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A remark on the corona problem for plane domains

Dedicated to Professor Y. Kusunoki on his 60th birthday

By

Junichiro NARITA

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1. Introduction.

Let D be a domain in the complex plane, and $H^\infty(D)$ be the Banach algebra of bounded analytic functions on D. We assume that $H^\infty(D)$ contains a nonconstant function. Then, by point evaluations, the domain D can be identified with an open subset of the maximal ideal space $\mathcal{M}(D)$ of $H^\infty(D)$. The corona problem asks whether D is dense in $\mathcal{M}(D)$.

Since L. Carleson [3] solved the corona problem for the unit disk affirmatively, several attempts are made to generalize this result to larger class of plane domains. Among them we are particularly interested in the results of T. Gamelin [6] and M. Behrens [2]. In [6] Gamelin proved the localization principle for $\mathcal{M}(D)$, and by use that he showed some class of plane domains for which the corona problem has an affirmative answer. In the same paper he also introduced some constants $C(D, m, \delta)$ associated with each open set D in $\boldsymbol{C}$, integer $m\geqq 1$, and $\delta>0$ (see § 2).

In § 2 we show the localization principle concerning the sort of Banach algebras used in the proof of Behrens [2] (Theorem 1). And as its corollary, we know that the Gamelin's constants are finite for the open sets considered in [6] (Theorem 2).

Following W. Deeb [4], we mean by a Δ-domain, a domain obtained from the open unit disk Δ by deleting the origin and a sequence of disjoint closed disks $\Delta_n=\Delta(c_n, r_n)=\{z\,;\,|z-c_n|\leqq r_n\}$ contained in $\Delta\setminus\{0\}$ with c_n tending to 0. In [2] Behrens showed that if the corona problem has a negative answer for some plane domain, then it has a negative answer even for some Δ-domain. Therefore the corona problem for general plane domains is reduced to the case of Δ-domains.

In § 3, using the result of § 2, we shall construct some new examples of Δ-domains for which the corona problem has still an affirmative answer. Actually there is a Δ-domain with $\Sigma|r_n|=+\infty$ for which the corona problem is affirmative.

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2. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of uniformly bounded open sets in $\boldsymbol{C}$ (i.e. there exists a bounded set which contains every D_n), and let $\bigcup_{n=1}^{\infty}(D_n\times\{n\})$ be the formal disjoint union of $\{D_n\}_{n=1}^{\infty}$. Let $H^{\infty}(\{D_n\})$ be the Banach algebra consisting of bounded functions on $\bigcup_{n=1}^{\infty}(D_n\times\{n\})$ which are analytic on each $D_n\times\{n\}$, and let $\mathcal{M}(\{D_n\})$ be the maximal ideal space of this algebra. We can identify $\bigcup_{n=1}^{\infty}(D_n\times\{n\})$ with a subset of $\mathcal{M}(\{D_n\})$ by usual point evaluations. Define a coordinate function $Z\in H^{\infty}(\{D_n\})$ by $Z(\lambda, n)=\lambda$

Now we show that the localization principle is valid for $\mathcal{M}(\{D_n\})$ with respect to Z. The proof is an analogue of that of Gamelin (cf. [6], [7] Chap. 2). In the following we denote the Gel'fand transform of $F\in H^{\infty}(\{D_n\})$ by $\hat{F}$.

Lemma. *Let $F\in H^{\infty}(\{D_n\})$, and extend each $F(\cdot, n)$ to $\boldsymbol{C}$ by taking 0 outside D_n. If the sequence of functions $\{F(\cdot, n)\}_{n=1}^{\infty}$ is equicontinuous at $\zeta\in\boldsymbol{C}$ and $F(\zeta, n)=0$ $(n=1, 2, \cdots)$, then $\hat{F}=0$ on $Z^{-1}(\{\zeta\})$.*

Proof. For each $m\in\boldsymbol{N}$, let g_m be a C^1-function with compact support in $\Delta\left(\zeta, \frac{1}{m}\right)=\left\{|z-\zeta|\leqq\frac{1}{m}\right\}$, $g_m=1$ in a neighborhood of ζ, and $\left\|\frac{\partial g_m}{\partial\bar{z}}\right\|_{\infty}\leqq 4m$. Extend each $F(\cdot, n)$ to $\boldsymbol{C}$ as above, and set

$$(T_{g_m}F)(w, n)=\frac{1}{\pi}\iint_C\frac{F(z, n)-F(w, n)}{z-w}\cdot\frac{\partial g_m}{\partial\bar{z}}dxdy\,,\quad w\in\boldsymbol{C},\quad z=x+iy.$$

