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On the cohomology mod 2 of E_8

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§1. Introduction

Let E_8 be the compact, connected, simply connected, simple Lie group of type E_8. As is well known that E_8 is a closed 248 dimensional manifold which is rational homotopy equivalent to

$$S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}.$$

The mod 2 cohomology ring of E_8 was determined by Araki and Shikata as follows:

Theorem (Araki-Shikata [1]). *As an algebra over the* mod 2 *Steenrod algebra*

$$H^*(E_8; F_2) = F_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4)$$
$$\otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29}),$$

where $\deg x_i = i$, $x_5 = Sq^2 x_3$, $x_9 = Sq^4 x_5$, $x_{17} = Sq^8 x_9$, $x_{23} = Sq^8 x_{15}$, $x_{27} = Sq^4 x_{23}$ *and* $x_{29} = Sq^2 x_{27}$.

They made elaborated calculations of the Bott Samelson K-cycles and so details of the proof is not published. The purpose of this paper is to give a simple proof of the above theorem.

First we determine $H^*(\tilde{E}_8; F_2)$ for $* \leq 31$, where $\tilde{E}_8$ is the 3-connective fibre space of E_8. Next we prove that $\dim H^*(E_8; F_2) \geq 2^{15}$. Finally using the cohomology Serre spectral sequence for the fibering $\tilde{E}_8 \xrightarrow{k} E_8 \rightarrow K(Z, 3)$, $H^*(E_8; F_2)$ is determined. To prove the above theorem, we use the following well known facts:

Theore 1.1 (Bott [5]). *If G is a compact, connected, simply connected Lie group, then $H_*(\Omega G; Z)$ is torsion free.*

Theorem 1.2 (Borel-Siebenthal [4]). *The group E_8 contains a closed, connected subgroup U of local type A_8.*

Theorem 1.3 (Cartan [7]). *The group E_8 contains a closed, connected subgroup V of local type D_8 satisfying*

(1) *the center of V is of order* 2,

(2) *E_8/V is the irreducible symmetric space $\boldsymbol{EVIII}$.*

§2. 3-connective fibre space of E_8

From now on the mod 2 cohomology and homology are simply denoted by $H^*(\ )$ and $H_*(\ )$. For a graded module $A=\sum A_i$ over F_2, P. S. $(A)=\sum_{i\geq 0}(\dim A_i)t^i\in Z[[t]]$ and for a graded algebra A over F_2, $A=\Delta(x_1, x_2,\dots, x_n)$ means $\{x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n};\ \varepsilon_i=0$ or $1\}$ is a basis of the vector space A. If $A=\Delta(x_1,\dots, x_n)$, then $\{x_1,\dots, x_n\}$ is called a simple system of generators of A. If $\boldsymbol{G}$ is a compact, connected Lie group, then $H^*(\boldsymbol{G})$ has a simple system of generators.

First recall the following fact: Since the universal covering of $\boldsymbol{U}$ is $\boldsymbol{SU}(9)$ and the center of $\boldsymbol{SU}(9)$ is of order 9, $H^*(\boldsymbol{U})$ is isomorphic to $H^*(\boldsymbol{SU}(9))$ as an algebra over the mod 2 Steenrod algebra.

Lemma 2.1. *If i: $\boldsymbol{U}\to\boldsymbol{E}_8$ is the inclusion, then the induced map i^*: $H^3(\boldsymbol{E}_8)\to H^3(\boldsymbol{U})$ is an isomorphism.*

Proof. Since $\boldsymbol{U}$ is a closed connected subgroup of maximal rank and $H^*(\boldsymbol{U};\boldsymbol{Z})$ is 2-torsion free, $H^*(\boldsymbol{E}_8/\boldsymbol{U};\boldsymbol{Z})$ is 2-torsion free and $H^*(\boldsymbol{E}_8/\boldsymbol{U})=0$ for $0<*\leq 5$ (cf. 13 of [3]). Since $\boldsymbol{E}_8/\boldsymbol{U}$ is 1-connected, Lemma 2.1 follows from the Serre exact sequence for the fibering $\boldsymbol{U}\to\boldsymbol{E}_8\to\boldsymbol{E}_8/\boldsymbol{U}$.

