i1}(\beta) \supset \bigcup_{j=1}^{m_i} G_{ij} \quad (i=1, \ldots, l)
$$

for $\omega \in \Omega_T$, by choosing $T$ so small that $bT^3 > \beta^{-1}$ holds. Moreover let us take $\beta$ large enough such that $\beta \geq a^{-1}C_1$ holds, and denote

$$
\mathcal{D}_{i}(\beta) = \{(t, x, y, \omega, \eta) \in (0, T) \times \mathbb{R}^2 \times \mathbb{R}^{n-2} \times \Omega_T \times \mathbb{R}^{n-1}; |\omega| \leq \beta |\eta| \leq |\omega| + 1\} \quad (i=0, \ldots, l),
$$
where
\[
\mathcal{D}_{u(\beta)} = \{(t, x, y, \omega, \eta) \in (0, T) \times \mathbb{R}^1 \times \mathbb{R}^{n-1} \times \Omega_T \times \mathbb{R}^{n-1}; \beta \eta \leq |\omega_1|, \]
\[
\mathcal{D}_{i(\beta)} = \{(t, x, y, \omega, \eta) \in (0, T) \times \mathbb{R}^1 \times \mathbb{R}^{n-1} \times \Omega_T \times \mathbb{R}^{n-1}; \beta \eta \geq |\omega_1|, \}
\]
then \((0, T) \times \mathbb{R}^1 \times \mathbb{R}^{n-1} \times \Omega_T \times \mathbb{R}^{n-1}\) is covered by \(\bigcup_{i=0}^{l} \mathcal{D}_{i(\beta)}\). Here we remark
\[
\{\xi; |\xi| \geq C_1|\eta|\} \supset \{\xi; |\xi| \geq a|\omega_1+1|\}
\]
for \((t, x, y, \omega, \eta) \in \mathcal{D}_{i(\beta)},\)
because it holds \(C_1|\eta| \leq a\beta|\eta| \leq a|\omega_1+1|\) in \(\mathcal{D}_{i(\beta)}\).

Lemma 2.2. There exist \(\beta > 0\) and \(T > 0\) such that the zeros of \(P_0(t, x, y; \omega, \xi, \eta)\) in \(\{\xi \geq \delta |\omega_1| \} \cap \{\xi \geq C_1|\eta|\}\) are separately contained in \(\{G_{ij}\}_{i_k < t < i_{k+1}}\) for \((t, x, y, \omega, \eta) \in \mathcal{D}_{i_0(\beta)}\), and those in \(\{\xi < \delta |\omega_1| \} \cap \{\xi \geq C_1|\eta|\}\) are contained in \(\{G_{ij}\}_{i_k \leq t \leq i_{k+1}}\) if \((t, x, y, \omega, \eta) \in \mathcal{D}_{i_0(\beta)} \cap \{C_1|\eta| \geq a|\omega_1|\}\).

Proof. Let \((t, x, y, \omega, \eta) \in \mathcal{D}_{i_0(\beta)}\) and \(\xi \in G_{ij}(i \geq i_0),\) that is,
\[
|\omega_1| \leq \beta|\eta| \leq |\omega_{i+1}| \leq |\omega_{i+2}| \leq \ldots, \quad |\omega_1| \leq \beta|\xi| \leq |\omega_{i+1}|,
\]
then we have
\[
T^{-|\omega_1|} \leq |\omega_1| \leq \beta(|\xi|^2 + |\eta|^2)^{1/2} \leq \sqrt{2} |\omega_{i+1}|,
\]
therefore we have \((t, x, y, \omega, \xi, \eta) \in D_i\{t_1, t_2\}\). Hence, from Lemma 1.1, there exist \(\beta > 0\) and \(T > 0\) such that
\[
|P^{(i)}_0(t, x, y; \omega, \xi, \eta) - A^{(i)}_0(t, x, y; \omega, \xi, \eta)| \leq \frac{1}{2} \xi_1(|\xi| + |\eta|)^{m_1 + \ldots + m_i}
\]
for \((t, x, y, \omega, \eta) \in \mathcal{D}_{i_0(\beta)}\), \(\xi \in G_{ij}(i \geq i_0)\).

On the other hand, we have from Lemma 2.1
\[
|A^{(i)}_0(t, x, y; \omega, \xi, \eta)| \geq \xi_1(|\xi| + |\eta|)^{m_1 + \ldots + m_i}
\]
for \(\xi \in \bigcup G_{ij}\) \(\cap \{\xi > C_1|\eta|\}\), and there is the only one zero of \(A^{(i)}_0(t, x, y; \omega, \xi, \eta)\) in \(G_{ij}\) if \(G_{ij} \subset \{\xi > C_1|\eta|\}\). Here we have \(P^{(i)}_0(t, x, y; \omega, \xi, \eta) \neq 0\) for \((t, x, y, \omega, \eta) \in \mathcal{D}_{i_0(\beta)}\), and \(\xi \in G_{ij}(i \geq i_0) \cap \{\xi > C_1|\eta|\}\), and there is the only one zero of \(P^{(i)}_0(t, x, y; \omega, \xi, \eta)\) in \(G_{ij}\) if \(G_{ij} \subset \{\xi > C_1|\eta|\}\) for \((t, x, y, \omega, \eta) \in \mathcal{D}_{i_0(\beta)}\) by using Rouché's theorem.

Now we shall have the local decomposition of \(P_0\), which is essential to get energy inequalities. We denote
\[
\mathcal{B}_n(X) = \left\{ f; \sup_{(t, x, y, \omega, \eta) \in X} |(tD_t)^s D_x^{i_1} \ldots D_y^{i_m} (|\omega_1| + |\eta|) D_\eta| \right\}^* \quad \text{for any } (k, a, v, \mu),
\]
where $X \subset (0, T) \times \mathbb{R}^1 \times \mathbb{R}^{n-1} \times \mathcal{Q}_T \times \mathbb{R}^{n-1}$.

**Lemma 2.3.** $P_0$ has the decomposition

$$P_0(t, x, y; \omega, \xi, \eta) = \mathcal{D}^{(i_0)}(t, x, y; \omega, \xi, \eta) \prod_{i=i_0+1}^{i_m} H_i(t, x, y; \omega, \xi, \eta),$$

$$H_i(t, x, y; \omega, \xi, \eta) = \prod_{j=1}^{m_i} (\xi - \xi_j(t, x, y; \omega, \eta)),$$

where $\mathcal{D}^{(i_0)}$, $H_i$ are polynomials with respect to $\xi$ and satisfy the following properties.

i) Set

$$\left| \omega_i \right|^{-1}(\xi - \xi_i(t, x, y; \omega, \eta)) = \left| \omega_i \right|^{-1} \xi - \xi_i'(t, x, y; \omega, \eta),$$

then

$$\xi_i' = a(t, x, y; \omega, \eta) + \sum_{k=1}^{i} \text{Im} \left( \frac{\omega_k}{|\omega_i|} \right) b_{ik}(t, x, y; \omega, \eta)$$

$$+ \sum_{k=i+1}^{i} \text{Im} \left( \left( \frac{\omega_k}{|\omega_i|} \right)^{-1} \right) b_{ik}(t, x, y; \omega, \eta),$$

where $a_{ij}, b_{ik} \in \tilde{\mathcal{D}}^{(i)}(\cup_{i \not= i_i} \mathcal{D}_i(t))$ are real valued and

$$\begin{cases} b_{ij}(t, x, y; \omega, \eta) < -\epsilon & (i=1, \ldots, \frac{1}{2} m_i), \\ b_{ij}(t, x, y; \omega, \eta) > \epsilon & (i=\frac{1}{2} m_i+1, \ldots, m_i) \end{cases}$$

in $(\cup_{i \not= i_i} \mathcal{D}_i(t)) \cap \{ \omega \in \mathbb{R}^1 \}$.

ii) Let $(x^0, y^0, \omega_{i_0}, \eta^0) \in \mathbb{R}^1 \times \mathbb{R}^{n-1} \times C^1 \times S^{n-2}(|\omega_{i_0}| \leq a^{-1} C_1)$, then there exist $\epsilon > 0$, $T > 0$, $\beta > 0$ such that

$$U_* = \left\{ (t, x, y, \omega, \eta) \in \mathcal{D}_0(t); \ |x-x^0| < \epsilon, \ |y-y^0| < \epsilon, \ \left| \frac{\omega_i}{|\eta|} - \omega_{i_0} \right| < \epsilon, \ \left| \frac{\eta}{|\eta|} - \eta^0 \right| < \epsilon \right\}$$

and

$$\mathcal{D}^{(i)}(t, x, y; \omega, \xi, \eta) = \mathcal{E}_+(t, x, y; \omega, \xi, \eta) \mathcal{E}_-(t, x, y; \omega, \xi, \eta)$$

$$\times \prod_{i=1}^{i_m} \mathcal{E}_i(t, x, y; \omega, \xi, \eta) (\epsilon(t, x, y) \equiv 0),$$

where

$$\mathcal{E}_+ (t, x, y; \omega, \xi, \eta) |\eta|^{-1/2\beta_0} = \sum_{n=0}^{1/2\beta_0} e_{\beta_0}(t, x, y; \omega, \eta) \left( \frac{\xi}{|\eta|} \right)^{1/2\beta_0 - n}$$

$$e_{\beta_0} = 1, e_{\beta_0} \in \tilde{\mathcal{D}}^{(i)}(U \epsilon),$$
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Moreover, zeros of $E_\pm$ (resp. $E_-$) with respect to $\xi$ have positive (resp. negative) imaginary parts in $U$, and

$$h_{j\ell} = a_{j\ell}(t, x, y; \omega, \eta) + \sum_{k=0}^{i_0} \frac{\omega_k}{|\eta|} b_{j\ell k}(t, x, y; \omega, \eta),$$

where $a_{j\ell}, b_{j\ell k} \in \mathcal{B}^\infty(U \epsilon)$ are real valued and $|b_{j\ell k}(t, x, y; \omega, \eta)| > c(>0)$.

Proof. i) We consider $m_{i+1} + \ldots + m_i$ zeros $\{\xi_{i+j}\}_{i+1 \leq j \leq i \leq m_i}$ of $P_0$ with respect to $\xi$ satisfying $|\xi| > a|\omega_{i+1}|$. remarking

$$P_0(t, x, y; \omega, \xi, \eta) = \omega_{i+1} \ldots \omega_i P_0^{(i)}(t, x, y; \omega, \xi, \eta),$$

we have the integral representation

$$\xi_{i+j}(t, x, y; \omega, \eta) = \frac{1}{2\pi i} \oint_{\gamma_{i+j}} \frac{\xi_{\partial \xi} P_0^{(i)}(t, x, y; \omega, \xi, \eta)}{P_0^{(i)}(t, x, y; \omega, \xi, \eta)} d\xi$$

therefore we have $\xi_{i+j} \in \mathcal{B}^\infty(\cup_{i \in I} \mathcal{D}_{i}(\gamma))$.

