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On the fractional derivative of Brownian local times

By

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Introduction.

Let B_t be a one dimensional Brownian motion and L_t^a denote its local time at a. As usual, we take a version of L which is jointly continuous in (a, t).

It is well known that the process $a \mapsto L_t^a$, for fixed t, satisfies the Hölder condition of order β-almost surely, where $0<\beta<1/2$([6], [7], [8], [11]).

According to a result by M. Yor, the Hilbert transform of local times $\mathcal{H}(L_t^{\cdot})(a) =(1/\pi)(\text{v. p.}(1/x)*L_t^{\cdot}(a))=\tilde{L}_t^a$ can be represented as an additive functional which corresponds to Cauchy's principal value. ([12], [13]). Since the Hölder's continuity property of functions remains invariably under the Hilbert transform, the process $a \mapsto \tilde{L}_t^a$ satisfies the Hölder condition of order β.

On the other hand, referring to a result of Hardy and Littlewood on the fractional derivative, the continuity property of local times implies that its fractional derivative of order α satisfies the Hölder condition of order $\beta-\alpha$ where $0<\alpha<\beta<1/2$.

In the present paper, we are concerned with a representation of the fractional derivative of local times. In the representation, additive functionals which will be defined via Hadamard's finite part as well as the Hilbert transform will play important roles.

§ 1. Definitions and preparatory lemmas.

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a complete probability space with right continuous increasing family $(\mathcal{F}_t)_{t\geq 0}$ of σ-fields of $\mathcal{F}$. Let B_t denote a continuous $\mathcal{F}_t$-martingale such that

(i) $E[(B_t-B_s)^2/\mathcal{F}_s]=t-s$, for $t \geqq s \geqq 0$,

(ii) the initial distribution μ has a compact support. That is to say. B_t is a one dimensional Brownian motion with compact initial distribution.

Let L_t^a be a version of local time of the Brownian motion B_t which is chosen to be jointly continuous in (t, a).

We shall introduce additive functionals which correspond to Hadamard's finite part. Consider

(1.1)
$$F_a(x)=\frac{(x-a)_+^{1-\alpha}}{(1-\alpha)(-\alpha)}\begin{cases} 0 & x<a \\ \dfrac{(x-a)^{1-\alpha}}{(1-\alpha)(-\alpha)} & x\geqq a\,, \end{cases}$$

where $0<\alpha<1/2$.

Then, the derivative of $F_a(x)$

(1.2)
$$\frac{d}{dx}F_a(x)=F_a'(x)=\frac{(x-a)_+^{-\alpha}}{(-\alpha)}=\begin{cases} 0 & x<a \\ \dfrac{(x-a)^{-\alpha}}{-\alpha} & x\geqq a\,, \end{cases}$$

belongs to $L^2_{\text{loc}}(R^1)$. Let p.f.$(x-a)_+^{-1-\alpha}$ be the second derivative of $F_a(x)$ in the sense of Schwartz's distribution.

Definition. (Brownian additive functional defined via Hadamard's finite part.) Let $0<\alpha<1/2$. Put

(1.3)
$$H^a(-1-\alpha,\, t)=2F_a(B_t)-2F_a(B_0)-2\int_0^t F_a'(B_s)dB_s\,,$$

where the stochastic integral is understood in the sense of Ito integral.

The right hand side of (1.3) is well defined, because $F_a'(x)$ belongs to $L^2_{\text{loc}}(R^1)$. We call $H^a(-1-\alpha, t)$ additive functional of B_t defined via Hadamard's finite part, p.f.$(x-a)_+^{-1-\alpha}$. (cf. [2] and [12]).

Henceforth in some cases, we will put

(1.4)
$$\int_0^t \text{p.f.}(x-a)_+^{-1-\alpha}(B_s)ds=H^a(-1-\alpha,\, t)\,.$$

Remark. It is known that the following formula holds: (cf. [12]).

