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The asymptotic distribution of eigenvalues for $\frac{\partial^2}{\partial x^2}+Q(x)\frac{\partial^2}{\partial y^2}$ in a strip domain

By

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§0. Let $Q(x)$ be a positive function defined for $x\geqq 0$. We consider the following boundary-value problem in a strip domain $\Omega=(0, \infty)\times(0, \pi)$

$$\begin{cases} -(u_{xx}+Q(x)u_{yy})=\lambda u & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega\,. \end{cases} \tag{0.1}$$

For simplicity, we sometimes denote by A the operator $-((\partial^2/\partial x^2)+Q(x)(\partial^2/\partial y^2))$. We say that a function $f\in C^k(G)$, if f has continuous derivatives up to order k in G. If $Q(x)\in C^2([0, \infty))$ satisfies

$$Q(x)\geqq Q_0>0\,, \qquad \lim_{x\to\infty} Q(x)=\infty\,, \tag{0.2}$$

then there exists in $L^2(\Omega)$ a complete orthonormal system of eigenfunctions $\{\varphi_n\}$ and corresponding eigenvalues $\{\lambda_n\}$ satisfying

$$A\varphi_n=\lambda_n\varphi_n \quad \text{in } \Omega\,, \qquad \varphi_n=0 \quad \text{on } \partial\Omega\,. \tag{0.3}$$

Let $N(\lambda)$ denote the number of eigenvalues not exceeding λ. We shall study the asymptotic behavior of $N(\lambda)$ as $\lambda\to\infty$.

When we regard $-Q(x)(\partial^2/\partial y^2)$ as a self-adjoint operator $\mathcal{Q}(x)$ with a parameter x and $u(x, y)$ as an $L^2(0, \pi)$-valued function $U(x)$, (0.1) is reduced to the Sturm-Liouville operator problem studied in Kostyuchenko-Levitan [8]

$$\begin{cases} U''+(\lambda-\mathcal{Q}(x))U=0 & \text{for } x>0 \\ U(0)=0\,. \end{cases} \tag{0.4}$$

Under certain conditions on $\mathcal{Q}(x)$, they obtained an asymptotic formula for $N(\lambda)$ in the form

$$N(\lambda)\sim\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{\lambda\geqq\alpha_n(x)}(\lambda-\alpha_n(x))^{1/2}dx\,, \tag{0.5}$$

where $\alpha_n(x)$ is the n-th eigenvalue of $\mathcal{Q}(x)$. Note that $\alpha_n(x)=n^2Q(x)$ in our case. If we set

$$U(x)=\sum_{n=1}^{\infty}\varphi_n(x)\sin ny\,,$$

then (0.4) is splitted into the following Strum-Liouville problems

$$\text{(0.6)}\qquad \begin{cases} \varphi_n''+(\lambda-n^2Q(x))\varphi_n=0 & \text{for } x>0 \\ \varphi_n(0)=0 & (n\geqq 1)\,. \end{cases}$$

Accordingly, our considerations in this article consist in studying the eigenvalues of the Sturm-Liouville problem (0.6) with particular care on the parameter n.

The eigenvalue problem for A may be interesting in itself, but it also plays an important role in studying the asymptotic distribution of eigenvalues for the Laplace operator with Dirichlet condition in an unbounded domain such as

$$G=\{(x,\ y)\in R^2\,|\,0<x<\infty,\ 0<y<b(x)\}\,.$$

$$\text{(0.7)}\qquad \begin{cases} \Delta u+\lambda u=0 & \text{in } G \\ u=0 & \text{on } \partial G\,. \end{cases}$$

We shall discuss the problem in § 4 of this article. Here we only mention that the large eigenvalues of (0.7) are, roughly speaking, asymptotically equal to the large eigenvalues of A with $Q(x)=\pi^2/b(x)^2$. H. Tamura [10] is the first one obtaining the asymptotic law of the distribution of eigenvalues in the from (0.5). In the previous paper F. Asakura [1], we also studied the problem by another means. In the course of the study, we obtained an asymptotic formula of the distribution of eigenvalues of A with remainder estimate in the form

$$\text{(0.8)}\qquad N(\lambda)=\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{\lambda\geq n^2Q(x)}(\lambda-n^2Q(x))^{1/2}dx+O(\lambda^{1/2})\,,$$

assuming $Q(x)\in C^4([0,\ \infty))$ to satisfy

$$\text{(0.9)}\qquad \begin{cases} \dfrac{A}{x}\leqq\dfrac{Q'(x)}{Q(x)}\leqq\dfrac{B}{x}, & \dfrac{|Q''(x)|}{Q'(x)}\leqq\dfrac{C}{x} \\[2ex] \dfrac{|Q'''(x)|}{Q'(x)}\leqq\dfrac{C}{x^2} & \text{for large } x\,. \end{cases}$$

Note that $Q(x)=x^{2k}$ satisfies (0.9).

In the present article, we study the asymptotic distribution of eigenvales of A by different two methods. One is using the zeta function of the eigenvalues defined as

$$\text{(0.10)}\qquad Z(\alpha,\ A)=\sum_{n=1}^{\infty}\lambda_n^{-\alpha}\,.$$

The other is adopting a uniform asymptotic expansion of the solution to the Sturm-Liouville problem (0.6) in a neighborhood of a turning point, which is employed in F. Asakura [1] and covers where the zeta function does not work.

In the first place, we review some basic spectral properties of A in a strip domain $\Omega=(0,\ \infty)\times(0,\ \pi)$. In § 2, assuming $Q(x)$ to satisfy

$$\left|\frac{Q'(x)}{Q(x)}\right| \leqq L, \qquad Q(x) \geqq (2L)^2, \tag{0.11}$$

we obtain some estmates of the resolvent kernel, which are crucial in studying the zeta function. We study the analytic extension of $Z(\alpha, A)$ in § 3. We may well expect that the infinite sum (0.10) converges for sufficiently large α. We shall show that (0.10) converges if and only if the integral

$$\int_0^\infty Q(x)^{-\alpha+1/2} dx \tag{0.12}$$

converges. Throughout this article we assume

$$\int_0^\infty Q(x)^{-1/2} dx = \infty . \tag{0.13}$$

This corresponds to the condition that the area of G is infinite in the case of the Laplace operator in unbounded domains (see (0.7)).

Let

$$\sigma = \inf\left\{\alpha \in R \left| \int_0^\infty Q(x)^{-\alpha+1/2} dx < \infty \right.\right\} \qquad (\sigma \geqq 1 \text{ by } (0.13)).$$

We obtain the analytic continuation of the zeta function as the following.

Theorem 0.1. *Let* $Q(x) \in C^2([0, \infty))$ *satisfy* (0.2), (0.11) *and* (0.13). *Then* $Z(\alpha, A)$ *has the analytic continuation across* $\operatorname{Re}\alpha = \sigma$ *of the form*

$$Z(\alpha, A) = \frac{\Gamma(\alpha-1/2)\zeta(2\alpha-1)}{2\sqrt{\pi}\,\Gamma(\alpha)} \int_0^\infty Q(x)^{-\alpha+1/2} dx + h(\alpha) \tag{0.14}$$

where $\Gamma(\alpha)$ *is the Gamma function,* $\zeta(\alpha)$ *the Riemann zeta function and* $h(\alpha)$ *is holomorphic in* $\operatorname{Re}\alpha > \sigma - \delta$ $(\delta > 0)$.

The form of the singularity of the zeta function with the largest real part reflects the asymptotic nature of the eigenvalues of A. We shall see later that $Z(\alpha, A)$ may have various types of singularities at $\alpha = \sigma$. For example when $Q(x) = x^{2\kappa}(\log x)^{-2\gamma}$ for large x, the singularity is of the form

$$\left(\alpha - \frac{\kappa+1}{2\kappa}\right)^{-(1+(\gamma/\kappa))} \sum_{n=0}^{\infty} \sum_{j=0}^{n} A_{nj}\left(\alpha - \frac{\kappa+1}{2\kappa}\right)^n \left(\log\left(\alpha - \frac{\kappa+1}{2\kappa}\right)\right)^j .$$

In such cases we introduce an Ikehara Tauberian theorem of the following form. Proof of the theorem is put off until § 6.

Theorem 0.2. *Let* $N(\lambda)$ *be a non-negative, non-decreasing function. If*

$$Z(\alpha) = \int_0^\infty \lambda^{-\alpha} dN(\lambda)$$

is convergent for $\operatorname{Re}\alpha > \sigma$ *and*

$$h(\alpha) = Z(\alpha) - (\alpha-\sigma)^{-(1+\rho)} \sum_{n=0}^{[1+\rho]} \sum_{j=0}^{n} A_{nj}(\alpha-\sigma)^n (\log(\alpha-\sigma))^j \quad (\rho \geqq 0) \tag{0.15}$$

can be extended to a continuous function in $\operatorname{Re}\alpha \geqq \sigma$, *then* $N(\lambda)$ *has the asymptotic form*

$$N(\lambda) \sim \frac{A_{00}}{\sigma \Gamma(1+\rho)} \lambda^{\sigma} (\log \lambda)^{\rho} \qquad \text{as } \lambda \to \infty . \tag{0.16}$$

In § 4, we turn our attention to the eigenvalue problem of the Laplace operator in unbounded domains (0.7). We show that if the zeta function of the eigenvalues of A with $Q(x)=\pi^2/b(x)^2$ has the same form as (0.15), then the asymptotic formula of the distribution of eigenvalues is described as (0.16).

Our results show by the way that if (0.12) is not finite for any α (for example $Q(x)=\log x$), $Z(\alpha, A)$ will not converge for any α, especially the growth of λ_n is slower than any small power of n. In § 5, we study these cases where the zeta functions are of no use in studying the asymptotic distribution of the eigenvalues. We shall find the methods in F. Asakura [1] still work there and obtain

Theorem 0.3. *Assume* $Q(x) \in C^4([0, \infty))$ *to satisfy, instead of* (0.9)

$$\begin{cases} \dfrac{A}{x \log x} \leqq \dfrac{Q'(x)}{Q(x)} \leqq \dfrac{B}{x \log x}, \qquad \left|\dfrac{Q''(x)}{Q'(x)}\right| \leqq \dfrac{C}{x}, \\ \left|\dfrac{Q'''(x)}{Q'(x)}\right| \leqq \dfrac{C}{x^2} \quad \text{for large } x \text{ with some constants } A, B \text{ and } C. \end{cases} \tag{0.17}$$

Then (0.8) *holds for large* λ.

Under the condition (0.17), we shall get a uniform asymptotic expansion in λ of the solution to the Sturm-Liouville problem (0.6) using the Airy functions. For $Q(x)=(\log x)^{2k}$ we obtain

$$N(\lambda) = \sqrt{\frac{k}{2\pi}} \lambda^{1/k-1/4k} \exp(\lambda^{1/k})(1+0(\lambda^{-1/2k})) . \tag{0.18}$$

This time, the operator A with $Q(x)=\pi^2/b(x)^2$ is not a good approximation of the Laplace operator in an unbounded domain $G=G_1 \cup G_2$ such that G_1 is bounded and $G_2=\{(x, y) \in R^2 \mid R<x<\infty, 0<y<(\log x)^{-k}\}$. Here we can merely obtain the estimates of $N(\lambda)$ from above and below as the following.

$$G_1 e^{(1-\varepsilon)\lambda^{1/2k}} \leqq N(\lambda) \leqq C_2 e^{(1+\varepsilon)\lambda^{1/2k}} \tag{0.19}$$

for any ε with certain constants C_1, C_2.

In conclusion, I would like to express my hearty thanks to Professor S. Mizohata and Professor N. Shimakura for valuable advice and incessant encouragement.

§ **1.** In this section we shall review some basic spectral properties of the operator A in a strip domain $\Omega=(0, \infty) \times (0, \pi)$, assuming (0.2).

We denote by $L^2(\Omega)$ the Hilbert space of square integrable (real valued) functions in Ω. In this article we shall work with real Hilbert spaces. Let us denote

$$\|u\|^2=\iint_\Omega u^2 dx\, dy$$

$$E(u)=\iint_\Omega u_x^2+Q(x)u_y^2 dx\, dy\,.$$

We denote by $C_0^1(\Omega, Q)$ the space of functions defined as

$$C_0^1(\Omega, Q)=\{u\in C^1(\Omega)\cap C^0(\bar{\Omega}) \mid u=0 \text{ on } \partial\Omega,\ \|u\|^2,\ E(u)<\infty\}.$$

The space $H_0^1(\Omega, Q)$ is defined as the completion of $C_0^1(\Omega, Q)$ with respct to $\|u\|_1^2=\|u\|^2+E(u)$.

In the first place we show the following inequality

Proposition 1.1. *If a function $Q(x)$ defined for $x\geqq 0$ satisfies* (0.2), *then for any $\varepsilon>0$ there exist $\omega_1, \cdots, \omega_N\in L^2(\Omega)$ so that*

$$\|u\|^2\leqq \sum_{j=1}^N |(\omega_j, u)|^2+\varepsilon E(u) \tag{1.1}$$

holds for any $u\in H_0^1(\Omega, Q)$.

Proof. We have only to show the inequality for $u\in C_0^1(\Omega, Q)$. Since $u(x, 0)=u(x, \pi)=0$ for any $x\geqq 0$, if follows

$$\int_0^\pi u_y(x, y)^2 dy\geqq\int_0^\pi u(x, y)^2 dy\,. \tag{1.2}$$

Multiplying $Q(x)$ to the both sides of (1.3) and integrating in x from R to infinity, we have

$$\int_R^\infty\int_0^\pi Q(x)u_y(x, y)^2 dy\, dx\geqq(\inf_{x\geq R} Q(x))\int_R^\infty\int_0^\pi u(x, y)^2 dy\, dx\,.$$

If we choose R such that $\inf Q(x)\geqq\varepsilon^{-1}$, we find

$$\int_{x\geq R} u(x, y)^2 dx\, dy\leqq\varepsilon\iint_{x\geq R} Q(x)u_y^2 dx\, dy \tag{1.3}$$

$$\leqq\varepsilon\iint_{x\geq R} u_x^2+Q(x)u_y^2 dx\, dy\,.$$

On the other hand, since $\Omega\cap\{x<R\}$ is a bounded domain, then for any ε there exist $\omega_1', \cdots, \omega_N'\in L^2(\Omega\cap\{x<R\})$ such that

$$\iint_{x\leq R} u(x, y)^2 dx\, dy\leqq\sum_{j=1}^N |(\omega_j', u)|^2+\varepsilon\iint_{x\leq R} u_x^2+Q(x)u_y^2 dx\, dy\,. \tag{1.4}$$

Combining (1.3) and (1.4), we get the inequality.

