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Boundary value problems for second order equations of variable type in a half space

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§ 1. Introduction and statements of results.

This paper is concerned with boundary value problems for homogeneous second order equations of the following type in a half space:

$$(P)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\displaystyle\sum_{j=1}^{n-1}\dfrac{\partial^2 u}{\partial y_j^2}+q(x)\dfrac{\partial^2 u}{\partial t^2}=0\,, & (x,\ y,\ t)\in(0,\ \infty)\times R^{n-1}\times R\,,\\ \lim\limits_{x\to 0} u(x,\ y,\ t)=g(y,\ t) \ \text{ and } \ \lim\limits_{x\to\infty} u(x,\ y,\ t)=0\,, & (y,\ t)\in R^{n-1}\times R\,, \end{cases}$$

where $n=1, 2, 3, \cdots$. The coefficient $q(x)$ satisfies

(C_0) $q(x)$ is real valued bounded and piecewise continnuous in $(0, \infty)$, which we assume throughout this paper. Remark that $q(x)$ may change its sign. We assume one of the following conditions:

$$(C_+)\qquad \varliminf_{x\to\infty} q(x)>0\,,$$

$$(C_-)\qquad \varlimsup_{x\to\infty} q(x)<0\,.$$

As for equations of mixed type, local boundary value problems such as Tricomi problems and Frankl problems were investigated intensively (cf. [1], [4] and [8]). Here we treat with global problems stated as (P) and obtain the integral representation of solutions such as Poisson formula, (see Example 1). This paper continues from [6] and [7]. Being different from problems for elliptic or hyperbolic equations of definite type, the method of localized energy estimates is not effective for our problem (P). Then what we can rely upon? Relating to this question we can clarify our purpose and method below. We see that (P) has at least the linear property: Let u_j be a solution of (P) for $g=e_j$, then $\Sigma\tilde{g}_j u_j$ is a solution of (P) for $g=\Sigma\tilde{g}_j e_j$, where $\tilde{g}_j$ is constant. Here Σ can be replaced by the integral symbol $\int$ with respect to some parameters. For example suppose $g(t)=\int_\Gamma e(t,\ \tau)\tilde{g}(\tau)d\tau$ in the case $n=1$, where $e(t,\ \tau)$ is a non vanishing function depending continuously on τ. Then the solution is given by $u(x,\ t)=\int_\Gamma e(t,\ \tau)E(x,\ t,\ \tau)\tilde{g}(\tau)d\tau$, where $e(t,\ \tau)E(x,\ t,\ \tau)$ is a solution of (P) for $g=e(t,\ \tau)$. Here

$$(P)_1 \qquad \begin{cases} \dfrac{\partial^2 E}{\partial x^2}+q(x)\Big(\dfrac{\partial}{\partial t}-\Big(\dfrac{\partial}{\partial t}\log e(t,\tau)\Big)\Big)^2 E=0\,, & x\in(0,\infty)\,,\\ E(0,t,\tau)=1 \quad\text{and}\quad \lim_{x\to\infty}E(x,t,\tau)=0\,, & t\in R\,. \end{cases}$$

We can see that the solution E does not depend on t if and only if $\frac{\partial}{\partial t}\log e(t,\tau)$ is independent of t. Then the equation in $(P)_1$ becomes an ordinary differential equation. Let us put $e(t,\tau)=\exp\{\theta(\tau)t+\theta_1(\tau)\}$. We can make $\theta_1(\tau)\equiv 0$ by modifying $\tilde{g}(\tau)$ suitably. Then after a change of variable we can suppose $e(t,\tau)=e^{it\tau}$ if we modify $\tilde{g}(\tau)$ again. Namely in order to reduce the equation in (P) to an ordinary differential equation we need to consider essentially the decomposition of Fourier-Laplace:

$$g(t)=\frac{1}{2\pi}\int_\Gamma e^{it\tau}\hat{g}(\tau)d\tau\,,$$

where Γ is a curve in complex plane C. Naturally we take Γ suitably so that the Fourier-Laplace inversion formula holds for $g(t)$, namely $\hat{g}(\tau)$ is given by $\hat{g}(\tau)=\int_R e^{-it\tau}g(t)dt$. In general case of $n\geqq 2$ we suppose

$$g(y,t)=\Big(\frac{1}{2\pi}\Big)^n\int_\Gamma e^{it\tau}\int_{R^{n-1}}e^{iy\eta}\hat{g}(\eta,\tau)d\eta d\tau\,, \tag{1.1}$$

$$\hat{g}(\eta,\tau)=\int_R e^{-it\tau}\int_{R^{n-1}}e^{-iy\eta}g(y,t)dydt\,. \tag{1.1$'$}$$

The solution of (P) will be given by

$$u(x,y,t)=\Big(\frac{1}{2\pi}\Big)^n\int_\Gamma e^{it\tau}\int_{R^{n-1}}e^{iy\eta}E(x,\eta,\tau)\hat{g}(\eta,\tau)d\eta d\tau\,. \tag{1.2}$$

In fact $u(x,y,t)$ becomes a geuine solution of (P) by virtue of the Lebesque theorem, if following conditions 1), 2) and 3) are fulfilled:

1) $E(x,\eta,\tau)$ satisfies

$$(\tilde{P}) \qquad \begin{cases} \dfrac{d^2}{dx^2}E=(|\eta|^2+q(x)\tau^2)E\,, & x\in(0,\infty)\\ E(0,\eta,\tau)=1 \quad\text{and}\quad \lim_{x\to\infty}E(x,\eta,\tau)=0\,, \end{cases}$$

for almost everywhere $(\eta,\tau)\in R^{n-1}\times\Gamma$.

2) The following type of estimate holds for all $x\in(0,\infty)$

$$(E) \qquad |E(x,\eta,\tau)|\leqq C(|\eta|+|\tau|+1)^k, \qquad (\eta,\tau)\in R^{n-1}\times\Gamma,$$

for a certain real number k.

3) $E(x,\eta,\tau)\hat{g}(\eta,\tau)$ is absolutely integrable in $R^{n-1}\times\Gamma$. Later we state Theorem 1, 2 and 3 relating to above requests.

Now let us look at some simple examples in order that we become familiar to statements of Theorems 1 and 2.

Example 1. Let $n=1$ and $q(x)\equiv 1$. Then $E(x,\tau)=e^{-\sqrt{\tau^2}x}$, where $\sqrt{\tau^2}=\pm\tau$ if $\mathrm{Re}\,\tau\lessgtr 0$ respectively. Put $E(x,0)\equiv 1$ and take $\Gamma=(-\infty,\infty)$. Then (1.2) equals Poisson formula in a half space. In fact, for $x>0$, we have

$$u(x, t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{it\tau}e^{-|\tau|x}\hat{g}(\tau)d\tau=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{i\tau(t-s)}e^{-|\tau|x}d\tau\right)g(s)ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+(t-s)^2}g(s)ds\,.$$

Example 2. Let $n=1$ and $q(x)\equiv-1$. Then $E(x, \tau)=e^{\mp i\tau x}$ if $\mathrm{Im}\,\tau\lessgtr 0$ respectively. Put $E(x, 0)\equiv 1$ and take $\Gamma_\pm=\{\tau\,;\tau=\sigma\mp i\gamma,\ \sigma\in R\}$, where γ is an arbitrary positive number. Then the following solutions $u_\pm(x, t)$ correspond to the forward and the backward waves respectively.

$$u_\pm(x, t)=\frac{1}{2\pi}\int_{\Gamma_\pm}e^{i\tau t}e^{\mp i\tau x}\hat{g}(\tau)d\tau=g(t\mp x)\,.$$

Example 3. Let $n=1$ and $q(x)=1$ in $(1, \infty)$ and $q(x)=-1$ for $x\in[0, 1]$. Then $E(x, \tau)=\tilde{E}_\pm(x, \tau)/\tilde{E}_\pm(0, \tau)$ if $\mathrm{Re}\,\tau\lessgtr 0$ respectively, where

$$\tilde{E}_\pm(x, \tau)=\begin{cases} e^{\mp\tau(x-1)}, & 1\leqq x\,, \\ \dfrac{1}{\sqrt{2}}\{e^{i(\tau x-\tau\pm(\pi/4))}+e^{-i(\tau x-\tau\pm(\pi/4))}\}, & 0\leqq x<1\,. \end{cases}$$

Since $\tilde{E}_\pm(0, \tau)=0$ at $\tau=\pm\tau_k=\pm\left(\frac{3}{4}\pi+k\pi\right)$, $k=0, 1, 2, 3, \cdots$, for every $x\in(0, \infty)$, $E(x, \tau)$ has a simple pole at $\tau=\pm\tau_k$. For $\mathrm{Re}\,\tau=0$, $(\tilde{P})$ has no solution. However, if we put $E(x, 0)\equiv 1$, $E(x, \tau)$ is continuous at $\tau=0$. Now let us remark that $E(x, \tau)$ has the following properties:

(1) for each $x\in(0, \infty)$, $E(x, \tau)$ is analytic in $\tau\in\mathcal{D}=C-iR\cup\bigcup_{k=0}^{\infty}\{\pm\tau_k\}$,

(2) for each $\tau\in\mathcal{D}\cup\{0\}$, $E(x, \tau)$ has a finite number of zeros in $(0, \infty)$,

(3) $E(x, \tau)$ is regarded as a function of τ^2 because it holds $E(x, \tau)=E(x, -\tau)$.

Taking account of above examples we put

$$\begin{cases} p=|\eta|^2, \qquad \alpha=\tau^2 \\ v(x, p, \alpha)=E(x, \eta, \tau)\,, \end{cases} \tag{1.3}$$

and study the following auxilliary problem:

$$(\tilde{P}_0)\qquad \begin{cases} \dfrac{d^2}{dx^2}v=(p+q(x)\alpha)v\,, \qquad x\in(0, \infty)\,, \\ v(0, p, \alpha)=1\,,\quad \lim_{x\to\infty}v(x, p, \alpha)=0 \quad \text{and for each } (p, \alpha)\,, \\ v(x, p, \alpha) \quad \text{has a finite number of zeros in } (0, \infty). \end{cases}$$

Using the results on $(\tilde{P}_0)$ we state Theorem 1 in terms of $E(x, \eta, \tau)$.

Theorem 1. *Suppose* (C_+). *Then there exists uniquely the solution* $E(x, \eta, \tau)$ *of* $(\tilde{P})$ *for all* (η, τ) *belonging to* $R^{n-1}\times D_\tau(\eta)$, *where* $D_\tau(\eta)=D(\eta)$ *is described as*

$$D(\eta)=C-\bigcup_{k=1}^{\infty}\{\pm\mu_k(|\eta|)\}\cup\bigcup_{j=1}^{m(|\eta|)}\{\pm\nu_j(|\eta|)\}\cup\{\tau:\mathrm{Re}\,\tau=0,\ |\tau|\geqq\tilde{\tau}(|\eta|)\}\,.$$

Here $m(|\eta|)$ *is a nonnegative integer or* ∞, *and* $D(\eta)$ *and* $E(x, \eta, \tau)$ *have the following properties:*

(D-1) $-\tilde{\tau}(|\eta|)$ *and* $i\nu_j(|\eta|)$, $(j=1, 2, \cdots, m(|\eta|))$ *are real valued decreasing functions satisfying*

$$-\tilde{\tau}\leqq i\nu_{m(|\eta|)}<i\nu_{m(|\eta|)-1}<\cdots<i\nu_1<0\,,\quad |\eta|\neq 0\,,\quad \text{and}\quad \tilde{\tau}(0)=0\,.$$

(D-2) $\mu_k(|\eta|)$, $(k=1, 2, 3, \cdots)$ *are increasing functions satisfying* $0<\mu_n(|\eta|)<\mu_{n+1}(|\eta|)$, $(n=1, 2, 3, \cdots)$, $\lim_{|\eta|\to\infty}\mu_1(|\eta|)=\infty$, *and* $\lim_{n\to\infty}\mu_n(|\eta|)=\infty$ *for all* $\eta\in R^{n-1}$. *Especially we remark* $\mu_1(0)>0$.

(D-3) μ_k *and* ν_j *are real analytic in* $\eta\in R^{n-1}-\{0\}$.

(D-4) *For* $(\eta, \tau)\in R^n\times\{\tau: \mathrm{Re}\,\tau=0,\ |\tau|>\tilde{\tau}(|\eta|)\}$, *all the solutions of* $v''=(|\eta|^2+q(x)\tau^2)v$ *have infinite zeros in* $(0, \infty)$.

(E-1) $E(x, \eta, \tau)$ *is analytic in* (η, τ) *at any point* (η_0, τ_0) *belonging to* $R^n\times D(\eta_0)$. $E(x, \eta, \tau)$ *is continuous at* $(\eta, \tau)=(0, 0)$, *if* (η, τ) *is restricted to* $\left\{(\eta, \tau),\ \eta\in R^{n-1},\ \left|\arg\left(\tau\pm\frac{\pi}{2}\right)\right|>\varepsilon>0\right\}$ *for any small* $\varepsilon>0$, *where we define* $E(x, 0, 0)=1$. $E(x, \eta, \tau)=E(x, \eta', -\tau)$ *holds if* $|\eta|=|\eta'|$.

(E-2) *For any* $(x, \eta)\in(0, \infty)\times R^{n-1}$, $E(x, \eta, \tau)$ *has simple poles at* $\tau=\pm\mu_k(|\eta|)$ *and* $\tau=\pm\nu_j(|\eta|)$, $(k=1, 2, 3, \cdots,\ j=1, 2, \cdots, m(|\eta|))$.

(E-3) *For* $(\eta, \tau)\in R^{n-1}\times D(\eta)$, $E(x, \eta, \tau)$ *belongs to* $L^2(0, \infty)$.

Remark 1. Assume (C_-). Then we have the same results as in Theorem 1 replacing τ by $i\tau$.

Remark 2. From Examples 2 and 3 we see that the uniqueness does not hold in general for problem (P), (see also example 4 below). However as we saw in Example 2, meaningful solutions are given corresponding to the path Γ_+ and Γ_-. We can say that the singularities of E in τ make the non-uniqueness and give informations on suitable function spaces, which we try to choose in Theorem 3.

We take the path Γ satisfying

$$\Gamma\subset D(\eta)\cup\{0\}\,.$$

In view of the analyticity of $E(x, \eta, \tau)$ the integral (1.2) depends on the equivalence class of paths in $D(\eta)\cup\{0\}$. Now for convenience we fix a path $\Gamma=\Gamma_{(\gamma)}$, $(\gamma>0)$, which is independent of $\eta\in R^{n-1}$ as follows:

$$\Gamma=\sum_{j=1}^{4}\Gamma_j \tag{1.4}$$

$$\Gamma_1=\{\tau: \tau=t-i\gamma,\ -\infty<t\leqq-\gamma\}$$

$$\Gamma_2=\{\tau: \tau=s+is,\ -\gamma\leqq s\leqq 0\}$$

$$\Gamma_3=\{\tau: \tau=s-is,\ 0\leqq s\leqq\gamma\}$$

$$\Gamma_4=\{\tau: \tau=t-i\gamma,\ \gamma\leqq t<\infty\}\,.$$

We also consider the conjugate path $\bar{\Gamma}=\{\tau: \bar{\tau}\in\Gamma\}$, (cf. Example 2).

Theorem 2. *Assume (C_+) or (C_-). Then there exists a positive constant C such that the solution $E(x, \eta, \tau)$ of $(\tilde{P})$ satisfies the following (E_+) in the case (C_+) and (E_-) in the case (C_-).*

$$(E_+)\qquad |E(x, \eta, \tau)|\leqq C\frac{|\tau|(|\eta|+|\tau|+1)^{1/2}}{(\gamma|\mathrm{Re}\,\tau|)^{1/2}}\varphi(x, (|\eta|^2+\delta\,\mathrm{Re}\,\tau^2)^{1/2}, x_0),$$

for all $(\eta, \tau)\in R^{n-1}\times\Gamma$, where $q(x)\geqq\delta>0$ in (x_0, ∞) and

$$\varphi(x, r, x_0)=\begin{cases} 1, & x\in[0, x_0],\\ e^{-r(x-x_0)}, & x\in(x_0, \infty).\end{cases}$$

$$(E_-)\qquad |E(x, \eta, \tau)|\leqq C\frac{|\tau|(|\eta|+|\tau|+1)^{1/2}}{\gamma}, \qquad \textit{for all}\quad (\eta, \tau)\in R^{n-1}\times\Gamma.$$

Incidentally we have more general statements.