Lemma 2.2 *If t_2 is a generator of $H^2(\Omega\boldsymbol{E}_8)=Z/2$, then $t_2^8\neq 0$.*

Proof. By Lemma 2.1, $t'=(\Omega i)^*(t_2)$ is a generator of $H^2((\Omega\boldsymbol{U})_0)$, where $(\Omega\boldsymbol{U})_0$ is the connected component of $\Omega\boldsymbol{U}$ containing the constant loop. Moreover $H^*((\Omega\boldsymbol{U})_0)$ is isomorphic to $H^*(\Omega\boldsymbol{SU}(9))$. Denote a generator of $H^2(\Omega\boldsymbol{SU}(9))$ by t. Then we need only show $t^8\neq 0$. Consider the fibering

$$\Omega\boldsymbol{SU}(9)\longrightarrow\Omega\boldsymbol{SU}(\infty)\longrightarrow\Omega(\boldsymbol{SU}(\infty)/\boldsymbol{SU}(9)),$$

where $\boldsymbol{SU}(\infty)=\underset{n}{\mathrm{Colim}}\,\boldsymbol{SU}(n)$. The algebra $H^*(\Omega\boldsymbol{SU}(\infty))$ is a polynomial algebra by the Bott periodicity theorem (cf. [6]) and the space $\boldsymbol{SU}(\infty)/\boldsymbol{SU}(9)$ is 18-connected. Therefore $t^8\neq 0$.

As is proved in [5], $\boldsymbol{E}_8$ is 2-connected and $\pi_3(\boldsymbol{E}_8)=\boldsymbol{Z}$. These are easy consequences of Theorem 1.1. Let j: $B\boldsymbol{E}_8\to K(\boldsymbol{Z}, 4)$ be a map representing a generator of $H^4(B\boldsymbol{E}_8;\boldsymbol{Z})=\boldsymbol{Z}$. Then Ωj and $\Omega^2 j$ are generator of $H^3(\boldsymbol{E}_8;\boldsymbol{Z})$ and $H^2(\Omega\boldsymbol{E}_8;\boldsymbol{Z})$ respectively. There are fiberings

(2.3) $$B\tilde{\boldsymbol{E}}_8\longrightarrow B\boldsymbol{E}_8\longrightarrow K(\boldsymbol{Z}, 4)$$

(2.4) $$\tilde{\boldsymbol{E}}_8\xrightarrow{k}\boldsymbol{E}_8\longrightarrow K(\boldsymbol{Z}, 3)$$

(2.5) $$\Omega\tilde{\boldsymbol{E}}_8\longrightarrow\Omega\boldsymbol{E}_8\longrightarrow K(\boldsymbol{Z}, 2)\simeq \boldsymbol{CP}^\infty$$

(2.6) $$K(\boldsymbol{Z}, 1)\simeq S^1\longrightarrow\Omega\tilde{\boldsymbol{E}}_8\longrightarrow\Omega\boldsymbol{E}_8,$$

where $B\tilde{\boldsymbol{E}}_8$ is the homotopy fibre of j. Note that (2.4) (resp. (2.5)) is a loop (resp a double loop) of (2.3) and so $\tilde{E}_8$ is the 3-connective fibre space of E_8 and $\Omega\tilde{E}_8$ is the 2-connective fibre space of ΩE_8.

Lemma 2.7 P. S. $(H^*(\Omega\tilde{E}_8))=(1+t^{14}+t^{28})(1+t^{22})(1+t^{26}) \mod (t^{31})$.

Proof. Since $H^*(\Omega E_8)$ is a Hopf algebra, $t_2^8\neq 0$ implies $t_2^{15}\neq 0$. Moreover there exists a graded algebra A over F_2 such that as an algebra

$$H^*(\Omega E_8)=F_2[t_2]\otimes A$$

for $*\leq 31$. Consider the cohomology Serre spectral sequence for the fibering (2.6). Then we can easily get $H^*(\Omega\tilde{E}_8)=A$ for $*\leq 30$. On the other hand by Theorem 1.1 P. S. $(A)=(1+t^{14}+t^{28})(1+t^{22})(1+t^{26}) \mod (t^{31})$ and so the lemma is proved.