Now let $\omega \in \mathcal{R}^i$, then $P_0(t, x, y; \omega, \xi, \eta)$ is a polynomial with respect to $\xi$ with real coefficients, and $\xi_{i+j}(t, x, y; \omega, \eta)$ is a simple zero of $P_0(t, x, y; \omega, \xi, \eta)$ for $\xi \in G_{ij}$ and $(t, x, y, \omega, \eta) \in \cup_{i \in I} \mathcal{D}_{i}(\gamma)$. Hence $\xi_{i+j}(t, x, y; \omega, \eta)$ is real in $\cup_{i \in I} \mathcal{D}_{i}(\gamma)$ if $\omega \in \mathcal{R}^i$. Since

$$P_0^{(i)}(t, x, y; \omega, \xi, \eta)|_{\omega_{i+1} = \ldots = \omega_i = 0} = A^{(i)}_0(t, x, y; \omega_i, \xi, \eta),$$

we have

$$A^{(i)}_0(t, x, y; \omega_i, \xi, \eta) = (\xi - \xi_{i+j}(t, x, y; \omega, \eta))(\xi(t, x, y; \omega, \xi, \eta)|_{\omega_1 = \ldots = \omega_{i-1} = 0}$$

where $\xi \neq 0$ for $\xi \in G_{i+j}$ and $(t, x, y, \omega, \eta) \in \cup_{i \in I} \mathcal{D}_{i}(\gamma)$. Owing to (iii) of Assumption (A),

$$\frac{\partial}{\partial \omega_i} A^{(i)}_0(t, x, y; \omega_i, \xi, \eta) \neq 0$$

follows from $A^{(i)}_0(t, x, y; \omega, \xi, \eta) = 0$ for $(\xi, \eta) \in \mathcal{R}^i - \{0\}$. Hence we have

$$\frac{\partial}{\partial \omega_i} (\xi - \xi_{i+j}(t, x, y; \omega, \eta))|_{\omega_1 = \ldots = \omega_{i-1} = 0} \neq 0$$

for $\omega \in \mathcal{R}^i$. Taking $T$ small enough, we have

$$\left| \frac{\partial}{\partial \omega_i} \xi_{i+j}(t, x, y; \omega, \eta) \right| > c(>0)$$
We consider $m_1 + \ldots + m_i$ zeros of $P_0$ with respect to $\xi$ satisfying $|\xi| \leq a|\omega_{i+1}|$. Let $(x^0, y^0, \omega_{i_0}^0, \xi, \eta^0) \in U^1 \times U^{n-1} \times \{|\omega_{i_2}| \leq |\omega_{i+1}| \} \times \{|\xi| \leq 1, |\eta| \leq 1\}$ be real zeros of $A_0^{(t_o)}(0, x^0, y^0; \omega_{i_0}, \xi, \eta^0)$, where $\xi^0$ is a $\sigma_j$-pole zero, then other $a_0$ zeros of $A_0^{(t_o)}(0, x^0, y^0; \omega_{i_0}, \xi, \eta^0)$ are non-real (pairs of complex conjugates). Then we can choose disjoint domains $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_i$ such that

$$\mathcal{G}_0 = \{\xi \in C^1; |\xi| < C, \text{Im} \frac{\xi}{|\eta|} \geq \pm \delta\} \quad (\delta > 0)$$

and

$$\mathcal{G}_j = \{\xi \in C^1; |\xi| - \xi_j |< \delta\},$$

where non-real zeros of $A^{(t_o)}(0, x^0, y^0; \omega_{i_0}, \xi, \eta^0)$ belong to $\mathcal{G}_0 \cup \mathcal{G}_i$. Therefore we can find $\varepsilon > 0, T > 0, \beta > 0$ such that the zeros of $\mathcal{G}_{(i)}(t, x, y; \omega, \xi, \eta)$ belong to $\mathcal{G}_0 \cup \mathcal{G}_i \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_i$ if $(t, x, y, \omega, \eta) \in U$. Then $a_{x_a}(t, x, y; \omega, \eta)$ are represented as polynomials of

$$\frac{1}{2\pi i} \int_{\partial \mathcal{G}_0} \frac{\xi \frac{d}{d\xi} P_0^{(i)}(t, x, y; \omega, \xi, \eta)}{P_0^{(i)}(t, x, y; \omega, \xi, \eta)} d\xi$$

by the Newton relations, and $(h_{ja}(t, x, y; \omega, \eta))$ are represented as polynomials of

$$\frac{1}{2\pi i} \int_{\partial \mathcal{G}_j} \frac{(\xi - \xi_j) \frac{d}{d\xi} P_0^{(i)}(t, x, y; \omega, \xi, \eta)}{P_0^{(i)}(t, x, y; \omega, \xi, \eta)} d\xi$$

also by the Newton relations, therefore $a_{x_a}|\eta|^{-a}, h_{ja}|\eta|^{-a} \in \mathbb{D}_0(U)$.

Now let $\omega \in R^l$, then coefficients of $P_0$ are real and we know

$$\{\text{zeros of } \mathcal{H}_j \} = \{\text{zeros of } P_0 \text{ in } \mathcal{G}_j\}$$

in $U$, therefore coefficients of $\mathcal{H}_j$ are real in $U \cap \{\omega \in R^l\}$. Since

$$P_0^{(i_0)}(t, x, y; \omega, \xi, \eta)|_{\omega_1 = \ldots = \omega_{i-1} = 0} = A_0^{(i_0)}(t, x, y; \omega_{i_0}, \xi, \eta),$$

we have

$$A_0^{(i_0)}(t, x, y; \omega_{i_0}, \xi, \eta)$$

$$= \mathcal{H}_j(t, x, y; \omega, \xi, \eta) \xi(t, x, y; \omega, \xi, \eta)|_{\omega_1 = \ldots = \omega_{i-1} = 0, \omega_{i+1} = \ldots = \omega_i = 0}$$

where $\xi \neq 0$ for $(t, x, y, \omega_{i_0}, \xi, \eta) = (0, x^0, y^0, \omega_{i_0}, \xi^0, \eta^0), \omega_1 = \ldots = \omega_{i-1} = 0, \omega_{i+1} = \ldots = \omega_i = 0$. Owing to(iii) of Assumption $(A)$, $\frac{\partial}{\partial \omega_{i_0}} A_0^{(i_0)}(0, x^0, y^0; \omega_{i_0}, \xi_{i_0}, \eta_{i_0}) = 0$.
follows from $A^{(i_0)}_0(0, x^0, y^0; \omega_i, \xi_j, \eta^0) = 0$, therefore we have
\[ \frac{\partial}{\partial \omega_{i_0}} g_i(0, x^0, y^0; \omega_i, \xi_j, \eta^0) \bigg|_{\omega_{i_0} = \ldots = \omega_{i_0-1} = 0, \omega_{i_0} = \omega_{i_0}^0} = 0. \]

Now we shall have

**Proof of Lemma 1.3.** Let $(t, x, y, \omega, \eta) \in D_{i_0}(0)$, then
\[ P^+_0(t, x, y; \omega, \xi, \eta) = \mathcal{D}^{(i_0)}_+(t, x, y; \omega, \xi, \eta), \]
where $H^+_t(t, x, y; \omega, \xi, \eta) = \prod_{j=1}^{1/2m_i} (\xi - \xi_i(t, x, y; \omega, \eta))$ and zeros of $\mathcal{D}^{(i_0)}_+$ with respect to $\xi$ are contained in $G$, where
\[ G = \{ \xi; |\xi| < \max(\delta|\omega_{i_0}|, C_i|\eta|) \}. \]

Hence we have
\[ R = \det \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_0(\xi) \xi^{1/2m_i - 1}}{\mathcal{D}^{(i_0)}_+(\xi)} \prod_{j=1}^{1/2m_i} H^+_t(\xi) \right)_{k=1, \ldots, 1/2m_i} \]
\[ = \det (M M_{i_0+1} \ldots M_i), \]
where
\[ M = \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_0(\xi) \xi^{1/2m_i + \ldots + 1/2m_i - k}}{\mathcal{D}^{(i_0)}_+(\xi)} \prod_{j=1}^{1/2m_i} H^+_t(\xi) \right)_{k=1, \ldots, 1/2m_i + \ldots + 1/2m_i}, \]
\[ M_i = \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_0(\xi)}{\mathcal{D}^{(i_0)}_+(\xi)} \prod_{j=1}^{1/2m_i} H^+_t(\xi) \prod_{j=1}^{1/2m_i} (\xi - \xi_j) \prod_{j=1}^{1/2m_i} (\xi - \xi) \right)_{k=1, \ldots, 1/2m_i} \]
\[ = \left( \frac{\tilde{Q}_0(\xi)(\xi - \xi)}{\mathcal{D}^{(i_0)}_+(\xi)} \prod_{j=1}^{1/2m_i} H^+_t(\xi) \prod_{j=1}^{1/2m_i} (\xi - \xi_j) \right)_{k=1, \ldots, 1/2m_i} \]
\[ M'_{i_0+1} = \tilde{Q}_0^{(i_0)}(\xi) \omega_{i_0-1/2m_i-1/2}^{m_1+\ldots+m_{i_0}-1/2}, \]
\[ M'_{i_0+2} = \tilde{Q}_0^{(i_0)}(\xi) \omega_{i_0-1/2m_i-1/2}^{m_1+\ldots+m_{i_0}-1/2}. \]

Denoting
\[ \phi = \prod_{i > i_0}^{1/2m_i} \frac{\mathcal{D}^{(i_0)}_+(\xi_i)}{\mathcal{D}^{(i_0)}_+(\xi)} \prod_{i > i_0}^{m_i} \prod_{j=1}^{1/2m_i} (\xi_j - \xi_i). \]
we have

\[ R = \varphi \det (M' (M'_{i_{0}+1} k_{1}, \ldots, 1/2 m_{i_{0}} + 1 \ldots (M'_{i_{k}}) s_{1}, \ldots, 1/2 m_{i_{k}})). \]

Since

\[ M' = \left( \frac{1}{2m_{i}} \int_{\omega_{i_{0}}^{1/2}}^{1/2 m_{i_{0}}+1/2 m_{i_{0}}} \frac{E_{0}^{(i_{0})}(\xi)}{A_{0}^{(i_{0})}(\xi)} d\xi d\eta - \frac{1}{2m_{i}} \right) + o(1), \]

\[ M'_{ij} = \bar{B}_{0}^{(i_{0})}(t, x, y; 1, \xi_{ij}, \omega_{i}, \omega_{j}) + o(1) = \bar{B}_{0}^{(i)}(t, x, y; 1, \kappa_{ij}, 0) + o(1). \]

we have

\[ M^{-1} = R^{(i_{0})}(t, x, y; \omega_{i_{0}}, \eta)|\eta|^{-s_{i_{0}}} + o(1). \]