Theorem 2′. *Suppose (C_+). Then we have for all $x>0$,*

$$(*)\qquad |E(x, \eta, \tau)|\leqq\begin{cases} C\dfrac{|\tau|(|\eta|+|\tau|+1)^{1/2}}{(|\mathrm{Im}\,\tau||\mathrm{Re}\,\tau|)^{1/2}}\varphi(x, (|\eta|^2+\delta\,\mathrm{Re}\,\tau^2)^{1/2}, x_0), & \\ & \textit{if}\quad |\mathrm{Im}\,\tau|\leqq|\mathrm{Re}\,\tau|,\ \eta\in R^{n-1},\\ C\dfrac{|\tau|(|\eta|+|\tau|+1)^{1/2}}{|\mathrm{Re}\,\tau|} & \textit{if}\quad |\mathrm{Im}\,\tau|\geqq|\mathrm{Re}\,\tau|,\ \eta\in R^{n-1}.\end{cases}$$

Remark. From (*) follows (E_-) if we replace τ by $i\tau$. By virtue of Theorem 2 the following u_+ and u_- defined by

$$(1.5)\qquad \begin{cases} u_+(x, y, t)=\left(\dfrac{1}{2\pi}\right)^n\displaystyle\int_\Gamma e^{i\tau t}\int_{R^{n-1}}e^{i\eta y}E(x, \eta, \tau)\hat{g}(\eta, \tau)d\eta d\tau,\\ u_-(x, y, t)=\left(\dfrac{1}{2\pi}\right)^n\displaystyle\int_{\bar{\Gamma}} e^{i\tau t}\int_{R^{n-1}}e^{i\eta y}E(x, \eta, \tau)\hat{g}(\eta, \tau)d\eta d\tau,\end{cases}$$

have definite meanings and become solutions of (P) if $\hat{g}(\eta, \tau)$ has a suitable decreasing order. To state it more exactly we introduce some spaces of locally summable functions: $L^p_{k,\gamma}$, $\tilde{L}^p_{k,\gamma}$ and $\hat{B}_{k,\gamma}$ with following norms respectively: $\gamma\geqq 0$, $p=1, 2, \cdots$, $k=0, 1, 2, \cdots$,

$$\|g\|_{L^p_{k,\gamma}}=\left[\sum_{j=1}^k\left\{\int_{-\infty}^0\left\|e^{-\gamma t}\frac{\partial^j}{\partial t^j}g(\cdot, t)\right\|^p_{L^p_{k-j}}dt+\int_0^\infty\left\|\frac{\partial^j}{\partial t^j}g(\cdot, t)\right\|^p_{L^p_{k-j}}dt\right]^{1/p},$$

$$\|h\|_{\tilde{L}^p_{k,\gamma}}=\left[\sum_{j=1}^k\left\{\int_{-\infty}^0\left\|\frac{\partial^j}{\partial t^j}h(\cdot, t)\right\|^p_{L^p_{k-j}}dt+\int_0^\infty\left\|e^{-\gamma t}\frac{\partial^j}{\partial t^j}h(\cdot, t)\right\|^p_{L^p_{k-j}}dt\right]^{1/p},$$

where
$$\|f\|^p_{L^p_j}=\sum_{|\alpha|\leqq j}\int_{R^{n-1}}\left|\frac{\partial^\alpha}{\partial y^\alpha}f(y)\right|^p dy,$$

$$\|h\|_{\hat{B}_{k,\gamma}}=\sum_{\substack{j=1\\ j+|\alpha|\leqq k}}\left\{\sup_{\substack{t\in(-\infty,0)\\ y\in R^{n-1}}}\left|\frac{\partial^\alpha}{\partial y^\alpha}\cdot\frac{\partial^j}{\partial t^j}h(y, t)\right|+\sup_{\substack{t\in(0,\infty)\\ y\in R^{n-1}}}\left|e^{-\gamma t}\frac{\partial^\alpha}{\partial y^\alpha}\cdot\frac{\partial^j}{\partial t^j}h(y, t)\right|\right\}.$$

Remark that we have $L^p_{k,0}=\tilde{L}^p_{k,0}=L^p_k$ and for $0<\gamma_1<\gamma_2$

$$L^p_{k,\gamma_2}\subset L^p_{k,\gamma_1}\subset L^p_k\subset\tilde{L}^p_{k,\gamma_1}\subset\tilde{L}^p_{k,\gamma_2}.$$

Using above notations we state

Theorem 3. *$u_+(x, y, t)$ defined by* (1.5) *satisfies the following estimates: For $\gamma>0$ and $k=0, 1, 2, \cdots$, there exist positive constants $C(k, \gamma)$, $C_+(k, \gamma)$ and $C_-(k, \gamma)$ such that we have the following estimates* (1), (2) *and* (3). *(For $k>2$ we assume the smoothness of $q(x)$.)*

$$(1)\quad \sup_{x\in(0,\infty)} \|u_+(x, \cdot, \cdot)\|_{\tilde{B}_{k,\gamma}} \leqq C(k, \gamma)\|g\|_{L^1_{k+n+2,\gamma}}$$

in the cases (C_+) and (C_-),

$$(2)\quad \sup_{x\in(0,\infty)} \|u_+(x, \cdot, \cdot)\|_{\tilde{L}_{k,\gamma}} \leqq C_+(k, \gamma)\|g\|_{L^2_{k+1,\gamma}}$$

in the cases (C_+),

$$(3)\quad \sup_{x\in(0,\infty)} \|u_+(x, \cdot, \cdot)\|_{\tilde{L}^2_{k,\gamma}} \leqq C_-(k, \gamma)\|g\|_{L^2_{k+2,\gamma}}$$

in the cases (C_-).
We have the same inequalities replacing u_+ and g by $\check{u}_-$ and $\check{g}$ respectively, where $\check{u}_-(x, y, t)=u_-(x, y, -t)$ and $\check{g}(y, t)=g(y, -t)$. Namely for $k\geqq 2$ $u_\pm(x, y, t)$ is a genuine solution of (P) in the case (1) *and $u(x, y, t)$ is a strong solution in local L^2 sense in the cases* (2) *and* (3). *Incidentally we remark that in the case (C_+) $u_\pm(x, y, t)$ is real analytic in y and t for any $x\in(x_0, \infty)$ if $q(x)>\delta>0$ holds in (x_0, ∞) for some $\delta>0$.*

We use Plancherel's theorem and Holmgren's kernel estimates to obtain Theorem 3 from the estimates (E_+) and (E_-) in Theorem 2.

We state some results concerning the boundary value problems for equations of definite type, which we can obtain in the course of the proof of above Theorem. We suppose one of the following conditions:

$$(C_E)\qquad 0<\inf_{x\in(0,\infty)} q(x)\leqq \sup_{x\in(0,\infty)} q(x)<\infty,$$

$$(C_H)\qquad -\infty<\inf_{x\in(0,\infty)} q(x)\leqq \sup_{x\in(0,\infty)} q(x)<0.$$

Theorem E. *Assume (C_E). Then we have the following results.*
1) *The same results as in Theorem* 1 *hold with*

$$D(\eta)=C-\bigcup_{j=1}^{m(|\eta|)}\{\pm\nu_j(|\eta|)\}\cup\{\tau: \mathrm{Re}\,\tau=0,\ |\tau|\geqq\tilde{\tau}(|\eta|)\}.$$

Namely we have $\bigcup_{j=1}^{\infty}\{\pm\mu_j(|\eta|)\}=\emptyset$. $\bigcup_{j=1}^{m(|\eta|)}\{\pm\nu_j(|\eta|)\}=\emptyset$ if $q(x)$ satisfies $q(x)\leqq \varliminf_{x\to\infty} q(x)$ for $x\in(0, \infty)$.
2) *It holds*

$$|E(x, \eta, \tau)|\leqq C\exp\{-x(|\eta|^2+\delta\,\mathrm{Re}(\tau^2))^{1/2}\},\quad (x, \eta, \tau)\in(0, \infty)\times R^n\times\Gamma,$$

where C is independent of $\gamma\geqq 0$ and $\delta=\inf_{x\in(0,\infty)} q(x)$.
3) *If g and $\check{g}$ belong to $L^1_{k+n+1,\gamma}$ or $L^2_{k,\gamma}$ for a certain $\gamma\geqq 0$ and non negative integer k, then we have $u_+(x, y, t)=u_-(x, y, t)$ which we write $u(x, y, t)$ and the following estimates*

$(1)_E$ $$\sup_{x\in(0,\infty)}\|u(x,\cdot,\cdot)\|_{\tilde{B}_{k,0}}\leqq C_1(k)\|g\|_{L^1_{k+n+1}},$$

$(2)_E$ $$\|u(x,\cdot,\cdot)\|_{L^2_k}=C_2(x,k)\|g\|_{L^2_k},\qquad x\in(0,\infty),$$

where we have

$$C_2(x_2,k)<C_2(x_1,k)<1,\qquad \text{for}\quad 0<x_1<x_2.$$

Theorem H. *Assume* (C_H) *and the smoothness of* $q(x)$. *Then*

1) *The same results as in Theorem E hold if we replace* τ *by* $i\tau$.

2) *There exists positive constant C such that*

$$|E(x,\eta,\tau)|\leqq C\frac{|\tau|}{\gamma},\qquad \text{for}\quad (x,\eta,\tau)\in(0,\infty)\times R^{n-1}\times\tilde{\Gamma},$$

where $\tilde{\Gamma}=\Gamma\cup\{\tau:\operatorname{Im}\tau=-\gamma\}$.

3) *For* $\gamma>0$, $k=0,1,2,\cdots$, $u_+(x,y,t)$ *satifies the followings,*

$(1)_H$ $$\sum_{|\alpha|+j\leq k}\sup_{\substack{x\in(0,\infty)\\(y,t)\in R^{n-1}\times R}}\left|e^{-\gamma t}\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}u_+(x,y,t)\right|$$
$$\leqq\frac{C}{\gamma^{3/4}}\sum_{|\alpha|+j\leq k+n+2}\left\|e^{-\gamma t}\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}g\right\|_{L^1},$$

$(2)_H$ $$\sum_{|\alpha|+j\leq k}\sup_{x\in(0,\infty)}\left\|e^{-\gamma t}\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}u_+(x,\cdot,\cdot)\right\|_{L^2}$$
$$\leqq\frac{C}{\gamma}\sum_{\substack{|\alpha|+j\leq k+1\\|\alpha|\leq k}}\|e^{-\gamma t}g\|_{L^2},$$

$(3)_H$ $$\sum_{\substack{l+|\alpha|+j\leq k+1\\j\leq k}}\left\|e^{-\gamma t}\frac{\partial^l}{\partial x^l}\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}u_+(\cdot,\cdot,\cdot)\right\|_{L^2(R^+\times R^{n-1}\times R)}$$
$$\leqq\frac{C(k)}{\gamma}\sum_{\substack{|\alpha|+j\leq k+2\\|\alpha|\leq k}}\left\|e^{-\gamma t}\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}g\right\|_{L^2}.$$

In the above estimates we can replace u_+ *and* g *by* $\check{u}_-$ *and* $\check{g}$ *respectively.*

Finally we consider on the non-uniqueness of solutions.

Remark 2. In Theorem 1 $\{\mu_k(|\eta|)\}_{k=1}^{\infty}$ appears if and only if $q(x)$ changes its sign really, and $\{\nu_j(|\eta|)\}_{j=1}^{m(|\eta|)}$ appears if $\sup_{(0,\infty)}q(x)>\overline{\lim}_{x\to\infty}q(x)$. For each pole the residue calculus gives null solutions of (P), i.e. non zero solutions satisfying zero data. For example

$$u_k(x,y,t;\eta)=e^{i\eta y}e^{i\mu_k(|\eta|)t}\operatorname*{Res}_{\tau=\mu_k(|\eta|)}E(x,\eta,\tau)$$

is a non zero solution of (P) with $g\equiv0$. μ_k can be replaced by ν_j. Thus we see

$$\int_{R^{n-1}}e^{iy\eta}\Big\{\sum_{\substack{k=-\infty\\k\neq0}}^{\infty}C_k(\eta)e^{i\mu_k(|\eta|)t}\operatorname*{Res}_{\tau=\mu_k(|\eta|)}E(x,\eta,\tau)$$
$$+\sum_{\substack{j=-m(|\eta|)\\j\neq0}}^{m(|\eta|)}d_k(\eta)e^{i\nu_k(|\eta|)}\operatorname*{Res}_{\tau=\nu_j(|\eta|)}E(x,\eta,\tau)\Big\}d\eta$$

is a null solution of (P) for given functions $C_k(\eta)$ and $d_j(\eta)$, where $\mu_{-k}=-\mu_k$ and

$\nu_{-j}=-\nu_j$.

Example 4. Here let us point out a simple hyperbolic case. Put $q(x)=-1$ in $(1, \infty)$ and $q(x)=-2$ on $[0, 1]$. Then $\nu_j^2(|\eta|)$ satisfies $\frac{|\eta|^2}{2}<\nu_j^2(|\eta|)<|\eta|^2$ and $\tan(-2\alpha_j-p)=\sqrt{2\alpha_j-p}/\sqrt{p-\alpha_j}$, where $p=|\eta|^2$ and $\alpha_j=\nu_j^2(|\eta|)$. Then we have $\operatorname{Res}_{\tau=\nu_j(|\eta|)} E(x, \eta, \tau)=e_j(|\eta|)\tilde{E}(x, \eta, \nu_j(|\eta|))$, $e_j(|\eta|)\neq 0$, where $\tilde{E}(x, \eta, \tau)$ satisfies $\tilde{E}''=(|\eta|^2+q(x)\tau^2)\tilde{E}$, $\tilde{E}(1, \eta, \tau)=1$ and $\tilde{E}'(1, \eta, \tau)=-\sqrt{|\eta|^2-\tau^2}$ for $|\eta|^2>\tau^2$. For an elliptic case where $q(x)$ is replaced by $-q(x)$, we have also similar null solutions replacing $\nu_j(|\eta|)$ by $i\nu_j(|\eta|)$.

§ 2. Plan of the proof.

In this section we explain the outline of proofs of Theorems in several steps. Detailed proofs are given in later sections.

(I) **Construction of $E(x, \eta, \tau)$ and structure of $D(\eta)$.** In order to obtain Theorem 1 we consider the problem $(\tilde{P}_0)$. If $(\tilde{P}_0)$ has a unique solution for (p, α), then for sufficiently large number $x(p, \alpha)$ there exists a unique solution of the following problem.

$$(2.1)\qquad \begin{cases} \tilde{v}_{xx}=(p+q(x)\alpha)\tilde{v}\,, & x\in(x(p, \alpha), \infty)\,, \\ \tilde{v}(x(p, \alpha), p, \alpha)=1\,, & \lim_{x\to\infty}\tilde{v}(x, p, \alpha)=0\,, \\ \tilde{v}(x, p, \alpha)\neq 0\,, & x\in(x(p, \alpha), \infty)\,. \end{cases}$$

Conversely suppose that (2.1) has a unique solution for a certain $x(p, \alpha)$. Then extending $\tilde{v}(x, p, \alpha)$ as a solution of the linear equation $\tilde{v}_{xx}=(p+q(x)\alpha)\tilde{v}$ we obtain the unique solution $v(x, p, \alpha)$ of (P_0) if $\tilde{v}(0, p, \alpha)\neq 0$. In fact it suffices to put

$$v(x, p, \alpha)=\tilde{v}(x, p, \alpha)/\tilde{v}(0, p, \alpha)\,.$$

Suppose (C_+). Then we need to show the following facts (1) and (2).

(1) For $p\in[0, \infty)$, there exists a non-positive number $\tilde{\alpha}(p)$ satisfying the following conditions: The problem (2.1) has the unique solution for sufficiently large $x(p, \alpha)$ if α belongs to $C-(-\infty, \tilde{\alpha}(p)]$, and (2.1) has no solution for any $x(p, \alpha)$ if α belongs to $(-\infty, \tilde{\alpha}(p))$.

(2) For $p\in[0, \infty)$ the set $Z(p)=\{\alpha\in C-(-\infty, \tilde{\alpha}(p)]\,;\, \tilde{v}(0, p, \alpha)=0\}$ is a set of real discrete points $\{\alpha_k(p)\}_{k=1}^{\infty}\cup\{\alpha_{-j}(p)\}_{j=1}^{m(p)}$, where it holds $\tilde{\alpha}(p)\leqq\alpha_{-m(p)}(p)<\cdots \alpha_{-1}(p)<0<\alpha_1(p)<\alpha_2(p)<\cdots$. For every $(x, p)\in[0, \infty)\times[0, \infty)$, $v(x, p, \alpha)$ has simple poles at all $\alpha_{-j}(p)$ and $\alpha_k(p)$.

Put $\tilde{\alpha}(p)=-\tilde{\tau}(|\eta|)^2$, $\alpha_k(p)=\mu_k(|\eta|)^2$ and $\alpha_{-j}(p)=\nu_j(|\eta|)^2$, and define $D(\eta)$ as in Theorem 1. In later steps we see more precise properties of $D(\eta)$ and $E(x, \eta, \tau)=v(x, |\eta|^2, \tau^2)$.