From Proposition 1.1, it follows that any sequence $u_j\in H_0^1(\Omega, Q)$ such that $E(u_j)\leqq L$ has a convergent subsequence in $L^2(\Omega)$. Then employing variational methods to the form $E(u)$ in $H_0^1(\Omega, Q)$ (see Courant-Hilbert text book), we get a complete orthonormal system of eigenfunctions $\{\varphi_n\}$ of A such that $\varphi_n \in C^2(\Omega) \cap H_0^1(\Omega, Q)$ and corresponding eigenvalues $\{\lambda_n\}$ with

(1.5) $$A\varphi_n = \lambda_n \varphi_n \quad \text{in } \Omega, \qquad \varphi_n = 0 \quad \text{on } \partial\Omega.$$

Later we shall make use of the boundary-value problem

(1.6) $$\begin{cases} -(u_{xx} + Q(x)u_{yy}) = \lambda u & \text{in } \Omega \\ u = 0 & \text{for } y = 0, \pi \\ u_x = 0 & \text{for } x = 0. \end{cases}$$

Employing the function spaces $\tilde{C}_0^1(\Omega, Q)$ and $\tilde{H}_0^1(\Omega, Q)$ defined as

$$\tilde{C}_0^1(\Omega, Q) = \{u \in C^1(\Omega) \cap C^0(\Omega) \mid u = 0 \quad \text{for } y = 0, \pi, \; \|u\|, \; E(u) < \infty\},$$

$$\tilde{H}_0^1(\Omega, Q) = \text{the completion of } \tilde{C}_0^1(\Omega, Q) \text{ with respect to } \|u\|_1,$$

we obtain a complete system of eigenfunctions $\{\psi_n\}$ of A such that $\psi_n \in C^2(\Omega) \cap \tilde{H}_0^1(\Omega, Q)$ and corresponding eigenvalues μ_n with

(1.7) $$A\psi_n = \mu_n \psi_n \quad \text{in } \Omega, \qquad \psi_n = 0 \quad \text{for } y = 0, \pi, \qquad \psi_{nx} = 0 \quad \text{for } x = 0.$$

§2. Let $R_n(x, y, \xi, \eta, \mu)$ be the kernel function of $(A+\mu)^{-n}$. R_n is, in general, a certain distribution in (x, y, ξ, η) and represented as

$$R_n(x, y, \xi, \eta) = \sum_{j=1}^{\infty} \frac{\varphi_n(x, y)\varphi_n(\xi, \eta)}{(\lambda_n + \mu)^n}$$

using the eigenfunctions $\{\varphi_n\}$ and corresponding eigenvalues $\{\lambda_n\}$. We may expect R_n to be a smooth function for large n. In this section we shall obtain certain estimates of

$$R_n(x, y, x, y) = \sum_{j=1}^{\infty} \frac{\varphi_n(x, y)^2}{(\lambda_n + \mu)^n}.$$

We seek out the eigenfunctions in the form $\varphi(x, y) = \varphi_{nj}(x) \sin ny$. Then $\varphi_{nj} \in L^2(0, \infty)$ and satisfies

(2.1) $$\begin{cases} \varphi_{nj}'' + (\lambda - n^2 Q(x))\varphi_{nj} = 0 & \text{for } x > 0 \\ \varphi_{nj}(0) = 0 & \end{cases}$$

Let $\{\varphi_{nj}\}$ be a complete orthonormal system of eigenfunctions, $\{\lambda_{nj}\}$ be corresponding eigenvalues. We observe that $\{\varphi_{nj}(x) \sin ny\}_{n,j=1}^{\infty}$ constitutes a complete orthonormal system in $L^2(\Omega)$ with $\{\lambda_{nj}\}$ corresponding eigenvalues.

To proceed further, we assume $Q(x)$ to satisfy (0.11) and (0.13) in addition to (0.2).

Remark 2.1. Condition (0.2) guarantees the existence and regularity of complete eigenfunctions. If $\int_0^\infty Q(x)^{-1/2}dx<\infty$, spectral properties of A are somewhat very similar to an elliptic operator in a bounded domain. We shall put aside such cases in the present article.

It is easy to verify the following properties of $Q(x)$.

Proposition 2.2. *If* $Q(x)$ *satisfies* (0.11), *then there exists a constant* A *independent of* n, x *and* y, *such that*

$$|Q(x+y)-Q(x)|\leqq A|y|Q(x) \quad \text{for } |y|\leqq 1, \tag{2.2}$$

$$n^2Q(x+y)\leqq Ae^{n|y|\sqrt{Q(x)}/2} \quad \text{for } |y|\geqq 1. \tag{2.3}$$

Proof. Set $R(x)=\log Q(x)$. Then it follows by the mean-value theorem

$$\left|\log\frac{Q(x+y)}{Q(x)}\right|=|R(x+y)-R(x)|\leqq L|y|.$$

Hence

$$e^{-L|y|}\leqq\frac{Q(x+y)}{Q(x)}\leqq e^{L|y|},$$

and then

$$e^{-L|y|}-1\leqq\frac{Q(x+y)-Q(x)}{Q(x)}\leqq e^{L|y|}-1.$$

In this way we can choose a constant A so that (2.5) holds.

Since $Q(x+y)\leqq Ae^{L|y|}Q(x)$, we shall prove, instead of (2.3), a stronger inequality

$$n^2Q(x)e^{L|y|}\leqq Ae^{n|y|\sqrt{Q(x)}/2}. \tag{2.4}$$

Setting $z=n\sqrt{Q(x)}$, we show that

$$Ae^{yz/2}\geqq z^2e^{Ly} \quad \text{holds for } y\geqq 1 \text{ and } z\geqq 2L. \tag{2.5}$$

By Taylor expansion of e^x, we find

$$\begin{aligned}Ae^{yz/2}-z^2e^{Ly}&=e^{Ly}\{Ae^{(z/2-L)y}-z^2\}\\&\geqq e^{Ly}\left\{A+A\left(\frac{1}{2}z-L\right)y+\frac{A}{2}\left(\frac{1}{2}z-L\right)^2y^2-z^2\right\}\\&\geqq e^{Ly}\left\{A+\frac{A}{2}\left(\frac{1}{2}z-L\right)^2y^2-z^2\right\}.\end{aligned}$$

Then we can see that there exists a constant A so that (2.5) holds. (2.2) and (2.3) corresponds to (17.10.3) and (17.10.4) in Chap. XVII of E. C. Titchmarsh [12].

We now give a brief review of the Fourier sine transformation. Let $f \in C^1(0, \infty)$ satisfying $f(0)=0$, $x^i f^{(j)} \in L^2(0, \infty)$ for i, $j=0, 1$. We define the Fourier sine transform as

$$f^{\wedge}(\xi)=\int_0^{\infty} \sin x\xi \ f(x)dx . \tag{2.6}$$

Then $f(x)$ is expressed by the Fourier inversion formula as

$$f(x)=\frac{2}{\pi}\int_0^{\infty} \sin x\xi \ f^{\wedge}(\xi)d\xi . \tag{2.7}$$

Moreover the Parseval formula holds as

$$\int_0^{\infty} |f(x)|^2 dx=\frac{2}{\pi}\int_0^{\infty} |f^{\wedge}(\xi)|^2 d\xi . \tag{2.8}$$

We set

$$\begin{aligned} E_n(x, y, \mu) &= \frac{2}{\pi}\int_0^{\infty} \frac{\sin x\xi \sin y\xi}{(\xi^2+n^2Q(x)+\mu)} d\xi \qquad (\mu>0) \\ &= \frac{1}{2\kappa_n}\{e^{-\kappa_n|x-y|}-e^{-\kappa_n(x+y)}\} \end{aligned} \tag{2.9}$$

where $\kappa_n^2=n^2Q(x)+\mu$. Then $E_n(x, y, \mu)$ satisfies

$$\begin{cases} -\dfrac{\partial^2}{\partial y^2} E_n(x, y, \mu)+\kappa_n^2 E_n(x, y, \mu)=\delta(x-y) \\ E_n(x, 0, \mu)=0 \end{cases} \tag{2.10}$$

where $\delta(x)$ is the Delta function.

Remark 2.3. For the equation (1.6), we obtain by separation of variables

$$\begin{cases} \varphi''+(\lambda-n^2Q(x))\varphi=0 \qquad \text{for } x>0 \\ \varphi'(0)=0 \\ \varphi, \varphi' \in L^2(0, \infty) . \end{cases} \tag{2.1$'$}$$

In order to study (2.1)′, we adopt the cosine transform instead, then the following arguments are the same.

Lemma 2.4. *Let $\varphi_{nj}(x)$ an eigenfunction of* (2.1) *with the eigenvalue λ_{nj}. Then $\varphi_{nj}(x)$ is expressed as*

$$\varphi_{nj}(x)=\int_0^{\infty} \{n^2Q(x)-n^2Q(y)+\lambda_{nj}+\mu\} E_n(x, y, \mu)\varphi_{nj}(y)dy . \tag{2.11}$$

Proof. Since φ_{nj} is an eigenfunction with the eigenvalue λ_{nj}, we find

$$\begin{cases} -\dfrac{d^2}{dy^2}\varphi_{nj}(y)+\kappa_n^2\varphi_{nj}(y)=\{n^2Q(x)-n^2Q(y)+\lambda_{nj}+\mu\}\varphi_{nj}(y) \\ \varphi_{nj}(0)=0 , \qquad \text{where } \kappa_n^2=n^2Q(x)+\mu . \end{cases} \tag{2.12}$$

Substituting $y-x$ for y in (2.3), we have

$$n^2Q(y)\leqq Ae^{n\sqrt{Q(x)}|x-y|/2} \qquad \text{for } y\geqq 1+x\,.$$

Then if follows

$$E_n(x, y, \mu)^2 n^4 Q(y)^2 \leqq \frac{A^2 e^{n\sqrt{Q(x)}|x-y|-2\sqrt{n^2Q(x)+\mu}|x-y|}}{n^2Q(x)+\mu}$$

$$\leqq \frac{A^2 e^{-\sqrt{n^2Q(x)+\mu}|x-y|}}{n^2Q(x)+\mu}=\frac{A^2}{\kappa_n^2}e^{-\kappa_n|x-y|}\,.$$

Hence $E_n(x, y, \mu)\{n^2Q(x)-n^2Q(y)+\lambda_{nj}+\mu\}\varphi_{nj}(y)$ is integrable in y for all $x>0$.

Since $E_n(x, y, \mu)$ satisfies (2.10), we have

$$\int_0^\infty E_n(x, y, \mu)\Big\{-\frac{d^2}{dy^2}\varphi_{nj}(y)+\kappa_n^2\varphi_{nj}(y)\Big\}dy$$

$$=\int_0^\infty \{n^2Q(y)-n^2Q(y)+\lambda_{nj}+\mu\}E_n(x, y, \mu)\varphi_{nj}(y)dy$$

$$=\varphi_{nj}(x)\,.$$

By (2.11), we find

$$\frac{\varphi_{nj}(x)}{\lambda_{nj}+\mu}=\int_0^\infty E_n(x, y, \mu)\varphi_{nj}(y)dy \tag{2.13}$$

$$+\frac{n^2}{\lambda_{nj}+\mu}\int_0^\infty (Q(x)-Q(y))E_n(x, y, \mu)\varphi_{nj}(y)dy\,.$$

Differentiating m times in μ the both sides of (2.16), we obtain

Lemma 2.5. *There exist constants C_{mlk} snch that*

$$\frac{(-1)^m\varphi_{nj}(x)}{(\lambda_{nj}+\mu)^{m+1}} \tag{2.14}$$

$$=\frac{1}{m!}\int_0^\infty \Big(\frac{\partial}{\partial\mu}\Big)^m E_n(x, y, \mu)\varphi_{nj}(y)dy$$

$$+\sum_{l=0}^{m}\sum_{k=0}^{l}\frac{C_{mlk}}{(\lambda_{nj}+\mu)^{m-l+1}\kappa_n^{2l+1}}(K_{njk}(x, \mu)-L_{njk}(x, \mu))\,,$$

where

$$K_{njk}(x, \mu)=n^2\kappa_n^k\int_0^\infty (Q(x)-Q(y))|x-y|^k e^{-\kappa_n|x-y|}\varphi_{nj}(y)dy$$

$$L_{njk}(x, \mu)=n^2\kappa_n^k\int_0^\infty (Q(x)-Q(y))(x+y)^k e^{-\kappa_n(x+y)}\varphi_{nj}(y)dy$$

and $\kappa_n=\kappa_n(x, \mu)=\sqrt{n^2Q(x)+\mu}$.

Proof. Recall

$$E_n(x, y, \mu)=\frac{1}{2\kappa_n}\{e^{-\kappa_n|x-y|}-e^{-\kappa_n(x+y)}\}.$$

We observe that there exist constants C_{jk} independent of α and z such that

$$\left(\frac{\partial}{\partial\mu}\right)^j\frac{e^{-z\sqrt{\alpha+\mu}}}{\sqrt{\alpha+\mu}}=\frac{e^{-z\sqrt{\alpha+\mu}}}{(\alpha+\mu)^{j+1/2}}\sum_{k=0}^{j}C_{jk}z^k(\alpha+\mu)^{k/2}. \tag{2.15}$$

Then by (2.15) together with the Leibniz formula, we obtain the lemma.

Now we carry out the estimates of (2.17). Firstly we settle the estimate of the first term in (2.14).

Proposition 2.6.

$$\frac{1}{(m!)^2}\sum_{j=1}^{\infty}\left(\int_0^\infty\left(\frac{\partial}{\partial\mu}\right)^m E_n(x, y, \mu)\varphi_{nj}(y)dy\right)^2 \tag{2.16}$$

$$=\frac{\Gamma(2m+3/2)}{2\sqrt{\pi}\Gamma(2m+2)}\kappa_n^{-4m-3}+\frac{e^{-2\kappa_n x}}{\kappa_n^{4m+3}}\sum_{k=0}^{2m+1}C_{mk}(x\kappa_n)^k$$

where C_{mk} *are constants independent of* x *and* μ.

Proof. Since $\{\varphi_{nj}\}$ is a complete orthonormal system in $L^2(0, \infty)$, we find

$$\sum_{j=1}^{\infty}\left(\int_0^\infty\left(\frac{\partial}{\partial\mu}\right)^m E_n(x, y, \mu)\varphi_{nj}(y)dy\right)^2=\int_0^\infty\left|\left(\frac{\partial}{\partial\mu}\right)^m E_n(x, y, \mu)\right|^2 dy.$$

Differentiating (2.9) in μ m-times, we have

$$\left(\frac{\partial}{\partial\mu}\right)^m E_n(x, y, \mu)=\frac{2(-1)^m m!}{\pi}\int_0^\infty\frac{\sin x\xi\sin y\xi}{(\xi^2+\kappa_n^2)^{m+1}}d\xi.$$

By the Parseval formula for sine transform, we find

$$\frac{1}{(m!)^2}\int_0^\infty\left|\left(\frac{\partial}{\partial\mu}\right)^m E_n(x, y, \mu)\right|^2 dy \tag{2.17}$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\sin^2 x\xi}{(\xi^2+\kappa_n^2)^{2m+2}}d\xi$$

$$=\frac{1}{\pi}\int_0^\infty\frac{1}{(\xi^2+\kappa_n^2)^{2m+2}}d\xi-\frac{1}{\pi}\int_0^\infty\frac{\cos 2x\xi}{(\xi^2+\kappa_n^2)^{2m+2}}d\xi$$

$$=\frac{B(2m+3/2, 1/2)}{2\pi}\kappa_n^{-4m-3}-\frac{1}{\pi(2m+1)!}\left(\frac{\partial}{\partial\mu}\right)^{2m+1}\int_0^\infty\frac{\cos 2x\xi}{(\xi^2+\kappa_n^2)}d\xi$$

$$=\frac{\Gamma(2m+3/2)}{2\sqrt{\pi}\Gamma(2m+2)}\kappa_n^{-4m-3}+\frac{1}{(2m+1)!}\left(\frac{\partial}{\partial\mu}\right)^{2m+1}\left(\frac{e^{-2\kappa_n x}}{\kappa_n}\right).$$

Employing (2.15) again, we obtain the proposition.

For the second terms in (2.14), we can readily verify

(2.18)
$$\frac{|K_{njk}(x,\ \mu)|}{(\lambda_{nj}+\mu)^{m-l+1}\kappa_n^{2l+1}}$$
$$\leqq\max\left\{\frac{G_{njk}(x,\ \mu)}{(\lambda_{nj}+\mu)^{m+1}\kappa_n},\ \frac{G_{njk}(x,\ \mu)}{(\lambda_{nj}+\mu)^{1/2}\kappa_n^{2m+2}}\right\}$$
$$\leqq\max\left\{\frac{G_{njk}(x,\ \mu)}{(\lambda_{nj}+\mu)^{m+1}\kappa_n},\ \frac{G_{njk}(x,\ \mu)}{\mu^{1/2}\kappa_n^{2m+2}}\right\}$$

where

$$G_{njk}(x,\ \mu)=n^2\kappa_n^k\left|\int_0^\infty(Q(x)-Q(y))|x-y|^k e^{-\kappa_n|x-y|}\varphi_{nj}(y)dy\right|.$$

We shall estimate G_{njk} in different two ways.