Finally we remark

\[ \varphi = \left( \frac{|\eta|}{\omega_{i_{0}+1}} \right)^{s_{i_{0}} - \sum_{\omega_{i}}^{1/2} \frac{s_{i_{0}}}{\omega_{i} + 1}} \cdots \left( \frac{1}{\omega_{i}} \right)^{s_{i_{1}} - 1} \omega_{i}^{s_{i_{1}}}(\epsilon + o(1)) \quad (\epsilon \not= 0). \]

2.2. Representation of \( P \) (resp. \( Q \)) by means of \( \{ V \} \) (resp. \( \{ W \} \)).

Let us assume \((t, x, y, \omega, \eta) \in U_{*}\) (in Lemma 2.3), which may be omitted in the following. Denoting

\[ E_{\pm}(\xi)|\eta|^{-1/2} \sum_{a=0}^{1/2} e^{a} \left( \frac{\xi}{|\eta|} \right)^{1/2} = \sum_{a=0}^{1/2} e^{a} \left( \frac{\xi}{|\eta|} \right)^{1/2}, \]

we define

\[ E_{\pm k}(\xi) = \sum_{a=0}^{k} e^{a} \left( \frac{\xi}{|\eta|} \right)^{1/2} \quad (0 \leq k \leq \frac{1}{2} \sigma_{0}). \]

In the same way, denoting

\[ H_{j}(\xi)|\eta|^{-j} = \sum_{a=0}^{j} h_{a} \left( \xi \left( \frac{\xi}{|\eta|} \right) \right)^{j-k} \quad (0 \leq k \leq \sigma_{j}). \]

Then we have

\[ \frac{1}{2m_{i}} \int_{\omega_{i_{0}}^{1/2}}^{1/2 m_{i_{0}}+1/2 m_{i_{0}}} \frac{E_{\pm k}(\xi)}{E_{\pm k}(\xi)} d\xi d\eta = \delta_{k}, \]

\[ \frac{1}{2m_{i}} \int_{\omega_{i_{0}}^{1/2}}^{1/2 m_{i_{0}}+1/2 m_{i_{0}}} \frac{H_{j}(\xi)}{H_{j}(\xi)} d\xi d\eta = \delta_{k}. \]

Here we define

\[ \{ V \} = \{ U_{i}^{(i)}(1 \leq k \leq \frac{1}{2} \sigma_{0}), U_{j}^{(j)}(1 \leq j \leq r, 1 \leq k \leq \sigma), V_{i_{0}+1}^{(i_{0}+1)}(1 \leq i \leq l, 1 \leq j \leq m_{i_{0}}) \} \]

in \( U_{*} \), where

\[ U_{i}^{(i)}(\xi) = \frac{E_{\pm k}(\xi)}{E_{\pm k}(\xi)} P_{0}^{(i_{0})}(\xi), \quad U_{j}^{(j)}(\xi) = \frac{E_{\pm k}(\xi)}{E_{\pm k}(\xi)} P_{0}^{(i_{0})}(\xi), \]
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\[ V_{ij}(\xi) = (|\omega_i|^{-1} \xi - \xi_i')^{-1} P^{(1)}_0(\xi). \]

**Lemma 2.4.** We have the representation of \( P^{(1)}_0(\xi) \) by means of \(|V|\):

\[
P^{(1)}_0(\xi) = \sum_{k=1}^{1/2} a_k^i U^i_k(\xi) + \sum_{k=1}^{1/2} a_k^j U^j_k(\xi) + \sum_{j=1} r \sum_{k=1}^a \beta_j V_{ij}(\xi) + \sum_{i'\neq i} \sum_{j=1}^m \beta_{ij} V_{ij}(\xi)
\]

in \( U \), where \( a_k^i, a_k^j, \beta_{ij}, \ldots \in \tilde{S}^\infty(U) \) and

\[
a_k^i = \frac{1}{2\pi i} \int_{\partial G^a_k} \frac{P^{(1)}_0(\xi)}{P^{(1)}_0(\xi)} \left( \frac{\xi}{|\eta|} \right)^{k-1} \frac{d\xi}{|\eta|},
\]

\[
a_k^j = \frac{1}{2\pi i} \int_{\partial G^a_j} \frac{P^{(1)}_0(\xi)}{P^{(1)}_0(\xi)} \frac{d\xi}{|\eta|},
\]

\[
\beta_{ij} = \frac{1}{2\pi i} \int_{\partial G_{ij}} \frac{P^{(1)}_0(\xi)}{P^{(1)}_0(\xi)} \frac{d\xi}{|\omega_i|}.
\]

Moreover we define

\[
\{ W \} = \{ W_0^k (1 \leq k \leq \frac{1}{2} a_0), W_{jk} (1 \leq j \leq r, 1 \leq k \leq \sigma_j) \},
\]

\[
W_{ij}(\xi_0 + 1 \leq l \leq m, 1 \leq j \leq m_i) \}
\]

in \( U \), where

\[
W_0^k = |\eta|^{-1/2} U_0^k, W_{jk} = |\eta|^{-1/2} U_{jk}, W_{ij} = |\omega_i|^{-1/2} V_{ij},
\]

then we have

**Lemma 2.5.** We have the representation of \( Q^{(1)}_0(\xi) \) by means of \( \{ W \} \) in \( U \):

\[
Q^{(1)}_0(\xi) = \sum_{k=1}^{1/2} \beta_k^i U^i_k(\xi) + \sum_{k=1}^{1/2} \beta_k^j U^j_k(\xi) + \sum_{j=1} r \sum_{k=1}^a \beta_j W_{jk}(\xi)
\]

\[
+ \sum_{i'=i+1}^m \sum_{j=1}^m q_{ij} W_{ij}(\xi),
\]

where, \( \beta_k^i, \beta_j^k, q_{ij} \in \tilde{S}^\infty(U) \) and

\[
\beta_k^i = \frac{1}{2\pi i} \int_{\partial G^a_k} \frac{Q^{(1)}_0(\xi)}{P^{(1)}_0(\xi)} \left( \frac{\xi}{|\eta|} \right)^{k-1} \frac{d\xi}{|\eta|},
\]

\[
\beta_{jk} = \frac{1}{2\pi i} \int_{\partial G_j} \frac{Q^{(1)}_0(\xi)}{P^{(1)}_0(\xi)} \frac{d\xi}{|\eta|},
\]

\[
q_{ij} = \frac{1}{2\pi i} \int_{\partial G_{ij}} \frac{Q^{(1)}_0(\xi)}{P^{(1)}_0(\xi)} \frac{d\xi}{|\omega_i|}.
\]

Let us denote

\[
\{ W \} = \{ W^+, W^- \}, \{ W^\pm \} = \{ W_0^+, W_0^-, W^+_{ij}, W^-_{ij} \},
\]

where
where \( \sigma_i^+ = \frac{1}{2} \sigma_j \) if \( \sigma_j \) is even and \( \sigma_i^+ = \frac{1}{2} \sigma_j \pm \frac{1}{2} \text{ sgn} (-b_j \sigma_j i) \) if \( \sigma_j \) is odd. Now let

\[
\tilde{Q}_0 = t^{-\sigma_1 + \cdots + \sigma_m + 1/2} B_0,
\]

then we have

\[
\tilde{Q}_0^{(i_0)}(\xi) = C^+ W_+^i(\xi) + C^- W_-^i(\xi)
\]

\[
= (C_0^+ C_1^+ \cdots C_i^+ \cdots C_r^+) \left[ \begin{array}{c}
W_0^+
\vdots
W_r^+
\vdots
W_{i_r+1}^+
\end{array} \right]
\]

\[
+ (C_0^- C_1^- \cdots C_i^- \cdots C_r^-) \left[ \begin{array}{c}
W_0^-
\vdots
W_r^-
\vdots
W_{i_r+1}^-
\end{array} \right]
\]

in \( U \), in the sense of Lemma 2.5. Moreover, since

\[
C_0^+ = |\eta|^{-1/2} \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_0^{(i_0)}(\xi)}{P_0^{(i)}(\xi)} d\xi \right)_{\sigma_1 = 1 \ldots , 1/2, 0},
\]

\[
C_0^- = |\eta|^{-1/2} \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_0^{(i_0)}(\xi) A_j(\xi)}{P_0^{(i)}(\xi)} d\xi \right)_{\sigma_1 = 1 \ldots , \sigma_j^+},
\]

and

\[
C_r^+ = |\omega|^{-1/2} \left( \frac{1}{2\pi i} \int \frac{\tilde{Q}_r^{(i_0)}(\xi)}{A_r^{(i)}(\xi)} d\xi \right)_{\sigma_1 = 1 \ldots , 1/2, 0},
\]

we have

\[
C_i^+ = |\eta|^{-1/2} \omega_i^2 \left( \frac{1}{2\pi i} \int \frac{B_i^{(i_0)}(\xi)}{A_i^{(i)}(\xi)} d\xi \right) + o(1) = C_i^{(i)} + o(1),
\]

\[
C_i^- = |\eta|^{-1/2} \omega_i^2 \left( \frac{1}{2\pi i} \int \frac{B_i^{(i_0)}(\xi) A_j(\xi)}{A_i^{(i)}(\xi)} d\xi \right) + o(1) = C_i^{(i)} + o(1),
\]
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\[ C_t^\dagger = |\omega_t|^{-1/2} \omega_t^{1/2} \left( \frac{1}{2\pi i} \int_{\partial G_1} \frac{\tilde{B}_0^{(i)}(\xi)}{A_0^{(i)}(\xi)} \, d\xi \right) + o(1) = C_t^\dagger + o(1) \]

as $\beta T^4 + \beta^{-1} \to 0$, where we denote

\[ C^\dagger = (C_0^\dagger C_1^\dagger \ldots C_j^\dagger C_{j+1}^\dagger \ldots C_i^\dagger). \]

**Lemma 2.6.** We assume the assumption $(B)$, then there exist $\varepsilon > 0$, $T > 0$ and $\beta > 0$ such that $|\det C^\dagger| > \varepsilon$ in $U_\varepsilon$.