(1.5)
$$H^a(-1-\alpha,\, t)=\int_0^t \text{p.f.}(x-a)_+^{-1-\alpha}(B_s)ds$$
$$=\lim_{\varepsilon\downarrow 0}\left\{\frac{\varepsilon^{-\alpha}}{(-\alpha)}L_t^a+\int_0^t I_{[a+\varepsilon,\infty)}(B_s)(B_s-a)^{-1-\alpha}ds\right\}.$$

By the definition, it is clear that the additive functional is continuous in t, but it is not of bounded variation with respect to t. (cf. [2]).

As the process $a\mapsto H^a(-1-\alpha, t)$ shall play essential roles in the future, we investigate for the present some regularity properties of the process.

Lemma 1. *The relation*

(1.6)
$$\lim_{|b-a|\downarrow 0} E[|H^a(-1-\alpha,\, t)-H^b(-1-\alpha,\, t)|^2]=0$$

holds.

For the proof of this lemma, we prepare the following;

Lemma 2. (A) *The family of random variables defined by*

$$\{(F_a(B_t)-F_b(B_t))^2\,;\ a,\, b\in[c,\, d]\}$$

is uniformly integrable, where $-\infty<c<d<+\infty$ *and* $0<t<+\infty$.

(B) *The family of random variables defined by*

$$\{(F_a(B_0)-F_b(B_0))^2;\ a,\ b\in[c,\ d]\}$$

is uniformly integrable, where $-\infty<c<d<+\infty$.

(C) *The family of random variables defined by*

$$\{(F_a'(B_s)-F_b'(B_s))^2;\ a,\ b\in[c,\ d],\ s\in[0,\ T]\}$$

is uniformly integrable with respect to $(\Omega\times[0,\ T],\ \mathcal{F}\otimes\mathcal{B}_{[0,T]},\ P\otimes ds)$ *where* $-\infty<c<d<+\infty$ *and* $0<T<+\infty$.

Proof of (*A*). Simple calculus shows that

$$\begin{aligned}&E[(F_a(B_t)-F_b(B_t))^4]\\&\quad\leqq 8E[(F_a(B_t))^4]+8E[(F_b(B_t))^4]\\&\quad\leqq\frac{8}{(1-\alpha)^4\alpha^4}\sup_{a\in[c,d]}\int_{R^1}\mu(dx)\int_a^\infty\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-x)^2}{2t}\right)(y-a)^{4(1-\alpha)}dy\\&\quad\quad+\frac{8}{(1-\alpha)^4\alpha^4}\sup_{b\in[c,d]}\int_{R^1}\mu(dx)\int_b^\infty\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-x)^2}{2t}\right)(y-b)^{4(1-\alpha)}dy\\&\quad<+\infty\end{aligned}$$

The above inequalities imply immediately that the family is uniformly integrable.

Proof of (*B*). Note that

$$\begin{aligned}&E[(F_a(B_0)-F_b(B_0))^4]\\&\quad\leqq\frac{8}{(1-\alpha)^4\alpha^4}\sup_{a\in[c,d]}\int_{R^1}\mu(dx)(x-a)^{4(1-\alpha)}\\&\quad\quad+\frac{8}{(1-\alpha)^4\alpha^4}\sup_{b\in[c,d]}\int_{R^1}\mu(dx)(x-b)^{4(1-\alpha)}<+\infty\,.\end{aligned}$$

Then, these inequalities imply the desired fact.

Proof of (*C*). Choose a number δ such that $0<\delta<\max((1/2\alpha)-1,\ 1)$. Observe that