(I) We divide the integral as the following

$$G_{njk}(x,\ \mu)\leqq n^2\kappa_n^k\int_0^\infty|Q(x)-Q(y)||x-y|^k e^{-\kappa_n|x-y|}|\varphi_{nj}(y)|dy$$
$$=n^2\kappa_n^k\left(\int_{|x-y|\leqq 1}+\int_{|x-y|\geqq 1}\right)$$
$$=G_{njk}^{(1)}(x,\ \mu)+G_{njk}^{(2)}(x,\ \mu).$$

Employing (2.2) and by the Schwarz inequality, we find

$$G_{njk}^{(1)}(x,\ \mu)$$
$$\leqq An^2Q(x)\kappa_n^k\int_{|x-y|\leqq 1}|x-y|^{k+1}e^{-\kappa_n|x-y|}|\varphi_{nj}(y)|dy$$
$$\leqq A\kappa_n^{k+2}\left(\int_{|x-y|\leqq 1}|x-y|^{2k+2}e^{-2\kappa_n|x-y|}dy\right)^{1/2}\left(\int_{|x-y|\leqq 1}|\varphi_{nj}(y)|^2dy\right)^{1/2}$$
$$\leqq A\kappa_n^{1/2}\left(\int_0^\infty t^{2k+2}e^{-2t}dt\right)^{1/2}\left(\int_{|z|\leqq 1}|\varphi_{nj}(x+z)|^2dz\right)^{1/2}.$$

Then we obtain

(2.19)
$$|G_{njk}^{(1)}(x,\ \mu)|^2\leqq C\kappa_n\int_{|z|\leqq 1}|\varphi_{nj}(x+z)|^2dz.$$

Next applying (2.3), we find

$$G_{njk}^{(2)}(x,\ \mu)\leqq n^2Q(x)\kappa_n^k\int_{|x-y|\geqq 1}|x-y|^k e^{-\kappa_n|x-y|}|\varphi_{nj}(y)|dy$$
$$+n^2\kappa_n^k\int_{|x-y|\geqq 1}Q(y)|x-y|^k e^{-\kappa_n|x-y|}|\varphi_{nj}(y)|dy$$
$$\leqq\kappa_n^{k+2}\left(\int_{|z|\geqq 1}|z|^{2k}e^{-2\kappa_n|z|}dz\right)^{1/2}$$
$$+\kappa_n^k\left(\int_{|z|\geqq 1}n^4Q(x+z)^2|z|^{2k}e^{-2\kappa_n|z|}dz\right)^{1/2}$$

$$\leqq \kappa_n^{1/2}\Big(\int_{\kappa_n}^{\infty} t^{2k}e^{-2t}dt\Big)^{1/2}$$

$$+A\kappa_n^k\Big(\int_{|z|\geq 1}|z|^{2k}e^{n\sqrt{Q(x)}|z|-2\sqrt{n^2Q(x)+\mu}|z|}dz\Big)^{1/2}$$

$$\leqq \kappa_n^{1/2}\Big(\int_{\kappa_n}^{\infty} t^{2k}e^{-2t}dt\Big)^{1/2}+A\kappa_n^{-1/2}\Big(\int_{|z|\geq 1}|z|^{2k}e^{-\kappa_n|z|}dz\Big)^{1/2}$$

$$\leqq C\kappa_n^{1/2}\Big(\int_{\kappa_n}^{\infty} t^{2k}e^{-2t}dt\Big)^{1/2}.$$

Hence for any N, we can choose a constant C_N which may depend on N, such that

(2.20) $$|G_{njk}^{(2)}(x, \mu)|^2\leqq C_N\kappa_n^{-2N-2}.$$

Combining (2.19) and (2.20), we obtain

(2.21) $$|G_{njk}(x, \mu)|^2\leqq C\kappa_n\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz+C_N\kappa_n^{-2N-2}.$$

(II) By the Bessel inequality, we find for any J

$$\sum_{j=1}^{J}|G_{njk}(x, \mu)|^2\leqq n^4\kappa_n^{2k}\int_0^{\infty}|Q(x)-Q(y)|^2|x-y|^{2k}e^{-2\kappa_n|x-y|}dy.$$

Then similar computations as in (I) show that

$$n^4\kappa_n^{2k}\int_0^{\infty}|Q(x)-Q(y)|^2|x-y|^{2k}e^{-2\kappa_n|x-y|}dy=\int_{|x-y|\leq 1}+\int_{|x-y|\geq 1}$$
$$\leqq C\kappa_n+C_N\kappa_n^{-2N-2}.$$

Hence we obtain

(2.22) $$\sum_{j=1}^{J}|G_{njk}(x, \mu)|^2\leqq C\kappa_n+C_N\kappa_n^{-2N-2}.$$

Employing (2.21) and (2.22), we get the estimate of $(\lambda_{nj}+\mu)^{-(m-l+1)}K_{njk}(x, \mu)$ in the following way

$$\sum_{j=1}^{J}\frac{K_{njk}(x, \mu)^2}{(\lambda_{nj}+\mu)^{2(m-l+1)}}\kappa_n^{-4l-2}$$

$$\leqq 2\kappa_n^{-2}\sum_{j=1}^{J}\frac{G_{njk}(x, \mu)^2}{(\lambda_{nj}+\mu)^{2(m+1)}}+2\mu^{-1}\kappa_n^{-4(m+1)}\sum_{j=1}^{J}|G_{njk}(x, \mu)|^2$$

$$\leqq C\kappa_n^{-1}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}}\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz$$

$$+C_N\kappa_n^{-2N-4}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}}+C\mu^{-1}\kappa_n^{-4m-3}+C_N\mu^{-1}\kappa_n^{-2N-4m-6}.$$

Then we obtain

Proposition 2.7.

(2.23)
$$\sum_{j=1}^{J}\frac{K_{njk}(x,\mu)^2}{(\lambda_{nj}+\mu)^{2(m-l+1)}}\kappa_n^{-4l-2}$$
$$\leqq C\mu^{-1}\kappa_n^{-4m-3}+C\mu^{-1/2}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}}\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz$$
$$+C_N\kappa_n^{-2N-4}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}}.$$

Let us come back to (2.14). As before we have

(2.24)
$$\frac{|L_{njk}(x,\mu)|}{(\lambda_{nj}+\mu)^{m-l+1}\kappa_n^{2l+1}}\leqq\max\left\{\frac{H_{njk}(x,\mu)}{(\lambda_{nj}+\mu)^{m+1}\kappa_n},\ \frac{H_{njk}(x,\mu)}{\mu^{1/2}\kappa_n^{2m+2}}\right\}$$

where
$$H_{njk}(x,\mu)=n^2\kappa_n^k\left|\int_0^\infty(Q(x)-Q(y))(x+y)^k e^{-\kappa_n(x+y)}\varphi_{nj}(y)dy\right|.$$

(III) We divide the integral as before
$$H_{njk}(x,\mu)\leqq n^2\kappa_n^k\int_0^\infty|Q(x)-Q(y)|(x+y)^k e^{-\kappa_n(x+y)}|\varphi_{nj}(y)|dy$$
$$=n^2\kappa_n^k\left(\int_{|x-y|\leq 1}+\int_{|x-y|\geq 1}\right)$$
$$=H_{njk}^{(1)}(x,\mu)+H_{njk}^{(2)}(x,\mu).$$

We find by (2.2)
$$H_{njk}^{(1)}(x,\mu)\leqq An^2Q(x)\kappa_n^k\int_{|x-y|\leq 1}|x-y|(x+y)^k e^{-\kappa_n(x+y)}|\varphi_{nj}(y)|dy$$
$$\leqq A\kappa_n^{k+2}\left(\int_0^\infty(x+y)^{2k+2}e^{-\kappa_n(x+y)}dy\right)^{1/2}\left(\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz\right)^{1/2}$$
$$\leqq A\kappa_n^{1/2}\left(\int_{x\kappa_n}^\infty t^{2k+2}e^{-2t}dt\right)^{1/2}\left(\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz\right)^{1/2}.$$

Then we get

(2.25)
$$|H_{njk}^{(1)}(x,\mu)|^2\leqq A\kappa_n e^{-\kappa_n x}\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz$$

Next applying (2.3), we find
$$H_{njk}^{(2)}(x,\mu)\leqq n^2Q(x)\kappa_n^k\int_{|x-y|\geq 1}(x+y)^k e^{-\kappa_n(x+y)}|\varphi_{nj}(y)|dy$$
$$+\kappa_n^k\int_{|x-y|\geq 1}(x+y)^k n^2Q(y)e^{-\kappa_n(x+y)}|\varphi_{nj}(y)|dy$$
$$\leqq\kappa_n^{k+2}\left(\int_{|x+y|\geq 1}(x+y)^{2k}e^{-2\kappa_n(x+y)}dy\right)^{1/2}$$
$$+\kappa_n^k\left(\int_{x+y\geq 1}(x+y)^{2k}e^{-n\sqrt{Q(x)}|x-y|-2\sqrt{n^2Q(x)+\mu}(x+y)}dy\right)^{1/2}$$

$$\leqq C\kappa_n^{3/2}\Big(\int_{\kappa_n}^{\infty} t^{2k}e^{-2t}dt\Big)^{1/2} + C\kappa_n^{-1/2}\Big(\int_{\kappa_n}^{\infty} t^{2k}e^{-t}dt\Big)^{1/2}.$$

Hence we have for any N

$$H_{njk}^{(2)}(x, \mu)^2 \leqq C_N \kappa_n^{-2N-2} \tag{2.26}$$

with a constant C_N depending on N.

Combining (2.25) and (2.26), we obtain

$$H_{njk}(x, \mu)^2 \leqq C\kappa_n e^{-\kappa_n x}\int_{|z|\leq 1} |\varphi_{nj}(x+z)|^2 dz + C_N\kappa_n^{-2N-2}. \tag{2.27}$$

(IV) By the Bessel inequality, we find

$$\sum_{j=1}^{J} |H_{njk}(x, \mu)|^2 \leqq n^4\kappa_n^{2k}\int_0^{\infty} |Q(x)-Q(y)|^2(x+y)^{2k}e^{-2\kappa_n(x+y)}dy.$$

Repeating the similar arguments as above, we obtain

$$\sum_{j=1}^{J} |H_{njk}(x, \mu)|^2 \leqq C\kappa_n e^{-\kappa_n x} + C_N\kappa_n^{-2N-2}. \tag{2.28}$$

Employing (2.27) and (2.28), we obtain

Proposition 2.8.

$$\sum_{j=1}^{J} \frac{L_{njk}(x, \mu)^2}{(\lambda_{nj}+\mu)^{2(m-l+1)}\kappa_n^{4l+2}} \tag{2.29}$$

$$\leqq C\mu^{-}\kappa_n^{-4m-3}e^{-\kappa_n x} + C\mu^{-1/2}e^{-\kappa_n x}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}}\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2 dz$$

$$+ C_N\kappa_n^{-2N-2}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}}.$$

Comparing (2.23) to (2.29), we observe that L_{njk} has better estimates than K_{njk} in the point where $e^{-\kappa_n x}$ is multiplied to certain terms. But in this article, we shall gain no advantages from this observation in the following arguments.

We recall (2.14). Square the both sides of (2.14) and set

$$\frac{\varphi_{nj}(x)^2}{(\lambda_{nj}+\mu)^{2(m+1)}} = \frac{1}{(m!)^2}\Big(\int_0^{\infty}\Big(\frac{\partial}{\partial\mu}\Big)^m E_n(x, y, \mu)\varphi_{nj}(y)dy\Big)^2 + R_{nmj}(x, \mu).$$

Then for any δ satisfying $0<\delta<1$, we have

$$|R_{nmj}(x, \mu)| \leqq \mu^{-\delta}\Big(\int_0^{\infty}\Big(\frac{\partial}{\partial\mu}\Big)^m E_n(x, y, \mu)\varphi_{nj}(y)\Big)^2$$

$$+ C\mu^{\delta}\sum_{l=0}^{m}\sum_{k=0}^{l}\frac{(K_{njk}(x, \mu)^2 + L_{njk}(x, \mu)^2)}{(\lambda_{nj}+\mu)^{2(m-l+1)}\kappa_n^{4l+2}}.$$

Using Proposition 2.6, 2.7 and 2.8, we find

$$\sum_{j=1}^{J} |R_{nmj}(x, \mu)|$$

$$\leqq \mu^{-\delta} \sum_{j=1}^{\infty} \Big(\int_0^\infty \Big(\frac{\partial}{\partial \mu}\Big)^m E_n(x, y, \mu) \varphi_{nj}(y) dy\Big)^2$$

$$+C\mu^{-1+\delta}\kappa_n^{-4m-3} + C\mu^{1/2+\delta} \sum_{j=1}^{J} \frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}} \int_{|z|\leq 1} |\varphi_{nj}(x+z)|^2 dz$$

$$+C_N \mu^\delta \kappa_n^{-2N-2} \sum_{j=1}^{J} \frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}}$$

$$\leqq C(\mu^{-\delta}+\mu^{-1+\delta})\kappa_n^{-4m-3}$$

$$+C\mu^{-1/2+\delta} \sum_{j=1}^{J} \frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}} \int_{|z|\leq 1} |\varphi_{nj}(x+z)|^2 dz$$

$$+C_N \kappa_n^{-2N-2} \sum_{j=1}^{J} \frac{1}{(\lambda_{nj}+\mu)^{2(m+1)}} .$$

Setting $\delta=1/4$, we have obtained the following lemma which is crucial in studying the zeta function.

Lemma 2.9. *Assume* $Q(x)$ *to satisfy* (0.11). *Then we have*

$$\sum_{j=1}^{J} \frac{\varphi_{nj}(x)^2}{(\lambda_{nj}+\mu)^{2(m+1)}} = \frac{1}{(m\,!)^2} \sum_{j=1}^{J} \Big(\int_0^\infty \Big(\frac{\partial}{\partial \mu}\Big)^m E_n(x, y, \mu)\varphi_{nj}(y) dy\Big)^2$$

$$+ \sum_{j=1}^{J} R_{nmj}(x, \mu),$$

where $R_{nmj}(x, \mu)$ *has the following estimate*

(2.30) $$\sum_{j=1}^{J} |R_{nmj}(x, \mu)|$$

$$\leqq C_m \mu^{-1/4} \kappa_n^{-(4m+3)}$$

$$+C_{mN} \sum_{j=1}^{J} \frac{\mu^{-1/4}}{(\lambda_{nj}+\mu)^{2(m+1)}} \Big\{\int_{|z|\leq 1} |\varphi_{nj}(x+z)|^2 dz + \kappa_n^{-2N}\Big\}$$

$$(\kappa_n = \kappa_n(x, \mu) = \sqrt{n^2 Q(x)+\mu}) .$$

§ **3.** In this section we study the zeta function defined as (0.10). In our case

$$Z(\alpha, A) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{nj}^{-\alpha} .$$

In the first place we show

Theorem 3.1. *Let* $Q(x)$ *satisfy the conditions* (0.11) *and* (0.13). *Then*

(3.1) $$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \lambda_{nj}^{-\beta} < \infty \qquad (\beta > 1),$$

if and only if

(3.2) $$\int_0^\infty Q(x)^{-\beta+1/2} dx < \infty .$$

Proof. Firstly assume (3.2). Pick an integer m such that $2m+2\geqq\beta$. Then it follows $\int_0^\infty Q(x)^{-2m-3/2}dx<\infty$. By Lemma 2.9 and Proposition 2.6, we find

$$(3.3)\quad \sum_{j=1}^{J}\frac{\varphi_{nj}(x)^2}{(\lambda_{nj}+\mu)^{2m+2}}$$

$$\leqq\frac{1}{(m!)^2}\Big(\int_0^\infty\Big(\frac{\partial}{\partial\mu}\Big)^m E_n(x,y,\mu)\varphi_{nj}(y)dy\Big)^2+\sum_{j=1}^{J}R_{nmj}(x,\mu)$$

$$\leqq\frac{\Gamma(2m+3/2)}{2\sqrt{\pi}\Gamma(2m+2)}\kappa_n^{-(4m+3)}+C_m\kappa_n^{-(4m+3)}e^{-\kappa_n x}$$

$$+C_{mN}\sum_{j=1}^{J}\frac{\mu^{-1/4}}{(\lambda_{nj}+\mu)^{2(m+1)}}\Big\{\int_{|z|\leq 1}|\varphi_{nj}(x+y)|^2dz+\kappa_n^{-2N}\Big\}$$

$$\leqq\frac{C_m}{(n^2Q(x)+\mu)^{2m+3/2}}+C_m\sum_{j=1}^{J}\frac{\mu^{-1/4}}{(\lambda_{nj}+\mu)^{2m+2}}\Big\{\int_{|z|\leq 1}|\varphi_{nj}(x+z|^2dz+\frac{1}{n^{2N}Q(x)^N}\Big\}$$