(II) $\tilde{v}(x, p, \alpha)$ **for** $\alpha\in C-(-\infty, 0]$. Suppose

$$(2.2)\qquad q(x)>\delta>0 \quad \text{for} \quad x\in[x_0, \infty).$$

For $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, 0]\}$ we put $x(p, \alpha)=x_0$ in the problem (2.1). Then $\tilde{v}(x, p, \alpha)$ is the solution of (2.1) if and only if

$$w(x, p, \alpha)=\tilde{v}'(x, p, \alpha)/\tilde{v}(x, p, \alpha) \tag{2.3}$$

satisfies the following (2.4) and (2.5), where $w'(x, p, \alpha)=\frac{\partial}{\partial x}w(x, p, \alpha)$.

$$w'=(p+q(x)\alpha)-w^2\,, \tag{2.4}$$

$$\mathrm{Re}\int_{x_0}^{\infty}w(s, p, \alpha)ds=-\infty\,. \tag{2.5}$$

Using the solution of (2.4) and (2.5) we construct $\tilde{v}(x, p, \alpha)$ by

$$\tilde{v}(x, p, \alpha)=\exp\int_{x_0}^{x}w(s, p, \alpha)ds\,. \tag{2.6}$$

Now remark that (2.4) is equivalent to

$$\tilde{w}'=1-(p+q(x)\alpha)\tilde{w}^2, \qquad \tilde{w}=1/w\,, \tag{2.4$'$}$$

if $w\neq 0$. Since $\tilde{v}$ and $\tilde{v}'$ do not vanish simultaneously, (2.4) is regarded as an ordinary differential equation with values on Riemann sphere with two local coordinates w and $1/w$. Moreover identifying C and R^2 we can regard (2.4) as a system of differential equations with values on a real compact manifold S^2. Here in short we have *an heuristic argument.* To obtain (2.5), $\mathrm{Re}\,w<\delta<0$ for all $x\in(0, \infty)$ is a sufficient condition. For example, if q is constant we may take $w(s, p, \alpha)=-(p+q\alpha)^{1/2}$ with negative real part. Now for trial take a small circle C_ε with center at $-(p+q\alpha)^{1/2}$. We can see that the vector field $(p+q(x)\alpha)-w^2$ faces the exterion at $w\in C_\varepsilon$ i.e. $\mathrm{Re}\{(p+q\alpha)-w^2\}\nu>0$ on $w\in C_\varepsilon$ where ν stands for outer normal. Evidently this property holds even if q is replaced by a function $q(x)$ which is sufficiently close to the constant q. From this fact it is possible for us to imagine that there exists a solution $w(x)$ of $w'=(p+q(x)\alpha)-w^2$ staying in the interior of C_ε for all $x\in[0, \infty)$. This reasoning guides us to a simple existence theorem stated later from a topological viewpoint. Then this method becomes useful also in the case where the variation of $q(x)$ is not small, if we regard (2.4) as an equation in a compact manifold as we explained above. This fact will be seen below.

Now let us consider a little more on the case where α belongs to $C-(-\infty, 0]$. (2.6) means that the intervals $\{x\,;\mathrm{Re}\,w(x, p, \alpha)<0\}$ is more effective than $\{x\,;\mathrm{Re}\,w(x, p, \alpha)>0\}$ in the integral $\int_{x_0}^{x}w(s, p, \alpha)ds$. The following lemma assures later that $w(x, p, \alpha)$ has this property.

Lemma 2.1. *Suppose that a piecewise continuous function $q(x)$ satisfies $\mathrm{Im}\,(\tilde{q}(x)\bar{\alpha})<0$ for all $x\in[0, \infty)$, where α is a constant with positive imaginary part. Then $w'=\tilde{q}(x)-w^2$ has a solution $w(x)$ satisfying $\mathrm{Im}(w\bar{\alpha})>0$ for all $x\in[0, \infty)$.*

We apply Lemma 2.1 with $\tilde{q}(x)=p+q(x)\alpha$ replacing $(0, \infty)$ by (x_0, ∞). In order to obtain other detailed properties of $\tilde{v}$ we need more precise lemmas in

§3. However the proofs of these lemmas rely upon the same principle stated in Lemma A in § 3.

(III) Definition of $\tilde{\alpha}(p)$ and its properties. Let us fix an arbitrary sequence $\{x_n\}_{n=1}^{\infty}$ satisfying $x_0<x_1<x_2<\cdots<x_n<\cdots$, $\lim x_n=\infty$, where x_0 satisfies (2.2). For $p\geqq 0$ and natural number n we define the set $A_n(p)$ of the real numbers α satisfying the following properties: There exists a solution $v(x)$ of $v''=(p+q(x)\alpha)v$ in (x_n, ∞) satisfying $v(x_n)=1$, $0<v(x)$ in (x_n, ∞) and $\lim_{x\to\infty} v(x)=0$. Moreover $v(x)$ is a unique bounded solution in (x_n, ∞) satisfying $v(x_n)=1$. Put $B_n(p)=R-A_n(p)$. Then $A_n(p)$ and $B_n(p)$ are upper and lower sets called by Dedekind respectively. This follows from the next lemma.

Lemma 2.2. *Assume $\tilde{q}(x)>q_1(x)$ for all $x\in[0, \infty)$. Let $v(x)$ be the unique bounded solution of $v''=q_1(x)v$ in $(0, \infty)$ and $v(0)=1$. Suppose $0<v(x)$ in $(0, \infty)$. Then $u''(x)=\tilde{q}(x)u$ has the unique bounded solution $u(x)$ in $(0, \infty)$ satisfying $u(0)=1$. $u(x)$ satisfies also $u'(0)<v'(0)$ and $0<u(x)<v(x)$ in $(0, \infty)$.*

Suppose (C_+). Apply Lemma 2.2 with $\tilde{q}(x)=p+q(x)\alpha$, $q_1(x)\equiv 0$ and $v\equiv 1$ in $(0, \infty)$. Then we have $A_n(p)\supset(-p/\sup_{(x_n,\infty)} q(x), \infty)$. At the same time it follows $A_n(p)\subset(-p/\inf_{(x_n,\infty)} q(x), \infty)$ if we use Lemma 2.2 by the method of contradiction. $A_n(p)\subset A_{n+1}(p)$ by definition. Note $\tilde{\alpha}_n(p)=\inf A_n(p)$ then it holds

$$\frac{-p}{\inf_{(x_{n+1},\infty)} q(x)}\leqq\tilde{\alpha}_{n+1}(p)\leqq\tilde{\alpha}_n(p)\leqq\frac{-p}{\sup_{(x_n,\infty)} q(x)}, \qquad 0\leqq p. \tag{2.7}$$

Let $0\leqq p_1<p$ and $\varepsilon>0$. Then $p_1+q(x)(\tilde{\alpha}_n(p_1)-\varepsilon)>p+q(x)(\tilde{\alpha}_n(p)+\varepsilon)$ and $p_1+q(x)(\tilde{\alpha}_n(p_1)+\varepsilon)<p+q(x)(\tilde{\alpha}_n(p)-\varepsilon)$ do not hold for all $x\in(x_n, \infty)$ in view of Lemma 2.2. Therefore it holds

$$\frac{p_1-p}{\inf_{(x_n,\infty)} q(x)}\leqq\tilde{\alpha}_n(p)-\tilde{\alpha}_n(p_1)\leqq\frac{p_1-p}{\sup_{(x_n,\infty)} q(x)}, \qquad 0\leqq p_1<p. \tag{2.8}$$

Define $\tilde{\alpha}(p)=\lim_{n\to\infty}\tilde{\alpha}_n(p)$. From (2.7) $\tilde{\alpha}(p)$ is independent of the sequence $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} x_n=\infty$. Then we have

$$\begin{cases} -p/\underline{\lim}\, q(x)\leqq\tilde{\alpha}(p)\leqq-p/\overline{\lim}\, q(x), & 0\leqq p, \\ (p_1-p)/\underline{\lim}\, q(x)\leqq\tilde{\alpha}(p)-\tilde{\alpha}(p_1)\leqq(p_1-p)/\overline{\lim}\, q(x), & 0\leqq p_1<p. \end{cases} \tag{2.9}$$

(IV) $\tilde{v}(x, p, \alpha)$ for $\alpha\in C-(-\infty, \tilde{\alpha}(p)]$. From the above step it follows

$$C-(-\infty, \tilde{\alpha}(p)]=\bigcup_{n=1}^{\infty}\{C-(-\infty, \tilde{\alpha}_n(p)]\}.$$

In the problem (2.1) we put

$$x(p, \alpha)=x_n \qquad \text{for} \quad (p, \alpha)\in[0, \infty)\times\{C-(-\infty, \tilde{\alpha}_n(p)]\}.$$

Using Lemmas 3.4 and 3.5 which are extensions of Lemmas 2.1 and 2.2 we can construct the unique solution $\tilde{v}_n(x, p, \alpha)$ of (2.1). Extend $\tilde{v}_n(x, p, \alpha)$ as a solution of $\tilde{v}_n''=(p+q(x)\alpha)\tilde{v}_n$ in $(0, \infty)$. Then the uniqueness implies

$$(2.10)_1 \qquad \frac{\tilde{v}_n(x, p, \alpha)}{\tilde{v}_n(0, p, \alpha)}=\frac{\tilde{v}(x, p, \alpha)}{\tilde{v}(0, p, \alpha)} \quad \text{for} \quad [p, \alpha)\in[0, \infty)\times\{C-(-\infty, 0]\}$$

and

$$(2.10)_2 \qquad \frac{\tilde{v}_n(x, p, \alpha)}{\tilde{v}_n(0, p, \alpha)}=\frac{\tilde{v}_m(x, p, \alpha)}{\tilde{v}_m(0, p, \alpha)} \quad \text{for} \quad (p, \alpha)\in[0, \infty)\times\{C-(-\infty, \tilde{\alpha}_n(p)]\},$$

where $\tilde{v}_n(0, p, \alpha)\neq 0$ is supposed and m is large than n.

Therefore we can extend $\dfrac{\tilde{v}(x, p, \alpha)}{\tilde{v}(0, p, \alpha)}$ for $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, \tilde{\alpha}(p)]\}$. Then we define $v(x, p, \alpha)$ by

$$(2.11) \qquad v(x, p, \alpha)=\lim_{n\to\infty}\frac{\tilde{v}_n(x, p, \alpha)}{\tilde{v}_n(0, p, \alpha)}$$

for $(x, p, \alpha)\in[0, \infty)\times[0, \infty)\times\{C-(-\infty, \tilde{\alpha}(p)]\}$ if $\tilde{v}_n(0, p, \alpha)\neq 0$ for some n.

(V) **The analyticity of** $v(x, p, \alpha)$ **in** (p, α)**.** At first we consider the continuity of $v(x, p, \alpha)$ in (p, α). To verify it, we use the continuous dependence of $w(x)$ on the initial data $w(0)$ and Heine-Borel theorem on a compact set of S^2, (see §4). To prove the analyticity in $\alpha\in C-(-\infty, 0]$, we describe $(w(x, p, \alpha+h)-w(x, p, \alpha))/h$ making use of L^2 integrals of $v(x, p, \alpha)$ and $v(x, p, \alpha+h)$ in x. This L^2-integrability for $\alpha\in\{\alpha : \operatorname{Im}\alpha\neq 0\}$ is assured by

$$(2.11) \qquad w(x)|v(x)|^2-w(x_1)|v(x_1)|^2=\int_{x_1}^{x}|v'(s)|^2ds+\int_{x_1}^{x}(p+q(s)\alpha)|v(s)|^2ds\,,$$

which follows from the integration by parts of $\int v''\bar{v}dx=\int(p+q(x)\alpha)|v|^2dx$. Tend the complex number h to zero then we obtain the explicit form of $\dfrac{\partial w}{\partial\alpha}$ for $\operatorname{Im}\alpha\neq 0$. Thus $w(x, p, \alpha)$ and $v(x, p, \alpha)$ are verified to be analytic in $\{\alpha : \operatorname{Im}\alpha\neq 0\}$. From the continuity of $v(x, p, \alpha)$ in α belonging to $\mathscr{D}(p)$, $v(x, p, \alpha)$ is analytic with respect to α in $\mathscr{D}(p)$ by virtue of Painlevé theorem. By (2.11) and Fatou theorem we obtain the explicite form of $\dfrac{\partial w}{\partial\alpha}$ for $\alpha\in\{\alpha : \operatorname{Im}\alpha=0\}\cap\mathscr{D}(p)$, which show the L^2-integrability in x for all $(p, \alpha)\in[0, \infty)\times\mathscr{D}(p)$. From this fact we can prove the analyticity of $v(x, p, \alpha)$ in $p\in(0, \infty)$.

The analyticity of $\alpha_k(p)$ and $\alpha_{-j}(p)$ follows from the implicit function theorem applied to $\tilde{v}(0, p, \alpha_k(p))=0$ and $\tilde{v}(0, p, \alpha_{-j}(p))=0$. Moreover the explicit formula of $\dfrac{\partial\alpha_k}{\partial p}=\dfrac{\partial\tilde{v}}{\partial p}(0, p, \alpha_k(p))\Big/\dfrac{\partial\tilde{v}}{\partial\alpha}(0, p, \alpha_k(p))$ gives not only the analyticity but also some informations on the monotone properties stated in Theorem 1.

(VI) **The method for the estimates in Theorem 2.** The main idea is to introduce the oblique coordinates depending on α as follows.

$$(2.12)_1 \qquad \alpha_1=\begin{cases}-\alpha & \text{in the case } (C_+),\\ \alpha & \text{in the case } (C_-),\end{cases}$$

$(2.12)_2$ $$\beta^2=-\alpha_1 \quad \text{and} \quad (\operatorname{Im}\alpha_1)(\operatorname{Im}\beta)>0\,.$$

Remark $\operatorname{Re}\beta>0$. For any complex number w we can denote uniquely

(2.13) $$w=w_{\alpha_1}\alpha_1+w_\beta\beta\,,$$

where w_{α_1} and w_β are real numbers:

(2.13)′ $$\begin{cases} w_{\alpha_1}=\operatorname{Im}(w\bar\beta)/\operatorname{Im}(\alpha_1\bar\beta)\,, \\ w_\beta=\operatorname{Im}(w\bar\alpha_1)/\operatorname{Im}(\beta\bar\alpha_1)\,. \end{cases}$$

We say simply that w_β is β component of w. Take the β component of (2.11), then

(2.14) $$w_\beta(x,\,p,\,\alpha)|v(x,\,p,\,\alpha)|^2=w_\beta(x_1,\,p,\,\alpha)|v(x_1,\,p,\,\alpha)|^2 +1_\beta\left\{\int_{x_1}^x|v'(s,\,p,\,\alpha)|^2ds+p\int_{x_1}^x|v(s,\,p,\,\alpha)|^2ds\right\},$$

where $1_\beta=-\dfrac{1}{|\alpha|}\dfrac{\operatorname{Im}\alpha_1}{\operatorname{Im}\beta}<0$. Put $x_1=0$, then

(2.15) $$|v(x,\,p,\,\alpha)|\leqq(w_\beta(0,\,p,\,\alpha)/w_\beta(x,\,p,\,\alpha))^{1/2}.$$

Therefore for the estimate of $v(x,\,p,\,\alpha)$ it suffices to know the minimum and the maximum of w_β in $(0,\,\infty)$. For $x\in[x_0,\,\infty)$ the behavior of w_β is evaluated from Lemmas in §3. In $(0,\,x_0)$ we use the equation

(2.16) $$w'_\beta(x)=\{\operatorname{Im}(p\bar\alpha_1)-\operatorname{Im}((w_{\alpha_1}\alpha_1+w_\beta\beta)^2\bar\alpha_1)\}/\operatorname{Im}(\beta\bar\alpha_1)$$

and estimate w_β by a comparison method in Section 5.

(VII) Existence theorem for (P). Using Theorem 2 the integral form (1.2) has an exact meaning in some function spaces. In order to obtain L^2 estimates in Theorem 3 it is convenient to modify the path Γ in a neighbourhood of $\tau=0$ in view of the analyticity of $v(x,\,p,\,\alpha)$. Remark that the modified paths are taken differently in the cases (C_+) and (C_-). Taking the partition of unity of Γ we use Plancherel's identity or Holmgren's kernel estimates in each parts.

§3. Some lemmas.

Here let us state some lemmas concerning $w'=\tilde q(x)-w^2$ and $u''=\tilde q(x)u$, where $\tilde q(x)$ is complex valued and piecewise continuous. The proofs of these lemmas result in the principle stated later in Lemma A.

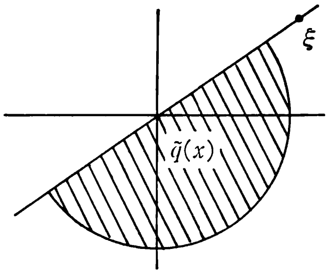
Lemma 3.1. *Suppose* $0<m<\operatorname{Re}\tilde q(x)$ *and* $|\tilde q(x)|<M$ *for all* $x\in[0,\,\infty)$. *Then there exists a solution* $w(x)$ *of* $w'=\tilde q(x)-w^2$ *satisfying* $\operatorname{Re}w(x)<-m^{1/2}$ *and* $|w(x)|<2M^{1/2}$ *for all* $x\in[0,\,\infty)$. *Moreover any other solution of* $w'=\tilde q(x)-w^2$ *satisfies* $\varliminf_{x\to\infty}\operatorname{Re}w(x)>0$.