If s_{14} is a generator of $H_{14}(\Omega\tilde{E}_8)$, then there are two positibilities:

(2.8) $$s_{14}^2=0,$$

(2.9) $$s_{14}^2\neq 0.$$

Lemma 2.10. (1) *If* $s_{14}^2\neq 0$, *then as an algebra*

$$H^*(\tilde{E}_8)=\Lambda(a_{15}, a_{23}, a_{27})$$

for $*\leq 31$, *where* $\deg a_i=i$.

(2) *If* $s_{14}^2=0$, *then as an algebra over the* mod 2 *Steenrod algebra*,

$$H^*(\tilde{E}_8)=F_2[a_{15}]\otimes\Delta(a_{23}, a_{27}, a_{29})$$

for $*\leq 31$, *where* $\deg a_i=i$, $a_{23}=Sq^8a_{15}$, $a_{27}=Sq^4a_{23}$ *and* $a_{29}=Sq^2a_{27}$.

Proof. (1) If $s_{14}^2\neq 0$, then as an algebra

$$H^*(\Omega\tilde{E}_8)=F_2[s_{14}]\otimes\Delta(s_{22}, s_{26})$$

for $*\leq 30$, where $\deg s_i=i$. Consider the Rothenberg-Steenrod spectral sequence (cf. [12])

$$E_2=\mathrm{Ext}_{H*(\Omega\tilde{E}_8)}(F_2, F_2)\Longrightarrow E_\infty=Gr(H^*(\tilde{E}_8)).$$

The E_2-term is isomorphic to

$$\Lambda(a'_{15}, a'_{23}, a'_{27})$$

for deg ≤ 31, where $\deg a'_i=i$. This spectral sequence clearly collapses for deg ≤ 31 by the dimensional reasons. Thus (1) is proved.

(2) If $s_{14}^2=0$, then as an algebra

$$H_*(\Omega\tilde{E}_8)=\Lambda(s_{14}, s_{22}, s_{26}, s_{28})$$

for $*\leq 30$, where $\deg s_i=i$. Thus the E_2-term of the Rothenberg-Steenrod spectral sequence is isomorphic to

$$F_2[a'_{15}]\otimes\Delta(a'_{23}, a'_{27}, a'_{29})$$

as an algebra for deg ≤ 31, where deg $a'_i=i$. Since a'_{15} and a'_{29} are elements of $E_2^{1,*}$, this spectral sequence also collapses for deg ≤ 31 by the dimensional reasons. Therefore as an algebra

$$H^*(E_8)=F_2[a_{15}]\otimes\Lambda(a_{23}, a_{27}, a_{29})$$

for $*\leq 31$, where deg $a_i=i$. Note that

$$Sq^1Sq^2Sq^4Sq^8a_{15}=Sq^{15}a_{15}=a_{15}^2\neq 0$$

by the Adem relations and so $a_{23}=Sq^8a_{15}$, $a_{27}=Sq^4a_{23}$ and $a_{29}=Sq^2a_{27}$.

§3. Proof of Theorem

The following is easily proved:

Lemma 3.1. *If G is a compact, connected Lie group and $\{x_1,\dots, x_n\}$ is a sample system of generators of $H^*(G)$, then*

(1) $$\dim G=\sum_{i=1}^n \deg x_i,$$

(2) $$P.S.(H^*(G))=\prod_{i=1}^n (1+t^{\deg x_i})$$

and

(3) $$\dim H^*(G)=2^n.$$

Now recall the following fact, which is a special case of Theorem 4.3 of [8]:

Lemma 3.2. *Let X be a compact $Z/2$-space and $X^{Z/2}$ be the fixed point set. Then* $\dim H^*(X)\geq\dim H^*(X^{Z/2})$.