Integrating in x the both sides of (3.3), we have

$$\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2m+2}}$$

$$\leqq C_m\int_0^\infty\frac{1}{(n^2Q(x)+\mu)^{2m+3/2}}dx+C_m\sum_{j=1}^{J}\frac{\mu^{-1/4}}{(\lambda_{nj}+\mu)^{2m+2}}\Big\{1+\frac{1}{n^{2N}}\int_0^\infty Q(x)^{-N}dx\Big\}$$

Hence we find that

$$(3.4)\qquad \sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+\mu)^{2m+2}}\leqq C_m\int_0^\infty\frac{1}{(n^2Q(x)+\mu)^{2m+3/2}}dx$$

holds with a constant C_m depending only on m for sufficiently large μ. Recall the identity

$$\int_s^\infty\frac{(\mu-s)^\gamma}{(t+\mu)^{m+1}}d\mu=\frac{\Gamma(1+\gamma)\Gamma(m-\gamma)}{(m+1)}\frac{1}{(t+s)^{m-\gamma}}\quad(-1<\gamma<m).$$

Multiplying $(\mu-s)^\gamma$ with $\gamma=2m+1-\beta$ to the both sides of (3.4) and integrating in μ from s to ∞, we find

$$\frac{\Gamma(2m+2-\beta)\Gamma(\beta)}{\Gamma(2m+2)}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+s)^\beta}\leqq C_m\frac{\Gamma(2m+3/2)\Gamma(\beta-1/2)}{(2m+2)}\int_0^\infty\frac{1}{(n^2Q(x)+s)^{\beta-1/2}}dx$$

$$\leqq C_m\frac{\Gamma(2m+3/2)\Gamma(\beta-1/2)}{\Gamma(2m+2)}\frac{1}{n^{2\beta-1}}\int_0^\infty Q(x)^{-\beta+1/2}dx\,.$$

Summing up the above expression from 1 to N, we have

$$\sum_{n=1}^{N}\sum_{j=1}^{J}\frac{1}{(\lambda_{nj}+s)^\beta}\leqq C_\beta\zeta(2\beta-1)\int_0^\infty Q(x)^{-\beta+1/2}dx\,.$$

Fix $s=s_0$ with a large constant s_0 and let $N, J\to\infty$. Then we obtain (3.1).

Conversely assume (3.1). Pick an integer m such that $2m+2\geqq\beta$. By Lemma 2.9 with $J=\infty$ and Proposition 2.6, we find

$$(3.5)\quad \sum_{j=1}^{\infty}\frac{\varphi_{nj}(x)^2}{(\lambda_{nj}+\mu)^{2m+2}}$$

$$=\frac{\Gamma(2m+3/2)}{2\sqrt{\pi}\Gamma(2m+2)(n^2Q(x)+\mu)^{2m+3/2}}$$

$$+\frac{e^{-2x\sqrt{n^2Q(x)+\mu}}}{(n^2Q(x)+\mu)^{2m+3/2}}\sum_{k=0}^{2m+1}C_{mk}(n^2Q(x)+\mu)^{k/2}+\sum_{j=1}^{\infty}R_{nmj}(x,\mu)$$

with

$$\sum_{j=1}^{\infty}|R_{nmj}(x,\mu)|$$

$$\leqq\frac{C_m\mu^{-1/4}}{(n^2Q(x)+\mu)^{2m+3/2}}$$

$$+C_{mN}\sum_{j=1}^{\infty}\frac{\mu^{-1/4}}{(\lambda_{nj}+\mu)^{2m+2}}\left\{\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz+\frac{1}{(n^2Q(x)+\mu)^N}\right\}.$$

Then it follows for $x\geqq x_0>0$

$$(3.6)\quad \sum_{j=1}^{\infty}\frac{\varphi_{nj}(x)^2}{(\lambda_{nj}+\mu)^{2m+2}}$$

$$\geqq\frac{\Gamma(2m+3/2)}{2\pi\Gamma(2m+2)(n^2Q(x)+\mu)^{2m+3/2}}(1-C_m\mu^{-1/4})$$

$$-C_{mN}\sum_{j=1}^{\infty}\frac{\mu^{-1/4}}{(\lambda_{nj}+\mu)^{2m+2}}\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz-C_{mN}\frac{\mu^{-1/4}}{(n^2Q(x)+\mu)^N}\sum_{j=1}^{\infty}\lambda_{nj}^{-\beta}.$$

Multiplying $(\mu-s)^{\gamma}$ $(\gamma=2m+1-\beta)$ to the both sides of (3.6) and then integrating from s to ∞, we find

$$(3.7)\quad \frac{\Gamma(2m+2-\beta)\Gamma(\beta)}{\Gamma(2m+2)}\sum_{j=1}^{\infty}\frac{\varphi_{nj}(x)^2}{(\lambda_{nj}+s)^{\beta}}$$

$$\geqq\frac{\Gamma(2m+2-\beta)\Gamma(\beta-1/2)}{2\pi\Gamma(2m+2)}\frac{1}{(n^2Q(x)+s)^{\beta-1/2}}(1-C_\beta s^{-1/4})$$

$$-C_\beta\sum_{j=1}^{\infty}\frac{s^{-1/4}}{(\lambda_{nj}+s)^{\beta}}\int_{|z|\leq 1}|\varphi_{nj}(x+z)|^2dz-C_\beta\frac{1}{(n^2Q(x)+s)^{N+\beta-1}}\sum_{j=1}^{\infty}\lambda_{nj}^{-\beta}.$$

Integrating (3.6) in x from x_0 to X, we have

$$(1-C_\beta s^{-1/4})\int_{x_0}^{X}\frac{1}{(n^2Q(x)+s)^{\beta-1/2}}dx\leqq C_\beta(1+C_\beta s^{-1/4})\sum_{j=1}^{\infty}\lambda_{nj}^{-\beta}.$$

Fix $s=s_0$ with a large constant s_0 and let $X\to\infty$. Then we obtain (3.2).

If we are more careful in computing the constants which appear in the arguments above, we can readily show the following formula.

Theorem 3.2. *Let* $Q(x)$ *satisfy the conditions* (0.2), (0.11) *and* (0.13). *Then it follows*

(3.8) $$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(\lambda_{nj}+s)^{\beta}}\sim\frac{\Gamma(\beta-1/2)}{2\sqrt{\pi}\Gamma(\beta)}\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{1}{(n^{2}Q(x)+s)^{\beta-1/2}}dx\qquad as\quad s\to\infty.$$

Applying the Keldysh Tauberian theorem to (3.8), we obtain the asymptotic formula of the distribution of the eigenvalues in the form

(3.9) $$N(\lambda)\sim\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{\lambda\geq n^{2}Q(x)}(\lambda-n^{2}Q(x))^{1/2}dx$$

(see for the details A. G. Kostyuchenko-B. M. Levitan [8]).

Next we consider the analytic continuation of the zeta function. Recall the identity

(3.10) $$\int_{0}^{\infty}\frac{\mu^{\gamma}}{(t+\mu)^{m+1}}d\mu=\frac{\Gamma(1+\gamma)\Gamma(m-\gamma)}{\Gamma(m+1)}t^{-m+\gamma}\quad(-1<\mathrm{Re}\,\gamma<m).$$

Lemma 3.3. *Let $\varphi_n(\mu)$ and $\Phi_n(\mu)$ be bounded measurable functions satisfying*
(i) $|\varphi_n(\mu)|\leqq\Phi_n(\mu)$,
(ii) *there exists a constant σ such that*

$$\sum_{n=1}^{\infty}\int_{0}^{\infty}\mu^{\beta}\Phi_n(\mu)d\mu<\infty\qquad for\ all\quad 0\leqq\beta<\sigma.$$

Then $F(\alpha)=\sum_{n=1}^{\infty}\int_{0}^{\infty}\mu^{\alpha}\varphi_n(\mu)d\mu$ is a holomorphic function of α in $0<\mathrm{Re}\,\alpha<\sigma$.

Proof. Let us denote

$$F_{n,R,\varepsilon}(\alpha)=\int_{\varepsilon}^{R}\mu^{\alpha}\varphi_n(\mu)d\mu,\qquad F_n=F_{n,\infty\,0}.$$

Then $F_{n,R,\varepsilon}(\alpha)$ is an entire function of α. For $R>1$, $0<\varepsilon<1$, $\alpha_1<\sigma$, we find that

$$|F_n(\alpha)-F_{n,R,\varepsilon}(\alpha)|\leqq\int_{0}^{\varepsilon}\Phi_n(\mu)d\mu+\int_{R}^{\infty}\mu^{\alpha_1}\Phi_n(\mu)d\mu$$

holds for all α such that $0\leqq\mathrm{Re}\,\alpha\leqq\alpha_1$. This shows that $F_{n,R,\varepsilon}$ is uniformly convergent to F_n for $R\to\infty$, $\varepsilon\to 0$ in $0\leqq\mathrm{Re}\,\alpha\leqq\alpha_1$ and then $F_n(\alpha)$ is holomorphic in $0<\mathrm{Re}\,\alpha<\alpha_1$. Moreover since

$$\left|F(\alpha)-\sum_{n=1}^{N-1}F_n(\alpha)\right|\leqq\sum_{n=N}^{\infty}\int_{0}^{\infty}\mu^{\mathrm{Re}\,\alpha}\Phi_n(\mu)d\mu$$

$$\leqq\sum_{n=N}^{\infty}\int_{0}^{\infty}\Phi_n(\mu)d\mu+\sum_{n=N}^{\infty}\int_{0}^{\infty}\mu^{\alpha_1}\Phi_n(\mu)d\mu,$$

$\sum_{n=1}^{N-1}F_n(\alpha)$ converges to $F(\alpha)$ uniformly in $0\leqq\mathrm{Re}\,\alpha\leqq\alpha_1$. Hence $F(\alpha)$ is holomorphic in $0<\mathrm{Re}\,\alpha<\alpha_1$.

Let $\sigma=\inf\left\{\alpha\,\middle|\int_{0}^{\infty}Q(x)^{-\alpha+1/2}dx<\infty\right\}$. Now we carry out the proof of Theorem 0.1.

Proof of Theorem 0.1. Choose an integer m such that $2m+2>\sigma$. We start with (3.5). Integrating in x the both sides of (3.5), we have

(3.12)
$$\sum_{j=1}^{\infty}\frac{1}{(\lambda_{nj}+\mu)^{2m+2}}$$
$$=\frac{\Gamma(2m+3/2)}{2\sqrt{\pi}\Gamma(2m+2)}\int_0^{\infty}\frac{1}{(n^2Q(x)+\mu)^{2m+3/2}}dx$$
$$+\sum_{k=0}^{2m+1}C_{mk}\int_0^{\infty}\frac{e^{-2x\sqrt{n^2Q(x)+\mu}}}{(n^2Q(x)+\mu)^{2m+3/2}}x^k(n^2Q(x)+\mu)^{k/2}dx$$
$$+\sum_{j=1}^{\infty}\int_0^{\infty}R_{nmj}(x,\ \mu)dx$$

Let us denote

(3.13)
$$\varphi_n(\mu)=\sum_{k=0}^{2m+1}C_{mk}\int_0^{\infty}\frac{e^{-2x\sqrt{n^2Q(x)+\mu}}}{(n^2Q(x)+\mu)^{2m+3/2}}x^k(n^2Q(x)+\mu)^{k/2}dx$$
$$+\sum_{j=1}^{\infty}\int_0^{\infty}R_{nmj}(x,\ \mu)dx\ ,$$

(3.14)
$$\Phi_n(\mu)=C_m\int_0^{\infty}\frac{e^{-x\sqrt{n^2Q(x)+\mu}}}{(n^2Q(x)+\mu)^{2m+3/2}}dx$$
$$+C_m\mu^{-1/4}\int_0^{\infty}\frac{1}{(n^2Q(x)+\mu)^{2m+3/2}}dx$$
$$+C_m\mu^{-1/4}\Big(1+\frac{1}{n^{2N}}\int_0^{\infty}Q(x)^{-N}dx\Big)\sum_{j=1}^{\infty}\frac{1}{(\lambda_{nj}+\mu)^{2m+2}}\ .$$

For sufficiently large C_m we can readily see $|\varphi_n(\mu)|\leqq\Phi_n(\mu)$. Multiplying μ^{γ} ($\gamma=2m+1-\alpha$) to the both sides of (3.12) and integrating in μ from 0 to ∞, we find by (3.10)

(3.15)
$$\frac{\Gamma(2m+2-\alpha)I\ (\alpha)}{\Gamma(2m+2)}\sum_{j=1}^{\infty}\lambda_{nj}^{-\alpha}$$
$$=\frac{\Gamma(2m+2-\alpha)\Gamma(\alpha-1/2)}{2\sqrt{\pi}I\ (2m+2)}\int_0^{\infty}\frac{1}{(n^2Q(x))^{\alpha-1/2}}dx+\int_0^{\infty}\mu^{2m+1-\alpha}\varphi_n(x,\ \mu)d\mu.$$

We want to use Lemma 3.3 to settle the last term of (3.15). Let α be real.

(3.16)
$$\int_0^{\infty}\mu^{2m+1-\alpha}\Phi_n(\mu)d\mu$$
$$=C_m\int_0^{\infty}\mu^{2m+1-\alpha}\int_0^{\infty}\frac{e^{-x\sqrt{n^2Q(x)+\mu}}}{(n^2G(x)+\mu)^{2m+3/2}}dxd\mu$$
$$+C_m\frac{\Gamma(2m+2-\alpha-1/4)\Gamma(\alpha-1/2+1/4)}{\Gamma(2m+2-1/2)}\int_0^{\infty}\frac{1}{(n^2Q(x))^{\alpha-1/2+1/4}}dx$$
$$+C_m\Big(1+\frac{1}{n^{2N}}\int_0^{\infty}Q(x)^{-N}dx\Big)\frac{\Gamma(2m+2-\alpha-1/4)\Gamma(\alpha+1/4}{\Gamma(2m+2)}\sum_{j=1}^{\infty}\lambda_{nj}^{-\alpha-1/4}\ .$$

For the first term of (3.16), we find

$$\int_0^\infty \mu^{2m+1-\alpha}\int_0^\infty \frac{e^{-x\sqrt{n^2Q(x)+\mu}}}{(n^2Q(x)+\mu)^{2m+3/2}}\,dx\,d\mu$$

$$\leqq\int_0^\infty \mu^{2m+1-\alpha}\int_0^\infty \frac{e^{-x\sqrt{n^2Q_0+\mu}}}{(n^2Q_0+\mu)^{2m+3/2}}\,dx\,d\mu$$

$$\leqq\frac{\Gamma(2m+2-\alpha)\Gamma(\alpha)}{\Gamma(2m+2)}\frac{1}{n^{2\alpha}Q_0^\alpha}$$

Hence

$$\sum_{n=1}^\infty\int_0^\infty \mu^{2m+1+\alpha}\Phi_n(\mu)d\mu$$

$$\leqq C_m\frac{\Gamma(2m+2-\alpha)\Gamma(\alpha)\zeta(2\alpha)}{\Gamma(2m+2)Q_0^\alpha}$$

$$+C_m\frac{\Gamma(2m+7/4-\alpha)\Gamma(\alpha-1/4)\zeta(2\alpha-1/2)}{\Gamma(2m+3/2)}\int_0^\infty Q(x)^{-\alpha+1/4}dx$$

$$+C_{mN}\Big(1+\zeta(2N)\int_0^\infty Q(x)^{-N}dx\Big)\frac{\Gamma(2m+3/4-\alpha)\Gamma(\alpha-1/4)}{\Gamma(2m+2)}Z(\alpha+1/4,\ A)\,.$$

In this way we have proved

$$\sum_{n=1}^\infty\int_0^\infty \mu^{2m+1-\alpha}\Phi_n(\mu)d\mu<\infty \qquad \text{for}\quad \sigma-1/4<\alpha\leqq 2m+1\,.$$

Since m is arbitrary, we find by Lemma 3.3 that

$$h(\alpha)=\sum_{n=1}^\infty\int_0^\infty \mu^{2m+1-\alpha}\varphi_n(\mu)d\mu$$

is holomorphic in $\sigma-1/4<\operatorname{Re}\alpha<\infty$.