**Proof,** Let us denote

\[ A_0^{(i)}(0, x^0, y^0; \omega_i^0, \xi, \eta^0) = c\mathcal{E}_+(\xi) \mathcal{E}_-(\xi) \prod_{k=1}^r (\xi - \xi_k^0)^{\sigma_k} \]

and

\[ A_0^{(i)}(0, x^0, y^0; \omega_i^0, \xi, \eta^0) = c\mathcal{E}_+(\xi) \mathcal{E}_-(\xi) \prod_{k=1}^r (\xi - \xi_k^0)^{\sigma_k}, \]

then we have

\[ C_j^\dagger(0, x^0, y^0; \omega_i^0, \eta^0) = \left( \frac{1}{2\pi i} \int_{\partial G_j} \frac{\omega_i^{1/2} \tilde{B}_0^{(i)}(0, x^0, y^0; \omega_i^0, \xi, \eta^0)}{\mathcal{E}_-(\xi) \prod_{k=1}^r (\xi - \xi_k^0)^{\sigma_k}} \, d\xi \right)_{k=1, \ldots, S} \]

\[ = \left( c\mathcal{E}_+(\xi_j^0) \prod_{k=1}^r (\xi^0 - \xi_k^0)^{\sigma_k} \right)^{-1} \]

\[ \times \left( \frac{1}{2\pi i} \int_{\partial G_j} \frac{\omega_i^{1/2} \tilde{B}_0^{(i)}(0, x^0, y^0; \omega_i^0, \xi, \eta^0)}{A_0^{(i)}(0, x^0, y^0; \omega_i^0, \xi, \eta^0)} \, d\xi \right)_{k=1, \ldots, S} \]

Together with the analogous consideration of $C_0^\dagger(0, x^0, y^0; \omega_i^0, \eta^0)$, we have

\[ \det C_+^\dagger(0, x^0, y^0, \omega_i^0, \eta^0) = c R(i\omega)(0, x^0, y^0; \omega_i^0, \eta^0) \quad (c \neq 0), \]

which is non-zero owing to the assumption $(B)$.

§ 3. Energy inequalities.

3.1. **Pseudo-differential operators.**

Let us consider a function $a(s, x, y; \tau, \eta) \in C_c(\mathbb{R}^{*+1} \times C_1^\dagger \times \mathbb{R}^{*+1})$ satisfying

\[ (*) \sup_{R^{*+1} \times C_1^\dagger \times \mathbb{R}^{*+1}} |D^\tau s, x, y| D^\tau a(s, x, y; \tau - i\gamma, \eta) |\sigma - i\gamma|^{\beta} < +\infty, \]

where $C_1^\dagger = \{ \tau \in C_1^\dagger; \Im \tau < 0 \}$, then $a(s, x, y; \sigma - i\gamma, \eta)$ is a symbol of a pseudo-differential operator with weight function $|\sigma - i\gamma|$ (see [2]). where variables are $(s, y) \in \mathbb{R}^n$ and dual
variables are \((\sigma, \eta) \in \mathbb{R}^n\) with parameters \((y, x) \in \mathbb{R}^1 \times \mathbb{R}^1\). Pseudo-differential operators are defined as follows:

\[
a(s, x, y; D_s - iy, D_y)u(s, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i(s + s')y} a(s, x, y; \sigma - iy, \eta) u(s + s', y + y') ds' dy' d\sigma d\eta,
\]

where \(a(s, x, y; D_s - iy, D_y; s, x, y) = a'(D_s - iy, D_y)u(s, y)\).

Then we have

**Lemma 3.1.** (Calderon-Vaillancourt[1], Kumano-go [2]) Let \(a(s, x, y; \tau, \eta)\) and \(b(s, x, y; \tau, \eta)\) satisfy (*), then

i) \(\|a(s, x, y; D_s - iy, D_y)u(s, x, y)\|_{H^k(\mathbb{R}^{n+1})} \leq C \|u(s, x, y)\|_{H^k(\mathbb{R}^{n+1})}\),

ii) \(\|\langle D_s - iy \rangle a(s, x, y; D_s - iy, D_y) - a'(D_s - iy, D_y)u(s, x, y)\|_{H^k(\mathbb{R}^{n+1})} \leq C \|u(s, x, y)\|_{H^k(\mathbb{R}^{n+1})}\),

iii) \(\|\langle D_s - iy \rangle b(s, x, y; D_s - iy, D_y) - b(s, x, y; D_s - iy, D_y)u(s, x, y)\|_{H^k(\mathbb{R}^{n+1})} \leq C \|u(s, x, y)\|_{H^k(\mathbb{R}^{n+1})}\),

where \(C\) is independent of \(\gamma > C(\gamma > 0)\).

Let us introduce an increasing function \(e(s) \in \mathbb{R}^n(\mathbb{R}^3)\) satisfying

\[
e(s) = \begin{cases} 
e' & \text{if } s < s_0, \\
e'' & \text{if } s > s_1 (s_0 < s_1),
\end{cases}
\]

then we have

\[
|D^k_s e(s)| \leq C_k e(s)
\]

Let us denote \(\omega_i(s, \tau) = \omega(q; \tau)^{-1} \tau\), then we have

\[
|D^k_s \omega_i(s, \tau)| \leq C_k |\omega_i(s, \tau)| |\tau|^{-1},
\]

and moreover

**Lemma 3.2.** Let \(a(t, x, y; \omega, \eta)\) satisfy (*'), that is, let \(a(t, x, y; \omega, \eta) \in \mathbb{F}(X)\)

\((X = (0, T) \times \mathbb{R}^1 \times \mathbb{R}^{n-1} \times \Omega_T \times \mathbb{R}^{n-1})\) and homogeneous with respect to \((\omega, \eta)\). Then \(a(e(s), x, y; \omega(s, \tau), \eta)\) satisfies (*).

**Proof.** We shall only see that \(|D_{t, \omega} a(e(s), x, y; \omega(s, \tau), \eta)|\) and \(|D_{t, \omega} a(e(s), x, y; \omega(s, \tau), \eta)| |\tau|\) are bounded. The rest of the proof will be shown analogously. First, we have
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\[ D_s[a(e(s), x, y; \omega(s, \tau), \eta)] = a(e(s), x, y; \omega(s, \tau), \eta) D_s e(s) + \sum_l a_u(e(s), x, y; \omega(s, \tau), \eta) D_s \omega(s, \tau), \]

where

\[ |a(e(s), x, y; \omega(s, \tau), \eta)| |D_s e(s)| \leq C, \]
\[ |a_u(e(s), x, y; \omega(s, \tau), \eta)| |D_s \omega(s, \tau)| \]
\[ \leq C |\omega(s, \tau)a_u(e(s), x, y; \omega(s, \tau), \eta)| \leq C', \]

therefore

\[ |D_s[a(e(s), x, y; \omega(s, \tau), \eta)]| \leq C. \]

Next, we have

\[ D_{s, r}[a(e(s), x, y; \omega(s, \tau), \eta)] = \sum_l a_u(e(s), x, y; \omega(s, \tau), \eta) D_{s, r} \omega(s, \tau), \]

where

\[ |a_u(e(s), x, y; \omega(s, \tau), \eta)| |D_{s, r} \omega(s, \tau)| \]
\[ \leq C |\omega(s, \tau)a_u(e(s), x, y; \omega(s, \tau), \eta)||\tau|^{-1} \leq C' |\tau|^{-1}, \]

therefore we have

\[ |D_{s, r}[a(e(s), x, y; \omega(s, \tau), \eta)]| |\tau| \leq C. \]

Finally we have

\[ D_{\eta}[a(e(s), x, y; \omega(s, \tau), \eta)] = \frac{1}{l} a_e(e(s), x, y; \omega(s, \tau), \eta), \]

where

\[ |a_e(e(s), x, y; \omega(s, \tau), \eta)| \leq C(1 + |\eta|)^{-1}, \]

therefore

\[ |D_{\eta}[a(e(s), x, y; \omega(s, \tau), \eta)]| |\tau| \leq C. \]

Hereafter we use following norms:

\[ \|u\|_s^2 = \sum_{|l| \leq s \leq t} \| (D_s - i\gamma)^{k-l} D_s^k \varrho e(\gamma^{m,s}) \cdots e(\gamma^{m,s}) \varrho \|_{L^2(R^{n+1})}^2, \]
\[ \|u\|_2^{s-1} = \sum_{|l| \leq s \leq t} \| (D_s - i\gamma)^{k-l} D_s^k \varrho e(\gamma^{m,s}) \cdots e(\gamma^{m,s}) \varrho \|_{L^2(R^{n+1})}^2, \]
\[ \langle u \rangle_s = \sum_{|l| \leq m \leq s} \langle (D_s - i\gamma)^{k-l} D_s^k \varrho e(\gamma^{m,s}) \cdots e(\gamma^{m,s}) \varrho \rangle_{L^2(R^{n+1})}^2. \]
\begin{align*}
\langle u \rangle^2_{k-(r)} &= \sum_{|x| \leq n \cdot (h, m-1)-r} \langle (D_1 - i\gamma)^{k-r-n} D^r_x e((x+i) \gamma_{r+1}) \cdots e((x+i) \gamma_{s}) u \rangle^2_{L^2(\mathbb{R}^n), x=0},
\end{align*}
where \( h \) and \( r \) are non-negative integers \( (r \leq h) \),
\[ R^{n+1}_+ = \{(x, y); (x, y) \in \mathbb{R}^n, x > 0\}, \]
and
\[ \gamma_j = \gamma_m \text{ for } j \geq m+1. \]

Now we consider localizing functions. Let \( \varphi_\alpha(\tau), \psi_\beta(\tau) \) be non-negative functions in \( \mathcal{B}^\infty(\mathbb{R}^2) \), satisfying
\[ \varphi_\beta(\tau) = 0 \text{ for } |\tau| > \beta (> 1), \quad \varphi_\beta(\tau) = 1 \text{ for } |\tau| < \beta - 1, \]
\[ \psi_\beta(\tau) = 0 \text{ for } |\tau| < \beta, \quad \psi_\beta(\tau) = 1 \text{ for } |\tau| > \beta + 1. \]

Let
\[ \Phi_i(\omega, \eta) = \varphi_{\beta_i} \left( \frac{\omega_i}{|\eta|} \right) \psi_{\beta_i+1} \left( \frac{\omega_i+1}{|\eta|} \right), \]
\[ \Psi_i(\omega, \eta) = \varphi_{\beta_i+1} \left( \frac{\omega_i}{|\eta|} \right) \psi_{\beta_i} \left( \frac{\omega_i+1}{|\eta|} \right) \]
for \( i = 0, 1, \ldots, l (\beta_0 < \beta_1 < \ldots < \beta_l) \), where
\[ \Phi_0 = \varphi_{\beta_0} \left( \frac{\omega_1}{|\eta|} \right), \quad \Phi_i = \Psi_{\beta_i+1} \left( \frac{\omega_i}{|\eta|} \right), \quad \Psi_0 = \varphi_{\beta_0+1} \left( \frac{\omega_1}{|\eta|} \right), \quad \Psi_l = \varphi_{\beta_l+1} \left( \frac{\omega_l}{|\eta|} \right). \]
Let \( \Phi_i(s; \tau, \eta) = \Phi_i(\omega(s, \tau), \eta) \), then we have

**Lemma 3.3.** Let \( h < m \), then there exist \( \epsilon_1 > 0, \epsilon_2 > 0, C_0 > 0 \) such that
\[ \epsilon_1 \|u\|_h \leq \sum_{i=0}^l \sum_{k=0}^m \|\Phi_i(s; D_1 - i\gamma, D_2) \left( \frac{D_x}{A} \right)^k A^{m_1+\cdots+m_i} \]
\[ \times (D_1 - i\gamma)^{k-(m_1+\cdots+m_i)} e(q_{13})^{m_1} \cdots e(q_{13})^{m_i} u \|_{L^2(\mathbb{R}^{n+1})} \]
\[ \leq \epsilon_2 \|u\|_h \]
for \( \gamma > C_0 \), where \( \Lambda = \Lambda(D_1 - i\gamma, D_2) \) and \( \Lambda(\tau, \eta) = (|\tau|^2 + |\eta|^2)^{1/2} \).