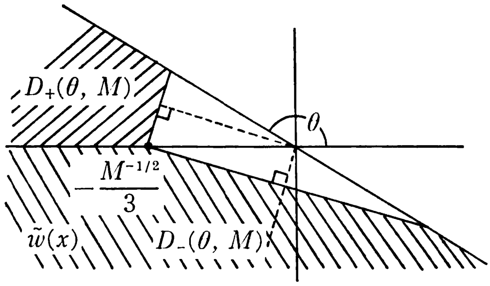
Lemma 3.2. *Suppose* $0<\arg\xi<\pi$. *Assume* $\operatorname{Im}(\tilde q(x)\bar\xi)\leqq0$, *and* $|\tilde q(x)|<M$ *for*

all $x\in[0, \infty)$, $\tilde{q}(x)\not\equiv 0$. *Then* $w'=\tilde{q}(x)-w^2$ *has a solution satisfying* $\mathrm{Im}(w(x)\bar{\xi})>0$ *and* $\tilde{w}(x)=1/w(x)\in D(\theta, M)$ *for all* $x\in[0, \infty)$, *where* $\theta=\pi-\arg\xi$ *and* $D(\theta, M)$ $=D_+(\theta, M)\cup D_-(\theta, M)$. *Here*

$$D_+(\theta, M)=\left\{\tilde{w} : \theta<\arg\tilde{w}\leqq\pi,\ \mathrm{Im}(\tilde{w}\ e^{-\theta i/2})>\frac{M^{-1/2}}{3}\sin\frac{\theta}{2}\right\},$$

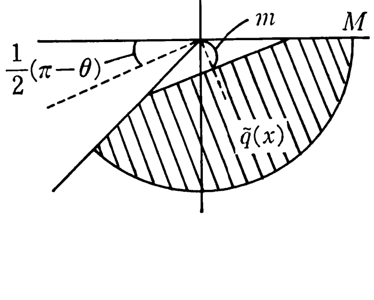
$$D_-(\theta, M)=\left\{\tilde{w} : \pi\leqq\arg\tilde{w}<\pi+\theta,\ \mathrm{Im}(\tilde{w}\ e^{(\pi-\theta)i/2})<\frac{M^{-1/2}}{3}\cos\frac{\theta}{2}\right\}.$$

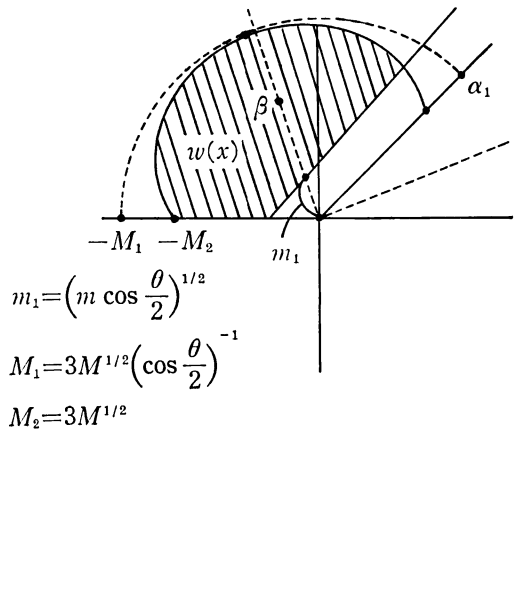




Lemma 3.3. *Suppose that* $\tilde{q}(x)$ *satisfies the same conditions as in Lemma* 3.2. *Let* x_1 *be an arbitrary number in* $(0, \infty)$. *Assume that* $1/w_0$ *belongs to* $D(\theta, M)$ *defined in Lemma* 3.2. *Then the solution* $w(x)$ *of* $w'=q(x)-w^2$ *and* $w(x_1)=w_0$ *stays in* $D(\theta, M)$ *for all* $x\in[0, x_1]$.

Lemma 3.4. *Let* $\tilde{q}(x)$ *satisfy* $-\pi<-\theta\leqq\arg\tilde{q}(x)<0$, $|\tilde{q}(x)|<M$ *and* $\mathrm{Im}\,\tilde{q}(x)e^{-i(\pi-\theta)/2}<-m$ *for all* $x\in[0, \infty)$. *Then there exists a solution* $w(x)$ *of* $w'=\tilde{q}(x)-w^2$ *satisfying* $\pi-\theta<\arg w(x)<\pi$, $1/w(x)\in D_-(\theta, M)$ *and* $w_\beta(x)=$ $\mathrm{Im}(w(x)\bar{\alpha}_1)/\mathrm{Im}(\beta\bar{\alpha}_1)\geqq\dfrac{1}{|\beta|}\left(m\cos\dfrac{\theta}{2}\right)^{1/2}$ *for all* $x\in[0, \infty)$, *where* α_1 *and* β *are complex numbers satisfying* $\arg\alpha_1=\pi-\theta$ *and* $(2.12)_2$. *Any other solution of* $w'=q(x)-w^2$ *satisfies* $\varlimsup_{x\to\infty} w_\beta(x)<0$.





Remark 3.1. $\overline{w}'=\overline{\tilde{q}(x)}-\overline{w}^2$ holds if $w'=\tilde{q}(x)-w^2$. Hence if $\overline{\tilde{q}(x)}$ satisfies the desired conditions in above Lemmas 3.2, 3.3 and 3.4 instead of $\tilde{q}(x)$, we have the same results replacing $w(x)$ by $\overline{w(x)}$.

Lemma 3.5. *Let $q_1(x)$ be a real valued function on $[0, \infty)$. Suppose that a real valued function $w_1(x)$ satisfies $w_1'=q_1(x)-w_1^2$ on $[0, \infty)$. Assume that another function $\tilde{q}(x)$ satisfies* $\mathrm{Re}\,\tilde{q}(x)\geqq q_1(x)$ *for all $x\in[0, \infty)$. $\tilde{q}(x)\not\equiv q_1(x)$. Then there exists a solution $w(x)$ of $w'=\tilde{q}(x)-w^2$ satisfying* $\mathrm{Re}\,w(x)<w_1(x)$ *for all $x\in[0, \infty)$.*

Trying to prove above Lemmas we can arrive the following general statements of Lemma A, which gives a common insight into these lemmas. We need some definitions to clarify the terminology. Let M be a real n-dimensional manifold.

Definition 1. Let Ω be an open precompact set in M. Ω is said to be contractible if there exists a continuous function $F(t, u): [0, 1]\times\Omega\to\Omega$ such that $F(0, U)=U$ and $F(1, U)=U_0$ for all $U\in\Omega$.

Definition 2. Let $\{\Omega(x)\}_{0\leq x<\infty}$ be a family of contractible set in M. We say that $\{\Omega(x)\}_{0\leq x<\infty}$ is a homeomorph family of contractible sets if there exists a smooth mapping $H(x, U): [0, \infty)\times M\to M$, such that for each $x\in[0, \infty)$ $H(x, U)$ is a one to one onto mapping from $\overline{\Omega(0)}$ to $\overline{\Omega(x)}$ and $H(0, U)=U$ in M.

Definition 3. A family $\{\Omega(x)\}_{0\leq x<\infty}$ of open sets in M is said to have a piecewise smooth boundary if there exist real valued functions $\varphi_j(x, U)$, $(j=1, 2, \cdots, k)$ defined on $[0, \infty)\times M$ such that

$$\Omega(x)=\{U:\varphi_j(x, U)<0\,, \quad j=1, 2, \cdots, k\}$$

$$\partial\Omega(x)=\{U:\varphi_j(x, U)\leqq 0\,, \quad j=1, 2, \cdots, k\}-\Omega(x)$$

for all $x\in[0, \infty)$, satisfying

$$\left(\frac{\partial\varphi_j}{\partial U_1}, \cdots, \frac{\partial\varphi_j}{\partial U_n}\right)\neq 0 \qquad \text{on } \partial\Omega(x), \quad \text{if} \quad \varphi_j(x, U)=0.$$

Definition 4. Let $\{\Omega(x)\}_{0\leq x<\infty}$ be a homeomorph family of contractible sets in M with a piecewise smooth boundary. Suppose that $Q(x, U)=(Q_1(x, U), Q_2(x, U), \cdots, Q_n(x, U))$ is a smooth vector field on M depending smoothly on $x\in[0, \infty)$: If $Q(x, U)$ and $\{\Omega(x)\}_{0\leq x<\infty}$ satisfy

$$D\varphi_l(x, U(x))\equiv\frac{\partial\varphi_l(x, U)}{\partial x}+\sum_{i=1}^{n}\frac{\partial\varphi_l(x, U)}{\partial U_i}Q_i(x, U)\geqq 0\,, \tag{3.1}$$

for all $(x, U, l)\in[0, \infty)\times\partial\Omega(x)\times\{l:\varphi_l(x, U)=0\}$, then we say that $Q(x, U)$ directs $\partial\Omega(x)$ to the exterior.

Lemma A. *Suppose that $Q(x, U)$ and $\{\Omega(x)\}_{0\leq x<\infty}$ satisfy the conditions stated in Definition 4. Then there exists a solution $U(x)$ of $U'(x)=Q(x, U)$ satisfying $U(x)\in\overline{\Omega(x)}$ for all $x\in[0, \infty)$. Moreover we have the following alternative:*

$U(x)\in\Omega(x)$ for all $x\in[0, \infty)$ or $U(x)\in\partial\Omega(x)$ for all $x\in[x_1, \infty)$ for a certain $x_1\in(0, \infty)$.

Remark. In our problems the latter case of the above alternative does not occur when we treat $w'=\tilde{q}(x)-w^2$ on Riemann sphere S^2.

First we prove Lemma 2.1 then proceed to verify Lemma A.

Proof of Lemma 2.1. We regard the solution $w(x)$ of $w'=\tilde{q}(x)-w^2$ as a function with values on Riemann sphere with two local coordinates: $\{w : w\in C\}$ and $\{\tilde{w} : \tilde{w}=1/w\in C\}$, since $\tilde{w}(x)$ satisfies $\tilde{w}'=1-\tilde{q}(x)\tilde{w}^2$. Put $\Omega=\{w : \mathrm{Im}(w\bar{\alpha})>0\}$ $=\{\tilde{w} : \mathrm{Im}(\tilde{w}\alpha)<0\}$. Then $\bar{\Omega}$ is compact and contractible. Remark that $\tilde{q}(x)-w^2$ and $1-\tilde{q}(x)\tilde{w}^2$ direct $\partial\Omega$ to the exterior. Suppose that the solution $w(x)$ of $w'=\tilde{q}(x)-w^2$ satisfies $w(x_1)\in\partial\Omega$ for a certain $x_1\in[0, \infty)$, then $w(x)$ belongs to $C\bar{\Omega}$ for $x\in(x_1, \infty)$. We prove Lemma 2.1 by the method of contradiction as follows. If for every $w_0\in\Omega$ there exists a positive number $x_0=x_0(w_0)$ such that the solution $w(x)$ of $w'=\tilde{q}(x)-w^2$ and $w(0)=w_0$ satisfies $w(x)\in\Omega$ for $x\in(0, x_0)$ and $w(x_0)\in\partial\Omega$. Put $x_0=0$ for $w_0\in\partial\Omega$. Denote by f the mapping from $\bar{\Omega}$ to $\partial\Omega : w_0\rightarrow w(x_0)$. Then f is continuous and $f|_{\partial\Omega}=$identity. This is a contradiction since Ω is contractible.

Proof of Lemma A. At first we restrict ourselves to the case where strict inequality holds in (3.1). If we suppose that for every $U_0\in\Omega(0)$ the solution $U(x)$ of $U'(x)=Q(x, U)$ and $U(0)=U_0$ satisfies $U(x_0)\in\partial\Omega(x_0)$, then we have a contradiction as follows. Denote $x_0=x_0(U_0)$ since x_0 is uniquely determined by U_0. Put $x_0=0$ for $U_0\in\partial\Omega(0)$. Note by H_x^{-1} the homeomorph mapping from $\Omega(x)$ to $\Omega(0)$. Consider the mapping $f : U_0\rightarrow H_{x_0}^{-1}U(x_0(U_0))$. Then f is continuous from $\overline{\Omega(0)}$ to $\partial\Omega(0)$ and $f|_{\partial\Omega(0)}=$identity, which contradicts to the fact that $\Omega(0)$ is contractible. In the general case we use a smooth vector field $\tilde{Q}(x, U)$ satisfying $\sum_{i=1}^{n}\frac{\partial\varphi_l}{\partial U_i}\tilde{Q}_i(x, U)>0$ for all $(x, U, l)\in[0, \infty)\times\partial\Omega(x)\times\{l : \varphi_l(x, U)=0\}$. $U'=Q(x, U)+\varepsilon\tilde{Q}(x, U)$ has a solution $U_\varepsilon(x)$ satisfying $U_\varepsilon(x)\in\Omega(x)$ for all $x\in[0, \infty)$ for $\varepsilon>0$. Since $\bar{\Omega}$ is compact there exists a sequence ε_j tending to zero such that $U_{\varepsilon_j}(0)$ has a limit $U_0\in\bar{\Omega}(0)$. The solution $U(x)$ of $U'=Q(x, U)$ with $U(0)=U_0$ satisfies $U(x)\in\overline{\Omega(x)}$ for all $x\in[0, \infty)$, because $U(x)=\lim_{j=\infty}U_{\varepsilon_j}(x)$ holds from the continuity of solutions and the uniqueness. In the same way we have the alternative in Lemma A.

Though Lemma 2.2 is found in [5] we give another proof applying Lemma A.

As for topological treatments for ordinary differential equations, we can point out historically [3] and [9]. We can consult [2]. Here we are lead to more general statements of Lemma A through the study on the problem $(\tilde{P})$ in Section 2 and the following sections.

Proof of Lemma 2.2. Put $w_1(x)=v'(x)/v(x)$, then $w_1'=q_1(x)-w_1^2$. Note M

$=\{w : w\in R\}\cup\{\tilde{w} : \tilde{w}=1/w\in R\}$ and $\Omega(x)=\{w\ ;\ -\infty<w<w_1(x)\}\cup\{\tilde{w} : \tilde{w}=1/w,\ -\infty<w<w_1(x),\ w\neq 0\}$. Since $w'=\tilde{q}(x)-w^2$ and $\tilde{w}'=1-\tilde{q}(x)\tilde{w}^2$ hold, they are regarded as an ordinary differential equation with values on M, which satisfies all the conditions in Lemma A. Thus $w'=\tilde{q}(x)-w^2$ has a solution $w(x)$ satisfying $-\infty<w(x)<w_1(x)$ for all $x\in[0, \infty)$. Then $u(x)=\exp\int_0^x w(s)ds$ satisfies $0<u(x)<v(x)$ in $(0, \infty)$ and $u'(0)<v'(0)$. Suppose $w_2'=\tilde{q}(w)-w_2^2$, $w_2(0)=w_1(0)+\varepsilon$. $w_3'=q_1(x)-w_3^2$ and $w_3(0)=w_1(0)+\varepsilon$ for any $\varepsilon>0$. Then $w_2(x)>w_3(x)$ for all $x\in[0, \infty)$. Therefore the desired uniqueness holds since $u_2(x)=\exp\int_0^x w_2(s)ds> v_3(x)=\exp\int_0^x w_3(s)ds$, which tends to ∞ as x tends to ∞.

Proof of Lemma 3.1. Let $w=w_1+iw_2$, where w_1 and w_2 are real. Put $M=\{(w_1, w_2)\ ;\ w=w_1+iw_2\in C\}\cup\{(\tilde{w}_1, \tilde{w}_2)\ ;\ \tilde{w}_1+i\tilde{w}_2=1/w\in C\}$. Let Ω be the interior domain surrounded by S_1 and S_2, where $S_1=\{w\ ;\ w=-m^{1/2}+is,\ s\in R\}$ and $S_2=\left\{\tilde{w}\ ;\ \tilde{w}=-\frac{t}{2M}\pm\frac{1-t}{2M}i,\ 0\leqq t\leqq 1\right\}$. Then $\Omega(x)=\Omega$ and $w'=\tilde{q}(x)-w^2$ satisfy all the conditions in Lemma A. Therefore $w'=\tilde{q}(x)-w^2$ has a solution $w(x)$ staying in Ω for all $x\in[0, \infty)$. Then it holds $\left|\exp\int_0^x w(s)ds\right|\leqq e^{-m^{1/2}x}$. It is evident that if $w_1(0)\in-\Omega=\{(w_1, w_2)\ ;\ (-w_1, -w_2)\in\Omega\}$. $u_1(x)=\exp\int_0^x w_1(s)ds$ satisfies $|u_1(x)|\geqq e^{m^{1/2}x}$. Any other solution of $w'=\tilde{q}(x)-w^2$ is described as $w_\xi(x)=\{u'(x)+\xi u_1'(x)\}/(u(x)+\xi u_1(x))$ for a certain $\xi\neq 0$, which approximates to $w_2(x)$ as x tends to ∞. Thus we have Lemma 3.1.

Proof of Lemma 3.2. Let $\Omega(x)=\Omega$ be the domain surrounded by

$$S_1=\{w\ ;\ \mathrm{Im}(w\bar{\xi})=0\},$$

$$S_2=\left\{\tilde{w}\ ;\ \tilde{w}=-\frac{1}{3}M^{-1/2}+te^{i\theta/2},\ 0\leqq t\leqq\frac{2}{3}M^{-1/2}\sin\frac{\theta}{2}\right\},$$

$$S_3=\left\{\tilde{w}\ ;\ \tilde{w}=-\frac{1}{3}M^{-1/2}t+\frac{1}{3}M^{-1/2}e^{-i(\pi-\theta)}(1-t),\quad 0\leqq t\leqq 1\right\}.$$

In order to apply Lemma A to $w'=\tilde{q}(x)-w^2$ and Ω we verify the following inequalities (1), (2) and (3).