Now we introduce some examples.

Example 3.5. (i) $Q(x)=x^{2\kappa}$ $(0<\kappa\leqq 1)$ for large x

$$Z(\alpha,\ A)=\begin{cases}\dfrac{\Gamma\left(\frac{1}{2\kappa}\right)\zeta\left(\frac{1}{\kappa}\right)}{4\sqrt{\pi}\,\kappa\Gamma\left(\frac{\kappa+1}{2\kappa}\right)\left(\alpha-\frac{\kappa+1}{2\kappa}\right)}+h(\alpha) & (0<\kappa<1)\,,\\[2ex] \dfrac{1}{8(\alpha-1)^2}+\dfrac{A_1}{(\alpha-1)}+h(\alpha) & (\kappa=1)\,,\end{cases}$$

(ii) $Q(x)=x^{2\kappa}(\log x)^{-2\gamma}$ $(0<\kappa<1,\ \gamma>0)$ for large x

$$Z(\alpha,\ A)=\frac{\Gamma(\alpha-1/2)\Gamma(2\alpha\gamma-\gamma+1)\zeta(2\alpha-1)}{2\sqrt{\pi}\Gamma(\alpha)(2\kappa\alpha-\kappa-1)^{2\alpha\gamma-\gamma+1}}+h(\alpha)$$

$$=\frac{\Gamma\left(\frac{1}{2\kappa}\right)\Gamma\left(1+\frac{\gamma}{\kappa}\right)\zeta\left(\frac{1}{\kappa}\right)}{2(2\kappa)^{1+\gamma/\kappa}\sqrt{\pi}\,\Gamma\left(\frac{\kappa+1}{2\kappa}\right)\left(\alpha-\frac{\kappa+1}{2\kappa}\right)^{1+\gamma/\kappa}}$$

$$\times\left\{\sum_{n=0}^{[1+\gamma/\kappa]}\sum_{j=0}^{n}A_{nj}\left(\alpha-\frac{\kappa+1}{2\kappa}\right)^{n}\left(\log\left(\alpha-\frac{\kappa+1}{2\kappa}\right)\right)^{j}+\tilde{h}(\alpha)\right..$$

where $\tilde{h}(\alpha)$ is holomorphic in $\mathrm{Re}\,\alpha>\frac{\kappa+1}{2\kappa}$ and continuous in $\mathrm{Re}\,\alpha\geqq\frac{\kappa+1}{2\kappa}$.

When we know the form of the largest singularity of the zeta function, we can get the asymptotic form of $N(\lambda)$ via Tauberian theorems. For me, it seems difficult to describe beforehand what sort of singularity the zeta function has. But anyhow concerning Example 3.5, we obtain the following by employing Theorem 0.2. Proof of the Tauberian theorem is put off until § 6.

(i) $Q(x)=x^{2\kappa}$ $(0<\kappa\leqq 1)$ for large x

$$N(\lambda)\sim\begin{cases}\dfrac{\Gamma\left(\frac{1}{2\kappa}\right)\zeta\left(\frac{1}{\kappa}\right)}{2\sqrt{\pi}(\kappa+1)\Gamma\left(\frac{\kappa+1}{2\kappa}\right)}\lambda^{1/2+1/2\kappa} & (0<\kappa<1),\\[2ex] \dfrac{1}{8}\lambda\log\lambda & (\kappa=1)\end{cases}$$

(ii) $Q(x)=x^{2\kappa}(\log x)^{-2\gamma}$ $(0<\kappa<1,\ \gamma>0)$

$$N(\lambda)\sim\frac{\Gamma\left(\frac{1}{2\kappa}\right)\zeta\left(\frac{1}{\kappa}\right)}{(2\kappa)^{1+\gamma/\kappa}\sqrt{\pi}\left(\frac{\kappa+1}{\kappa}\right)\Gamma\left(\frac{\kappa+1}{2\kappa}\right)}\lambda^{1/2+1/2\kappa}(\log\lambda)^{\gamma/\kappa}.$$

§ 4. In this section, we discuss the eigenvalue problem for the Laplace operator in an unbounded domain (see H. Tamura [10] and F. Asakura [1]). Let G be a domain R^2. We consider the following Dirichlet problem

$$\begin{cases}\Delta u+\lambda u=0 & \text{in } G\\ u=0 & \text{on } \partial G\end{cases}\tag{4.1}$$

where $\Delta u=u_{xx}+u_{yy}$ is the Laplace operator.

We assume G to satisfy the following conditions.

(4.2) (i) G is divided as $G=G_1\cup G_2$ where G_1 is a bounded domain with C^2 boundary and G_2 has the form

$$G_2=\{(x,\ y)\in R^2\mid A<x<\infty,\ a_1(x)<y<a_2(x)\}$$

with C^2 functions $a_j(x)$ $(j=1,\ 2)$,

(4.3) (ii) $a_1(x)$, $a_2(x)$ satisfy

(1) $b(x)=a_2(x)-a_1(x)\to 0$ as $x\to\infty$,

(2) $a_1'(x)$, $a_2'(x)\to 0$ as $x\to\infty$,

(3) $|a_1''(x)|$, $|a_2''(x)|\leqq M$ with a constant M,

(4) $\dfrac{|b'(x)|}{|b(x)|}\leqq M$ with a constant M,

(5) $\int_A^\infty b(x)dx=\infty$.

We denote by $L^2(G)$ the Hilbert space of square integrable functions in G and $H^j(G)$, $H_0^j(G)$ usual Sobolev spaces of order j. We can show the next proposition just in the same manner as Proposition 1.1.

Proposition 4.1. *Under the conditions* (i) *and* (ii)-(1), *the inclusion map from* $H_0^1(G)$ *to* $L^2(G)$ *is compact.*

We regard the Dirichlet integral $D(u)$ as a quadratic form in $H_0^1(G)$. Employing the variational method (see for example Courant-Hilbert text book), we find that there exists a complete orthonormal system of eigenfunctions of the Laplace operator in G with the Dirichlet condition (4.1).

Now let us consider the Laplace operator as a symmetric operator from $C^2(G)\cap C_0^0(\bar{G})$ to $L^2(G)$. We can show that the self-adjoint extension is unique in certain cases.

Theorem 4.2. *Assume the conditions* (i) *and* (ii) *and assume* $b'(x)\leqq 0$ *in addition. Then the closure of the Laplace operator defined in* $C^2(G)\cap C_0^0(\bar{G})$ *is a strictly self-adjoint operator with the domain* $H^2(G)\cap H_0^1(G)$.

A proof of the theorem is found in F. Asakura [2].

For sufficiently large R we set

$$G_R^{(1)}=\{(x,\ y)\in G\,|\,x<R\}\ ,$$

$$G_R^{(2)}=\{(x,\ y)\in G\,|\,x>R\}\ .$$

For each j let $\bar{\lambda}_n^{(j)}$ the n-th eigenvalue of the problem

$$(4.4)_j \qquad \begin{cases} \Delta u+\lambda u=0 & (x,\ y)\in G_R^{(j)} \\ \quad u=0 & (x,\ y)\in\partial G_R^{(j)}\ . \end{cases}$$

In a similar fashion let $\underline{\lambda}_n^{(j)}$ the n-th eigenvalue of the problem

$$(4.5)_j \qquad \begin{cases} \Delta u+\lambda u=0 & (x,\ y)\in G_R^{(j)} \\ \quad u=0 & (x,\ y)\in\partial G_R^{(j)}\cap\partial G \\ \quad u_x=0 & (x,\ y)\in\partial G_R^{(j)} \quad \text{with} \quad x=R\ . \end{cases}$$

We denote

$$\bar{N}_R^{(j)}(\lambda)={}^{\#}\{n\,|\,\bar{\lambda}_n^{(j)}\leqq\lambda\}$$

$$\underline{N}_R^{(j)}(\lambda)={}^{\#}\{n\,|\,\underline{\lambda}_n^{(j)}\leqq\lambda\}\ .$$

Then by the Courant mini-max principle, we find

Proposition 4.3. *Let* $N(\lambda)$ *be the number of eigenvalues of* (4.1) *not exceeding* λ. *Then* $N(\lambda)$ *is estimated as*

(4.6) $$\bar{N}_R^{(1)}(\lambda)+\bar{N}_R^{(2)}(\lambda)\leqq N(\lambda)\leqq \underline{N}_R^{(1)}(\lambda)+\underline{N}_R^{(2)}(\lambda).$$

Consider the following change of the variables and functions

$$\begin{cases} \xi=x \\ \eta=\dfrac{\pi}{b(x)}(y-a_1(x)), \quad \varphi=\dfrac{b(x)}{\pi}u \end{cases}$$

and set $G_R^{(2)}=G_R$, $\Omega_R=(R, \infty)\times(0, \pi)$. Then we find

$$\iint_{G_R} u(x, y)^2 dx dy=\iint_{\Omega_R} \varphi(\xi, \eta)^2 d\xi d\eta .$$

Moreover denote

$$D_R(u)=\iint_{G_R} u_x^2+u_y^2 dx dy$$

$$E_R(\varphi)=\iint_{\Omega_R} \varphi_\xi^2+\frac{\pi^2}{b(\xi)^2}\varphi_\eta^2 d\xi d\eta .$$

Then we obtain

Proposition 4.4. *For any small $\varepsilon>0$, there exist constants R and L such that*

(4.7) $$(1-\varepsilon)E_R(\varphi)-L\|\varphi\|^2\leqq D_R(u)\leqq(1+\varepsilon)E_R(\varphi)+L\|\varphi\|^2 .$$

Proof is carried out in the same manner as Proposition 3.2 in F. Asakura [1].

Let $\bar{\mu}_n$ denote the n-th eigenvalue of

(4.8) $$\begin{cases} \varphi_{\xi\xi}+\dfrac{\pi^2}{b(\xi)^2}\varphi_{\eta\eta}+\mu\varphi=0 & (\xi, \eta)\in\Omega_R \\ \varphi=0 & (\xi, \eta)\in\partial\Omega_R , \end{cases}$$

similarly $\underline{\mu}_n$ the n-th eigenvalue of

(4.9) $$\begin{cases} \varphi_{\xi\xi}+\dfrac{\pi^2}{b(\xi)^2}\varphi_{\eta\eta}+\mu\varphi=0 & (\xi, \eta)\in\Omega_R \\ \varphi=0 & (\xi, \eta)\in\partial\Omega_R \quad \text{with} \quad \eta=0, \pi \\ \varphi_\xi=0 & (\xi, \eta)\in\partial\Omega_R \quad \text{with} \quad \xi=R , \end{cases}$$

By virtue of Proposition 4.4, we find employing the Courant min-max principle

(4.10) $$\begin{aligned} &(1-\varepsilon)\bar{\mu}_n-L\leqq\bar{\lambda}_n^{(2)}\leqq(1+\varepsilon)\bar{\mu}_n+L \\ &(1-\varepsilon)\underline{\mu}_n-L\leqq\underline{\lambda}_n^{(2)}\leqq(1+\varepsilon)\underline{\mu}_n+L \end{aligned}$$

Let us denote

$$\bar{C}(\lambda)={}^{\#}\{n\,|\,\bar{\mu}_n\leqq\lambda\}$$

$$\underline{C}(\lambda)={}^{\#}\{n\,|\,\underline{\mu}_n\leqq\lambda\} .$$

We obtain

Proposition 4.5. *$N(\lambda)$ is estimated as*

(4.11) $$-C_1\lambda+\bar{C}\left(\frac{\lambda-L}{1+\varepsilon}\right)\leqq N(\lambda)\leqq C_2\lambda+\underline{C}\left(\frac{\lambda+L}{1-\varepsilon}\right).$$

Proof. It follows from (4.10) that if $\underline{\lambda}_n^{(2)}\leqq\lambda$, then $\underline{\mu}_n\leqq\frac{\lambda+L}{1-\varepsilon}$. This shows

$$N_R^{(2)}(\lambda)\leqq\underline{C}\left(\frac{\lambda+L}{1-\varepsilon}\right).$$

Similarly we have

$$\bar{N}_R^{(2)}(\lambda)\geqq\bar{C}\left(\frac{\lambda-L}{1+\varepsilon}\right)$$

Since $G_R^{(1)}$ is bounded, we know that $\bar{N}_R^{(1)}(\lambda)\leqq C_1\lambda$ and $\underline{N}_R^{(1)}(\lambda)\leqq C_2\lambda$ hold with constants C_1, C_2. Hence we obtain the proposition.

Set $Q(\xi)=\frac{\pi^2}{b(\xi)^2}$. We observe that for large R

(i) $Q(\xi)\in C^2([R,\infty))$,

(ii) $\left|\frac{Q'(\xi)}{Q(\xi)}\right|\leqq2\left|\frac{b'(\xi)}{b(\xi)}\right|\leqq2M$, $\quad Q(\xi)\geqq\frac{\pi^2}{\inf_{\xi\leq R}(b(\xi))^2}\geqq(4M)^2$,

(iii) $\int_R^\infty Q(\xi)^{-1/2}d\xi=\frac{1}{\pi}\int_R^\infty b(x)dx=\infty$.

Let $A_G\varphi=-\left(\varphi_{\xi\xi}+\frac{\pi^2}{b(\xi)^2}\varphi_{\eta\eta}\right)$, $\Omega_R=(R,\infty)\times(0,\pi)$. Thus we have seen that the operator A_G falls into the previous considerations.

Theorem 4.6. *Let $N(\lambda)$ be the number of eigenvalues of the Laplace operator in* (4.1) *not exceeding λ. Assume that the zeta function $Z(\alpha, A_G)$ has the analytic continuation of the form*

(4.12) $$Z(\alpha, A_G)=\frac{A}{(\alpha-\sigma)^{1+\rho}}\sum_{n=1}^{[1+\rho]}\sum_{j=0}^{n}A_{nj}(\alpha-\sigma)^n(\log(\alpha-\delta))^j+g(\alpha)$$
$$(A_{00}=1,\ \rho\geqq0)$$

where $g(\alpha)$ is holomorphic in $\mathrm{Re}\,\alpha>\sigma$ and continuous in $\mathrm{Re}\,\alpha\geqq\sigma$. Then $N(\lambda)$ has the asymptotic form

(4.13) $$N(\lambda)\sim\frac{A}{\sigma\Gamma(1+\rho)}\lambda^\sigma(\log\lambda)^\rho.$$

Proof. By Theorem 0.2, we find

$$\bar{C}(\lambda)\sim\underline{C}(\lambda)\sim\frac{A}{\sigma\Gamma(1+\rho)}\lambda^\sigma(\log\lambda)^\rho.$$

Since $C\left(\frac{\lambda\mp L}{1\pm\varepsilon}\right)$ has the asymptotic behavior

$$C\left(\frac{\lambda\mp L}{1\pm\varepsilon}\right)\sim\frac{A}{\sigma\Gamma(1+\rho)(1\pm\varepsilon)}\lambda^\sigma(\log\lambda)^\rho,$$

then it follows from Proposition 4.6 that

$$\limsup_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\sigma}(\log\lambda)^{\rho}}\leqq\frac{A}{(1-\varepsilon)^{\sigma}\sigma\Gamma(1+\rho)}$$

$$\liminf_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\sigma}(\log\lambda)^{\rho}}\geqq\frac{A}{(1+\varepsilon)^{\sigma}\sigma\Gamma(1+\rho)}.$$

Since ε is arbitrary we obtain

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\sigma}(\log\lambda)^{\rho}}=\frac{A}{\sigma\Gamma(1+\rho)}.$$

§5. In this section we study the distribution of eigenvalues of the operator (0.1) in certain cases where the integral (0.12) does not converge for any α (for example $Q(x)=(\log x)^{2k}$). In this case, we can not make use of the zeta function any more in studying the distribution of eigenvalues. Here we shall adopt the methods in F. Asakura [1], which is based on a uniform asymptotic expansion of the solution to the Sturm-Liouville problem developped in E. C. Titchmarsh [11].