**Proof.** We shall show the first inequality. It is obvious that
\[ \|u\|_h \leq C \sum_{i=0}^l \sum_{k=0}^m \|\Phi_i(s; D_1 - i\gamma, D_2) \left( \frac{D_x}{A} \right)^k A^{m_1+\cdots+m_j} \]
\[ \times (D_1 - i\gamma)^{k-(m_1+\cdots+m_j)} e(q_{13})^{m_1} \cdots e(q_{13})^{m_j} u \|. \]
Since
\[ \Phi_i(\omega, \eta) \Lambda(\tau, \eta)^{m_1+\cdots+m_i} \omega_{i+1}^{m_i+1} \cdots \omega_{i}^{m_i} \]
\[ = \delta_{ij} \Phi_i(\omega, \eta) \Lambda(\tau, \eta)^{m_1+\cdots+m_i} \omega_{i+1}^{m_i+1} \cdots \omega_{i}^{m_i}, \]
where
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\[ \Psi_i(\omega, \eta) \left( \frac{\omega_{i+1}}{A(\tau, \eta)} \right)^n \cdots \left( \frac{\omega_{i}}{A(\tau, \eta)} \right) \] 

if \( j < i \),

\[ \Psi_i(\omega, \eta) \left( \frac{\omega_{i+1}}{A(\tau, \eta)} \right)^{-m} \cdots \left( \frac{\omega_{j}}{A(\tau, \eta)} \right)^{-m} \] 

if \( j > i \),

where \( \delta_{ij} \) satisfies (\( * \)), we have

\[ \sum_{j=1}^{i} \left\| \Phi_i(D_s - iy, D_y, s) \left( \frac{D_s}{A} \right)^k \right\| \left( A^{m_1 + \cdots + m_k} (D_s - iy)^{(m_1 + \cdots + m_k)} \right) \times e(q_{1s}) \cdots e(q_{is})^m i u \]

\[ \leq C \left\| \left( \frac{D_s}{A} \right)^k \right\| A^{m_1 + \cdots + m_k} (D_s - iy)^{(m_1 + \cdots + m_k)} e(q_{1s}) \cdots e(q_{is})^m i u \]

In the following, we have to use more fine localizing functions defined by

\[ \Phi_i(x, y; \omega, \eta) = \Phi_i(\omega, \eta) \rho_j(x, y; \omega, \eta), \quad \sum_j \rho_j(x, y; \omega, \eta) = 1. \]

So we use \( \Phi_i \) as one of \( \{ \Phi_i \} \) hereafter.

Now we consider the modification \( A_0 \) of \( A_0 \) as follows. In stead of

\[ A_0(t, x, y; \tau, \xi, \eta) = \sum_{k=0}^{m} \alpha_{0}(t, x, y; \tau, \xi, \eta) \text{e}^{i \tau \eta} \text{e}^{i \xi \eta} \eta_{m-k} \]

\[ = P^{(i)}_0(t, x, y; \omega, \xi, \eta) \eta_{m+1} \cdots \eta_{i+1} \text{e}^{i \xi \eta} \text{e}^{i \xi \eta} \eta_{m} \]

and

\[ P^{(i)}_0(t, x, y; \omega, \xi, \eta) = \sum_{k+l=m} \alpha_{k+l}(t, x, y; \xi, \eta) \text{e}^{i \tau \eta} \text{e}^{i \xi \eta} \eta_{m} \]

we consider modified symbols

\[ A_0(s, x, y; \tau, \xi, \eta) = \sum_{k=0}^{m} \alpha_{k}(s, x, y; \tau, \xi, \eta) \text{e}^{i \tau \eta} \text{e}^{i \xi \eta} \eta_{m-k} \]

and

\[ P^{(i)}_0(s, x, y; \tau, \xi, \eta) = P^{(i)}_0(s, x, y; \omega, \tau, \xi, \eta), \]

then we have

\[ A_0(s, x, y; \tau, \xi, \eta) = P^{(i)}_0(s, x, y; \tau, \xi, \eta) \text{e}^{i \tau \eta} \text{e}^{i \xi \eta} \eta_{m} \times e(q_{1s}) \cdots e(q_{is})^m i. \]

Lemma 3.4.

\[ \left\| (\Phi_i(D_s - iy, D_y, s) \right\| A_0(D_s - iy, D_y, D_x; s, x, y) \]

\[ = \left( \Phi_i A_0 \right)(D_s - iy, D_x, D_y; s, x, y) \| \leq C \| u \| \| m-1. \]

Proof. Set
\[ \Phi_1(D_x - iy, D_y; s)A_0(D_x - iy, D_x, D_y; s, x, y) \]
\[ = \Phi_1(D_x - iy, D_y; s) \sum_{k=0}^{m} (D_x - iy)^{m-k} \hat{a}_{0k}(D_x, D_y; s, x, y)e(y_{1s})...e(y_{ks}) \]
\[ = \sum_{k=0}^{m} (D_x - iy)^{m-k} \Phi_1(D_x - iy, D_y; s)\hat{a}_{0k}(D_x, D_y; s, x, y)e(y_{1s})...e(y_{ks}) \]
\[ + \sum_{k=0}^{m} \left[ \Phi_1(D_x - iy, D_y; s)(D_x - iy)^{m-k} \Phi_1(D_x - iy, D_y; s) \right] \]
\[ \times \hat{a}_{0k}(D_x, D_y; s, x, y)e(y_{1s})...e(y_{ks}) \]
\[ = I_1 + I_2, \]
then we have
\[ I_1 = (\Phi_1 = \hat{A}_0)(D_x - iy, D_x, D_y; s, x, y) \]
and
\[ \|I_2u\| \leq C \sum_{k=0}^{m} \|(D_x - iy)^{m-1-k}D_x^{k}e(y_{1s})...e(y_{ks})u\| \leq C\|u\|^{m-1}, \]
where we used Lemma 3.1.

Now we consider
\[ \Psi_i(\omega, \eta)P_0^{(i)}(t, x, y; \omega, \xi, \eta + (1 - \Psi_i(\omega, \eta))P_0^{(i)}(t, x, y; \omega, \xi, \eta, 0, ..., 0) \]
and hereafter we denote it also by \( P_0^{(i)} \) then \( P_0^{(i)}(s, x, y; \tau, \xi, \eta)A(\tau, \eta)^{-m_1-...-m_i} \)
\( i=1,...,l \) are polynomials with respect to \( \frac{\xi}{A(\tau, \eta)} \) with coefficients satisfying (\(*\)).

Denoting
\[ P_0^{(i)}(s, x, y; \tau, \xi, \eta)A(\tau, \eta)^{-m_1-...-m_i} = \sum_{k=0}^{m} \epsilon_k(s, x, y; \tau, \eta) \left( \frac{\xi}{A(\tau, \eta)} \right)^k, \]
we define
\[ P_0^{(i)}(s, x, y; D_x - iy, D_x, D_y) \]
\[ = \sum_{k=0}^{m} \epsilon_k(s, x, y; D_x - iy, D_y) \left( \frac{D_x}{(D_x - iy, D_y)} \right)^k A(D_x - iy, D_y)^{m_1+...+n_i}. \]
Since
\[ \Phi_1(s, \tau, \eta)A_0(s, x, y; \tau, \xi, \eta) \]
\[ = \Phi_1(s, \tau, \eta)P_0^{(i)}(s, x, y; \tau, \xi, \eta)A(\tau, \eta)^{-m_1-...-m_i}e(q_1s)^{m_1}...e(q_is)^{m_i}, \]
we have using Lemma 3.1

**Lemma 3.5.**
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$$||((\Phi_{i,n} A_0)(D_s - i\gamma, D_x, D_y; s, x, y)$$

$$- P_0^{(i)}(s, x, y; D_s - i\gamma, D_x, D_y)$$

$$\times (D_s - i\gamma)^{m_1 + \ldots + m_k} e(q_{i,3}) e(q_{i,2}) e(q_{i,1}) u||$$

$$\leq C ||u||_{m-1}.$$  

3.2 Basic energy estimates.

Considering $P_0^{(i)}$ as $P_0^{(i)}$ in §2, we have the decomposition

$$P_0^{(i)}(t, x, y; \omega, \xi, \eta)$$

$$= P^{(i)}(t, x, y; \omega, \xi, \eta) \sum_{j=i+1}^{i} H_j(t, x, y; \omega, \xi, \eta),$$

where

$$P^{(i)}(t, x, y; \omega, \xi, \eta) = c(t, x, y) E_x(t, x, y; \omega, \xi, \eta) E_x(t, x, y; \omega, \xi, \eta)$$

$$\times \prod_{j=1}^{r} \mathcal{A}_j(t, x, y; \omega, \xi, \eta),$$

where $E_x, \mathcal{A}_j, H_j$ are defined globally and have similar properties to those in §2.

Lemma 3.6.

i) $$\sum_{k=0}^{1/2s-1} <A^{-1/2}\left(\frac{D_s}{A}\right)^k u>^2 + \sum_{k=0}^{1/2s-1} \left(\frac{D_s}{A}\right)^k u \|^2$$

$$\leq C ||A^{-1/2} e_x(s, x, y; D_s - i\gamma, D_x, D_y) u||^2 + \ldots,$$

ii) $$\sum_{k=0}^{1/2s-1} \left(\frac{D_s}{A}\right)^k u \|$$

$$\leq C ||A^{-1/2} e_x(s, x, y; D_s - i\gamma, D_x, D_y) u||^2 + \sum_{k=0}^{1/2s-1} A^{-1/2} \left(\frac{D_s}{A}\right)^k u \| \ldots,$$

iii) $$\sum_{k=0}^{1/2s-1} \sum_{j=s_j}^{s_j-1} \sum_{k=0}^{s_j-1} <D_s - i\gamma|^{-1/2} A^{-1/2} \left(\frac{D_s}{A} - \xi_j^2\right)^k u>^2 + \gamma \sum_{k=0}^{s_j-1} <D_s - i\gamma|^{-1/2} \left(\frac{D_s}{A} - \xi_j^2\right)^k u \|^2$$

$$\leq C \frac{1}{\gamma} ||A^{-1/2} e_x(s, x, y; D_s - i\gamma, D_x, D_y) u||^2$$

$$+ \mu \sum_{k=0}^{s_j-1} <D_s - i\gamma|^{-1/2} A^{-1/2} \left(\frac{D_s}{A} - \xi_j^2\right)^k u> + \ldots$$

$$(\gamma, \mu > 0, j = 1, \ldots, r),$$

iv) $$<|D_s - i\gamma|^{-1/2} e_x|^{-1/2} u>^2 + \gamma ||D_s - i\gamma|^{-1/2} u||^2$$
\[ \frac{1}{\gamma} \langle \psi_{\omega}^{-1} D_{x} - \xi_{x} \rangle u \rangle \geq C \frac{1}{\gamma} \| \langle \psi_{\omega}^{-1} D_{x} - \xi_{x} \rangle u \rangle \|^{2} + \ldots \quad \left( k = \frac{1}{2} m_{j} + 1, \ldots, m_{j}, \frac{1}{2} m_{j} + 1, \ldots, l \right), \]

v) \[ \gamma \| D_{x} - i \gamma \|^{-1} \| u \|^2 \leq C \left( \frac{1}{\gamma} \langle \psi_{\omega}^{-1} D_{x} - \xi_{x} \rangle u \rangle \right)^{2} + \ldots \]

where \( \ldots \) means lower order terms.