(1) $\mathrm{Im}\{(\tilde{q}(x)-w^2)\bar{\xi}\}=\mathrm{Im}(\tilde{q}(x)\bar{\xi})-|w|^2\mathrm{Im}\,\xi\leqq 0$ for $w\in S_1$,

(2) $\mathrm{Im}\{(1-\tilde{q}(x)\tilde{w}^2)e^{-i\theta/2}\}<0$, for $\tilde{w}\in S_2$,

(3) $\mathrm{Im}\{(1-\tilde{q}(x)\tilde{w}^2)e^{i(\pi-\theta)/2}\}\geqq\cos\frac{\theta}{2}+\frac{|\tilde{q}(x)|}{9M}J>0$, for $\tilde{w}\in S_3$,

where $J=\mathrm{Im}\{t^2e^{i(\varphi+(3/2)\pi-(3/2)\theta)}+2t(1-t)e^{i(\varphi+(3/2)\pi-(1/2)\theta)}+(1-t)^2e^{i(\varphi+(3/2)\pi+(1/2)\theta)}\}$ and $-\tilde{q}(x)=|\tilde{q}(x)|e^{i(\pi-\theta)}e^{i\varphi}$, $0<\varphi=\varphi(x)<\pi$. The proof of (2) is quite similar to that of (3). To prove (3) we substitute

$$\sin\left(\phi-\frac{3}{2}\theta\right)=\sin\phi\cos\frac{3}{2}\theta-\left(1+\sin\frac{3}{2}\theta\right)\cos\phi+\cos\phi$$

$$\sin\left(\phi\mp\frac{1}{2}\theta\right)=\sin\phi\cos\frac{1}{2}\theta\pm\left(1-\sin\frac{\theta}{2}\right)\cos\phi\mp\cos\phi$$

with $\psi=\varphi+\frac{3}{2}\pi$ in J, then it holds

$$J=\cos\left(\varphi+\frac{3}{2}\pi\right)\{t^2-2t(1-t)+(1-t)^2\}+B(\varphi,\ \theta,\ t)\cos\frac{\theta}{2}, \tag{3.2}$$

where
$$B(\varphi,\ \theta,\ t)=B_1(\varphi,\ \theta,\ t)+B_2(\varphi,\ \theta,\ t),$$

$$B_1(\varphi,\ \theta,\ t)=\sin\left(\varphi+\frac{3}{2}\pi\right)\left\{t^2\left(\cos\theta-2\sin\frac{2\theta}{2}\right)+2t(1-t)+(1-t)^2\right\},$$

$$B_2(\varphi,\ \theta,\ t)=B-B_1=-2t^2\cos\left(\varphi+\frac{3}{2}\pi\right)\sin\theta$$
$$-(2t-1)^2\cos\left(\varphi+\frac{3}{2}\pi\right)\Big/\left(\tan\frac{\theta}{2}+\left(\cos\frac{\theta}{2}\right)^{-1}\right).$$

We have $|B(\varphi,\ \theta,\ t)|<8$ for $(\varphi,\ \theta,\ t)\in(0,\ \pi)\times(0,\ \pi)\times[0,\ 1]$. Since the first term of (3.2) is non negative, (3) holds and Lemma 3.2 is proved.

Proof of Lemma 3.3 is quite similar to the above proof.

Proof of Lemma 3.4. Let $\Omega(x)=\Omega$ be the domain surrounded by the following curves S_k, $(k=1,\ 2,\ 3,\ 4)$:

$$S_1=\{w:\operatorname{Im} w=0\},$$

$$S_2=\left\{w:\ w=m_1e^{i(\pi-(\theta/2))}-te^{i(\pi-\theta)/2},\ 0\leqq t\leqq m_1\tan\frac{\theta}{2}\right\},$$

$$S_3=\left\{w:\ w_\beta=\frac{\operatorname{Im}(w\bar{\alpha}_1)}{\operatorname{Im}(\beta\bar{\alpha}_1)}=\left(m\cos\frac{\theta}{2}\right)^{1/2}/|\beta|,\ \pi-\theta<\arg w\leqq\pi-\frac{\theta}{2}\right\},$$

$$S_4=\left\{\tilde{w}:\ \tilde{w}=-\frac{1}{3}M^{-1/2}t+\frac{1}{3}M^{-1/2}e^{-i(\pi-\theta)}(1-t),\ 0\leqq t\leqq 1\right\}.$$

Notice that S_4 is equal to S_3 in the proof of Lemma 3.2. In order to apply Lemma A we verify the following inequalities:

(1) $\operatorname{Im}(\tilde{q}(x)-w^2)=\operatorname{Im}\tilde{q}(x)<0$, for $w\in S_1$,

(2) $\operatorname{Im}\{(\tilde{q}(x)-w^2)e^{-i(\pi-\theta)/2}\}\leqq -m+m_1^2\cos\frac{\theta}{2}+2m_1t\sin\frac{\theta}{2}-t^2\cos\frac{\theta}{2}$
$$\leqq -m+m_1^2\left(\cos\frac{\theta}{2}\right)\left(1+\tan^2\frac{\theta}{2}\right)=0,\quad \text{for } w\in S_2,$$

(3) $\arg(\tilde{q}(x)-w^2)\in(-\theta,\ \pi-\theta)$ holds for $w\in S_3$, since $\arg\tilde{q}(x)\in(-\theta,\ 0)$ and $\arg(-w^2)\in(\pi-2\theta,\ \pi-\theta)$ for $w\in S_3$,

(4) $\operatorname{Im}\{(1-\tilde{q}(x)\tilde{w}^2)e^{i(\pi-\theta)/2}\}>0$ for $\tilde{w}\in S_4$.

In view of Lemma A $w'=\tilde{q}(x)-w^2$ has a solution $w(x)$ staying in Ω for all $x\in[0,\ \infty)$. Evidently the solution $w_1(x)$ of $w_1'=\tilde{q}(x)-w_1^2$ with $w_1(0)\in-\Omega$ satisfies $w_1(x)\in-\Omega$ for all $x\in[0,\ \infty)$. Put $v(x)=\exp\int_0^x w(s)ds$ and $v_1(x)=\exp\int_0^x w_1(s)ds$. Then from (2.14) for $v(x)$ and $v_1(x)$, we see that $|v(x)|$ stays bounded and $|v_1(x)|$ tends to ∞ as x tends to ∞. Therefore any other solution $w_2(x)=\dfrac{w_1(x)v_1(x)+cw(x)v(x)}{v_1(x)+cv(x)}$ approximates to $w_1(x)$ as x tends to ∞. Thus we have

Lemma 3.4.

Proof of Lemma 3.5. Define $\Omega(x)$ by $\{w : -\infty<\mathrm{Re}\, w<w_1(x)\}$ in S^2. Then we can apply Lemma A to the equation $w'=\tilde{q}(x)-w^2$ to obtain Lemma 3.5. In fact $\partial\Omega(x)=\{w : \mathrm{Re}\, w=w_1(x)\}\cup\{\tilde{w} : \tilde{w}=1/w,\ \mathrm{Re}\,(1/\tilde{w})=1/w_1(x)\}$ satisfies the conditions in Lemma A since $\mathrm{Re}(\tilde{q}(x)-w^2)\geqq q_1(x)-w_1(x)^2$ on $\{w : \mathrm{Re}\, w=w_1(x)\}$ and $\mathrm{Re}(1-\tilde{q}(x)\tilde{w}^2)=1>0$ at $\tilde{w}=0$. Therefore we have Lemma 3.5.

§4. Proof of Theorem 1.

Here we complete the proof of Theorem 1. Remark that the reasoning in this section is continued from that in Section 2.

4.1. $\tilde{v}(x, p, \alpha)$ **for** $\alpha\in C-(-\infty, 0]$. Consider the solution $\tilde{v}(x, p, \alpha)$ of (2.1) with $x(p, \alpha)=x_0$ for $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, 0]\}$, where x_0 satisfies (2.2). Put $\sup_{(0,\infty)}|q(x)|=M_0$ and $\tilde{q}(x)=p+q(x)\alpha$. If $\mathrm{Re}\,\alpha>0$ we apply Lemma 3.1 replacing $(0, \infty)$, m and M respectively by (x_0, ∞), $p+\delta\,\mathrm{Re}\,\alpha$ and $p+M_0|\alpha|$. Then the solution $w(x)=w(x, p, \alpha)$ of $w'=\tilde{q}(x)-w^2$ exists satisfying

$$(4.1)\qquad \begin{cases} |w(x)|<2(p+M_0|\alpha|^{1/2}), \\ \mathrm{Re}\, w(x)<-(p+\delta\,\mathrm{Re}\,\alpha)^{1/2}, \qquad x_0\leqq x<\infty, \quad \mathrm{Re}\,\alpha\geqq 0. \end{cases}$$

For $\mathrm{Im}\,\alpha\neq 0$ we apply Lemma 3.4 and Remark 3.1 replacing $(0, \infty)$ by (x_0, ∞) and putting $\theta=|\arg\alpha|$, $m=(p+\delta|\alpha|)\cos\frac{\theta}{2}$ and $M=p+M_0|\alpha|$. Then there exists a solution $w(x)=w(x, p, \alpha)$ of $w'=\tilde{q}(x)-w^2$ satisfying the following (4.2) for $x\in[x_0, \infty)$:

$$(4.2)\qquad \begin{cases} (1)\quad |w(x)|\leqq 3M^{1/2}\left(\cos\frac{\theta}{2}\right)^{-1}, \quad \theta=|\arg\alpha|, \\ (2)\quad \frac{1}{|\beta|}\left(m\cos\frac{\theta}{2}\right)^{1/2}\leqq w_\beta(x, p, \alpha)=\frac{\mathrm{Im}(w(x, p, \alpha)\bar{\alpha}_1)}{\mathrm{Im}(\beta\bar{\alpha}_1)}, \qquad \alpha_1=-\alpha. \\ (3)\quad \pi-\theta<\pm\arg w(x)<\pi \qquad \text{if} \quad 0<\mp\arg\tilde{q}(x)\leqq\theta<\pi. \end{cases}$$

From (4.2) (1) we have

$$(4.3)\qquad w_\beta(x)\leqq\frac{6}{|\beta|}M^{1/2}\left(\cos\frac{\theta}{2}\right)^{-1}, \qquad x\in[x_0, \infty).$$

Using above $w(x)=w(x, p, \alpha)$ we define

$$\tilde{v}(x)=\tilde{v}(x, p, \alpha)=\exp\int_{x_0}^{x} w(s, p, \alpha)ds, \qquad \text{for} \quad x_0<x.$$

Integrate $\tilde{v}''(x)\bar{\tilde{v}}(x)$ by parts then for $x_0\leqq x_1<x$

$$(4.4)\qquad -w(x_1)|\tilde{v}(x_1)|^2=-w(x)|\tilde{v}(x)|^2+\int_{x_1}^{x}(|\tilde{v}'(s)|^2+(p+q(s)\alpha)|\tilde{v}(s)|^2)ds.$$

Put $x_1=x_0$ and take the real part of (4.4) or β component of (4.4) in (α_1, β) coordinate, then

$$(4.4)_1 \qquad -\mathrm{Re}\, w(x_0)=-\mathrm{Re}\, w(x)|\tilde{v}(x)|^2+\int_{x_0}^{x}(|\tilde{v}'(s)|^2+(p+q(s)\mathrm{Re}\,\alpha)|\tilde{v}(s)|^2)ds\,,$$

$$(4.4)_2 \qquad w_\beta(x_0)=w_\beta(x)|\tilde{v}(x)|^2+(-1)_\beta\int_{x_0}^{x}(|\tilde{v}'(s)|^2+p|\tilde{v}(s)|^2)ds\,,$$

where $(-1)_\beta=-1_\beta=\dfrac{\mathrm{Im}\,\alpha_1}{|\alpha|\mathrm{Im}\,\beta}>0$. We can verify from (4.2) (2), (4.4), $(4.4)_1$ and $(4.4)_2$

$$(4.5) \qquad \lim_{x\to\infty}\tilde{v}(x,\ p,\ \alpha)=0 \qquad \text{for} \quad (p,\ \alpha)\in[0,\ \infty)\times\{C-(-\infty,\ 0]\}\,,$$

$$(4.5)' \qquad \int_{x_0}^{x}(|\tilde{v}(s,\ p,\ \alpha)|^2+|\tilde{v}'(s,\ p,\ \alpha)|^2)ds<\infty\,, \qquad \text{for above} \quad (p,\ \alpha).$$

Remark that $\tilde{v}(x, p, \alpha)$ is the unique solution of (2.1) with $x(p, \alpha)=x_0$ for $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, 0]\}$. In fact we can verify the uniqueness using Lemmas 3.1 and 3.4, $(4.4)_1$ and $(4.4)_2$.

Now we extend $\tilde{v}(x)=\tilde{v}(x, p, \alpha)$ as a solution of $\tilde{v}''=(p+q(x)\alpha)\tilde{v}$ to the region $(0, \infty)\ni x$. Then by Lemma 3.3 $w(x, p, \alpha)=\dfrac{\tilde{v}'(x,\ p,\ \alpha)}{\tilde{v}(x,\ p,\ \alpha)}$ belongs to $D(\theta, M)$. Therefore $\tilde{v}(x, p, \alpha)\neq 0$ so we can define $v(x, p, \alpha)=\tilde{v}(x, p, \alpha)/\tilde{v}(0, p, \alpha)$ for $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, 0]\}$.

4.2. The continuity of $\tilde{v}(x, p, \alpha)$ with respect to (p, α). Let us prove that the above $\tilde{v}(x, p, \alpha)$ is continuous in (p, α) for fixed $x\in[x_0, \infty)$. Note $w(p, \alpha)=w(x_0, p, \alpha)$. It suffices to prove that $w(p, a)$ is continuous at each point $(p_0, \alpha_0)\in[0, \infty)\times\{C-(-\infty, 0]\}$. At first suppose $\mathrm{Re}\,\alpha_0>0$. Then $\mathrm{Re}\,w(p_0, \alpha_0)<0$ from Lemma 3.1. Denote by $w(x, p, \alpha; w_0)$ the solution of $w'=(p+q(\alpha)\alpha)-w^2$ with $w(x_0)=w_0$. For given $\varepsilon>0$ we put $U_\varepsilon=\{w: |w-w(p_0, \alpha_0)|<\varepsilon,\ \mathrm{Re}\,w<0\}$. Note $\bar{\Omega}=\{w: \mathrm{Re}\,w\leqq 0\}$. By virtue of Lemma 3.1, for every $w_0\in\bar{\Omega}-U_\varepsilon$ there exists a positive number $x(w_0)$ uniquely satisfying $w(x, p_0, \alpha_0: w_0)\notin\bar{\Omega}$ for all $x\in(x(w_0), \infty)$. From the continuity of solutions, there exists a positive number $\delta=\delta(w_0)$ such that for $(p, \alpha, w_1)\in V(p_0, \delta)\times w(\alpha_0: \delta)\times U(w_0; \delta)$, $w(x, p, \alpha, w_1)\notin\bar{\Omega}$ for sufficiently large x, where $V(p_0; \delta)=\{p: p\geqq 0,\ |p-p_0|<\delta\}$, $w(\alpha_0; \delta)=\{\alpha: \mathrm{Re}\,\alpha>0,\ |\alpha-\alpha_0|<\delta\}$ and $U(w_0; \delta)=\{w\in\bar{\Omega}-U_\varepsilon; \min\{|w-w_0|,\ |1/w-1/w_0|\}<\delta\}$. Remark that we can take $\delta(w_0)$ as a lower semi-continuous and positive valued function on $\bar{\Omega}-U_\varepsilon$. Thus $0<\delta_1\leqq\delta(w_0)$ for all $w_0\in\bar{\Omega}-U_\varepsilon$. For $(p, \alpha)\in V(p_0; \delta_1)\times w(\alpha_0; \delta_1)$ and all $w_0\in\bar{\Omega}-U_\varepsilon$ we have $w(x, p, \alpha; w_0)\notin\bar{\Omega}$ for sufficiently large x. Hence by Lemma 3.1, $w(p, \alpha)$ belongs to U_ε for $(p, \alpha)\in V(p_0; \delta_1)\times w(\alpha_0; \delta_1)$. So we have the desired continuity if $\mathrm{Re}\,\alpha_0>0$. For $\mathrm{Im}\,\alpha_0<0$ we can verify the same continuity if we replace Lemma 3.1 and $\bar{\Omega}$ in the above proof by Lemma 3.4 and

$$\bar{\Omega}_{\alpha_0}=\left\{w: \frac{\pi-\theta}{2}\leqq\arg w\leqq\pi\right\}\cup\left\{\tilde{w}=1/w \text{ or } \tilde{w}=0,\ \frac{\pi-\theta}{2}\leqq\arg w\leqq\pi\right\},$$

where $\theta=|\arg\alpha_0|$. If $\mathrm{Im}\,\alpha_0>0$ we may replace w by $\bar{w}$. Thus we have the continuity of $w(x, p, \alpha)$ and $v(x, p, \alpha)$ in (p, α).

4.3. The continuity of $\tilde{v}_n(x, p, \alpha)$ in $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, \alpha_n(p)]\}$.