We come back to the problem (2.1). Let $\Phi_n(x,\lambda)$ be the solution of

$$(5.1)\qquad \begin{cases}\Phi_n''+(\lambda-n^2Q(x))\Phi_n=0 & \text{for}\quad x>0\\ \displaystyle\int_0^\infty \Phi_n(x,\lambda)^2dx<\infty .\end{cases}$$

We observe that the solution is determined uniquely up to constant multiples. We also find that λ_{nj} is an eigenvalue of the problem (2.1) if and only if $\Phi_n(0,\lambda_{nj})=0$.

Remark 5.1. For the problem (2.1)′ we can see that λ_{nj} is an eigenvalue if and only if $\Phi_n'(0,\lambda_{nj})=0$. Hence we need uniform asymptotic expansions of both $\Phi_n(x,\lambda)$ and $\Phi_n'(x,\lambda)$.

Now, we assume $Q(x)$ to satisfy (0.17). To make the explanations simpler, we may study the equation (2.1) in the interval (R,∞) with the condition $\varphi(R)=0$ after we shift the interval $(0,\infty)$ to (R,∞) by translation of the variable. Set $Q_n(x,\lambda)=\frac{n^2}{\lambda}Q(x)$. Then $Q_n(x,\lambda)$ itself satisfies

(5.2) (i) $Q_n(x,\lambda)\in C^4([R,\infty))$ in x, $Q_n(x,\lambda)>0$, $\lim_{x\to\infty}Q_n(x,\lambda)=\infty$,

(5.3) (ii) $\dfrac{A}{x\log x}\leqq\dfrac{Q_n'(x,\lambda)}{Q_n(x,\lambda)}\leqq\dfrac{B}{x\log x}$ for $x\geqq R$,

(5.4) (iii) $\left|\dfrac{Q_n''(x,\lambda)}{Q_n'(x,\lambda)}\right|\leqq\dfrac{C}{x}$, $\left|\dfrac{Q_n'''(x,\lambda)}{Q_n'(x,\lambda)}\right|\leqq\dfrac{C}{x^2}$ for $x\geqq R$,

where A, B, C are independent of n and λ.

We observe by (5.3) that

(5.5) $$1\leqq\frac{Q_n(\alpha x,\lambda)}{Q(x,\lambda)}\leqq\alpha^B \quad \text{holds for} \quad 1\leqq\alpha\leqq\alpha_0 .$$

Set $P_n(x, \lambda)=1-Q_n(x, \lambda)$. Then the equation is expressed as

(5.6) $$\Phi_n''+\lambda P_n(x,\lambda)\Phi_n(x,\lambda)=0 .$$

Since $P_n'(x, \lambda)<0$, there exists a unique point $x=X_n(\lambda)$ satisfying $P_n(X_n(\lambda), \lambda)=0$ and $P_n(x, \lambda)<0$ for $x>X_n(\lambda)$, $P_n(x, \lambda)>0$ for $x<X_n(\lambda)$.

We introduce a function $\phi_n(x, \lambda)$ in the following fashion.

(5.7) $$\begin{cases} \frac{2}{3}\phi_n(x,\lambda)^{3/2}=\int_{X_n}^{x}(Q_n(t,\lambda)-Q_n(X_n(\lambda),\lambda))^{1/2}dt \\ \qquad =\int_{X_n}^{x}(-P_n(t,\lambda))^{1/2}dt \quad \text{for} \quad x\geqq X_n(\lambda), \\ \frac{2}{3}(-\phi_n(x,\lambda))^{3/2}=\int_{x}^{X_n}(Q_n(X_n(\lambda),\lambda)-Q_n(t,\lambda))^{1/2}dt \\ \qquad =\int_{x}^{X_n}(P_n(t,\lambda))^{1/2}dt \quad \text{for} \quad x\leqq X_n(\lambda). \end{cases}$$

We can readily see that $\phi_n(x, \lambda)$ is class C^3 in x and satisfies

(5.8) $$\phi_n(\phi_n')^2=-P_n .$$

Let Ai and Bi the Airy functions defined by

$$Ai(z)=\frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3+zt\right)dt$$

$$Bi(z)=\frac{1}{\pi}\int_0^\infty e^{-t^3/3+zt}+\sin\left(\frac{1}{3}t^3+zt\right)dt ,$$

which are linearly independent solutions to $y''=xy$. Set

$$A_n(x,\lambda)=\phi_n'(x,\lambda)^{-1/2}Ai(\lambda^{1/3}\phi_n(x,\lambda))$$

$$B_n(x,\lambda)=\phi_n'(x,\lambda)^{-1/2}Bi(\lambda^{1/3}\phi_n(x,\lambda)) .$$

Then A_n and B_n are linearly independent solution of the equation

(5.9) $$Y''+\lambda P_n(x,\lambda)Y+\frac{1}{2}\{\phi_n, x\}Y=0$$

where $\{\phi, x\}$ is the Schwarzian derivative of ϕ defined by

$$\{\phi, x\}=\frac{\phi'''}{\phi'}-\frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2 .$$

In our case

(5.10) $$\frac{1}{2}\{\phi_n, x\}=\frac{P_n''}{4P_n}-\frac{5}{16}\left\{\frac{P_n}{\phi_n^3}+\left(\frac{P_n'}{P_n}\right)^2\right\} .$$

We shall take $A_n(x, \lambda)$ to be the first approximation to $\Phi_n(x, \lambda)$ as λ tends to ∞. Set

$$K_n(x,t,\lambda)=-\pi\lambda^{1/3}[A_n(x,\lambda)B_n(t,\lambda)-A_n(t,\lambda)B_n(x,\lambda)] .$$

Then the equation (5.1) is equivalent to the integral equation

$$\Phi_n(x, \lambda)=A_n(x, \lambda)-\frac{1}{2}\int_x^\infty K_n(x, t, \lambda)\{\phi_n, t\}\Phi_n(t, \lambda)dt. \tag{5.11}$$

For the solution of the integral equation (5.11), we can show

Proposition 5.1. *Assume*

$$\int_R^\infty |\{\phi_n, t\}||P_n(t, \lambda)|^{-1/2}dt\leqq L \tag{5.12}$$

with a constant L *which is independent of* n *and* λ. *Then* Φ_n *has the following asymptotic forms*

$$\Phi_n(x, \lambda)=\begin{cases} \phi_n'(x, \lambda)^{-1/2}Ai(\lambda^{1/3}\phi_n(x, \lambda))\{1+O(\lambda^{-1/2})\} & \text{for } x\geqq X_n(\lambda), \quad (5.13) \\ \phi_n'(x, \lambda)^{-1/2}Ai(\lambda^{1/3}\phi_n(x, \lambda))\{1+O(\lambda^{-1/2})\} \\ \quad +O(\lambda^{-1/2}\phi_n'(x, \lambda)^{-1/2}Bi(\lambda^{1/3}\phi_n(x, \lambda))) & \text{for } x\leqq X_n(\lambda), \quad (5.14) \end{cases}$$

as $\lambda\to\infty$, *which hold uniformly in* n.

Proof. We observe that $A_n(x, \lambda)\neq 0$ for $x\geqq X_n(\lambda)$. Set

$$Z_n(x, \lambda)=\frac{\Phi_n(x, \lambda)}{A_n(x, \lambda)}.$$

Then $Z_n(x, \lambda)$ is the solution of the equation

$$Z_n(x, \lambda)=1-\frac{1}{2}\int_x^\infty K_n(x, t, \lambda)\{\phi_n, t\}\frac{A_n(t, \lambda)}{A_n(x, \lambda)}Z_n(t, \lambda)dt.$$

Employing certain estimates of the Airy functions, we find

$$\left|K_n(x, t, \lambda)\{\phi_n, t\}\frac{A_n(t, \lambda)}{A_n(x, \lambda)}\right|\leqq\frac{C|\{\phi_n, t\}|}{\lambda^{1/2}\phi_n'(t, \lambda)|\phi_n(t, \lambda)|^{1/2}}$$

$$\leqq\frac{C|\{\phi_n, t\}|}{\lambda^{1/2}|P_n(t, \lambda)|^{1/2}} \qquad \text{(see A. Erdélyi [6]).}$$

Then if we assume (5.12), we obtain

$$Z_n(x, \lambda)=1+O(\lambda^{-1/2}),$$

which shows (5.13).

For $x\leqq X_n(\lambda)$, inserting (5.13) into (5.11), we find that the equation is expressed as

$$\Phi_n(x, \lambda)=\Phi_n^{(0)}(x, \lambda)-\frac{1}{2}\int_x^{X_n(\lambda)}K_n(x, t, \lambda)\{\phi_n, t\}\Phi_n(t, \lambda)dt$$

where

$$\Phi_n^{(0)}(x, \lambda)=A_n(x, \lambda)-\frac{1}{2}\int_{X_n(\lambda)}^\infty K_n(x, t, \lambda)\{\phi_n, t\}\Phi_n(t, \lambda)dt$$

$$=A_n(x, \lambda)+O(\lambda^{-1/2}A_n(x, \lambda))+O(\lambda^{-1/2}B_n(x, \lambda)).$$

Set $W_n(x, \lambda)=(1+\lambda^{1/6}|\phi_n(x, \lambda)|^{1/4})\phi'_n(x, \lambda)^{1/2}\Phi_n(x, \lambda)$. Then by the similar discussions as above, we obtain (5.13) (see for the details A. Erdélyi [6] and F. Asakura [3]).

For (5.12) we can show the following lemma which is crucial in our discussion.

Lemma 5.2. *Assume* $Q(x)$ *to satisfy* (0.17). *Then we have*

$$\int_R^\infty |\{\phi_n, t\}||P_n(t, \lambda)|^{-1/2}dt \leqq CX_n(\lambda)^{-1}(\log X_n(\lambda))^{1/2} \tag{5.15}$$

with a constant not depending on n, λ.

Proof. For simplicity we omit λ to denote $P_n(t)=P_n(t, \lambda)$, $X_n=X_n(\lambda)$ e.t.c. Pick $\alpha_0>1$. For $1<\alpha\leqq\alpha_0$, divide the integral as

$$\begin{aligned}\int_R^\infty &= \int_{\alpha X_n}^\infty + \int_{X_n}^{\alpha X_n} + \int_{X_n/\alpha}^{X_n} + \int_R^{X_n/\alpha}\\ &= I_1+I_2+I_3+I_4 .\end{aligned}$$

In the following discussion, we abuse C to denote any constant which is independent of n and λ.

(I) Estimates of I_1 $(x\geqq\alpha X_n)$

Recall

$$\begin{aligned}\frac{1}{2}\{\phi_n, t\} &= \frac{P''_n}{4P_n} - \frac{5}{16}\left\{\frac{P_n}{\phi_n^3} + \left(\frac{P'_n}{P_n}\right)^2\right\}\\ &= -\frac{Q''_n}{4P_n} - \frac{5}{16}\left\{\frac{P_n}{\phi_n^3} + \left(\frac{Q'_n}{P_n}\right)^2\right\}.\end{aligned} \tag{5.16}$$

We carry out the estimates of (5.16) term by term.

$$\begin{aligned}&\int_{\alpha X_n}^\infty \left|\frac{Q''_n}{P_n}\right| |P_n|^{-1/2}dt\\ &= \int_{\alpha X_n}^\infty |Q''_n(t)|(Q_n(t)-Q_n(X_n))^{-3/2}dt\\ &\leqq \frac{C}{X_n}\int_{\alpha X_n}^\infty Q'_n(t)(Q_n(t)-Q_n(X_n))^{-3/2}dt\\ &\leqq \frac{C}{X_n}(Q_n(\alpha X_n)-Q_n(X_n))^{-1/2} .\end{aligned} \tag{5.17}$$

By the mean value theorem, we have

$$=\frac{C}{X_n^{3/2}}Q'_n(\xi) \qquad \text{for some} \quad X_n\leqq\xi\leqq\alpha X_n .$$

Since by (5.3)′

$$\inf_{X_n\leqq\xi\leqq\alpha X_n} Q'_n(\xi) \geqq \inf_{X_n\leqq\xi\leqq\alpha X_n} \frac{AQ_n(\xi)}{\xi\log\xi} \geqq \frac{AQ_n(X_n)}{\alpha X_n\log\alpha X_n} \geqq \frac{C}{X_n\log X_n}, \tag{5.18}$$

we obtain

(5.19) $$\int_{\alpha X_n}^{\infty}\left|\frac{Q_n''}{P_n}\right||P_n|^{-1/2}dt\leqq\frac{C(\log X_n)^{1/2}}{X_n}.$$

(5.20) $$\int_{\alpha X_n}^{\infty}\left|\frac{P_n}{\phi_n^3}\right||P_n|^{-1/2}dt=\int_{\alpha X_n}^{\infty}(-P_n(t))^{1/2}\phi_n(t)^{-3}dt.$$

Set $\zeta_n=\frac{2}{3}\phi_n^{3/2}$, then we find $\zeta_n'=(-P_n(t))^{1/2}$ and the integral (5.20) is expressed as

$$\frac{9}{4}\int_{\alpha X_n}^{\infty}\zeta_n'(t)\zeta_n(t)^{-2}dt=\frac{9}{4}\zeta_n(\alpha X_n)^{-1}.$$

We observe

$$\zeta_n(\alpha X_n)=\int_{X_n}^{\alpha X_n}(Q_n(t)-Q_n(X_n))^{1/2}dt$$
$$\geqq\inf_{X_n\leqq\xi\leqq\alpha X_n}Q_n'(\xi)^{1/2}\int_{X_n}^{\alpha X_n}(t-X_n)^{1/2}dt.$$

By (5.18), we have

$$\zeta_n(\alpha X_n)\geqq\frac{CX_n}{(\log X_n)^{1/2}}.$$

Hence we find

(5.21) $$\int_{\alpha X_n}^{\infty}\left|\frac{P_n}{\phi_n^3}\right||P_n|^{-1/2}dt\leqq\frac{C(\log X_n)^{1/2}}{X_n}.$$

(5.22) $$\int_{\alpha X_n}^{\infty}\left|\frac{P_n'}{P_n}\right|^2|P_n|^{-1/2}dt$$
$$=\int_{\alpha X_n}^{\infty}Q_n'(t)^2(Q_n(t)-Q_n(X_n))^{-5/2}dt$$
$$\leqq\frac{C}{X_n}\int_{\alpha X_n}^{\infty}Q_n'(t)Q_n(t)(Q_n(t)-Q_n(X_n))^{-5/2}dt$$
$$\leqq\frac{C}{X_n}\int_{\alpha X_n}^{\infty}Q_n'(t)(Q_n(t)-Q_n(X_n))^{-3/2}dt$$
$$+\frac{C}{X_n}Q_n(X_n)\int_{\alpha X_n}^{\infty}Q_n'(t)(Q_n(t)-Q(X_n))^{-5/2}dt$$
$$\leqq\frac{C}{X_n}(Q_n(\alpha X_n)-Q_n(X_n))^{-1/2}+\frac{C}{X_n}(Q_n(\alpha X_n)-Q_n(X_n))^{-3/2}$$
$$\leqq\frac{C}{X_n^{3/2}}\inf_{X_n\leqq\xi\leqq\alpha X_n}|Q_n'(\xi)|^{-1/2}+\frac{C}{X_n^{5/2}}\inf_{X_n\leqq\xi\leqq\alpha X_n}|Q_n'(\xi)|^{-3/2}.$$

Then it follows from (5.18)

(5.23) $$\int_{\alpha X_n}^{\infty}\left|\frac{P_n'}{P_n}\right|^2|P_n|^{1/2}dt\leqq\frac{C(\log X_n)^{1/2}}{X_n}.$$

In this way we have proved

$$|I_1|\leqq\frac{C(\log X_n)^{1/2}}{X_n}.$$

(II) Estimates of I_2 $(X_n \leqq x \leqq \alpha X_n)$.