**Proof.** i), ii) are shown from the ellipticity of \( E \) and iii), iv), v) are shown from the hyperbolicity of \( H_{j} \) and \( H_{j} \) in usual techniques. For example, let us show iv). We consider

\[ I = 2 \text{Im} \langle \psi_{\omega}^{-1} D_{x} - \xi_{x} \rangle u, \| D_{x} - i \gamma \|^{-1} u \rangle \]

\[ = \langle \psi_{\omega}^{-1} u, \| D_{x} - i \gamma \|^{-1} u \rangle - 2 \text{Im} \xi_{x} u, \| D_{x} - i \gamma \|^{-1} u \rangle + \ldots \]

\[ \left( k = \frac{1}{2} m_{j} + 1, \ldots, m_{j} \right). \]

Since we have

\[ \text{Im} \xi_{x} = \sum_{k=1}^{i} \text{Im} \left( \frac{\xi_{k}}{\omega_{x}} \right) - \sum_{k=1}^{i} \text{Im} \left( \frac{\xi_{k}}{\omega_{y}} \right) \]

\[ = \frac{1}{\gamma} \left( \sum_{k=1}^{i} \frac{e^{\xi_{k}}}{\omega_{x}} - \sum_{k=1}^{i} \frac{e^{\xi_{k}}}{\omega_{y}} \right) \left( b_{j,k} > c > 0 \right) \]

for \( k = \frac{1}{2} m_{j} + 1, \ldots, m_{j} \) from Lemma 2.3, we have

\[ I \geq \langle \psi_{\omega}^{-1} \| D_{x} - i \gamma \|^{-1} u \rangle^{2} + \gamma \| D_{x} - i \gamma \|^{-1} u \rangle^{2} + \ldots \]

**Remark.** As a corollary of Lemma 3.6, we have

\[ i') \sum_{k=1}^{1/2m_{j} - 1} \langle D_{x} - i \gamma \rangle^{-1/2} A^{-1/2} \left( \frac{D_{x}}{A} \right)^{k} u \rangle^{2} + \gamma \sum_{k=1}^{1/2m_{j} - 1} \| D_{x} - i \gamma \|^{-1} \left( \frac{D_{x}}{A} \right)^{k} u \rangle^{2} \]

\[ \leq C \frac{1}{\gamma} \| A^{-1/2} \xi_{x} \langle \psi_{\omega}^{-1} \rangle u \rangle \|^{2} + \ldots, \]

\[ ii') \sum_{k=1}^{1/2m_{j} - 1} \langle D_{x} - i \gamma \rangle^{-1} \left( \frac{D_{x}}{A} \right)^{k} u \rangle^{2} \]

\[ \leq C \left( \frac{1}{\gamma} \| A^{-1/2} \xi_{x} \langle \psi_{\omega}^{-1} \rangle u \rangle \right)^{2} + \sum_{k=1}^{1/2m_{j} - 1} \langle D_{x} - i \gamma \rangle^{-1} \left( \frac{D_{x}}{A} \right)^{k} u \rangle^{2} + \ldots. \]

Corresponding to the decomposition of \( P_{0}^{(i)} \), we can define \( \{ V \} \) and \( \{ W^{*} \} \), in the same way as in \( \S 2 \). Then \( \{ V(s, x, y; \tau, \xi, \eta) A(\tau, \eta)^{-(m_{j} + \cdots + m_{j} - 1/2)} \} \), \( \{ W(s, x, y; \tau, \xi, \eta) A(\tau, \eta)^{-(m_{j} + \cdots + m_{j} - 1/2)} \} \) are polynomials with respect to \( A(\tau, \eta) \) of order less than \( m \) with coefficients satisfying (v). Hence we can define \( \{ V(s, x, y; D_{x} - i \gamma, D_{x}, D_{y}) \}, \{ W(s, x, y; D_{x} - i \gamma, D_{x}, D_{y}) \} \) and we have

**Corollary of Lemma 3.6.**

\[ \langle D_{x} - i \gamma \rangle^{-1/2} \langle \psi_{\omega}^{-1} \rangle u \rangle \]
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\[ + \mu y \| (D_s - iy)^{-1} \dot{V}(s, x, y; D_s - iy, D_x, D_y) u \|^2 \]

\[ \leq C \mu \langle |D_s - iy|^{-1/2} \dot{V}(s, x, y; D_s - iy, D_x, D_y) u \rangle^2 \]

\[ + C \mu \left( \frac{1}{\gamma} \| P_\delta(s, x, y; D_s - iy, D_x, D_y) u \| \right)^2 \]

\[ + \sum_{k=0}^{m} \| (D_s - iy)^{-1} \left( \frac{D_s}{A} \right)^k A^{m_1 + \cdots + m_k - u} \|^2 \].

Owing to Lemma 2.5, we have

\[ \langle |D_s - iy|^{-1/2} \dot{V} \Phi ; u \rangle \leq C \left\{ \langle |D_s - iy|^{-1/2} \dot{V} \Phi ; u \rangle + \sum_{k=0}^{m-1} \langle |D_s - iy|^{-1/2} \left( \frac{D_s}{A} \right)^k A^{m_1 + \cdots + m_k - u} \rangle \right\} , \]

therefore

\[ \langle |D_s - iy|^{-1/2} \dot{V} \Phi ; u \rangle^2 + \gamma \| (D_s - iy)^{-1} \dot{V} \Phi ; u \|^2 \]

\[ \leq C \left\{ \langle |D_s - iy|^{-1/2} \dot{Q}(i) \Phi ; u \rangle^2 + \frac{1}{\gamma} \| P(i) \Phi ; u \|^2 \right\} \]

\[ + \sum_{k=0}^{m} \| (D_s - iy)^{-1} \left( \frac{D_s}{A} \right)^k A^{m_1 + \cdots + m_k - u} \|^2 \].

Hence we have, summing up these energy estimates on localized operators,

**Proposition 3.7.** Under the assumptions (A) and (B), there exists \( C_0 > 0 \) such that

\[ \langle u \rangle_{m-1}^2 + \gamma \| u \|_{m-1}^2 \leq C_0 \left( \frac{1}{\gamma} \| A u \|^2 + \sum_{j=1}^{1/2m} \langle B(j) u \rangle_{m-1-[r_j]}^2 \right) \]

for \( \gamma \gg C_0 \).

Moreover, energy estimates of higher order follows from Proposition 3.7, that is,

**Proposition 3.8.** Under the assumptions (A) and (B), there exists \( C_h > 0 \) such that

\[ \langle u \rangle_{m-1+h}^2 + \gamma \| u \|_{m-1+h}^2 \]

\[ \leq C_h \left( \frac{1}{\gamma} \| A u \|_{m+h-[m]}^2 + \sum_{j=1}^{1/2m} \langle B(j) u \rangle_{m-1+h-[r_j]}^2 \right) \]

for \( \gamma \gg C_h(h \geq 0) \).

**Proof.** We shall use the method of mathematical induction with respect to \( h \).

We get the case of \( h = 0 \) from Proposition 3.10. Let us assume that the case of \( h \) holds, then we have

\[ \langle u \rangle_{m+h}^2 + \gamma \| u \|_{m+h}^2 \]

\[ \leq C_h \left( \frac{1}{\gamma} \| A ((D_s - iy) u) \|_{m+h-[m]}^2 + \| A(e(ym)s)D_x, y u) \|_{m+h-[m]}^2 \right) \]

\[ + \sum \langle \dot{B}(j) ((D_s - iy) u) \rangle_{m-1+h-[r_j]}^2 \]
for $\gamma \geq C_1$. On the other hand, we have
\[
\|\dot{A}(D_s - iy)u - (D_s - iy)\dot{A}u\|_{m+h-[m]} + \|\dot{A}e^{(y,m)s}D_x,y u - e^{(y,m)s}D_x,y \dot{A}u\|_{m+h-[m]} \leq C_h\|u\|_{m+h},
\]
and
\[
\|\dot{B}(j)(D_s - iy)u - (D_s - iy)\dot{B}(j)u\|_{m-1+h-[r], j} + \|\dot{B}(j) e^{(y,m)s}D_x,y u - e^{(y,m)s}D_x,y \dot{B}(j) u\|_{m-1+h-[r], j} \leq C_h \|u\|_{m-1+h}.
\]
Hence we have
\[
\|u\|_{m+h}^2 + \gamma \|u\|_{m+h}^2 \leq C_{h+1} \left( \frac{1}{\gamma} \|\dot{A}u\|_{m+h+1-[m]}^2 + \sum_j \|\dot{B}(j)u\|_{m+h-[r], j}^2 \right)
\]
for $\gamma \geq C_{h+1}$, which is the case of $h+1$.

Now we consider energy estimates in $-\infty < s < s_0$ instead of $-\infty < s < +\infty$. Since $(\dot{A}, \dot{B})$ defines a strictly hyperbolic initial boundary value problem with uniform Lopatinski condition in $[s_0, \infty) \times R^+$, we have, denoting
\[
\|u(s)\|_h^2 = \int_{-\infty}^s \sum_{k=m} \|(D_s - iy)^{k-1} D_x \cdots e^{(y,m)s} u\|_{L^2_R(d_s)}^2 ds,
\]
\[
\|u(s)\|_{m-1+h}^2 = \int_{-\infty}^s \sum_{k=m} \|(D_s - iy)^{k-1} D_x \cdots e^{(y,m)s} u\|_{L^2_R(d_s)}^2 ds,
\]
and so on.

Corollary 1 of Proposition 3.8.
\[
\|u(s)\|_{m-1+h}^2 + \gamma \|u(s)\|_{m-1+h}^2 \leq C_h \left( \frac{1}{\gamma} \|\dot{A}u(s)\|_{m+h-[m]}^2 + \sum \|\dot{B}(j)u(s)\|_{m-1+h-[r], j}^2 \right)
\]
for $\gamma \geq C_h$ ($h \geq 0$).