Then $T_{g_m}F\in H^{\infty}(\{D_n\})$ and

$$\begin{aligned}\|T_{g_m}F\|_{\infty}&\leqq\sup_n\sup_{w\in C}|(T_{g_m}F)(w, n)|\\&\leqq 8\sup_n\sup\left\{|F(z, n)-F(z', n)|\,;\,z, z'\in\Delta\left(\zeta, \frac{1}{m}\right)\right\}\\&\rightarrow 0\,,\quad\text{as }m\rightarrow\infty\end{aligned}$$

(cf. [7] p. 4-5). Further, letting $F_m(z, n)=F(z, n)-(T_{g_m}F)(z, n)+(T_{g_m}F)(\zeta, n)$, each $F_m(\cdot, n)$ is analytic in a neighborhood $\{z\in\boldsymbol{C}\,;\,g_m(z)=1\}$ of ζ which is independent of n, and $F_m(\zeta, n)=0$. So $(Z-\zeta)^{-1}F_m\in H^{\infty}(\{D_n\})$ and consequently $\phi(F_m)=\phi(Z-\zeta)\phi((Z-\zeta)^{-1}F_m)=0$ for $\phi\in\hat{Z}^{-1}(\{\zeta\})$. Since $\|F-F_m\|_{\infty}\rightarrow 0$ as $m\rightarrow\infty$, $\phi(F)=0$ as desired.

Theorem 1. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of uniformly bounded open sets in $\boldsymbol{C}$, and let U be an open set in $\boldsymbol{C}$. Let $\Phi:\mathcal{M}(\{D_n\cap U\})\rightarrow\mathcal{M}(\{D_n\})$ be the projection induced by the restriction map $H^{\infty}(\{D_n\})\rightarrow H^{\infty}(\{D_n\cap U\})$. Then the restriction of Φ to $\mathcal{M}(\{D_n\cap U\})\cap\hat{Z}^{-1}(U)$ is a homeomorphism onto $\mathcal{M}(\{D_n\})\cap\hat{Z}^{-1}(U)$. Moreover, for $\zeta\in U$ and $\varphi\in\mathcal{M}(\{D_n\cap U\})\cap\hat{Z}^{-1}(\{\zeta\})$, φ belongs to $\overline{\bigcup_{n=1}^{\infty}((D_n\cap U)\times\{n\})}$ in $\mathcal{M}(\{D_n\cap U\})$ if and only if $\Phi(\varphi)$ belongs to $\overline{\bigcup_{n=1}^{\infty}(D_n\times\{n\})}$ in $\mathcal{M}(\{D_n\})$.*

Proof. Obviously Φ is continuous and $\hat{Z}(\Phi(\varphi))=\hat{Z}(\varphi)$ for any $\varphi\in\mathcal{M}(\{D_n\cap U\})$. We shall now construct the inverse Ψ of $\Phi|\mathcal{M}(\{D_n\cap U\})\cap\hat{Z}^{-1}(U)$. For $\zeta\in U$ let $\psi\in\mathcal{M}(\{D_n\})$ for which $\hat{Z}(\psi)=\zeta$ and let $f\in H^\infty(\{D_n\cap U\})$. And let g be a C^1-function with compact support in U such that $g=1$ in a neighborhood of ζ. Extending each $f(\cdot, n)$ to $\boldsymbol{C}$ by taking 0 outside $D_n\cap U$, we define $T_g f$ as in the proof of lemma. Put $F(z, n)=(T_g f)(z, n)-(T_g f)(\zeta, n)+f(\zeta, n)$. Then we see that $F\in H^\infty(\{D_n\})$, each $(F-f)(\cdot, n)$ is analytic in a neighborhood $\{g=1\}$ of ζ, and that $(F-f)(\zeta, n)=0$. Hence $F-f$ satisfies the condition of lemma. Now we define $(\Psi(\psi))(f)=\psi(F)$. Clearly $\Psi(\psi)$ is linear. Let $f_1, f_2\in H^\infty(\{D_n\cap U\})$ and let F_1, F_2, F_3 be defined as above corresponding to f_1, f_2, f_1f_2 respectively. Then $F_1F_1-f_1f_2$, and hence $F_1F_2-F_3$ also, satisfies the condition of lemma. Consequently $(\Psi(\psi))(f_1f_2)=\psi(F_3)=\psi(F_1F_2)=\psi(F_1)\psi(F_2)=(\Psi(\psi))(f_1)(\Psi(\psi))(f_2)$, i.e. $\Psi(\psi)$ is also multiplicative, and so $\Psi(\psi)$ is a complex homomorphism on $H^\infty(\{D_n\cap U\})$. Now it is immediate that Ψ is the desired inverse of $\Phi|\mathcal{M}(\{D_n\cap U\})\cap\hat{Z}^{-1}(U)$ and this correspondence is a homeomorphism between $\mathcal{M}(\{D_n\cap U\})\cap\hat{Z}^{-1}(U)$ and $\mathcal{M}(\{D_n\})\cap\hat{Z}^{-1}(U)$. The last assertion follows from this fact.