Integrating (5.7) by parts twice, we find

$$\frac{2}{3}\phi_n^{3/2} = -(-P_n)^{3/2}(P_n')^{-1}\left\{1+\frac{2}{5}P_n''(P_n')^{-2}P_n+S_n\right\} \tag{5.24}$$

where $S_n = \frac{2}{5}P_n'(-P_n)^{-3/2}\int_{X_n}^{x}(-P_n)^{5/2}\{P_n'''(P_n')^{-3}-3(P_n'')^2(P_n')^{-4}\}\,dt.$

First we show that we can make the magnitude of $P_n''P_n(P_n')^{-2}$ and S_n arbitrarily small, if we choose α sufficiently close to 1.

$$\begin{aligned}&|P_n''(P_n')^{-2}P_n| \\ &= |Q_n''|(Q_n')^{-2}(Q_n(x)-Q_n(X_n)) \\ &= \frac{|Q_n''(x)|}{Q_n'(x)}\frac{Q_n'(\xi)}{Q_n(x)}(x-X_n) \qquad (X_n \leqq \xi \leqq \alpha X_n) \\ &\leqq \frac{C}{X_n}\frac{\sup Q_n'(\xi)}{\inf Q_n'(x)}(x-X_n).\end{aligned} \tag{5.25}$$

Since

$$\sup_{X_n \leqq \xi \leqq \alpha X_n} Q_n'(\xi) \leqq B \sup_{X_n \leqq \xi \leqq \alpha X_n} \frac{Q_n(\alpha X_n)}{\xi \log \xi} \leqq \frac{CQ_n(\alpha X_n)}{X_n \log X_n}, \tag{5.26}$$

we find together with (5.18)

$$|P_n''(P_n')^{-2}P_n| \leqq \frac{C}{X_n}\frac{Q_n(\alpha X_n)}{Q_n(X_n)}(x-X_n). \tag{5.27}$$

Then by virtue of (5.5), we find

$$|P_n''(P_n')^{-2}P_n| \leqq \frac{C}{X_n}(x-X_n) \leqq C(\alpha-1). \tag{5.28}$$

$$\begin{aligned}|S_n| &\leqq CQ_n'(x)(Q_n(x)-Q_n(X_n))^{-3/2}\int_{X_n}^{x}\left(\left|\frac{Q_n'''(t)}{Q_n'(t)}\right|+\left|\frac{Q_n''(t)}{Q_n'(t)}\right|^2\right) \\ &\quad Q_n(t)^{-2}(Q_n(t)-Q_n(X_n))^{5/2}dt \\ &\leqq \frac{C}{X_n^2}Q_n'(x)Q_n(x)-Q_n(X_n))^{-3/2}\int_{X_n}^{x}Q_n(t)^{-2}(Q_n(t)-Q_n(X_n))^{5/2}dt \\ &\leqq \frac{C}{X_n^2}\left(\frac{\sup Q_n'(\xi)}{\inf Q_n'(\xi)}\right)^{7/2}(x-X_n)^{-3/2}\int_{X_n}^{x}(t-X_n)^{5/2}dt \\ &\leqq \frac{C}{X_n^2}\left(\frac{Q_n(\alpha X_n)}{Q_n(X_n)}\right)^{7/2}(x-X_n)^2 \leqq \frac{C}{X_n^2}(x-X_n)^2.\end{aligned} \tag{5.29}$$

Hence we have

$$|S_n| \leqq C(\alpha-1)^2. \tag{5.30}$$

By (5.28) and (5.30) we find

$$\phi_n^{3/2} = -(-P_n)^{3/2}(P_n')^{-1}(1+\theta), \qquad \theta = O(\alpha-1).$$

If we choose α sufficiently close to 1, we obtain

(5.31) $$\phi_n^{-3}=(-P_n)^{-3}(P_n')^2\left\{1-\frac{4}{5}P_n''(P_n')^{-2}P_n+O(S_n)+O((P_n'')^2P_n')^{-4}(P_n)^2\right\}.$$

Hence we find by (5.31) that the Schwarzian derivative is estimated as

(5.32) $$|\{\phi_n, x\}|\leqq C\{(P_n'')^2(P_n')^{-2}+P_n^{-2}(P_n')^2S_n\}.$$

We carry out the estimates of the right side of (5.ȝ2).

(5.33) $$\left(\frac{P_n''}{P_n'}\right)^2=\left(\frac{Q_n''}{Q_n'}\right)^2\leqq\frac{C}{X_n^2}$$

(5.34) $$\begin{aligned}|P_n^{-2}(P_n')^2S_n|&\leqq Q_n'(x)^2(Q_n(x)-Q_n(X_n))^{-2}|S_n|\\&\leqq C\frac{(\sup Q_n'(x))^2}{(\inf Q_n'(\xi))^2}(x-X_n)^{-2}|S_n|\end{aligned}$$

Then it follows from (5.29)

$$|P_n^{-2}(P_n')^2S_n|\leqq\frac{C}{X_n^2}.$$

Hence we obtain

(5.35) $$|\{\phi_n, x\}|\leqq\frac{C}{X_n^2}.$$

It follows from (5.35)

$$\begin{aligned}&\int_{X_n}^{\alpha X_n}|\{\phi_n, t\}||P_n(t)|^{-1/2}dt\\&\leqq\frac{C}{X_n^2}\int_{X_n}^{\alpha X_n}(Q_n(t)-Q_n(X_n))^{-1/2}dt\\&\leqq\frac{C}{X_n^2(\inf Q_n'(\xi)^{1/2})}\int_{X_n}^{\alpha X_n}(t-X_n)^{-1/2}dt.\end{aligned}$$

In this way we obtain

$$|I_2|\leqq\frac{C(\log X_n)^{1/2}}{X_n}.$$

The estimates of I_3 and I_4 are carried out just in the same manner as above. Hence we obtain the lemma.

Employing Proposition 5.1 and Lemma 5.2, we have shown

Theorem 5.3. *The solution* Φ_n *of the problem* (5.1) *has the asymptotic form*

(5.36) $$\Phi_n(x, \lambda)=\phi_n'(x, \lambda)^{-1/2}\{Ai(\lambda^{1/3}\phi_n(x, \lambda))+O(\lambda^{-1/2})\}$$

which holds uniformly in x *and* n *as* $\lambda\to\infty$.

For the derivatives, we have

Proposition 5.4. *Assume, in addition to* (5.12)

(5.37) $$\left|\frac{\phi_n''}{(\phi_n')^2}\right|\leqq M$$

with a constant M which is independent of λ and n. Then the derivative of Φ_n has the asymptotic form

$$\Phi_n'(x,\lambda)=\begin{cases}\lambda^{1/3}\phi_n'(x,\lambda)^{1/2}Ai'(\lambda^{1/3}\phi_n(x,\lambda))\{1+O(\lambda^{-1/3})\} & \text{for } x\geqq X_n, \quad (5.38)\\ \lambda^{1/3}\phi_n'(x,\lambda)^{1/2}Ai'(\lambda^{1/3}\phi_n(x,\lambda))\{1+O(\lambda^{-1/3})\} & \quad (5.39)\\ \quad +O(\phi_n'(x,\lambda)^{1/2}Ai(\lambda^{1/3}\phi_n(x,\lambda)))+O(\lambda^{-1/6}Bi'(\lambda^{1/3}\phi_n(x,\lambda))) & \\ \quad +O(\lambda^{-1/2}\phi_n'(x,\lambda)^{1/2}Bi(\lambda^{1/3}\phi_n(x,\lambda))) & \text{for } x\leqq X_n.\end{cases}$$

Outline of proof. Differentiating the both sides of (5.11), we find

$$\Phi_n'(x,\lambda)=A_n'(x,\lambda)-\frac{1}{2}\int_x^\infty\frac{\partial}{\partial x}K_n(x,t,\lambda)\{\phi_n,t\}\Phi_n(t,\lambda)dt. \tag{5.40}$$

Note

$$A_n'(x,\lambda)=\lambda^{1/3}\phi_n'(x,\lambda)^{1/2}\left\{Ai'(\lambda^{1/3}\phi_n(x,\lambda))-\frac{\lambda^{-1/3}\phi_n''(x,\lambda)}{2\phi_n'(x,\lambda)^2}Ai(\lambda^{1/3}\phi_n(x,\lambda))\right\}$$
$$=\lambda^{1/3}\phi_n'(x,\lambda)^{1/2}\{Ai'(\lambda^{1/3}\phi_n(x,\lambda))+O(\lambda^{-1/3}Ai(\lambda^{1/3}\phi_n(x,\lambda))\}.$$

Inserting (5.13) and (5.14) into (5.40) and estimating the integral, we obtain the result.

In our case we can show

Lemma 5.5. *Assume $Q(x)$ to satisfy* (5.3) *and* (5.4). *Then we have*

$$\left|\frac{\phi_n''(x,\lambda)}{\phi_n'(x,\lambda)^2}\right|\leqq\frac{C(\log X_n)^{1/3}}{X_n^{2/3}} \tag{5.41}$$

with a constant C which is independent of x, λ and n.

Proof. We observe

$$\frac{\phi_n''}{(\phi_n')^2}=-\frac{1}{2\phi_n}-\frac{1}{2}\left(-\frac{\phi_n}{P_n}\right)^{3/2}\frac{P_n'}{\phi_n}. \tag{5.42}$$

Choose $\alpha_0>1$, consider the case $x\geqq\alpha X_n$ for $1<\alpha\leqq\alpha_0$.

$$\frac{2}{3}\phi_n(x,\lambda)^{3/2}=\int_{X_n}^x(-P_n(t))^{1/2}dt \tag{5.43}$$
$$=\int_{X_n}^{\alpha X_n}(Q_n(t)-Q_n(X_n))^{1/2}dt$$
$$\geqq\inf_{X_n\leqq\xi\leqq\alpha X_n}Q_n'(\xi)^{1/2}\int_{X_n}^{\alpha X_n}(t-X_n)^{1/2}dt.$$

Then it follows from (5.18)

$$\frac{1}{|\phi_n(x)|}\leqq\frac{C(\log X_n)^{1/3}}{X_n}. \tag{5.44}$$

$$\frac{2}{3}\phi_n(x,\lambda)^{3/2}=\int_{X_n}^x\frac{1}{Q_n'(t)}(Q_n(t)-Q_n(X_n)^{1/2}Q_n'(t)dt \tag{5.45}$$

$$\leqq \frac{1}{\inf_{X_n \leqq \xi \leqq x} Q'_n(\xi)} \int_{X_n}^{x} (Q_n(t)-Q_n(X_n))^{1/2} Q'_n(t)dt$$

$$\leqq C(x \log x)(-P_n(x, \lambda))^{3/2}.$$

Hence we have

$$\left|\left(-\frac{\phi_n(x, \lambda)}{P_n(x, \lambda)}\right)\right| \leqq C x^{2/3} (\log x)^{2/3}. \tag{5.46}$$

Using this we find

$$\left|\left(-\frac{\phi_n}{P_n}\right)^{3/2} \frac{P'_n}{\phi_n}\right| = \left|\frac{\phi_n}{P_n}\right|^{1/2} \left|\frac{P'_n}{P_n}\right|$$

$$\leqq C(x \log x)^{1/3} \frac{Q'_n(x)}{Q_n(x)-Q_n(X_n)}$$

$$\leqq \frac{C}{(x \log x)^{2/3}} \frac{Q_n(x)}{Q_n(x)-Q_n(X_n)}$$

$$\leqq \frac{C}{(X_n \log X_n)^{2/3}} \frac{Q_n(\alpha X_n)}{Q_n(\alpha X_n)-Q_n(X_n)}$$

$$\leqq \frac{C}{X_n^{5/3} (\log X_n)^{2/3}} \frac{Q_n(\alpha X_n)}{\inf Q'_n(\xi)} \leqq \frac{C Q_n(\alpha X_n)(\log X_n)^{1/3}}{Q_n(X_n) X_n^{2/3}}.$$

Then it follows from (5.5)

$$\left|\left(-\frac{\phi_n}{P_n}\right)^{3/2} \frac{P'_n}{\phi_n}\right| \leqq \frac{C(\log X_n)^{1/3}}{X_n^{2/3}}.$$

In this way we have proved (5.37) for $x \geqq \alpha X_n$. We can proceed in the same way in the case $R \leqq x \leqq \alpha^{-1} X_n$.

Next consider the case $X_n \leqq x \leqq \alpha X_n$. Recall the proof of Lemma 5.2, then it follows from (5.24)

$$\frac{\phi''_n}{(\phi'_n)^2} = -\frac{1}{2\phi_n} \left\{\frac{2}{5} P''_n (P'_n)^{-2} P_n + S_n\right\}. \tag{5.47}$$

and

$$\phi_n^{-1} = -(P'_n)^{3/2} P_n^{-1} \left\{1 - \frac{4}{15} P''_n (P'_n)^{-2} P_n + O(S_n) + O((P''_n)^2 (P'_n)^{-4} P_n^2)\right\}. \tag{5.48}$$

Inserting (5.48) into (5.47), we find

$$\frac{\phi''_n}{(\phi'_n)^2} = O(P''_n (P'_n)^{-4/3}) + O((P'_n)^{2/3} P_n^{-1} S_n)). \tag{5.49}$$

We carry out the estimates of the right side of (5.48).

$$\left|\frac{P''_n}{(P'_n)^{4/3}}\right| = \left|\frac{Q''_n}{(Q'_n)^{4/3}}\right| \leqq \frac{C}{x Q'_n(x)^{1/3}} \tag{5.50}$$

$$\leqq \frac{C}{X_n \inf Q'_n(x)^{1/3}}.$$

Then by (5.18) we find

(5.51) $$\left|\frac{P_n''}{(P_n')^{4/3}}\right| \leqq \frac{C(\log X_n)^{1/3}}{X_n^{2/3}}.$$

(5.52) $$\left|\frac{(P_n')^{2/3}S_n}{P_n}\right| \leqq \frac{Q'(x)^{2/3}|S_n|}{Q_n(x)-Q_n(X_n)}$$

$$\leqq C\frac{(\sup Q_n'(x))^{2/3}}{(\inf Q_n'(\xi))}\frac{|S_n|}{(x-X_n)}.$$

By (5.18), (5.26) and (5.29), we have

$$\left|\frac{(P_n')^{2/3}S_n}{P_n}\right| \leqq \frac{C(\log X_n)^{1/3}}{X_n^{5/3}}\frac{Q_n(\alpha X_n)^{2/3}}{Q_n(X_n)}(x-X_n)$$

$$\leqq \frac{C(\log X_n)^{1/3}}{X_n^{2/3}}.$$

In this way we have proved (5.40) in the case $X_n \leqq x \leqq \alpha X_n$. We can treat the case $\alpha^{-1}X_n \leqq x \leqq X_n$ in the same way and then we obtain the lemma.

Employing Lemma 5.5 and Proposition 5.4, we obtain the asymptotic form of the derivative of Φ_n.