Let us prove the continuity of $\tilde{v}_n'(x_n, p, \alpha)/\tilde{v}_n(x_n, p, \alpha)$ at $(p, \alpha)=(p_0, \alpha_0)\in[0, \infty)\times(\alpha_n(p_0), \infty)$. From the continuity of $\alpha_n(p)$ we fix (p_1, α_1) satisfying $p_1=cp_0$, $(0<c<1)$ and $\alpha_0>\alpha_1>\alpha_n(p_0)$. Since $\alpha_1>\alpha_n(p_1)$ there exists uniquely the solutions of (2.1) with $x(p_1, \alpha_1)=x_n$. Then $\tilde{v}_n(x, p_1, \alpha_1)\neq 0$ in (x_n, ∞) and $\lim_{x\to\infty}\tilde{v}_n(x, p_1, \alpha_1)=0$. Put $w_1(x)=\tilde{v}_n'(x, p_1, \alpha_1)/\tilde{v}_n(x, p_1, \alpha_1)$ and

$$\Omega(x)=\{w : \mathrm{Re}\, w<w_1(x)\}\cup\{\tilde{w} : \tilde{w}=1/w,\ \mathrm{Re}\, w<w_1(x)\}.$$

Here we denote by $w(x, p, \alpha; w_0)$ the solution of $w'=p+q(x)\alpha-w^2$ with $w(x_n)=w_0$. By virtue of Lemma 3.5 there exists the unique initial data $w(p, \alpha)$ such that $\mathrm{Re}\, w(x, p, \alpha; w(p, \alpha))<w_1(x)$ i.e. $w(x, p, \alpha; w(p, \alpha))\in\Omega(x)$ for all $x\in[x_n, \infty)$ if $p>p_1$ and $\alpha>\alpha_n(p_1)$. Put $U_\varepsilon=\{w\in\Omega(x_n); |w-w(p_0, \alpha_0)|<\varepsilon\}$ for given $\varepsilon>0$. For any $w_0\in\overline{\Omega(x_n)}-U_\varepsilon$, $w(x, p_0, \alpha_0; w_0)\notin\overline{\Omega(x)}$ for sufficiently large x. As in the above step we can find a positive number δ_1, $(0<\delta_1<\min(p_0-p_1, \alpha_0-\alpha_1)$, such that $w(x, p, \alpha; w_1)\notin\overline{\Omega(x)}$ for sufficiently large x if p, α and w_1 satisfy $|p-p_0|<\delta_1$, $|\alpha-\alpha_0|<\delta_1$ and $|w_1-w_0|<\delta_1$ respectively. Therefore $w(p, \alpha)\in U_\varepsilon$ for $|p-p_0|<\delta_1$ and $|\alpha-\alpha_0|<\delta_1$. This means the continuity of $w(p, \alpha)$ at $(p, \alpha)=(p_0, \alpha_0)$. Here $\tilde{v}_n(x, p, \alpha)$ is continuous at $(p_0, \alpha_0)\in[0, \infty)\times\{C-(-\infty, \alpha_n(p_0)]\}$ for every fixed $x\in[0, \infty)$. $v(x, p, \alpha)=\tilde{v}_n(x, p, \alpha)/\tilde{v}_n(0, p, \alpha)$ has the same properties if $\tilde{v}_n(0, p_0, \alpha_0)\neq 0$.

4.4. The analyticity of $\tilde{v}(x, p, \alpha)$ in (p, α).

To verify analyticity of $w(x, p, \alpha)$ in α, we take the difference of $w'(x, p, \alpha+h)=p+q(x)(\alpha+h)-w(x, p, \alpha+h)^2$ and $w'(x, p, \alpha)=p+q(x)\alpha-w(x, p, \alpha)^2$. Put $w_{(h)}(x, p, \alpha)=\frac{1}{h}\{w(x, p, \alpha+h)-w(x, p, \alpha)\}$. It follows

$$\frac{d}{dx}w_{(h)}(x, p, \alpha)=q(x)-(w(x, p, \alpha+h)+w(x, p, \alpha))w_{(h)}(x, p, \alpha). \tag{4.6}$$

Since $\tilde{v}_n(x, p, \alpha)=\exp\int_{x_n}^{x} w(s, p, \alpha)ds$ holds for $\alpha\in\{C-(-\infty, \alpha_n(p))]\}$, the relation (4.6) yields

$$\begin{aligned} w_{(h)}(x, p, \alpha) &\\ =-(\tilde{v}_n(x, p, \alpha)\tilde{v}_n(x, p\alpha+h))^{-1}&\int_x^\infty\tilde{v}_n(s, p, \alpha)\tilde{v}_n(s, p, \alpha+h)q(s)ds \end{aligned} \tag{4.7}$$

for $(x, p, \alpha)\in[x_0, \infty)\times[0, \infty)\times\{C-(-\infty, 0]\}$ and $\alpha+h\in C-(-\infty, 0]$, $(n=0, 1, 2, 3, \cdots$, See $(2.10)_2)$. Remark that (4.5)$'$ assures the integrability of the right hand side of (4.7). Making h tend to zero we have

$$\frac{\partial}{\partial\alpha}w(x, p, \alpha)=-\frac{1}{\tilde{v}_n(x, p, \alpha)^2}\int_x^\infty\tilde{v}_n(s, p, \alpha)^2q(s)ds\,, \tag{4.8}$$

for $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, 0]\}$. Since the right hand side of (4.8) is continuous in α from Lebesque theorem, $w(x, p, \alpha)$ is analytic with respect to α in $C-(-\infty, 0]$ for $x\in[x_n, \infty)$. Extend $\tilde{v}(x, p, \alpha)$ as a solution of $\tilde{v}_n'=(p+q(x)\alpha)\tilde{v}_n$ to $[0, \infty)$, then $\tilde{v}_n(x, p, \alpha)$ is analytic with respect to α in $C-(-\infty, 0]$ for every fixed $x\in[0, \infty)$. Now the continuity of $\tilde{v}_n(x, p, \alpha)$ in α yields the analyticity with respect to α belonging to $C-(-\infty, \alpha_n(p)]$ if we use Painlevé theorem.

Now we prove (4.5)′ replaced $\tilde{v}$ by $\tilde{v}_n$ for $(p, \alpha)\in[0, \infty)\times\{C-(-\infty, \alpha_n(p)]\}$. From (4.4) and (4.5)′ we have for all $x\in(0, \infty)$,

$$w(x)|\tilde{v}_n(x)|^2=-\int_x^\infty|\tilde{v}_n'(s)|^2ds-\int_x^\infty(p+q(x)\alpha)|v_n(s)|^2ds\,, \tag{4.9}$$

where $w(x)=w(x, p, \alpha)$, $\tilde{v}_n(x)=\tilde{v}_n(x, p, \alpha)$ and $\operatorname{Im}\alpha\neq 0$. Take the imaginary part of (4.9) and devide it by $\operatorname{Im}\alpha$, then

$$-\frac{\operatorname{Im} w(x, p, \alpha)}{\operatorname{Im}\alpha}=\frac{\int_x^\infty|\tilde{v}_n(s, p, \alpha)|^2q(s)ds}{|\tilde{v}_n(x, p, \alpha)|^2}, \qquad \operatorname{Im}\alpha\neq 0\,. \tag{4.10}$$

As $\operatorname{Im}\alpha$ tends to zero, Cauchy-Riemann relation shows that the left hand side converges to $-\dfrac{\partial w}{\partial\alpha}(x, p, \alpha)$ for α satisfying $\operatorname{Im}\alpha=0$ and $\tilde{v}_n(x, p, \alpha)\neq 0$, since $\operatorname{Im} w(x, p, \alpha)=0$ for $\operatorname{Im}\alpha=0$. Apply Fatou theorem, then

$$-\frac{\partial w}{\partial\alpha}(x, p, \alpha)\geqq\frac{\int_x^\infty\tilde{v}_n(s, p, \alpha)^2q(s)ds}{\tilde{v}_n(x, p, \alpha)^2}, \qquad \alpha\in(\alpha_n(p), \infty)\,, \tag{4.11}$$

if $\tilde{v}_n(x, p, \alpha)\neq 0$. Taking $x=x_n$ we have

$$\int_0^\infty|\tilde{v}_n(s, p, \alpha)|^2ds<\infty\,, \qquad \text{for}\quad (p, \alpha)\in[0, \infty)\times\{C-(-\infty, \alpha_n(p)]\}, \tag{4.12}$$

since $\tilde{v}_n$ is a solution of the linear equation $v''=(p+q(x)\alpha)v$. From (4.11) and (4.12) we can obtain (4.7) again even for $\alpha\in C-(-\infty, \alpha_n(p)]$. Therefore (4.11) holds with equality, i.e. (4.8) holds for (p, α) belonging to $[0, \infty)\times\{C-(-\infty, \alpha_n(p)]\}$.

Finally we prove the analyticity in p. Here we remark that Lemma 3.5 permits us to consider $w(x, p, \alpha)$ for complex value p. $w^{(h)}(x, p, \alpha)=\dfrac{1}{h}\{w(x, p+h, \alpha)-w(x, p, \alpha)\}$ satisfies

$$\frac{d}{dx}w^{(h)}(x, p, \alpha)=1-\{w(x, p+h, \alpha)+w(x, p, \alpha)\}\,w^{(h)}(x, p, \alpha)$$

for all $(x, p, \alpha)\in[x_n, \infty)\times[0, \infty)\times\{C-(-\infty, \alpha_n(p)]\}$, then it follows similarly to (4.8)

$$\frac{\partial}{\partial p}w(x, p, \alpha)=-\frac{\int_x^\infty|\tilde{v}_n(s, p, \alpha)|^2ds}{|\tilde{v}_n(x, p, \alpha)|^2}, \tag{4.13}$$

which means $w(x, p, \alpha)$ is analytic in p for above (x, p, α).

In conclusion $\tilde{v}_n(x, p, \alpha)$ is analytic in (p, α) if (x, p, α) belongs to $[0, \infty)\times[0, \infty)\times\{C-(-\infty, \tilde{\alpha}_n(p)]\}$. Therefore $v(x, p, \alpha)$ is analytic in (p, α) for every $(x, p, \alpha)\in[0, \infty)\times[0, \infty)\times\{C-(-\infty, \tilde{\alpha}(p)]\}$ if $\alpha\notin Z(p)=\cup Z_n(p)$, where

$$Z_n(p)=\{\alpha\in(\alpha_n(p), \infty),\ \tilde{v}_n(0, p, \alpha)=0\}.$$

4.5. Properties of $Z(p)$. Now we see the structure of $Z_n(p)$. Let $\alpha_0 \in Z_n(p_0)$ i.e. $v_n(0, p_0, \alpha_0)=0$. (4.8) and (4.13) make

$$\frac{\partial}{\partial\alpha}\tilde{v}_n(0, p_0, \alpha_0)=\tilde{v}_n'(0, p_0, \alpha_0)^{-1}\int_0^\infty \tilde{v}_n(s, p_0, \alpha_0)^2 q(s)ds \tag{4.14}$$

$$\frac{\partial}{\partial p}\tilde{v}_n(0, p_0, \alpha_0)=\tilde{v}_n'(0, p_0, \alpha_0)^{-1}\int_0^\infty \tilde{v}_n(s, p_0, \alpha_0)^2 ds, \tag{4.15}$$

respectively, because $\left(\frac{\partial}{\partial\alpha}w\right)\tilde{v}_n^2=-\tilde{v}_n'\frac{\partial}{\partial\alpha}\tilde{v}_n$ holds if $\tilde{v}_n=0$. Since $\tilde{v}_n''(x, p, \alpha)=(p+q(x)\alpha)\tilde{v}_n$ is square integrable in $(0, \infty)$, we have from the integration by parts of $\tilde{v}_n''\tilde{v}_n$

$$\int_0^\infty \tilde{v}_n'(s, p_0, \alpha_0)^2 ds+\int_0^\infty (p_0+q(s)\alpha_0)\tilde{v}_n(s, p_0, \alpha_0)^2 ds=0.$$

We rewrite (4.14) as

$$\frac{\partial}{\partial\alpha}\tilde{v}_n(0, p_0, \alpha_0) =-\alpha_0^{-1}\tilde{v}_n'(0, p_0, \alpha_0)^{-1}\int_0^\infty (\tilde{v}_n'(s, p_0, \alpha_0)^2+p_0\tilde{v}_n(s, p_0, \alpha_0)^2)ds. \tag{4.14$'$}$$

From (4.14)$'$ $Z_n(p)$ is a set of discrete points for every $p\in[0, \infty)$. Since $\frac{\partial\tilde{v}_n}{\partial\alpha}(0, p_0, \alpha_0)\neq 0$ from (4.14)$'$, $\tilde{v}_n(0, p, \alpha)=0$ is solved locally by $\alpha=\alpha(p)$ satisfying $\alpha_0=\alpha(p_0)$ and

$$\frac{d\alpha(p)}{dp}=-\frac{\partial\tilde{v}_n(0, p, \alpha(p))}{\partial p}\Big/\frac{\partial\tilde{v}_n(0, p, \alpha(p))}{\partial\alpha}. \tag{4.16}$$

From (4.15) and (4.14)$'$ replaced (p_0, α_0) by $(p, \alpha(p))$ we have

$$\frac{d\alpha(p)}{dp}=\alpha(p)\frac{\int_0^\infty \{\tilde{v}_n'(s, p, \alpha(p))^2+p\tilde{v}_n(s, p, \alpha(p))^2\}ds}{\int_0^\infty \tilde{v}_n(s, p, \alpha(p))^2 ds}. \tag{4.17}$$

We can see that $\alpha(p)$ is increasing if $\alpha(p)>0$ and decreasing if $\alpha(p)<0$. Let us consider the case where p is close to zero. Since $\delta_0\leqq q(x)<M$ in (x_0, ∞) we have from Lemma 3.1,

$$-2(p+M\alpha)^{1/2}<w(x_0, p, \alpha)<-(p+\delta_0\alpha)^{1/2} \qquad \text{for} \quad \alpha>0.$$

Since $\tilde{v}(x, 0, 0)\equiv 1$ is the solution of $\tilde{v}''=0$, $\tilde{v}(x_0)=1$ and $\tilde{v}'(x_0)=0$, the solution $\tilde{v}(x, p, \alpha)$ of $\tilde{v}''=(p+q(x)\alpha)\tilde{v}$, $\tilde{v}(x_0)=1$ and $\tilde{v}'(x_0)=w(x_0, p, \alpha)$ is close to $\tilde{v}(x, 0, 0)\equiv 1$ if p and α is sufficiently small and $\alpha>0$. Therefore we can find a positive constant $\alpha_1(0)$ satisfying

$$\tilde{v}(0, 0, \alpha)\neq 0 \qquad \text{for} \quad 0<\alpha<\alpha_1(0).$$

On the other hand we can verify from Sturm's separation theorem that for $p=0$ and $\alpha<0$ all the solutions of $v''=q(x)\alpha v$ oscillate infinite times in (x_0, ∞). Therefore $\tilde{\alpha}(0)=\tilde{\alpha}_n(0)=0$, $n=1, 2, 3, \cdots$. For $p>0$, $\tilde{\alpha}_n(p)<0$ for $n\geqq 1$ and (2.8) holds. Similarly to the continuity of $\tilde{v}(0, p, \alpha)$ at $(p, \alpha)=(0, 0)$, $\tilde{v}_n(0, p, \alpha)$ is close to 1 if (p, α) is sufficiently small. Considering the case of $n=1$ for example we have

$$\sup_{p>0} \alpha_{-1}(p)<0$$

if we take care the monotone property of $\alpha_{-1}(p)$. Thus for sufficiently small $p>0$, $\alpha_{-1}(p)$ does not appear in $(\tilde{\alpha}(p), 0)$. From Sturm's separation theorem we see that $\alpha_{-1}(p)$ appears in $(\tilde{\alpha}(p), 0)$ for sufficiently large p if

$$\sup_{(0,\infty)} q(x)>\overline{\lim_{x\to\infty}}\, q(x)\,. \tag{4.18}$$

More precisely $\{\alpha_{-j}(p)\}_{j=1}^{m(n,p)}$ appear in $(\alpha_n(p), 0)$ for large p if it holds $\sup_{(0,x_n)} q(x) > \sup_{(x_n,\infty)} q(x)$, where $m(n, p)$ is a non negative integer tending ∞ as p tends to ∞. We note $m(p)=\lim_{n\to\infty} m(n, p)$. Then $m(p)<\infty$ or $m(p)=\infty$, that depends on the given $q(x)$. Thus we have

$$Z_n(p)=\bigcup_{k=1}^{\infty}\{\alpha_k(p)\}\cup\bigcup_{j=1}^{m(n,p)}\{\alpha_{-j}(p)\}\,.$$

Here we remark that $\{\alpha_k(p)\}_{k=1}^{\infty}$ appears in $(0, \infty)$ if and only if $q(x)$ becomes negative in $(0, x_0)$, which is verified by comparison theorem. From (2.9) we have $p+(\underline{\lim}\, q(x))\tilde{\alpha}(p)\geqq 0$. Suppose that

$$\sup_{(0,\infty)} q(x)=\underline{\lim_{x\to\infty}}\, q(x) \quad (=\lim_{x\to\infty} q(x)) \tag{4.18$'$}$$

holds. Then (4.18)$'$ implies $p+q(x)\alpha\geqq 0$ for $(x, \alpha)\in[0, \infty)\times(\tilde{\alpha}(\alpha), 0)$. Hence $\bigcup_{j=1}^{m(p)}\{\alpha_{-j}(p)\}=\emptyset$ if $q(x)$ satisfies (4.18)$'$.

4.6. Solutions of $v''=(p+q(x)\alpha)v$ **for** $\alpha\in(-\infty, \tilde{\alpha}(p))$.

Here we prove (D-4) using the following lemma.

Lemma 4.1. *Suppose that* $p(x)$ *and* $q(x)$ *satisfy* (C_0). *Let* $q(x)$ *satisfy* $0<\delta<q(x)$ *in* $(0, \infty)$. *Then there exists a real number* α_0 *satisfying the following* 1) *and* 2).

1) *For* $\alpha>\alpha_0$, $u''=\{p(x)+q(x)\alpha\}u$ *has a unique solution* $u(x)$ *satisfying* $u(0)=1$, $\lim_{x\to\infty} u(x)=0$ *and* $0<u(x)$ *in* $(0, \infty)$.

2) *For* $\alpha<\alpha_0$ *all the solutions of* $u''=\{p(x)+q(x)\alpha\}u$ *has at least one zero in* $(0, \infty)$.