For each open subset D of $\boldsymbol{C}$, positive integer m and $\delta>0$, let $C(D, m, \delta)$ be the smallest constant such that for given $f_1, \cdots, f_m\in H^\infty(D)$ with $\|f_i\|_\infty\leqq 1$ $(i=1, \cdots, m)$ and $\sum_{i=1}^m|f_i|\geqq\delta$ there exists $g_1, \cdots, g_m\in H^\infty(D)$ satisfying $\sum_{i=1}^m f_ig_i=1$ with $\|g_i\|_\infty\leqq C(D, m, \delta)$ $(i=1, \cdots, m)$. And if no such constant exists we set $C(D, m, \delta)=\infty$.

As a consequence of the well known equivalent form of the corona problem, if $C(D, m, \delta)$ is finite for each $\delta>0$ and $m\geqq 1$, then D is dense in $\mathcal{M}(D)$. But in general the converse is not known. So the following theorem is a slight improvement of Gamelin's result ([6] Th. 3.2). For the proof, note that $\bigcup_{n=1}^\infty(D_n\times\{n\})$ is dense in $\mathcal{M}(\{D_n\})$ if and only if each D_n is dense in $\mathcal{M}(D_n)$ and $\limsup_{n\to\infty} C(D_n, m, \delta)<\infty$ for all $m\in\boldsymbol{N}$ and $\delta>0$ (cf. [2] Th. 8.1).

Theorem 2. *Let $\{D_n\}_{n=1}^\infty$ be a sequence of uniformly bounded open sets in $\boldsymbol{C}$. If there exists a positive constant ε such that the diameter of every component of $\boldsymbol{C}\backslash D_n$ exceeds ε for all n, then $\sup_n C(D_n, m, \delta)<\infty$ for all $m\in\boldsymbol{N}$ and $\delta>0$.*

Proof. Let $\{W_k\}_{k=1}^\infty$ be a sequence of open sets such that each W_k is equal to some D_n and every D_n appears infinitely often in this sequence. For each $\lambda\in\boldsymbol{C}$, let $\Delta\left(\lambda, \frac{\varepsilon}{2}\right)=\left\{z\in\boldsymbol{C}\,;\,|z-\lambda|<\frac{\varepsilon}{2}\right\}$. Then by hypothesis, each component of $W_k\cap\Delta\left(\lambda, \frac{\varepsilon}{2}\right)$ is simply connected for all k. Therefore, by Carleson's result (which asserts that for the unit disk Δ, $C(\Delta, m, \delta)$ is finite for all $m\in\boldsymbol{N}$ and $\delta>0$), $\bigcup_{k=1}^\infty\left(\left(W_k\cap\Delta\left(\lambda, \frac{\varepsilon}{2}\right)\right)\times\{k\}\right)$ is dense in $\mathcal{M}\left(\left\{W_k\cap\Delta\left(\lambda, \frac{\varepsilon}{2}\right)\right\}\right)$. Then it follows from Theorem 1 that $\bigcup_{k=1}^\infty(W_k\times\{k\})$ is also dense in $\mathcal{M}(\{W_k\})$. Therefore

$\limsup_{k\to\infty} C(W_k, m, \delta)<\infty$ and hence $\sup_n C(D_n, m, \delta)<\infty$ for all $m\in\boldsymbol{N}$ and $\delta>0$.