By Theorem 1.3, E_8 has an involution τ such that E_8^τ, the fixed point set of τ, is V. Since the local type of V is D_8 and the centor of V is of order 2, V is isomorphic to $SO(16)$ or $Ss(16)$. Moreover $H^*(SO(16))$ is isomorphic to $H^*(Ss(16))$ as an algebra and so $\dim H^*(V)=2^{15}$ (cf. [2]). Then using Lemma 3.2 we have the following:

Lemma 3.3. $\dim H^*(E_8)\geq 2^{15}$

Lemma 3.4. *If $s_{14}^2\neq 0$, then* $\dim H^*(E_8)\leq 2^{14}$.

Proof. Consider the cohomology Serre spectral sequence for the fibering (2.4). Since as an algebra

$$H^*(K(Z, 3))=F_2[u_3, u_5, u_9, u_{17}]$$

for $*\leq 32$, where deg $u_i=i$, $u_5=Sq^2u_3$, $u_9=Sq^4u_5$ and $u_{17}=Sq^8u_9$ (cf. Serre [14]). the E_2-term is isomorphic to

$$F_2[u_3, u_5, u_9, u_{17}]\otimes\Lambda(a_{15}, a_{23}, a_{27})$$

as an algebra for deg ≤ 31. Since this spectral sequence is a Hopf algrbra spectral sequence, $1\otimes a_{15}$ and $1\otimes a_{27}$ are permanent cycles and a_{23} is transgressive with $\tau(a_{23})=au_3^8$ for some $a\in \boldsymbol{F}_2$. Thus the E_∞-term is isomorphic to E_2 if $a=0$ or $E_2/(a_{23}, u_3^8)$ if $a=1$ for deg ≤ 31. There is an element $x_i\in H^i(\boldsymbol{E}_8)$ such that $k^*(x_i)=a_i$ for $i=15$, 23, and 27 if $a=0$ or $i=15$ and 27 if $a=1$. Put $x_3=(\Omega j)^*(u_3)$, $I_1=\{3, 6, 12, 5, 10, 20, 9, 18, 17, 15, 27\}$ and $I_0=I_1\cup\{23, 24\}$. Then as an algebra $H^*(\boldsymbol{E}_8)$ is isomorphic to $\Delta(x_i;\ i\in I_a)$ for $*\leq 31$, where $x_{2^k i}=(x_i)^{2^k}$ and $a=0$ or 1. Note that $248-(\sum_{i\in I_0} i)=59$ and $248-(\sum_{i\in I_1} i)=106$. Since the degrees of the other elements of the simple system are greater than 31, there is only one other element if $a=0$ or there are at most three other elements if $a=1$ in the simple system by (1) of Lemma 3.1. Therefore dim $H^*(\boldsymbol{E}_8)\leq 2^{14}$.

Proof of Theorem. By Lemma 3.3 and Lemma 3.4, $s_{14}^2=0$. Using Lemma 2.10, we can easily show that the cohomology Serre spectral sequence collapses for deg ≤ 31, since it is a Hopf algebra spectral sequence. Thus there is an element x_{15} such that $k^*(x_{15})=a_{15}$. Put $x_3=(\Omega j)^*(u_3)$, $x_5=Sq^2x_3$, $x_9=Sq^4x_5$, $x_{17}=Sq^8x_9$, $x_{23}=Sq^8x_{15}$, $x_{27}=Sq^4x_{23}$, $x_{29}=Sq^2x_{27}$, $x_2{}^k=(x_i)_{2^k i}$ and $I_2=I_0\cup\{29, 30\}$. Then

$$H^*(\boldsymbol{E}_8)=\Delta(x_i;\ i\in I_2)$$

for $*\leq 31$. Moreover since $(\sum_{i\in I_2} i)=248$, $\{x_i;\ i\in I_2\}$ is a simple system of generators by (1) of Lemma 3.1. Using the fact that $H^*(\boldsymbol{E}_8)$ is a Hopf algebra, we have the relations

$$x_3^{16}=x_5^8=x_9^4=x_{17}^2=x_{15}^4=x_{23}^2=x_{27}^2=x_{29}^2=0$$

by the dimensional reasons.

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