Proof of Lemma 4.1. Denote by A the set of real numbers satisfying the property 1). $A\supset\left(\frac{\sup p(x)}{\inf q(x)}, \infty\right)$, $CA\supset\left(-\infty, \frac{\inf p(x)}{\sup q(x)}\right)$ and $\alpha_0=\inf A$. Then $A-\{\alpha_0\}=(\alpha_0, \infty)$ from Lemma 2.2. Since 1) holds we prove 2) by the method of contradiction. Take $\alpha<\alpha_0$ and suppose that there exists a number $\alpha_1\in(\alpha, \alpha_0)$ such that $u_1''=(p(x)+q(x)\alpha_1)u_1$ has a solution $u_1(x)$ satisfying $u_1(0)=1$ and $0<u_1(x)$ in $(0, \infty)$. (Otherwise 2) holds for α by comparison theorem.) Denote by $u_1(x\,;t)$ the solution of $u_1''=(p(x)+q(x)\alpha_1)u_1$ satisfying $u_1(0\,;t)=1$ and $u_1'(0\,;t)=t$. Put $t_1=\inf\{t: u_1(x\,;t)>0 \text{ in } (0, \infty)\}$. Then we have

(4.19) $$0<\overline{\lim_{x\to\infty}}\, u_1(x\,;\,t_1)\leqq\infty$$

which we prove later. At first using (4.19) we prove that 2) holds for α. Put $w_1(x)=\frac{u_1'(x\,;\,t_1)}{u_1(x\,;\,t_1)}$ then $w_1'=(p(x)+q(x)\alpha_1)-w_1^2$. Now we can show

(4.20) $$-M\leqq w_1(x)\leqq\max\{w_1(0),\ M\},\qquad\text{for}\quad x\in[0,\ \infty)\,,$$

where $$M=\sup|p(x)+q(x)\alpha_1|^{1/2}.$$

The proof is given as follows. Suppose $w_1(x_1)<-M-\varepsilon$ for a certain positive number x_1, then $w_1'(x)<0$ and $w_1(x)<-M-\varepsilon$ follow as long as $w_1'=(p(x)+q(x)\alpha_1)-w_1^2$ holds for $x>x_1$. Then $\tilde{w}_1'\geqq 1-M^2/(M+\varepsilon)^2>0$ since $\tilde{w}_1$ satisfies $\tilde{w}_1'=1-(p(x)+q(x)\alpha_1)\tilde{w}_1^2$. Therefore $\tilde{w}_1(x_2)=0$ for some $x_2>x_1$, which means $u_1(x_2)=0$. This is a contradiction, thus $-M\leqq w_1(x)$ for all $x\in[0,\ \infty)$. If $w_1(x)>M$ then $w_1'(x)=(p(x)+q(x)\alpha_1)-w_1^2(x)<0$. Hence we have $w_1(x)\leqq\max\{w_1(0),\ M\}$ and (4.20). Denote by $w(x\,;\,t)$ and $w_1(x\,;\,t)$ the solutions of $w'=(p(x)+q(x)\alpha)-w^2$ and $w_1'=(p(x)+q(x)\alpha_1)-w_1^2$ with $w(0\,;\,t)=t$ and $w_1(0\,;\,t)=t$ respectively. Remark $w(x\,;\,t)<w_1(x\,;\,t)$ holds for $x>0$. Notice $w_1(x)=w_1(x\,;\,t_1)$. From the definition of t_1, for any $t<t_1$ there exists a positive number $x_1=x_1(t)$ satisfying $w_1(x_1\,;\,t)=u_1'(x_1\,;\,t)/u_1(x_1\,;\,t)=-\infty$. This implies $w(x_2\,;\,t)=-\infty$ for a certain $x_2\in(0,\ x_1)$, which equals $u(x_2\,;\,t)=0$. Here $u(x\,;\,t)$ is the solution of $u''=(p(x)+q(x)\alpha)u$ satisfying $u(0\,;\,t)=1$ and $u'(0\,;\,t)=t$. Even for $t\geqq t_1$, we can find positive numbers x_1 and ε satisfying $w(x_1\,;\,t)<w_1(x_1\,;\,t_1-\varepsilon)$ as follows. Note $w(x)=w(x\,;\,t)$ in short. Suppose $w(x)\geqq w_1(x)=w_1(x\,;\,t_1)$ for all $x>0$. Then we are led to the following contradiction: Note $u(x)=u(x\,;\,t)=\exp\int_0^x w(s)ds$ and $u_1(x)=u_1(x\,;\,t_1)=\exp\int_0^x w_1(s\,;\,t_1)ds$. We have

(4.21) $$\{u(x)u_1(x)(w(x)-w_1(x))\}'=u(x)u_1(x)q(x)(\alpha-\alpha_1)\,.$$

(4.23) $$u(x)u_1(x)(w(x)-w_1(x))=(\alpha-\alpha_1)\int_0^x u(s)u_1(s)q(s)ds+(t-t_1)$$

follows. Since the left hand side of (4.22) is positive we have

(4.23) $$\int_0^x u(s)u_1(s)q(s)ds\leqq\frac{t-t_1}{\alpha_1-\alpha}\qquad\text{for all}\quad x\in(0,\ \infty)\,.$$

On the other hand, since $u(x)\geqq u_1(x)$ holds in $(0,\ \infty)$, (4.19) and (4.20) make

(4.24) $$\lim\int_0^x u(s)u_1(s)ds=\infty\,,$$

which contradicts to (4.23). Thus we have $w(x_1\,;\,t)<w_1(x_1\,;\,t_1)$ for a certain $x_1>0$, which implies $w(x_1\,;\,t)<w_1(x\,;\,t_1-\varepsilon)$ for small positive ε from the continuity of solutions on initial data. Therefore as we have considered in the case $t<t_1$, 2) holds for α also in the case $t\geqq t_1$. Finally we show (4.19). The definition of t_1 gives $u_1(x\,;\,t_1)>0$ in $(0,\ \infty)$. Suppose $\lim_{x\to\infty}u_1(x\,;\,t_1)=0$. Since $\alpha_1<\alpha_0$, from the definition of α_0 there exists another solution $u_1(x\,;\,t_2)$, $(t_2>t_1)$ satisfying $\lim_{x\to\infty}u_1(x\,;\,t_2)=0$. Then we have $\{u_1(x\,;\,t_1)u_1(x\,;\,t_2)(w_1(x\,;\,t_1)-w_1(x\,;\,t_2))\}'$

$=0$ similarly to (4.21), and $u_1(x\,;t_1)u_1(x\,;t_2)(w_1(x\,;t_1)-w_1(x\,;t_2))=t_1-t_2$. Therefore it follows

$$\lim_{x\to\infty}(w_1(x\,;t_2)-w_1(x\,;t_1))=\infty\,.$$

Since we can verify that $w_1(x\,;t_1)=w_1(x)$ satisfies (4.20), $w_1(x\,;t_2)$ must diverge to ∞. However $w_1'(x\,;t_2)=(p(x)+q(x)\alpha_1)-w_1(x\,;t_2)^2$ shows that $w_1(x\,;t_2)$ must decrease if $w_1(x\,;t_2)>M$. This is a contradiction. Hence (4.19) holds. Thus the proof of Lemma 4.1 is completed.

We use Lemma 4.1 replaced $(0, \infty)$ and $p(x)$ by (x_n, ∞) and p, then α_0 corresponds to $\tilde{\alpha}_n(p)$. Therefore for $\alpha<\tilde{\alpha}_n(p)$, the solution of $v''=(p+q(x)\alpha)v$ has zero in (x_n, ∞). This implies $(D\text{-}4)$ in Theorem 1 for $\alpha<\tilde{\alpha}(p)$, since $E(x, \eta, \tau)=v(x, |\eta|^2, \tau^2)$. The proof of Theorem 1 is finished.

§5. Proof of Theorem 2.

In order to estimate $v(x, p, \alpha)$ we use (2.15). Therefore it is important to evaluate the maximum and the minimum of $w_\beta(x, p, \alpha)$ in $(0, \infty)$, where $w_\beta(x, p, \alpha)$ is the β component of $w(x, p, \alpha)$ in (α_1, β) coordinate. It is convenient to devide the estimate for $(p, \alpha)\in[0, \infty)\times\Gamma$ into local estimates in three parts: a neighbourhood of origin, the region satisfying $p\geqq 2M_0|\alpha|$, where $M_0=\sup_{0<x}|q(x)|$, and other general region.

5.1. The case where (p, α) belongs to a neighbourhood of origin.

First we consider $\tilde{v}(x, p, \alpha)$ in (x_0, ∞). From (4.2), (4.3) and

$$|\tilde{v}(x, p, \alpha)|\leqq\left|\frac{w_\beta(x_0, p, \alpha)}{w_\beta(x, p, \alpha)}\right|^{1/2} \qquad \text{for} \quad x\in[x_0, \infty)\,,$$

which follows from $(4.4)_2$, there exists a positive constant C_1 such that

$$|\tilde{v}(x, p, \alpha)|\leqq C_1 \qquad \text{for} \quad (x, p, \alpha)\in[x_0, \infty)\times(U(0)\cap[0, \infty))\times(U(0)\cap\Gamma^2)\,,$$

where $U(0)$ is a neighbourhood of origin zero in complex plane and C_1 depends on $U(0)$. Then (4.2) (1) gives

$$|\tilde{v}'(x, p, \alpha)|\leqq C_2(p+|\alpha|) \tag{5.1}$$
$$\text{for} \quad (x, p, \alpha)\in[x_0, \infty)\times(U(0)\cap[0, \infty))\times(U(0)\cap\Gamma^2)\,,$$

where C_2 depends on $U(0)$. From the continuity of the solutions of $w'=(p+q(x)\alpha)-w^2$ with respect to initial data and coefficients, we see that the solution $\tilde{v}(x, p, \alpha)$ is close to 1 for $x\in[0, x_0]$. Because $v\equiv 1$ satisfies $v''=(p+q(x)\alpha)v$ with $p=\alpha=0$, $v(x_0)=1$ and $v'(x_0)=0$. Hence if we refine $U(0)$ it holds with a certain positive constant C

$$C^{-1}\leqq|\tilde{v}(x, p, \alpha)|\leqq C\,, \tag{5.2}$$
$$\text{for} \quad (x, p, \alpha)\in[0, \infty)\times(U(0)\cap[0, \infty))\times(U(0)\cap\Gamma^2)\,,$$

in both cases (C_+) and (C_-). As for the case (C_+) we also take account of $|\tilde{v}(x, p, \alpha)|\leqq\exp\int_{x_0}^{x}\mathrm{Re}\,w(s, p, \alpha)ds$ for $x\in[x_0, \infty)$ and (4.1). Thus we have

(E_+) and (E_-) for sufficiently small (η, τ) belonging to $R^{n-1}\times\Gamma$. We remark that Γ can be modified satisfying the condition concerning the continuity at (0, 0) indicated in $(E$-1) of Theorem 1.

5.2. The case where $p\geqq 2M_0|\alpha|$ holds. We apply Lemma 3.1 with $\tilde{q}(x)=p+q(x)\alpha$, $m=p-M_0|\alpha|$ and $M=p+M_0|\alpha|$. Then we have for $x\in[0, \infty)$ in both cases (C_+) and (C_-)

$$(5.3)\qquad \begin{cases} |w|\leqq 2M_0^{1/2}(p+|\alpha|)^{1/2} \\ \operatorname{Re} w<-\dfrac{1}{2}M_0^{1/2}(p+|\alpha|)^{1/2} \end{cases}$$

From (5.3) we have (E_+) and (E_-) for $p\geqq 2M_0|\alpha|$.

5.3. The general case involving $p<2M_0|\alpha|$. We take the β component of (4.4) in (α_1, β) coordinate, where $\alpha_1=-\alpha$ in (C_+) and $\alpha_1=\alpha$ in (C_-). Then we have for $0\leqq x_1<x$

$$(5.4)\qquad w_\beta(x_1)|\tilde{v}(x_1)|^2=w_\beta(x)|\tilde{v}(x)|^2+(-1)_\beta\int_{x_1}^{x}(|\tilde{v}'(s)|^2+p|\tilde{v}(s)|^2)ds\,,$$

where $w_\beta(x)=w_\beta(x, p, \alpha)$ and $\tilde{v}(x)=\tilde{v}(x, p, \alpha)$. Put $x_1=0$ then we have $|\tilde{v}(x)/\tilde{v}(0)|\leqq(w_\beta(0)/w_\beta(x))^{1/2}$ for all $x>0$ because $(-1)_\beta>0$. Therefore we have (2.15). The estimate of $w_\beta(x, p, \alpha)$ for $x\in[x_0, \infty)$ is given as follows, if we apply Lemma 3.4 with $\tilde{q}(x)=p+q(x)\alpha$, $M=P+M_0|\alpha|$, $m=(p+\delta|\alpha|)\cos\dfrac{\theta}{2}$ and $\theta=\pi-|\arg\alpha_1|$ replacing $(0, \infty)$ by (x_0, ∞): For both (C_+) and (C_-),

$$(5.5)\qquad w_\beta(x, p, \alpha)\geqq\frac{1}{|\beta|}(p+\delta|\alpha|)^{1/2}\sin\frac{|\arg\alpha_1|}{2},\qquad x\in[x_0, \infty)$$

$$(5.6)\qquad w_\beta(x, p, \alpha)<\frac{6}{|\beta|}(p+M_0|\alpha|)^{1/2}\Big(\sin\frac{|\arg\alpha_1|}{2}\Big)^{-1},\qquad x\in[x_0, \infty).$$

Now let us estimate $w_\beta(x, p, \alpha)$ in $(0, x_0)$. Apply Lemma 3.3 with $\xi=\alpha$ if $\operatorname{Im}\alpha>0$. Since $1/w(x_0, p, \alpha)$ belongs to $D(\theta, M)$, it holds

$$(5.7)\qquad 1/w(x, p, \alpha)\in D(\theta, M)\qquad \text{for}\quad x\in[0, x_0]\,,$$

in both cases (C_+) and (C_-). We may replace w by $\bar{w}$ if $\operatorname{Im}\alpha<0$. If $|\operatorname{Im}\alpha|\neq 0$, from the figure in Lemmas 3.2 and 3.4 we have for $x\in(0, \infty)$

$$(5.8)\qquad \begin{cases} w_\beta(x, p, \alpha)<\dfrac{3(p+M_0|\alpha|)^{1/2}}{|\beta|\Big(\sin\dfrac{|\arg\alpha_1|}{2}\Big)\cos\dfrac{|\arg\alpha_1|}{2}} & \text{in the case } (C_+), \\[3ex] w_\beta(x, p, \alpha)<\dfrac{6(p+M_0|\alpha|)^{1/2}}{|\beta|\Big(\sin\dfrac{|\arg\alpha_1|}{2}\Big)} & \text{in the case } (C_-). \end{cases}$$

Now we want to estimate $1/w_\beta(x, p, \alpha)$ for $x\in[0, x_0]$. Since $\dfrac{d}{dx}w_\beta=\Big(\dfrac{d}{dx}w\Big)_\beta$, we have

$$(5.9)\qquad \frac{d}{dx}w_\beta(x,\ p,\ \alpha)=(p+q(x)\alpha-w^2)_\beta=(p-w_{\alpha_1}^2\alpha_1^2-2w_{\alpha_1}w_\beta\alpha_1\beta)_\beta$$

using $(p_\beta)=p1_\beta=p\dfrac{-\mathrm{Im}\,\alpha_1}{|\alpha|\mathrm{Im}\,\beta}$, $(\alpha_1^2)_\beta=|\alpha|\dfrac{\mathrm{Im}\,\alpha_1}{\mathrm{Im}\,\beta}$, $(\alpha_1\beta)_\beta=|\alpha|$ and

$$(5.9)'\qquad \frac{d}{dx}w_\beta(x)=-\frac{p}{|\alpha|}\cdot\frac{\mathrm{Im}\,\alpha_1}{\mathrm{Im}\,\beta}+|\alpha|\frac{\mathrm{Im}\,\beta}{\mathrm{Im}\,\alpha_1}w_\beta^2-|\alpha|\frac{\mathrm{Im}\,\alpha_1}{\mathrm{Im}\,\beta}\Big(w_{\alpha_1}+\frac{\mathrm{Im}\,\beta}{\mathrm{Im}\,\alpha_1}w_\beta\Big)^2,$$

we have the differential inequality concerning $w_\beta(x)$:

$$(5.10)\qquad \frac{d}{dx}w_\beta(x)\leqq-\frac{p}{|\alpha|}\cdot\frac{\mathrm{Im}\,\alpha_1}{\mathrm{Im}\,\beta}+|\alpha|\frac{\mathrm{Im}\,\beta}{\mathrm{Im}\,\alpha_1}w_\beta^2,$$

which does not contain $q(x)$. We compare (5.9)′ and

$$\frac{d}{dx}f=\Big(\frac{|\alpha|\mathrm{Im}\,\beta}{\mathrm{Im}\,\alpha_1}\Big)f^2,\quad f(x_0)=\frac{1}{|\beta|}(p+\delta|\alpha|)^{1/2}\sin\frac{|\arg\alpha_1|}{2}.$$