$$\begin{aligned}(1.7)\quad&\int_0^T E[(F_a'(B_s)-F_b'(B_s))^{2(1+\delta)}]ds\\&\leqq\frac{8}{\alpha^{2(1+\delta)}}\sup_{a\in[c,d]}\int_0^T ds\int_{R^1}\mu(dx)\int_a^\infty\frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{(x-y)^2}{2s}\right)\frac{dy}{(y-a)^{2\alpha(1+\delta)}}\\&\quad+\frac{8}{\alpha^{2(1+\delta)}}\sup_{b\in[c,d]}\int_0^T ds\int_{R^1}\mu(dx)\int_b^\infty\frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{(x-y)^2}{2s}\right)\frac{dy}{(y-b)^{2\alpha(1+\delta)}}\\&\leqq\frac{8}{\sqrt{2\pi}\,\alpha^{2(1+\delta)}}\sup_{a\in[c,d]}\int_0^T\frac{ds}{\sqrt{s}}\int_{R^1}\mu(dx)\int_a^\infty\exp\left(-\frac{(x-y)^2}{2T}\right)\frac{dy}{(y-a)^{2\alpha(1+\delta)}}\\&\quad+\frac{8}{\sqrt{2\pi}\,\alpha^{2(1+\delta)}}\sup_{b\in[c,d]}\int_0^T\frac{ds}{\sqrt{s}}\int_{R^1}\mu(dx)\int_b^\infty\exp\left(-\frac{(x-y)^2}{2T}\right)\frac{dy}{(y-b)^{2\alpha(1+\delta)}}\\&\leqq\frac{16\sqrt{T}}{\sqrt{2\pi}\,\alpha^{2(1+\delta)}}\left\{\sup_{a\in[c,d]}\int_{R^1}\mu(dx)\int_a^\infty\exp\left(-\frac{(x-y)^2}{2T}\right)\frac{dy}{(y-a)^{2\alpha(1+\delta)}}\right.\end{aligned}$$

$$+\sup_{b\in[c,d]}\int_{R^1}\mu(dx)\int_b^\infty \exp\Big(-\frac{(x-y)^2}{2T}\Big)\frac{dy}{(y-b)^{2\alpha(1+\delta)}}\Big\}.$$

where $2\alpha(1+\delta)<1$.

Noting that both $(y-a)^{-2\alpha(1+\delta)}$ and $(y-b)^{-2\alpha(1+\delta)}$ belong to $L^1_{\mathrm{loc}}(R^1)$ and tend to zero as y goes to infinity, we can conclude from (1.7) that

$$\sup_{a,b\in[c,d]}\int_0^T E[(F_a'(B_s)-F_b'(B_s))^{2(1+\delta)}]ds<+\infty .$$

Then, by de la Vallée Poussin's theorem, we can see that the family is uniformly integrable. (cf. [1]).

Proof of Lemma 1. By the definition of the additive functional $H^a(-1-\alpha, t)$, we have

$$(1.8)\qquad E[|H^a(-1-\alpha, t)-H^b(-1-\alpha, t)|^2]$$

$$\leqq 12E\Big[\Big|\frac{1}{(1-\alpha)(-\alpha)}(B_t-a)_+^{1-\alpha}-\frac{1}{(1-\alpha)(-\alpha)}(B_t-b)_+^{1-\alpha}\Big|^2\Big]$$

$$+12E\Big[\Big|\frac{1}{(1-\alpha)(-\alpha)}(B_0-a)_+^{1-\alpha}-\frac{1}{(1-\alpha)(-\alpha)}(B_0-b)_+^{1-\alpha}\Big|^2\Big]$$

$$+12E\Big[\Big|\int_0^t\Big\{\frac{1}{(-\alpha)}(B_s-a)_+^{-\alpha}-\frac{1}{(-\alpha)}(B_s-b)^{-\alpha}\Big\}^2 ds\Big|\Big].$$

Observe that each integrand of the right hand side of the inequality (1.8) converges to zero respectively as a tends to b. Then, by virtue of Lemma 2, we can obtain the relation

$$(1.6)\qquad \lim_{|b-a|\to 0} E[|H^a(-1-\alpha, t)-H^b(-1-\alpha, t)|^2]=0 . \qquad \text{Q.E.D.}$$

The Lemma 1 implies the following;

Lemma 3. *There exists a version of* $H^a(-1-\alpha, t)=\int_0^t \mathrm{p.f.}(x-a)_+^{-1-\alpha}(B_s)ds$ *which is measurable with respect to* $(t, a, \omega)\in[0, \infty)\times R^1\times\Omega$.

Henceforth, we mean by $H^a(-1-\alpha, t)$ a measurable version of $(t, a, \omega)\mapsto H^a(-1-\alpha, t)(\omega)$.