3. The sequence of disks $\{\Delta_n\}_{n=1}^{\infty}$, $\Delta_n=\Delta(c_n, r_n)=\{|z-c_n|\leqq r_n\}\subset\Delta\setminus\{0\}$ is called "hyperbolically-rare" if there exists numbers $R_n>r_n$ such that $\sum_{n=1}^{\infty}\frac{r_n}{R_n}<\infty$ and the disks $W_n=\Delta(c_n, R_n)$ are contained in Δ and mutually disjoint. In the following we fix a Δ-domain $D'=\Delta\setminus\overline{\bigcup_{n=1}^{\infty}\Delta'_n}$ with hyperbolically-rare sequence $\{\Delta'_n\}_{n=1}^{\infty}$. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of uniformly bounded domains and E_n be a closed set in Δ'_n whose complement in the extended plane is conformally equivalent to D_n for each n. Behrens [2] proved that $\Delta\setminus\overline{\bigcup_{n=1}^{\infty}E_n}$ is dense in $\mathcal{M}(\Delta\setminus\overline{\bigcup_{n=1}^{\infty}E_n})$ if and only if $\bigcup_{n=1}^{\infty}(D_n\times\{n\})$ is dense in $\mathcal{M}(\{D_n\})$. And as a corollary he showed that if the corona problem has a negative answer for some plane domain, there exists a Δ-domain $D=\Delta\setminus\overline{\bigcup_{m=1}^{\infty}\Delta_m}$ such that each $\Delta_m=\Delta(c_m, r_m)$ is contained in some Δ'_n and D is not dense in $\mathcal{M}(D)$. If $\{\Delta_m\}$ itself is hyperbolically-rare, or satisfies the following condition given by Deeb and Wilken [5]

$$\lim_{k\to\infty}\sum_{\substack{m=1\\ m\neq k}}^{\infty}\frac{r_m}{|c_m|}\cdot\frac{|c_k|}{|c_k-c_m|}=0 \qquad (*)$$

(which contains the hyperbolically-rare case), then D is dense in $\mathcal{M}(D)$.

The result of §2 offers some new examples of Δ-domains for which the corona problem is affirmative. Let D', D_n, E_n be as above. By Theorem 2 if the diameters of the complements of D_n are uniformly bounded from below by a positive constant, then $\Delta\setminus\overline{\bigcup_{n=1}^{\infty}E_n}$ is dense in $\mathcal{M}(\Delta\setminus\overline{\bigcup_{n=1}^{\infty}E_n})$. In the case that each E_n is the union of a finite number of disjoint disks, these are the Δ-domains for which the corona problem has an affirmative answer. We note that the above condition (*) requires certain smallness and mutually rareness of $\{\Delta_m\}$, for example (*) implies $\sum_{m=1}^{\infty}\frac{r_m}{|c_m|}<\infty$. So it will be meaningful to show the existence of a Δ-domain for which the corona problem is affirmative and satisfies the opposite condition: $\sum_{m=1}^{\infty}\frac{r_m}{|c_m|}=\infty$. In fact, we have the following example which even satisfies $\sum_{m=1}^{\infty}r_m=\infty$.

Example. Now we construct a Δ-domain for which the sum of radii of the deleting disks is infinite and the corona problem has an affirmative solution.

For positive integers j, k, μ, ν, we set

$$E_{\mu,j,k}=\left\{z=x+iy\,;\ y=\mu x+\frac{2\pi k}{\mu},\ -\frac{2j}{\mu}\leqq x\leqq-\frac{2j-1}{\mu}\right\}$$

$$E_{\nu,\mu}=\bigcup_{j=1}^{\nu}\bigcup_{k=1}^{\mu}E_{\mu,j,k}$$

$$E'_{\mu,j,k}=\{e^z\,;\ z\in E_{\mu,j,k}\}$$

$$E'_{\nu,\mu}=\{e^z\,;\, z\in E_{\nu,\mu}\}$$
$$D'_{\nu,\mu}=\Delta\setminus E'_{\nu,\mu}\,.$$