Here the nonlinearity concerns and from $f(x_0)\leqq w_\beta(x_0)$ we have

$$(5.11)\qquad f(0)=1/\Big(\frac{1}{b}+ax_0\Big)\leqq f(x)\leqq w_\beta(x,\ p,\ \alpha)\qquad\text{for}\quad x\in[0,\ x_0],$$

where $a=|\alpha|\mathrm{Im}\,\beta/\mathrm{Im}\,\alpha_1$ and $b=f(x_0)$. From (2.15), (5.8) and (5.11) we have

$$(5.12)\qquad |v(x,\ p,\ \alpha)|^2\leqq\Big(A\frac{3(p+M_0|\alpha|)^{1/2}}{|\beta|\sin(|\arg\alpha_1|/2}\Big)\Big(|\alpha|\frac{\mathrm{Im}\,\beta}{\mathrm{Im}\,\alpha_1}x_0+\frac{|\beta|(p+\delta|\alpha|)^{-1/2}}{\sin(|\arg\alpha_1|/2)}\Big),$$

where $A=2$ for (C_-) and $1/\cos(|\arg\alpha_1|/2)$ for (C_+). Since it holds

$$(5.13)_1\qquad \begin{cases}\sin\dfrac{\arg\alpha_1}{2}=\begin{cases}\cos\dfrac{|\arg\alpha|}{2}=\dfrac{|\mathrm{Re}\,\tau|}{|\tau|} & \text{in the case } (C_+)\\[2mm] \sin\dfrac{|\arg\alpha|}{2}=\dfrac{\gamma}{|\tau|} & \text{in the case } (C_-)\end{cases}\\[6mm] \cos\dfrac{|\arg\alpha_1|}{2}=\sin\dfrac{|\arg\alpha|}{2}=\dfrac{\gamma}{|\tau|} & \text{in the case } (C_+)\end{cases}$$

$$(5.13)_2\qquad \frac{\mathrm{Im}\,\beta}{\mathrm{Im}\,\alpha_1}=-\frac{1}{2\,\mathrm{Re}\,\beta}=\begin{cases}1/(2\,\mathrm{Re}\,\tau) & \text{in the case } (C_+)\\ 1/2\gamma & \text{in the case } (C_-),\end{cases}$$

from $p=|\eta|^2$, $\alpha=\tau^2$ and $\gamma=|\mathrm{Im}\,\tau|$, $E(x,\ \eta,\ \tau)=v(x,\ |\eta^2,\ \tau^2)$ is estimated as follows. There exists a positive numbers depending on M_0, δ and x_0 such that we have for $\gamma=|\mathrm{Im}\,\tau|\neq0$ and $x\in[0,\ \infty)$

$$(5.14)\qquad \begin{cases}|E(x,\ \eta,\ \tau)|\leqq C\gamma^{-1/2}|\tau|((|\eta|+|\tau|+1)/|\mathrm{Re}\,\tau|)^{1/2} & \text{in the case } (C_+),\\[2mm] |E(x,\ \eta,\ \tau)|\leqq C\dfrac{|\tau|}{\gamma}(|\eta|+|\tau|+1)^{1/2} & \text{in the case } (C_-).\end{cases}$$

Finally remark that in the case (C_+) we can use also (4.1) in $(x_0,\ \infty)$. In both cases (C_+) and (C_-) the estimate (5.3) is better than (5.14) if $|\eta|^2\geqq2M_0|\tau|^2$. Thus we have Theorem 2. Using these results we have Theorem 3 in next section.

§ 6. Proof of Theorem 3.

In order to obtain the estimates in Theorem 3, we modify slightly the path Γ according to the cases (C_+) and (C_-). For this purpose we consider at first the properties of $\hat{g}(\eta, \tau)$.

6.1. Function spaces. For $\gamma>0$ we note

$$(6.1) \qquad e(t, \gamma)=\begin{cases} e^{-\gamma t}, & t\leqq 0 \\ 1, & t>0 \end{cases} \qquad \tilde{e}(t, \gamma)=\begin{cases} 1, & t<0 \\ e^{-\gamma t}, & t\geqq 0. \end{cases}$$

Then we have for all $t\in R$

$$(6.1)' \qquad \begin{cases} (1) \quad \hat{e}(t, \gamma)\leqq 1\leqq e(t, \gamma), \quad \tilde{e}(t, \gamma)\leqq e^{-\gamma t}\leqq e(t, \gamma), \\ (2) \quad e(t, \gamma)^{-1}=\tilde{e}(-t, \gamma) \\ (3) \quad \tilde{e}(t, \gamma)\leqq |e^{-i\tau t}|\leqq e(t, \gamma), \qquad \text{if} \quad -\gamma\leqq \operatorname{Im}\tau\leqq 0, \end{cases}$$

and the norms in § 1 are written as follows

$$\|g\|_{L^p_{k,\gamma}}=\Big\{\sum_{j+|\alpha|\leqq k}\Big\|e(t, \gamma)\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}g(y, t)\Big\|^p_{L^p}\Big\}^{1/p},$$

$$\|v\|_{\tilde{L}^p_{k,\gamma}}=\Big\{\sum_{j+|\alpha|\leqq k}\Big\|\tilde{e}(t, \gamma)\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}v(y, t)\Big\|^p_{L^p}\Big\}^{1/p},$$

$$\|v\|_{\tilde{B}_{k,\gamma}}=\Big\{\sum_{j+|\alpha|\leqq k}\sup_{\substack{t\in R\\ y\in R^{n-1}}}\Big|\tilde{e}(t, \gamma)\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^j}{\partial t^j}v(y, t)\Big|\Big\}.$$

Suppose that $g(y, t)$ belongs to $L^1_{k,\gamma}$, $(k\geqq 0)$, then from (6.1)′ (3)

$$\hat{g}(\eta, \tau)=\int_R\int_{R^{n-1}}e^{-i\tau t}e^{-i\eta y}g(y, t)dydt$$

is continuous on $(\eta, \tau)\in R^n\times\{\tau: -\gamma\leqq\operatorname{Im}\tau\leqq 0\}$, where $(|\eta|+|\tau|+1)^k|\hat{g}(\eta, \tau)|$ is bounded uniformly. $\hat{g}(\eta, \tau)$ is analytic with respect to τ in $\{\tau: -\gamma<\operatorname{Im}\tau<0\}$. Therefore (1.1) holds if $k\geqq n+1$ and $\Gamma\subset\{\tau; -\gamma\leqq\operatorname{Im}\tau\leqq 0\}$. Suppose that $g(y, t)$ belongs to $L^2_{0,\gamma}$, i.e. $e(t, \gamma)g(y, t)\in L^2$. Since $L^2_{0,\gamma}\cap L^1_{n+1,\gamma}$ is dense in $L^2_{0,\gamma}$, we have

$$(6.2) \qquad g(y, t)=\underset{j\to\infty}{\text{l.i.m.}}\Big(\frac{1}{2\pi}\Big)^n\int_\Gamma e^{i\tau t}\int_{R^{n-1}}e^{i\eta y}\hat{g}_j(\eta, \tau)d\eta d\tau,$$

where $g_j(y, t)\in L^1_{n+1,\gamma}$ converges to $g(y, t)$ in $L^2_{0,\gamma}$.

6.2. The case where $g(y, t)$ belongs to $L^1_{k+n+2,\gamma}$. Then from Theorem 2 we have in both cases (C_+) and (C_-)

$$(6.3) \qquad |E(x, \eta, \tau)\hat{g}(\eta, \tau)|\leqq\frac{C\,C_\gamma\|g\|_{L^1_{k+n+2,\gamma}}}{(|\eta|+|\tau|+1)^{n+k+\varepsilon}},$$

where $\varepsilon=1$ in case (C_+) and $\varepsilon=1/2$ in case (C_-). Therefore from (6.1)′ we have in both case

$$|\tilde{e}(t, \gamma)u(x, y, t)|\leqq C(k, \gamma)\|g\|_{L^1_{k+n+2,\gamma}}.$$

This means (1) in Theorem 3.

6.3. The case (C_+) **for** $g\in L^2_{k+1,\gamma}$. In view of $(D\text{-}1)$, $(D\text{-}2)$ and $(E\text{-}1)$ in Theorem 1 we modify Γ as follows.

$$(1.4)_1 \quad \begin{cases} \Gamma=\sum_{k=1}^{5}\Gamma_k\,, \\ \Gamma_1=\left\{\tau\,;\,\tau\in R,\ -\frac{1}{2}\mu_1(0)\leqq t\leqq\frac{1}{2}\mu_1(0)\right\}, \\ \Gamma_2=\{\tau\,;\,\tau=s-i\gamma,\ -\infty<s\leqq-\mu_1(0)\}, \qquad \gamma>0\,, \\ \Gamma_3=\{\tau\,;\,\tau=s-i\gamma,\ -s-i\gamma\in\Gamma_2\}, \\ \Gamma_4=\left\{\tau\,;\,\tau=\tau(t),\ 0\leqq t\leqq1,\ \tau(0)=-\frac{1}{2}\mu_1(0),\ \tau(1)=-\mu_1(0)-i\gamma\right\}, \\ \Gamma_5=\{\tau\,;\,-\operatorname{Re}\tau+i\operatorname{Im}\tau\in\Gamma_4\}, \end{cases}$$

where Γ_4 is a smooth curve which is situated away from $(-\infty,\ -\mu_1(0)]$ and the imaginary axis. Now we take smooth functions on Γ as follows. $\sum_{j=1}^{5}\chi_j(\tau)\equiv1$ on Γ, $\chi_j(\tau)\geqq0$, $\operatorname{supp}\chi_j\subset\Gamma_j$, $j=1,2,3$ and $\operatorname{supp}\chi_4$ and $\operatorname{supp}\chi_5$ are contained respectively in neighbourhoods of Γ_4 and Γ_5 which have positive distances from $(-\infty,\ -\mu_1(0))\cup\mu_1(0),\ \infty)$ and the imaginary axis. Decompose $u(x,y,t)=u_+(x,y,t)$ as

$$u(x,y,t)=\sum_{j=1}^{5}u_j(x,y,t)$$

$$u_j(x,y,t)=\left(\frac{1}{2\pi}\right)^n\int_\Gamma e^{i\tau t}\int_{R^{n-1}}e^{i\eta y}\chi_j(\tau)E(x,\eta,\tau)\hat{g}(\eta,\tau)d\eta d\tau$$

for $g(y,t)\in L^2_{k+1,\gamma}\cap L^1_{n+k+2,\gamma}$. If the estimate (E_+) is proved, then from (6.2) we obtain the strong solution $u(x,y,t)$ for $g\in L^1_{k+n+2,\gamma}$. (E_+) is proved as follows. By virtue of Plancherel's theorem we have

$$(6.4) \qquad \|u_1(x,\cdot,\cdot)\|_{H^k(R^{n-1}\times R)}\leqq C_k\|g\|_{H^k(R^{n-1}\times R)}, \qquad x\in[0,\infty)\,,$$

since (5.3) makes

$$(6.6) \qquad (|\tau|+1)^k\chi_j(\tau)|E(x,\eta,\tau)|\leqq C_k\,, \qquad j=1,4,5\,.$$

For $j=2$ and 3, Plancherel's theorem gives

$$\left\|e^{-\gamma t}\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^l}{\partial t^l}u_j(x,\cdot,\cdot)\right\|^2_{L^2}=\left(\frac{1}{2\pi}\right)^{2n}\left\|\eta^\alpha\tau^l\chi_j(\tau)E(x,\cdot,\cdot)\hat{g}(\cdot,\cdot)\right\|^2_{L^2},$$

where the right hand side is L^2 norm in $(\eta,\operatorname{Re}\tau)\in R^{n-1}\times R$. From (6.1)′ (1) we have

$$(6.6) \qquad \|u_j(x,\cdot,\cdot)\|_{\tilde{L}^2_{k,\gamma}}\leqq C_j(k,\gamma)\|g\|_{L^2_{k+1,\gamma}}, \qquad j=2,3\,.$$

For $j=4$ and 5 we rewrite $u_j(x,y,t)$ as

$$u_j(x,y,t)=\left(\frac{1}{2\pi}\right)^n\int_{R^{n+1}}e^{j\eta y}\int_{-\infty}^{\infty}K_j(x,\eta,t,s)(\mathcal{F}_y g)(\eta,s)dsd\eta\,,$$

where $K_j(x, \eta, t, s)=\int_\Gamma e^{i\tau(t-s)}\chi_j(\tau)E(x, \eta, \tau)d\tau$

$$=\frac{1}{(t-s)^k}\int_\Gamma e^{i\tau(t-s)}\Big(i\frac{\partial}{\partial\tau}\Big)^k(\chi_j(\tau)E(x, \eta, \tau))d\tau\,,$$

$k=0, 1, 2, \cdots$.

It holds

$$\tilde{e}(t, \gamma)\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^l}{\partial t^l}u_j(x, y, t)$$

$$=\Big(\frac{1}{2\pi}\Big)^n\int_{R^{n-1}}e^{i\eta y}\int_{-\infty}^\infty \tilde{K}_j^{(l)}(x, \eta, t, s)e(s, \gamma)(i\eta)^\alpha(\mathcal{F}_y g)(\eta, s)dsd\eta\,,$$

where $\tilde{K}_j^{(l)}(x, \eta, t, s)=\tilde{e}(t, \gamma)\frac{\partial^l}{\partial t^l}K_j(x, \eta, t, s)e(s, \gamma)^{-1}$.

(6.7) $$|\tilde{K}_j^{(l)}(x, \eta, t, s)|\leqq\frac{C_l}{|t-s|^2+1}, \qquad j=4, 5\,, \quad l=0, 1, 2, \cdots,$$

holds for some constants C_l if we take $k=0$ and 2 in above formula. Apply Plancherel's theorem then

$$\Big\|\tilde{e}(t, \gamma)\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^l}{\partial t^l}u_j(x, y, t)\Big\|_{L^2(y, t)}$$

$$=\Big(\frac{1}{2\pi}\Big)^{n+1/2}\Big\|\int_{-\infty}^\infty \tilde{K}_j^{(l)}(x, \eta, t, s)e(s, \gamma)(i\eta)^\alpha(\mathcal{F}_y g)(\eta, s)ds\Big\|_{L^2(\eta, t)}.$$

Then the square of the right hand side is estimated by

$$\iint\|\tilde{K}_j^{(l)}(x, \eta, t, \cdot)\|_{L^2}\Big(\int_{-\infty}^\infty|\tilde{K}_j^{(l)}(x, \eta, t, s)|\,|e(s, \gamma)(i\eta)^\alpha(\mathcal{F}_y g)(\eta, s)|^2ds\Big)dtd\eta$$

$$\leqq(C_l\pi)^2\|g\|^2_{L^2_{|\alpha|, \gamma}},$$

if we use Schwarz inequality, (6.7) and Fubini theorem. Thus we have

(6.8) $$\|u_j(x, \cdot, \cdot)\|_{L^2_{k,\gamma}}\leqq C_j(k, \gamma)\|g\|_{L^2_{k,\gamma}}, \qquad j=4, 5\,.$$

The estimate (2) in Theorem 3 follows from (6.4), (6.6) and (6.8).

6.4. The case (C_-) **for** $g\in L^2_{k+2}$. Remark that in the case (C_-) we have the results in Theorem 1 replacing τ by $i\tau$. Here we modify the path Γ such that Γ is situated away from the real axis and $(i\mu_1(0), i\infty)\cup(-i\infty, -i\mu_1(0))$ on the imaginary axis. For example we take $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$, where

$$\Gamma_1=\Big\{\tau\,;\,\tau=\tau(t),\ -1\leqq t\leqq 1,\ \tau(\mp 1)=\mp\gamma-i\gamma,\ \tau(0)=\frac{\mu_1(0)}{2}i\Big\},$$

$$\Gamma_2=\{\tau\,;\,\tau=\sigma-i\gamma,\ -\infty<\sigma\leqq-\gamma\},$$

$$\Gamma_3=\{\tau\,;\,-\mathrm{Re}\,\tau-i\gamma\in\Gamma_2\},$$

where $\tau(t)$ is a smooth curve in $\{\tau\,;\,\mathrm{Im}\,\tau<0\}$ which does not pass $(-\infty, \infty)$, $(-i\infty, -\mu_1(0)i]$ and $[\mu_1(0)i, i\infty)$. Decompose $u(x, y, t)$ as before $u(x, y, t)=\sum_{j=1}^3 u_j(x, y, t)$ for $g\in L^2_{k+2,\gamma}\cap L^1_{n+k+2,\gamma}$ using smooth functions $\chi_j(\tau)$, $(j=1, 2, 3)$, where $\mathrm{supp}\,\chi_j(\tau)\subset\Gamma_j$, $j=1, 2$ and $\mathrm{supp}\,\chi_1(\tau)$ is involved in a small neighborhood of Γ_1. $u_1(x, y, t)$ is estimated similarly to $u_4(x, y, t)$ in the case (C_+). As for $u_j(x, y, t)$, $(j=2, 3)$ we have similarly

$$\|u_j(x, y, t)\|_{\tilde{L}^2_{k,\gamma}} \leqq C_j(k, \gamma)\|g\|_{L^2_{k+2,\gamma}}.$$

Thus by completion we have the estimate (3) in Theorem 3.

§7. Some comments to Theorems E and H.

Theorem E and H are proved in the same way of proofs of Theorems 1, 2 and 3. Namely it suffices to regard $x_0=0$ in previous sections. Especially the estimates in Theorems E and H follow from (5.5), (5.6) and $(5.13)_1$. We can prove estimate $(3)_H$ from $(4.4)_2$, (5.6), $(5.13)_1$ and $(5.13)_2$. In fact we have

$$\sum_{j=0}^{1}\int_0^\infty |\eta|^{1-j}\frac{\partial^j}{\partial x^i}E(x, \eta, \tau)|^2 dx \leqq \frac{|\tau|^2}{2\gamma^2}$$

which yields $(3)_H$.

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