In the rest of this section, we discuss some properties of the fractional calculus. For the purpose, we shall prepare some definitions.

Definition (left compact). We say that a function g has a left compact support if there exists a number c such that $g(x)=0$ for $\forall x\leqq c$.

Definition (The Hölder condition of order β in the global sense). We say that a function f satisfies the condition (H) of order β if there exists a number $K>0$ such that

$$(1.9)\qquad |f(x+h)-f(x)|\leqq K|h|^\beta, \qquad \text{for } \forall h, \forall x\in R^1 .$$

Definition (fractional derivative of order β). (cf. [3], [9]) Let g be a function and have a left compact support. We say that the convolution in the sense of Schwartz's distribution

$$D^{\alpha}g=\frac{1}{\Gamma(-\beta)}\text{p.f.}(x^{-1-\beta})_{+}*g \tag{1.10}$$

is the fractional derivative of order β of the function g, where $\Gamma(\cdot)$ stands for the Gamma function.

The following lemma is due to Hardy and Littlewood. (cf. Theorem 19 and 20 in [4] and cf. also [14]).

Lemma 4. (G.H. Hardy and J.E. Littlewood) *Let g be a function whose support is compact. Suppose that g satisfies the condition (H) of order β. Then the fractional derivative of order α of the function g satisfies the followings, where $0<\alpha<\beta\leqq 1$.*

(i) *$(D^{\alpha}g)(x)$ satisfies the condition (H) of order $\beta-\alpha$.*

(ii) *$(D^{\alpha}g)(x)$ belongs to $L^{2}(R^{1})\cap L^{1}(R^{1})$.*

(iii)

$$(D^{\alpha}g)(x)=\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{x}\frac{\{g(x)-g(a)\}}{(x-a)^{1+\alpha}}da \tag{1.11}$$

holds.

§2. Fractional derivative of Brownian local times.

The first topic that we are going to take up is a representation of Brownian additive functionals which correspond to $D^{\alpha}g$.

Theorem 1. *Suppose that $0<\alpha<1/2$ and $\alpha<\beta\leqq 1$.*

Let $g(x)$ be a function whose support is compact and satisfy the condition (H) of order β. Then

$$\int_{0}^{t}(D^{\alpha}g)(B_{s})ds=\frac{1}{\Gamma(-\alpha)}\int_{R^{1}}H^{\alpha}(-1-\alpha,\ t)g(a)da \tag{2.1}$$

holds.

In order to give the proof of the above theorem, we shall prepare the following Fubini type lemma;

Lemma 5. *Let $g(x)$ be a continuous function and have a compact support. Then, the Fubini type relation*

$$\int_{0}^{t}\left\{\int_{R^{1}}F_{a}'(B_{s})g(a)da\right\}dB_{s}=\int_{R^{1}}\left\{\int_{0}^{t}F_{a}'(B_{s})dB_{s}\right\}g(a)da \tag{2.2}$$

holds.

One can complete the proof of this lemma following essentially the same way as in the proof of Lemma 4.1 of Chap. 3 in Ikeda-Watanabe's book [5]. So we omit the proof. (cf. also [12]).

Proof of the Theorem 1. Put

$$F(x)=F_0(x)=\frac{1}{(1-\alpha)(-\alpha)}(x^{1-\alpha})_+$$

and

$$F'(x)=F'_0(x)=\frac{1}{(-\alpha)}(x^{-\alpha})_+ .$$

Note that $g(x)$ has a compact support and belongs to $L^2(R^1)$. Then, by the fact that $F(x)$ belongs to $L^2_{\mathrm{loc}}(R^1)$, we can see the followings;

$$(F*g)'=(F'(x)*g)=\left(\frac{(x^{-\alpha})_+}{(-\alpha)}*g\right) \tag{2.3}$$

and

$$(F*g)''=(\mathrm{p.f.}(x^{-1-\alpha})_+*g) \tag{2.4}$$

where $*$ stands for the convolution operator.