Then the set of domains $\{D'_{\nu,\mu}\,;\,\mu\geqq\nu\}$ satisfies the hypothesis of Theorem 2. And each $D'_{\nu,\mu}$ is conformally equivalent to a domain $D_{\nu,\mu}=\Delta\setminus\bigcup_{j=1}^{\nu}\bigcup_{k=1}^{\mu}\Delta_{\mu,j,k}$ by a conformal map φ with $\varphi(0)=0$, where $\Delta_{\mu,j,k}$ $(j=1,\cdots,\nu\,;\,k=1,\cdots,\mu)$ are mutually disjoint closed disks in Δ and each $\Delta_{\mu,j,k}$ corresponds to $E'_{\mu,j,k}$ through the boundary correspondence. Let $\sigma'(z)=e^{2\pi i/\mu}z$. Then $\sigma=\varphi\circ\sigma'\circ\varphi^{-1}$ is a conformal automorphism of $D_{\nu,\mu}$ with order μ, $\sigma(\partial\Delta)=\partial\Delta$, and $\sigma(0)=0$. It is easy to show that σ can be extended, by the repetition of reflections, to a conformal map σ_1 on a domain W where $W\in O_{AD}$ ([1] Chap. IV Theorem 16D), and then σ_1 is a restriction of a Möbius transformation to W ([1] Chap. IV Theorem 2D). So $\sigma(z)=e^{2\pi i/\mu}z$. Since $\sigma(\Delta_{\mu,j,k})=\Delta_{\mu,j,k+1}$ $(k=1,\cdots,\mu-1)$, $\sigma(\Delta_{\mu,j,\mu})=\Delta_{\mu,j,1}$, we can put $\Delta_{\mu,j,k}=\Delta(s_{\mu,j}e^{2\pi ki/\mu},\, b_{\mu,j})$ $(\mu\geqq1\,;\,j=1,\cdots,\nu\,;\,k=1,\cdots,\mu)$.

We want to show that for each ν, there exists an integer $\mu=\mu(\nu)\geqq\nu$ such that

(i) $D_{\nu,\mu(\nu)}\supset\left\{|z|<\dfrac{1}{2}\right\}$ and

(ii) $\mu(\nu)\sum_{j=1}^{\nu}b_{\mu(\nu),j}\geqq\nu$.

For this purpose, we utilize the module of quadrilaterals and of ring domains (for definitions see [8]). Now we fix ν, and set

$$Q_{\mu,j}=\left\{z=x+iy\,;\,\mu x+\frac{2\pi}{\mu}<y<\mu x+\frac{4\pi}{\mu},\ -\frac{2j}{\mu}<x<-\frac{2j-1}{\mu}\right\}$$
$$Q'_{\mu,j}=\{e^z\,;\,z\in Q_{\mu,j}\}\,.$$

We regard $Q_{\mu,j}$ as a quadrilateral with its "a-sides" $E_{\mu,j,1}$ and $E_{\mu,j,2}$. Then the module $M(Q_{\mu,j})$ of $Q_{\mu,j}$ is independent of j, and tends to ∞ as $\mu\to\infty$. Since the a-sides of the image quadrilateral $\varphi(Q'_{\mu,j})$ are contained in $\Delta_{\mu,j,1}$ and $\Delta_{\mu,j,2}$ respectively, we see from Chap. I lemma 6.5 of [8].

$$\lim_{\mu\to\infty}\max_{1\leqq j\leqq\nu}\frac{\mathrm{dist}(\Delta_{\mu,j,1},\,\Delta_{\mu,j,2})}{b_{\mu,j}}=0. \tag{1}$$

On the other hand, let $d_{\mu,j}$ be the largest number such that the ring domain

$$R_{\mu,j}=\{z\in \boldsymbol{C}\,;\,|s_{\mu,j}|+b_{\mu,j}<|z|<|s_{\mu,j}|+b_{\mu,j}+d_{\mu,j}\}$$

is contained in $D_{\nu,\mu}$. Then $\varphi^{-1}(R_{\mu,j})$ separates $E_{\mu,j',1}$ and $E_{\mu,j'-1,1}$ (or $\{|z|\geqq1\}$) for some j', and we see from Chap. I lemma 6.2 of [8],

$$\lim_{\mu\to\infty}\max_{1\leqq j\leqq\nu}d_{\mu,j}=0. \tag{2}$$

Finally it is obvious that

$$\lim_{\mu\to\infty}\max_{1\leqq j\leqq\nu}b_{\mu,j}=0. \tag{3}$$

From (1), (2), (3), it is easily concluded that for sufficiently large $\mu=\mu(\nu)$ the above conditions (i), (ii) are satisfied.

Now let $\{\Delta'_n\}_{n=1}^{\infty}$, $\Delta'_n=\Delta(c'_n,\, r'_n)$ be a hyperbolically-rare sequence of closed disks in Δ with $c'_n \to 0$ as before. Then by taking appropriate subsequence $\nu(n)$ satisfying the condition $\sum_{n=1}^{\infty} r'_n \nu(n) = \infty$, and conformal mappings $\varphi_n(z) = \frac{r'_n}{2z} + c_n$, we find that

$$G = \Delta \backslash \overline{\bigcup_{n=1}^{\infty} \varphi_n(\hat{C} \backslash D_{\nu(n),\, \mu(\nu(n))})}$$

gives a desired Δ-domain.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

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