Now we come back to the variable $t$ from $s$. Since
\[
e^{-rt} A(e(s), x, y; D_s, D_x, D_y) u = A(e(s), x, y; D_s - iy, D_x, D_y)(e^{-rt} u), \ldots,
\]
and
\[
\int_{-\infty}^s \|e^{-rt} u(t)\|_{L^2}^2 ds = \int_0^s \int_0^{t-1} |v(t)|^2 dt
\]
we have
Corollary 2 of Proposition 3.8.

\[ \|u(t)\|_{m-1+h, 2r+1} + \gamma \|u(t)\|_{m-1+h, 2r+1} \]
\[ \leq C_k \left( \frac{1}{\gamma} \|Au(t)\|_{m+h-[m], 2r+1} + \sum_j \langle B(j)u(t) \rangle_{m-1+h-[r], 2r+1} \right) \]

for \(0 < t < T, \gamma \geq C_k(h > 0)\).

§ 4. Generalization.

In §1–§3, we considered the degenerate hyperbolic mixed problem, whose degeneracy was measured by \(t\). In §4, we shall consider the degenerate hyperbolic mixed problem whose degeneracy will be measured by some weight functions \(\rho(t)\) and \((\gamma_1(t), \ldots, \gamma_r(t))\). We shall be satisfied to state the framework of the generalized problem, because the proof can be accomplished just in the same way as the original one, stated in §1–§3.

4.1. Weight functions.

First we introduce a weight function \(\rho(t) \in C^\infty(0, T]\), satisfying \(\rho(t) > 0\) and

\[ \left| \rho(t)^{k-1} \left( \frac{d}{dt} \right)^k \rho(t) \right| \leq C_k \quad (k=0, 1, \ldots), \]

\[ \int_0^T \frac{dt}{\rho(t)} = +\infty. \]

Let \(\sigma(t) \in C^\infty(0, T]\) be defined by

\[ \frac{d}{dt} \sigma(t) = \rho(t)^{-1} \sigma(t) \quad \text{(i.e. } \sigma(t) = Ce^{-\int_0^t \rho(s) ds}) \]

then \(\sigma(t)\) is strictly increasing function satisfying \(\sigma(+0)=0\). For example, we can take

\[ (k^{-1}(t), t^k), (k^{-1}t^{k+1}, e^{-t^k}), (k^{-1}t^{k+1}e^{-t-k}, e^{-t^{-k}}) \]

for \(k > 0\) as \((\rho, \sigma)\). We dealt with the case of \((\rho, \sigma) = (t, t)\) in §1–§3.

Now we introduce a variable transformation by

\[ s = \log \sigma(t) \]

then \((0, T)\) is transformed to \((-\infty, s_0)\) \((s_0 = \log C)\) and \(\rho(t)\frac{d}{dt} = \frac{d}{ds} \).

First we remark \(\rho'(t(s)) \in \mathcal{B}^m(-\infty, s_0)\), and moreover

**Lemma 4.1.**

\[ \rho(t)^k\left( \frac{d}{dt} \right)^k = \left( \frac{d}{ds} \right)^k + a_{k+1}(s) \left( \frac{d}{ds} \right)^k + \ldots + a_k(s), \]

where \(a_{k+1}(s) \in \mathcal{B}^m(-\infty, 0)\).

**Proof.** Let us assume

\[ \rho(t)^k\left( \frac{d}{dt} \right)^k = \left( \frac{d}{ds} \right)^k + a_{k+1}(s) \left( \frac{d}{ds} \right)^k + \ldots + a_k(s), \quad a_{k+1}(s) \in \mathcal{B}^m, \]
then we have, operating \( \rho(t) \frac{d}{dt} \) on both sides,

\[
\rho(t)+1 \left( \frac{d}{dt} \right)^{k+1} + kp(t) \rho(t) \left( \frac{d}{dt} \right)^k = \left( \frac{d}{ds} \right)^{k+1} + a_k(s) \left( \frac{d}{ds} \right)^k \\
+ (a_k(s) + a_{k+1}(s)) \left( \frac{d}{ds} \right)^{k-1} + \ldots + (a_{k+1}(s) + a_{k+2}(s)) \frac{d}{ds} + a_{k+2}(s),
\]

that is,

\[
\rho(t)+1 \left( \frac{d}{dt} \right)^{k+1} = -kp'(t(s)) \left( \left( \frac{d}{ds} \right)^{k} + a_k(s) \left( \frac{d}{ds} \right)^{k-1} + \ldots + a_{k+2}(s) \right) \\
+ \left( \frac{d}{ds} \right)^{k+1} + a_k(s) \left( \frac{d}{ds} \right)^{k} + (a_k(s) + a_{k+1}(s)) \left( \frac{d}{ds} \right)^{k-1} \\
+ \ldots + (a_{k+1}(s) + a_{k+2}(s)) \frac{d}{ds} + a_{k+2}(s).
\]

### 4.2. Operators \( A \) and \( B \).

Let us introduce a system of positive functions \( (\gamma_1(t), \ldots, \gamma_m(t)) \in C^\infty(0, T) \) such that

\[
\gamma_1(t) = \ldots = \gamma_m(t) = q_1(t), \\
\gamma_{m_1+1}(t) = \ldots = \gamma_{m_1+m}(t) = q_2(t), \\
\ldots \\
\gamma_{m_1+\ldots+m_1+m}(t) = \ldots = \gamma_{m_1+\ldots+m_1+m}(t) = q_1(t) \ (m_1+\ldots+m_1=m),
\]

where we assume

i) \( \lim_{i \to +0} \frac{q_{i+1}(t)}{q_i(t)} = 0 \),

ii) \( \left| \rho(t) \left( \frac{d}{dt} \right)^k \left( q_i(t) \right)^{1/2} \right| \leq C_k \ (k=0, 1, \ldots) \).

i.e. \( q_i(t(s))^{1/2} \in \mathcal{G}(-\infty, 0) \).

Now, denoting \( \gamma_j(t) \rho(t)^{-1} = \delta_j(t) \), let us consider

\[
A = A_0 + \rho(t)^{-1} A_1 + \ldots + \rho(t)^{-m} A_m,
\]

where

\[
A_j = \sum_{i \in \mathbb{N}^m} a_{j_i}(t, x) \delta_1(t) \delta_2(t) \ldots \delta_m(t) D_1^{j-i} D_2^{j-i} \ldots D_m^{j-i},
\]

\[
a_{j_i}(t, x) \in \mathcal{G}((0, T] \times \mathbb{R}^n), \ a_{00} = 1, \ a_{0i}(t, x) \in C^0([0, T] \times \mathbb{R}^n).
\]

Then we have

\[
\rho(t)^m A = \sum_{j=0}^{m} \sum_{i \in \mathbb{N}^m} a_{j_i}(t, x) \gamma_1(t) \ldots \gamma_m(t) D_1^{j-i} D_2^{j-i} \ldots D_m^{j-i}
\]

\[
= \sum_{j=0}^{m} \sum_{i=0}^{m-j} a_{j_i}(t, x, D_x) \gamma_1(t) \ldots \gamma_i(t) \rho(t) D_i^{j-i},
\]

\[
a_{j_i}(t, x, \xi) = \sum_{i=0}^{m} a_{j_i}(t, x) \xi^i
\]
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that is,

\[ \rho(t)^{m}A = \sum_{i=0}^{m} A_j(t, x; \rho(t)D_t, D_x) = A(t, x; \rho(t)D_t, D_x), \]

where \( a_j(t, x) \in \mathcal{B}^e((-\infty, 0) \times \mathbb{R}^n) \) and \( a_0(t, x) = a_0(t, x). \n
Denoting

\[ A^{(i)}_{0}(t, x; \tau, \xi) = \sum_{k=0}^{m} a_{0m+\ldots+1} \tau^m \xi^{i-k}, \]

we assume assumption (A), stated just by the same words as in §1.

Let us denote

\[ q_i(t)^{-1} = \omega_i \quad (i = 1, \ldots, l), \]

and

\[ \gamma_i(t)^{-1} \gamma_i(t)^{-1} A_j(t, x; \tau, \xi), \]

\[ = \sum \alpha_j(t, x) \xi^{\frac{\tau}{\gamma_i+1}(t)} \left( \frac{\tau}{\gamma_i+2(t)} \ldots \frac{\tau}{\gamma_i(t)} \right) = P_i^{(i)}(t, x; \omega, \xi) \omega_i^{m+1} \ldots \omega_i^m. \]

Moreover, we denote

\[ \max_{t \leq T} \left| \frac{q_{i+1}^{(i)}(t)}{q_i(t)} \right| = \delta(T) \quad (\lim_{T \to 0} \delta(T) = 0), \]

\[ \Omega_T = \{ \omega \in C^1 ; |\omega_i| < \delta(T) |\omega_i+1| \ (i = 1, \ldots, l-1), \}

\[ D_{i(a, b)} = \{ (t, x, \omega, \xi) \in (0, T) \times \mathbb{R}^n \times \Omega_T \times C^a ; \frac{|\omega_i+1|}{|\xi|} \leq a, \frac{|\omega_i|}{|\xi|} \geq b \}, \]

then we have

**Lemma 4.2.**

\[ |P_0^{(i)}(t, x; \omega, \xi) - A_0^{(i)}(t, x; \omega_i, \xi)| \leq C \left( a \left( \frac{|\omega_i|}{|\xi|} \right)^{m+1} \right) \xi^{m+1} \ldots \xi^m \text{ in } D_{i(a, b)}. \]

Now we consider boundary operators, denoting

\[ \delta_i(t) = (\delta_j(t) \delta_{i+1}(t))^{1/2}, \quad \gamma_i(t) = (\gamma_j(t) \gamma_{i+1}(t))^{1/2}, \]

\[ B = B_0 + \rho(t)^{-1} B_1 + \ldots + \rho(t)^{-r} B_r, \]

where

\[ B_j = \sum_{|\nu| \leq j} b_{j+1}(t, x) \delta_0(t) \ldots \delta_\nu(t) D_x D_t^{i-j-|\nu|}, \]

\[ = \sum_{|\nu| \leq j} b_{j+1}(t, x) \gamma_0(t) \ldots \gamma_\nu(t) D_x D_t^{i-j-|\nu|}, \]
where \( b_{ij}(t, x) \in \mathcal{B}((-(\infty,0) \times R^n)) \) and \( b_{00}(t, x) \in C^0([0, T] \times R^n) \). Then we have
\[
\rho(t)' \mathbf{B} = \sum_j \sum_{i=|i|}^{|r-j|} b_{ij}(t, x) \hat{y}_n(t) ... \hat{y}_0(t) D_{ij}^x(t)^{r-j-i} \times D_{ij}^t \times \mathbf{D}_{ij}^{r-i-j-k}.
\]
\[
= \sum_j \sum_{k=0}^r b_{jk}(t, x, D_x) \hat{y}_n(t) ... \hat{y}_0(t) (\rho(t) D_{ij})^{r-j-k}, \quad b_{jk}(t, x, \xi) = \sum_{i=|i|}^{|r-j-k|} b_{ij}(t, x) \xi^i;
\]
that is,
\[
\rho(t)' \mathbf{B} = \sum_j B_j(t, x); \quad \rho(t) D_{ij} = B(t, x; \rho(t) D_{ij}, D_x),
\]
where \( b_{ij}(t, x) \in \mathcal{B}((-(\infty,0) \times R^n)) \) and \( b_{00}(t, x) = b_{00}(t, x) \).