From the definition of $H^a(-1-\alpha, t)$, we have

$$\begin{aligned}\int_{R^1}H^a(-1-\alpha, t)g(a)da=&2\int_{R^1}\{F_a(B_t)-F_a(B_0)\}g(a)da\\ &-2\int_{R^1}\left\{\int_0^t F'_a(B_s)dB_s\right\}g(a)da .\end{aligned} \tag{2.5}$$

By virtue of the Lemma 5, the relation (2.5) implies that

$$\begin{aligned}\int_{R^1}H^a(-1-\alpha, t)g(a)da=&2\int_{R^1}F(B_t-a)g(a)-2\int_{R^1}F(B_0-a)g(a)da\\ &-2\int_0^t\left\{\int_{R^1}F'(B_s-a)g(a)da\right\}dB_s\\ =&2(F*g)(B_t)-2(F*g)(B_0)-2\int_0^t(F'*g)(B_s)dB_s\\ =&2(F*g)(B_t)-2(F*g)(B_0)-2\int_0^t(F*g)'(B_s)dB_s\end{aligned} \tag{2.6}$$

holds.

By virtue of Lemma 4, we know that $d^2/dx^2(F*g)=\mathrm{p.f.}(x^{-1-\alpha})_+*g$ is a function satisfying the condition (H) of order $\beta-\alpha$. Then, by Ito formula we obtain from (2.6) the following relation,

$$\begin{aligned}\int_{R^1}H^a(-1-\alpha, t)g(a)da=&\int_0^t(\mathrm{p.f.}(x^{-1-\alpha})_+*g)(B_s)ds\\ =&\Gamma(-\alpha)\int_0^t(D^\alpha g)(B_s)ds .\end{aligned} \tag{2.7}$$

Thus we have proved the Theorem 1. Q.E.D.

It is now time for us to state the main result of the present paper.

Theorem 2. *Suppose that* $0<\alpha<1/2$. *Then*

$$(2.8)\qquad (D^{\alpha}L_{t}^{\cdot})(a)=\frac{1}{\Gamma(-\alpha)}\{-\cos(\pi(1+\alpha))H^{a}(-1-\alpha, t)+\sin(\pi(1+\alpha))\mathcal{H}(H^{\cdot}(-1-\alpha, t))(a)\}$$

holds, where $\mathcal{H}f$ *stands for the Hilbert transform of the function* f, $\mathcal{H}f=1/\pi$ (v.p. $1/x)*f$.

To begin the proof of the Theorem 2, we need some preparatory discussions. First, we recall some formulae on the Fourier transform. Following L. Schwartz, let $(\mathcal{F}S)(\phi)=\hat{S}(\phi)=S_{\xi}\left(\int e^{-2\pi i\xi x}\phi(x)dx\right)$, where $S\in\mathcal{S}'$ and $\phi\in\mathcal{S}$. (cf. [9]). Then, the following is a well known formula (cf. [3]);

$$(2.9)\qquad \mathcal{F}(\text{p.f.}(x^{-1-\alpha})_{+})(\xi)=\widehat{\text{p.f.}(x^{-1-\alpha})_{+}}(\xi)=\begin{cases} i\Gamma(-\alpha)(2\pi)^{\alpha}e^{-i(\alpha+1)\pi/2}|\xi|^{\alpha} & \xi<0. \\ -i\Gamma(-\alpha)(2\pi)^{\alpha}e^{i(\alpha+1)\pi/2}\xi^{\alpha} & \xi>0. \end{cases}$$

Put

$$(2.10)\qquad \psi(\xi)=\begin{cases} i(2\pi)^{\alpha}e^{-i(\alpha+1)\pi/2}|\xi|^{\alpha} & \xi<0 \\ -i(2\pi)^{\alpha}e^{i(\alpha+1)\pi/2}\xi^{\alpha} & \xi>0 \end{cases}$$

and

$$(2.11)\qquad \kappa_{\alpha}(\xi)=\frac{\psi(\xi)}{\bar{\psi}(\xi)}=\begin{cases} -e^{-i(\alpha+1)\pi} & \xi<0 \\ -e^{i(\alpha+1)\pi} & \xi>0. \end{cases}$$