Theorem 5.6. *The derivative of the solution to the problem* (5.1) *has the asymptotic form*

(5.53) $$\Phi_n'(x, \lambda)=\lambda^{1/3}\phi_n'(x, \lambda)^{1/2}\{Ai'(\lambda^{1/3}\phi_n(x, \lambda))+O(\lambda^{-1/3})\}.$$

The Airy function $Ai(z)$ has the asymptotic form

(5.54) $$Ai(-z)=\pi^{-1/2}z^{-1/4}\left\{\cos\left(\frac{2}{3}z^{3/2}-\frac{\pi}{4}\right)+O(z^{-3/2})\right\}$$

(5.55) $$Ai'(-z)=\pi^{-1/2}z^{-1/4}\left\{\sin\left(\frac{2}{3}z^{3/2}-\frac{\pi}{4}\right)+O(z^{-3/2})\right\}$$

(see A. Erdélyi [6]).

These asymptotic forms say that, for large n, $Ai(-z)$ has one and only one zero around $\left(\frac{3}{2}\left(n-\frac{1}{4}\right)\pi\right)^{3/2}$. The next lemma tells that it is really the n-th zero of $Ai(-z)$.

Lemma 5.7. *If n is sufficiently large, $Ai(-x)$ has exactly n zeros in the interval* $0<x<\left\{\frac{3}{2}\left(n+\frac{1}{4}\right)\pi\right\}^{2/3}$.

For the proof of the lemma, we notice $Ai(-z)=(1/3)z^{1/2}\{J_{1/3}(\zeta)+J_{-1/3}(\zeta)\}$, where $J_\nu(\zeta)$ is the Bessel function of order ν and $\zeta=(2/3)z^{3/2}$. The location of zeros of $z^{1/2}\{J_{1/3}(z)+J_{-1/3}(z)\}$ is well studied in §7.9 of E.C. Titchmarsh [11]. It is known that there exist exactly n zeros in the interval $0<x<(n+(1/4))\pi$, for sufficiently large integer n. Then the lemma follows.

Now we well prepared to show the asymptotic formula of the distribution of the eigenvalues. We can see easily that Theorem 5.3 and Theorem 5.6 together with (5.54), (5.55) and Lemma 5.7 are all of what we need to employ the arguments in § 7.8∼7.10 in E. C. Titchmarsh [11] and § 4 of F. Asakura [1]. We just follow the arguments there and then we obtain.

Proposition 5.8. *Let* $N_n(\lambda)$ *be the number of the eigenvalues of* (2.1) *not exceeding* λ. *Then we have*

$$N_n(\lambda)=\frac{1}{\pi}\int_R^{X_n}(\lambda-n^2Q(x))^{1/2}dx+O(1) \tag{5.56}$$

as $\lambda\to\infty$, *where the remainder estimate is valid uniformly in* n.

Now we can carry out the proof of Theorem 0.3.

Proof of Theorem 0.3. Let $N(\lambda)$ be the number of the eigenvalues of the original problem (0.1) not exceeding λ. We observe

$$N(\lambda)=\sum_{n=1}^{\infty}N_n(\lambda)\,.$$

Since $N_n(\lambda)=0$ for $n\geqq\lambda^{1/2}Q(R)^{-1/2}$, the summation is in fact finite and then we find

$$\begin{aligned}N(\lambda)&=\sum_{n=1}^{[\lambda^{1/2}Q(R)^{-1/2}]}N_n(\lambda)\\&=\frac{1}{\pi}\sum_{n=1}^{\infty}\int_R^{X_n}(\lambda-n^2Q(x))^{1/2}dx+O(\lambda^{1/2})\,.\end{aligned}$$

In this way we obtain the theorem.

For $Q(x)=(\log x)^{2k}$, it follows from above

$$N(\lambda)=\sqrt{\frac{k}{2\pi}}\lambda^{1/2-1/4k}e^{\lambda^{1/2k}}\{1+O(\lambda^{-1/2k})\}\,. \tag{5.57}$$

Finally I would like to make an additional remark on the eigenvalues of the Laplace operator in an unbounded domain. If $G=G_1\cup G_2$ where G_1 is bounded and G_2 is represented as

$$G_2=\left\{(x,\ y)\leqq R^2\,|\,R<x<\infty,\ 0<y<\frac{1}{(\log x)^k}\right\},$$

then the large eigenvalues of the Laplace operator with the Dirichlet condition are expected to behave like the large eigenvalues of the operator considered as the example above. But in this case we merely obtain

Proposition 5.9. $N(\lambda)$ *has the estimates*

$$C_1e^{(1-\varepsilon)\lambda^{1/2k}}\leqq N(\lambda)\leqq C_2e^{(1+\varepsilon)\lambda^{1/2k}} \tag{5.58}$$

for any ε *with certain constants* C_1, C_2 *depending on* ε.

To show (5.58), we shall follow the notations in the proof of Theorem 4.6. Set $Q(x)=\pi^2(\log x)^{2k}$. Then it follows from (5.57)

$$\bar{C}(\lambda) \sim \underline{C}(\lambda) \sim \sqrt{\frac{k}{2\pi}} \lambda^{1/2-1/4k} e^{\lambda^{1/2k}} . \tag{5.59}$$

Inserting (5.59) into (4.11), we have

$$\lambda^{1/2-1/4k} \exp\left(\left(\frac{\lambda}{1+\varepsilon}\right)^{1/2k}\right)(1+o(1))$$

$$\leqq N(\lambda) \leqq \lambda^{1/2-1/4k} \exp\left(\left(\frac{\lambda}{1-\varepsilon}\right)^{1/2k}\right)(1+o(1)) .$$

Then (5.58) follows.

§6. In this section we carry out the proof of Theorem 0.2. We follow the proof of the Ikehara Tauberiam in D. V. Widder [13].

A function $f(x)$ defined in $(-\infty, \infty)$ is said to be slowly decreasing, if

$$\liminf \{f(x+\delta)-f(x)\} = 0 \qquad (x \to \infty, \ \delta \to 0, \ \delta > 0) .$$

Let $K(x)$ be a smooth, positive and even function so that the Fourier transform $K^\wedge(\xi)$ is positive, even decreasing in $\xi \geqq 0$ satisfying $K^\wedge(0)=(1/2\pi)$, $\operatorname{supp} K^\wedge(\xi) \subset [-1, 1]$. Set $K_\lambda(x)=\lambda K(\lambda x)$, then the Fourier transform $\hat{K_\lambda}(\xi)$ of $K_\lambda(x)$ is $K^\wedge(\xi/\lambda)$. We can readily verify

Proposition 6.1 (Theorem 9, Chap. V, D. V. Widder [13]). *Let $f(x)$ be bounded and slowly decreasing satisfying*

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} K_\lambda(x-t) f(t) dt = A \qquad \textit{for all} \quad \lambda > 0 .$$

Then

$$\lim_{x\to\infty} f(x) = A .$$

Let $N(t)$ be non-negative, non-decreasing function and let

$$Z(\alpha) = \int_1^\infty t^{-\alpha} dN(t)$$

satisfy the hypotheses of Theorem 0.2. Without loss of generality we may assume $\sigma=1$ and $N(1)=0$. We set $t=e^s$. Since $N(e^s)=o(e^{\alpha s})$ for any $\alpha>1$, we find

$$Z(\alpha) = \int_0^\infty e^{-\alpha s} dN(e^s)$$

$$= \alpha \int_0^\infty e^{-\alpha s} N(e^s) ds .$$

We shall prove the Tauberian theorem of the form

Theorem 6.2 (see D. V. Widder [13], Chap. V, Theorem 17). *Let $N(x)$ be non-negative and non-decreasing. If*

$$L(s) = \int_0^\infty e^{-sx} N(x) dx \tag{6.1}$$

converges for Re $s>1$ *and with* $\rho\geqq 0$

$$h(s)=L(s)-(s-1)^{-(1+\rho)}\sum_{n=0}^{[1+\rho]}\sum_{j=0}^{n}A_{nj}(s-1)^{n}(\log(s-1))^{j} \tag{6.2}$$

can be extended to a continuous function in Re $s\geqq 1$, *then* $N(t)$ *has the asymptotic behavior*

$$N(t)\sim\frac{A_{00}}{\Gamma(1+\rho)}e^{t}t^{\rho} \quad \text{as} \quad t\longrightarrow\infty . \tag{6.3}$$

Proof. We may assume $\rho>0$. Because we can skip directly to (6.8), if $\rho=0$. For $1<t<A$, $x>0$, we find

$$\int_{t}^{2A}e^{-xs}(s-t)^{\rho-1}ds=\int_{t}^{\infty}-\int_{2A}^{\infty}$$

$$=\Gamma(\rho)x^{-\rho}e^{-xt}-e^{-xt}\int_{A-t}^{\infty}e^{-xu}u^{\rho-1}du .$$

We can see that the both sides of above make sense for $1<\mathrm{Re}\,t<A$, $x>0$. Then we have

$$\frac{1}{\Gamma(\rho)}\int_{t}^{2A}e^{-xs}(s-t)^{\rho-1}ds=x^{-\rho}e^{-xt}+e^{-2Ax}G_{A}(x, t) \tag{6.4}$$

where $G_{A}(x, t)$ is continuous and uniformly bounded in $1<x<\infty$, $1<\mathrm{Re}\,t<A$ and the integral is taken along the path satisfying Re $s\geqq\mathrm{Re}\,t$. In a similar fashion we have

$$\frac{1}{\Gamma(\rho)}\int_{t}^{2A}(s-t)^{\rho-1}(s-1)^{-1-\rho}ds=\frac{1}{\Gamma(1+\rho)(t-1)}+H_{A}(t) , \tag{6.5}$$

where $H_{A}(t)$ is continuous for $1\leqq\mathrm{Re}\,t\leqq A$. We find that the integral

$$\frac{1}{\Gamma(\rho)}\int_{t}^{2A}(s-t)^{\rho-1}(s-1)^{n-\rho-1}(\log(s-1))^{m}ds \tag{6.6}$$

converges for $1\leqq\mathrm{Re}\,t\leqq A$ and continuous there, if $n\geqq 2$. When $n=1$, we can see that the principal part of (6.6) as $t\rightarrow 1$ is $\sum_{j=1,2}B_{j}(\log(t-1))^{j}$. Then we find (6.6) is expressed as

$$\sum_{j=1,2}B_{j}(\log(t-1))^{j}+H_{A}(t) , \tag{6.7}$$

where $H_{A}(t)$ is continuous for $1\leqq\mathrm{Re}\,t\leqq A$ and $B_{j}=0(j=1, 2)$ for $n\geqq 2$.

For $\mathrm{Re}\,t>1$, multiplying to (6.2) $\Gamma(\rho)^{-1}(s-t)^{\rho-1}$ and integrating from t to $2A$, we find

$$f(t)=\int_{0}^{\infty}e^{-tx}x^{-\rho}N(x)dx \tag{6.8}$$

$$=\frac{A_{00}}{\Gamma(1+\rho)(t-1)}+\sum_{n=1,2}C_{n}(\log(t-1))^{n}+h_{A}(t) ,$$

where $h_{A}(t)$ is continuous in $1\leqq\mathrm{Re}\,t\leqq A$. Set

$$a(t)=e^{-t}t^{-\rho}N(t)\chi_{+}(t) ,$$

$$A(t)=\frac{A_{00}}{\Gamma(1+\rho)}\chi_+(t)$$

where $\chi_+(t)$ is the characteristic function of the interval $(0, \infty)$. Then it follows from the hypotheses that

$$e^{-\varepsilon t}a(t)=e^{-(1+\varepsilon)t}t^{-\rho}N(t)\chi_+(t)$$

is a summable function in $(-\infty, \infty)$. Set

$$\begin{aligned} I_\lambda^{(\varepsilon)}(x) &= \int_{-\infty}^{\infty} K_\lambda(x-t)(a(t)-A(t))e^{-\varepsilon t}dt \\ &= \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} \hat{K_\lambda}(\xi)e^{-i(x-t)\xi}d\xi\Big)(a(t)-A(t))e^{-\varepsilon t}dt \\ &= \int_{-\infty}^{\infty} \hat{K_\lambda}(\xi)e^{-ix\xi}\Big(\int_{-\infty}^{\infty}(a(t)-A(t))e^{-\varepsilon t+i\xi t}dt\Big)d\xi\,. \end{aligned}$$

Since

$$\begin{aligned} &\int_{-\infty}^{\infty}(a(t)-A(t))e^{-\varepsilon t+i\xi t}dt \\ &\quad = \int_0^{\infty} e^{-(1+\varepsilon)t+i\xi t}t^{-\rho}N(t)dt-\frac{A_{00}}{\Gamma(1+\rho)}\int_0^{\infty}e^{-\varepsilon t+i\xi t}dt \\ &\quad = f(1+\varepsilon-i\xi)-\frac{A_{00}}{\Gamma(1+\rho)(\varepsilon-i\xi)} \\ &\quad = \sum_{n=1,2} C_n(\log(\varepsilon-i\xi))^n+h_A(1+\varepsilon-i\xi) \end{aligned}$$

and $\operatorname{supp}\hat{K_\lambda}(\xi)\subset[-\lambda, \lambda]$, we find

$$I_\lambda^{(\varepsilon)}(x)=\int_{-\lambda}^{\lambda}\hat{K_\lambda}(\xi)e^{-ix\xi}\Big\{\sum_{n=1,2}C_n(\log(\varepsilon-i\xi))^n+h_A(1+\varepsilon-i\xi)\Big\}d\xi\,.$$

We observe that $h_A(1+\varepsilon-i\xi)$ is uniformly bounded for $|\xi|\leqq\lambda$, $0\leqq\varepsilon\leqq\varepsilon_0$ and $h_A(1+\varepsilon-i\xi)$ converges to $h_A(1-i\xi)$ as $\varepsilon\to 0$. Moreover, since

$$\begin{aligned} |\log(\varepsilon-i\xi)| &\leqq \frac{1}{2}|\log(\varepsilon^2+\xi^2)|+\frac{\pi}{2} \\ &\leqq C\max\{\log|\xi|, 1\}, \end{aligned}$$

$\log(\varepsilon-i\xi)$ is dominated by a summable function which does not depend on ε. Then it follows from the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\to 0} I_\lambda^{(\varepsilon)}(x)=\int_{-\lambda}^{\lambda}\hat{K_\lambda}(\xi)e^{-ix\xi}\Big\{\sum_{n=1,2}C_n(\log(i\xi))^n+h_A(1-i\xi)\Big\}d\xi\,.$$

Set $I_\lambda(x)=\lim I_\lambda^{(\varepsilon)}(x)$. Then we can express with a summable function Φ that

$$\begin{aligned} I_\lambda(x) &= \int_{-\lambda}^{\lambda}\Phi(\xi)e^{-ix\xi}d\xi \\ &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}K_\lambda(x-t)(a(t)-A(t))e^{-\xi t}dt\,. \end{aligned}$$

We observe that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} K_\lambda(x-t)a(t)e^{-\varepsilon t}dt$$

$$=\lim_{\varepsilon\to 0} I_\lambda^{(\varepsilon)}(x)+\frac{A_{00}}{\Gamma(1+\varepsilon)}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} K_\lambda(x-t)e^{-\varepsilon t}dt$$

$$=I_\lambda(x)+\frac{A_{00}}{\Gamma(1+\rho)}\int_{-\infty}^{x} K_\lambda(y)dy\,.$$

Since $K_\lambda(x-y)a(t)e^{-\varepsilon t}$ is positive, increasing in ε and converging to $K_\lambda(x-t)a(t)$, then it follows from the Beppo-Levi theorem

$$I_\lambda(x)=\int_{-\lambda}^{\lambda} \Phi(\xi)e^{-ix\xi}d\xi$$

$$=\int_{-\infty}^{\infty} K_\lambda(x-t)(a(t)-A(t))dt\,.$$

Employing the Riemann-Lebesgue lemma, we find

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} K_\lambda(x-t)(a(t)-A(t))dt=0\,,$$

which shows

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} K_\lambda(x-t)a(t)dt=\frac{A_{00}}{\Gamma(1+\rho)}$$

We can easily verify that $a(t)$ is bounded and slowly decreasing. Then it follows from Proposition 6.1 that

$$\lim_{t\to\infty} e^{-st}t^{-\rho}N(t)=\frac{A_{00}}{\Gamma(1+\rho)}\,.$$

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