Let us denote
\[
\gamma_1(t)^{-1} ... \gamma_m(t)^{-1} \tau_m^{-1-r+i+1/2} B_j(t, x; \tau, \xi)
\]
\[
= \sum_k \sum_{i=|i|}^{|r-j-k|} b_{ij}(t, x, \xi) \left( \frac{\tau}{\gamma_k+1} \right)^{1/2} \left( \frac{\tau}{\gamma_k+2} \right) ... \left( \frac{\tau}{\gamma_m} \right) = Q_j^{(i)}(t, x; \omega, \xi) \omega_{i+1}^{m+1} ... \omega_{i}^{m},
\]
and
\[
B_0^{(i)}(t, x; \tau, \xi) = \sum_{k=0}^{m-1} b_{0m+...+m_{i-1}+k}(t, x, \xi) \tau_{m-i-1}^{m-i-1-k},
\]
then we have

**Lemma 4.3.**
\[
|Q_0^{(i)}(t, x; \omega, \xi) - \omega_{i}^{1/2} B_0^{(i)}(t, x; \omega, \xi)| \leq C \left( a^{1/2} \left( \frac{1}{\xi} \right)^{m_i} + b^{-1/2} \right) \xi^{m_{i+1}+...+m_{i-1}/2}
\]
in \( D_{1(-\infty,0)} \).

### 4.3. Problem and Result.

Let us use the following notations of norms:
\[
\| u(t) \|_{L^2_{[a,b]}(x)} = \int_a^b \rho(t)^{-1} \sigma(t)^{-n} \sum_{|i|+|j|+|k|=a-r} \| y_{r+j}(t) ... y_{r-i}(t) \times D_{i,j,k}(\rho(t) D_{ij})^{i+j+k} u \|_{L^2(\omega)}^2 \ dt.
\]
\[
\| u(t) \|_{H^2_{[a,b]}(x)} = \| u(t) \|_{L^2_{[a,b]}(x)} + \| y_{r+j}(t) ... y_{r-i}(t) \times D_{i,j,k}(\rho(t) D_{ij})^{i+j+k} u \|_{L^2(\omega)}^2 \ dt.
\]
Moreover we use the following functional spaces:
\[
H_{k-\{r\},a}((0, T) \times \Omega) = \{ u; \ u=0 \ \text{for} \ t<0, \ \| u(T) \|_{k-\{r\},a} < +\infty \},
\]
\[
H_{k-\{r\},a}((0, T) \times \Omega) = \{ u; \ u=0 \ \text{for} \ t<0, \ <u(T)>_{k-\{r\},a} < +\infty \}.
\]
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Now let us consider \( \frac{1}{2} m \) boundary operators \( \{ B(t, x; D_{t}, D_{x}; j) \}_{j=1, \ldots, 1/2m} \) of orders \( \{ r_{j} \}_{j=1, \ldots, 1/2m} \), then we have \( \{ B(t, x; \rho(t) D_{t}, D_{x}; j) \} \).

Our problem is to seek a solution \( u \in H_{m-1+h, \epsilon}((0, T) \times \Omega) \) satisfying

\[
(P) \begin{cases}
A(t, x; D_{t}, D_{x}) u = f & \text{in } (0, T) \times \Omega, \\
B(t, x; D_{t}, D_{x}; j) u = g_{j} & \text{on } (0, T) \times \partial \Omega \quad (j=1, \ldots, \frac{1}{2} m),
\end{cases}
\]

for given \( \rho(t)^{m} f \in H_{m+h-[m], \epsilon}((0, T) \times \Omega) \) and \( \rho(t)^{r} g_{j} \in H_{m-1+h-[r_{j}], \epsilon}((0, T) \times \partial \Omega) \).

By our consideration in §4.2, \((P)\) is equivalent to

\[
(P) \begin{cases}
A(t, x; \rho(t) D_{t}, D_{x}) u = f & \text{in } (0, T) \times \Omega, \\
B(t, x; \rho(t) D_{t}, D_{x}; j) u = g_{j} & \text{on } (0, T) \times \partial \Omega \quad (j=1, \ldots, \frac{1}{2} m),
\end{cases}
\]

where \( f = \rho(t)^{m} f \) and \( g_{j} = \rho(t)^{r} g_{j} \).

Beside the Assumption \((A)\), we assume the uniform Lopatinski conditions for \( 0 < t \leq T \) (Assumption \((B)\)), which is stated by the same words as in §1, except redefining \( \tilde{B}_{0} \) by

\[
\tilde{B}_{0}(t, x, y; \tau, \xi, \eta; f) = (\tau - i \gamma_{r_{j}+1}(\tau) |\eta|) \ldots (\tau - i \gamma_{m-1}(\tau) |\eta|) \times B_{0}(t, x, y; \tau, \xi, \eta; f).
\]

Generalized Theorem. Under the assumptions \((A)\) and \((B)\), there exists a solution of \((P)\) and it holds the energy inequalities: there exists a positive constant \( C_{h} \) such that

\[
(E)_{h} \quad \kappa \| u(t) \|_{m-1+h, \epsilon}^{2} + \| u(t) \|_{m-1+h, \epsilon}^{2} \leq C_{h} \left\{ \frac{1}{\kappa} \| \rho(t)^{m}(Au)(t) \|_{m+h-[m], \epsilon}^{2} + \sum_{j=1}^{2^{m}} \langle \rho(t)^{r} (B_{j} u)(t) \rangle_{m-1+h-[r_{j}], \epsilon}^{2} \right\}
\]

for \( \kappa \geq C_{h} \) and \( u \in H_{m+h, \epsilon}((0, T) \times \Omega) \) \( (h=0, 1, \ldots) \).

4.4. \( C^{\infty}\)-well posedness.

Now we shall compare the orders of zero of weight functions. To make it easy, we assume \( \rho(t) \in C^{\infty}[0, T] \) and \( \rho'(t) \geq 0 \) hereafter.

Lemma 4.4. \( \rho'(0) = 0 \) is a necessary and sufficient condition for \( \sigma(t) \) to be zero of infinite order at \( t=0 \), that is, \( \sigma(t) = 0(t^{N}) \) as \( t\to 0 \) for any \( N \).

Proof. Let \( \rho'(0) = 0 \), then \( \rho(t) \leq ct^{2} \), therefore

\[
\int_{1}^{T} \frac{dt}{\rho(t)} = \frac{1}{c} \int_{1}^{T} \frac{dt}{t^{2}} = \frac{1}{c} \int_{1}^{T} (t - 1)
\]

therefore

\[
e^{-\int_{1}^{T} \frac{dt}{\rho(t)}} \leq C e^{-1/c \cdot 1/t} \quad (C = e^{1/c \cdot 1/T}).
\]
Let \( p'(0) > 0 \), then \( p(t) \geq at \), therefore

\[
\int_0^T \frac{dt}{p(t)} \leq \frac{1}{e} \int_1^T \frac{dt}{t} = \frac{1}{e} (\log T - \log t),
\]

therefore

\[
e^{-\int_0^t \frac{dt}{p(t)}} \geq C e^{\frac{1}{e} \log t} = C t^{1/e} \quad (C = e^{-1/e} \log T).
\]

**Lemma 4.5.** Let us assume \( p'(0) = 0 \), then \( \sigma(t) = 0(p(t)^N) \) as \( t \to 0 \) for any \( N > 0 \).

**Proof.** Since

\[
\sup_{0 < t < T_0} p'(t) = a(T_0) \to 0 \quad \text{as} \quad T_0 \to 0,
\]

we have

\[
\int_0^{T_0} \frac{dt}{p(t)} = \int_0^{T_0} \frac{dt}{p(t)} = \frac{1}{a(T_0)} \int_0^{T_0} \frac{dt}{p(t)} dt = \frac{1}{a(T_0)} (\log p(T_0) - \log p(t))
\]

for \( 0 < t < T_0 \), therefore

\[
e^{-\int_0^t \frac{dt}{p(t)}} \leq C T_0 e^{(T_0)^{-1} \log t} = C T_0 p(t)^{a(T_0)^{-1}} \quad (0 < t < T_0).
\]

Since

\[
\sigma(t) = C T_0 e^{-\int_0^t \frac{dt}{p(t)}} \leq C T_0 p(t)^{a(T_0)^{-1}},
\]

we have

\[
\frac{\sigma(t)}{p(t)^N} \leq C T_0 p(t)^{a(T_0)^{-1} - N}, \quad 0 < t < T_0.
\]

Hence there exist \( T_N > 0 \) and \( C_N > 0 \) such that

\[
\frac{\sigma(t)}{p(t)^N} \leq C_N \quad \text{for} \quad 0 < t < T_N.
\]

**Lemma 4.6.** We assume \( p'(0) = 0 \) and

\[
(*) \quad A(t, x; D_t, 0) = D_{t,m}^m + \epsilon_{m(t,x)} \frac{D_{t,m}^{m-1} + \ldots + \epsilon_{m(t,x)} \frac{D_{t,m}^{m-1}}{m}}{m}, \quad c(t, x) \in \mathbb{R}^m((0, T) \times \Omega).
\]

If there exists \( M > 0 \) such that

\[
(**) \quad \gamma_1(t)^m = 0(\sigma(t)) \quad \text{as} \quad t \to 0,
\]

then the solution \( u \) of (P) satisfies \( u \in H_{m, e}(0, T) \times \Omega) \) if \( f \in H_{m+\gamma-1}(0, T) \times \Omega \) and \( g \in H_{m-1+\gamma-1}(0, T) \times \partial \Omega \) (\( \gamma \): sufficiently large).

The proof of Lemma 4.6 is analogous to those in [7, 8, 9].

Here we assume

**Assumption(C)**

i) \[
\int_0^T \frac{\gamma_1(t)}{p(t)} dt < +\infty,
\]
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ii) in case of \( p'(0)=0 \), (*)& (**) are satisfied.

Owing to (i) of the assumption (C), we can show that solutions of (P) have finite dependence domains in the same way as in [6]. Then we have

**Corollary of Generalised Theorem.** Under the assumptions (A), (B) and (C), the problem (P) is \( C^\infty \)-well posed, that is, we have unique solution \( u \in C^\infty((-0, T) \times \Omega) \) satisfying (P) and \( u=0 \) for \( t<0 \), if given datas satisfy \( \sigma(t)^{-4} f \in C^\infty((-\infty, T) \times \Omega) \) and \( \sigma(t)^{-4} g \in C^\infty((-\infty, T) \times \partial \Omega) \).

**References**