Define a linear operator on $L^{2}(R^{1})$ such that

$$(2.12)\qquad P_{\alpha}f(x)=\int_{R^{1}}e^{2\pi ix\xi}\kappa_{\alpha}(\xi)\hat{f}(\xi)d\xi,\qquad f\in L^{2}(R^{1}).$$

Note that

$$\kappa_{\alpha}(\xi)=\begin{cases} -\cos(\pi(1+\alpha))+i\sin(\pi(1+\alpha)) & \xi<0 \\ -\cos(\pi(1+\alpha))-i\sin(\pi(1+\alpha)) & \xi>0, \end{cases}$$

and recall that

$$(\mathcal{F}\delta)(\xi)=1 \quad\text{and}\quad \mathcal{F}\left(\frac{1}{\pi}\text{v.p.}\frac{1}{x}\right)(\xi)=\begin{cases} i & \xi<0 \\ -i & \xi>0. \end{cases}$$

Then, we can see easily that

$$(2.13)\qquad P_{\alpha}f(x)=-\cos(\pi(1+\alpha))f(x)+\sin(\pi(1+\alpha))(\mathcal{H}f)(x),\qquad \forall f\in L^{2}(R^{1})$$

holds.

On the other hand, for any $f\in L^{2}(R^{1})$, we observe that

$$(2.14)\qquad P_{\alpha}(-\cos(\pi(1+\alpha))f(x)-\sin(\pi(1+\alpha))(\mathcal{H}f)(x))=f(x),$$

where we have used the fact that $-\mathscr{H}(\mathscr{H}f)=-\left(\frac{1}{\pi}\mathrm{v.p.}\frac{1}{x}\right)*\left(\frac{1}{\pi}\mathrm{v.p.}\frac{1}{x}\right)*f=\delta*f=f$ holds.

Moreover we have

$$(P_{\alpha}f,\ P_{\alpha}h)=\int_{R^1}\widehat{P_{\alpha}f}(\xi)\overline{\widehat{P_{\alpha}h}(\xi)}d\xi \tag{2.15}$$

$$=\int_{R^1}\kappa_{\alpha}(\xi)\hat{f}(\xi)\overline{\kappa_{\alpha}(\xi)\hat{h}(\xi)}d\xi=\int_{R^1}\hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi$$

$$=(f,\ h) \qquad \text{for} \quad \forall f,\ \forall h\in L^2(R^1),$$

since $\kappa_{\alpha}(\xi)\overline{\kappa_{\alpha}(\xi)}=1$ holds.

Thus we obtain the following lemma.

Lemma 7. *P_{α} is an unitary operator on $L^2(R^1)$ such that*

$$P_{\alpha}f(x)=-\cos(\pi(1+\alpha))f(x)+\sin(\pi(1+\alpha))(\mathscr{H}f)(x), \tag{2.13}$$

and

$$P_{\alpha}^{-1}f(x)=-\cos(\pi(1+\alpha))f(x)-\sin(\pi(1+\alpha))(\mathscr{H}f)(x) \tag{2.16}$$

hold for any $f\in L^2(R^1)$.

We are now in a position to prove the Theorem 2.

Proof of Theorem 2. Let g be a function having a compact support and satisfy the condition (H) of order β where $0<\alpha<\beta\leqq 1$. Then by Lemma 4, we know that $D^{\alpha}g$ satisfies the conditson (H) of order $\beta-\alpha$ and belongs to $L^2(R^1)$.

Note that the function $a\mapsto L_t^a$ has a compact support and satisfies the condition (H) of order γ $(0<\forall\gamma<1/2)$. (cf. [6], [7], [8], [11]). Then, again by Lemma 4, the fractional derivative of local times $(D^{\alpha}L_t^{\cdot})(a)$ satisfies the condition (H) of order $\gamma-\alpha$ and belongs to $L^2(R^1)$.

Thus we observe that

$$\int_0^t(D^{\alpha}g)(B_s)ds=\int_{R^1}L_t^a(D^{\alpha}g)(a)da=\int_{R^1}\widehat{L_t^{\cdot}}(\xi)\overline{\widehat{D^{\alpha}g}(\xi)}d\xi \tag{2.17}$$

$$=\int_{R^1}\overline{\widehat{L_t^{\cdot}}(\xi)\overline{\widehat{D^{\alpha}g}(\xi)}}d\xi=\int_{R^1}\overline{\widehat{L_t^{\cdot}}(\xi)\overline{\phi(\xi)\hat{g}(\xi)}}d\xi$$

$$=\int_{R^1}\frac{\phi(\xi)}{\bar{\phi}(\xi)}\hat{g}(\xi)\overline{\phi(\xi)\widehat{L_t^{\cdot}}(\xi)}d\xi=\int_{R^1}\kappa_{\alpha}(\xi)\hat{g}(\xi)\overline{\widehat{D^{\alpha}L_t^{\cdot}}(\xi)}d\xi$$

$$=\int_{R^1}(D^{\alpha}L_t^{\cdot})(a)(P_{\alpha}g)(a)da$$

holds.

In virtue of Theorem 1, we get from (2.17) that

$$\frac{1}{\Gamma(-\alpha)}\int_{R^1}H^{\alpha}(-1-\alpha,\ t)g(a)da=\int_{R^1}(D^{\alpha}L_t^{\cdot})(a)(P_{\alpha}g)(a)da \tag{2.18}$$

holds.

Define the set of functions $\mathcal{G}$ by

$\mathcal{G}=\{g\,;\ \text{suup}(g)$ is compact and g satisfies the condition (H) of order $\beta\}$ and $P_\alpha(\mathcal{G})$ by $P_\alpha(\mathcal{G})=\{P_\alpha g\,;\ g\in\mathcal{G}\}$.

Then, by Lemma 7, both the set $\mathcal{G}$ and the set $P_\alpha(\mathcal{G})$ are dense in $L^2(R^1)$. Recall that $(D^\alpha L_t^\cdot)(a)$ belongs to $L^2(R^1)$. Then we can see from (2.18) that $a\longmapsto H^a(-1-\alpha, t)$ belongs to $L^2(R^1)$.

Apply the Lemma 7 to the relation (2.18). Then we obtain

$$(2.19)\qquad \frac{1}{\Gamma(-\alpha)}\int_{R^1}P_\alpha(H^\cdot(-1-\alpha, t))(a)g(a)da$$
$$=\frac{1}{\Gamma(-\alpha)}\int_{R^1}H^a(-1-\alpha, t)(P_\alpha^{-1}g)(a)da=\int_{R^1}(D^\alpha L_t^\cdot)(a)g(a)da\,.$$

Note again that the set $\mathcal{G}$ is a dense subset of $L^2(R^1)$. Then we can conclude from (2.19) that

$$(D^\alpha L_t^\cdot)(a)=P_\alpha(H^\cdot(-1-\alpha, t))(a)$$

holds. Q.E.D.

Corollary. *Let* $0<\alpha<\beta<1/2$. *Then the process* $a\mapsto H^a(-1-\alpha, t)$ *satisfies the condition* (H) *of order* $\beta-\alpha$, *P-almost surely.*

Proof. By the Theorem 2, we know that

$$(2.20)\qquad H^a(-1-\alpha, t)=P_\alpha^{-1}(D^\alpha L_t^\cdot)(a)$$
$$=-\cos(\pi(1+\alpha))(D^\alpha L_t^\cdot)(a)-\sin(\pi(1+\alpha))\mathcal{H}((D^\alpha L_t^\cdot)(\cdot))(a)$$

holds.

By Lemma 4, it is known that the fractional derivative of local times $a\mapsto(D^\alpha L_t^\cdot)(a)$ satisfies the condition (H) of order $\beta-\alpha$.

Note that the Hölder continuity property of order γ of functions remains invariably with the same γ under the Hilbert transform. (cf. [10]). Then the relations (2.20) imply immediately the desired fact. Q.E.D.

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