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Stochastic control related to branching diffusion processes

By

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1. Introduction.

First of all we will recall a controlled stochastic differential equation (CSDE in short) and its Hamilton-Jacobi-Bellman equation (H-J-B eq. in short). [2, 4, 6, 13].

Let Γ be a compact subset of R^k. Let B be a d-dimensional Brownian motion. A Γ-valued process is called an admissible control, if it is progressively measurable with respect to B. $\mathfrak{A}$ denotes the totality of admissible controls.

Consider CSDE for $U\in\mathfrak{A}$,

$$(1.1)\qquad \begin{cases} d\xi(t)=\alpha(\xi(t),\ U(t))dB(t)+\gamma(\xi(t),\ U(t))dt \\ \xi(0)=x. \end{cases}$$

Under the mild conditions, we have a unique solution $\xi=\xi(\cdot,\ x,\ U)$ of (1.1). Define a pay-off function $V(t,\ x,\ \phi,\ U)$ by

$$(1.2)\qquad V(t,\ x,\ \phi,\ U)=E\int_0^t e^{-\int_0^s c(\xi(\theta),U(\theta))d\theta}f(\xi(s),\ U(s))ds + e^{-\int_0^t c(\xi(\theta),U(\theta))d\theta}\phi(\xi(t)),$$

where $\xi(t)=\xi(t,\ x,\ U)$. We want to maximize its value by a suitable choice of $U\in\mathfrak{A}$.

$$(1.3)\qquad V(t,\ x,\ \phi)=\sup_{U\in\mathfrak{A}}V(t,\ x,\ \phi,\ U)$$

is called a value function.

The operator $V(t)$ defined by

$$(1.4)\qquad V(t)\phi(x)=V(t,\ x,\ \phi)$$

becomes a semigroup on a Banach lattice of BUC(R^d)(=totality of bounded and uniformly continuous functions on R^d). Its generatory $\mathfrak{G}$ is given by

$$(1.5)\qquad \mathfrak{G}\phi=\sup_{u\in\Gamma}(A(u)\phi-c(x,\ u)\phi+f(x,\ u)),\qquad \text{for smooth } \phi,$$

where $A(u)=\sum_{ij} a_{ij}(x, u)\frac{\partial^2}{\partial x_i \partial x_j}+\sum_i \gamma_i(x, u)\frac{\partial}{\partial x_i}$ and $a(x, u)=\frac{1}{2}\alpha^2(x, u)$ [2, 9, 13]. Moreover $V(t)\phi(x)$ is a viscosity solution of H-J-B eq. [10],

$$(1.6)\quad \begin{cases} \dfrac{\partial V(t, x)}{\partial t}=\sup_{u\in\Gamma}(A(u)V(t, x)-c(x, u)V(t, x)+f(x, u)), \text{ in } (0, T)\times R^d \\ V(0, x)=\phi(x), \qquad x\in R^d. \end{cases}$$

Further smoothness of coefficients and non-degeneracy of α produce more regularity of solution [3, 7, 11].

Recently N. V. Krylov [7] and N. S. Trudinger [15] investigated more general Bellman equations, namely they extended $A(u)V-c(x, u)V+f(x, u)$ to some non-linear elliptic operator $F^u(V_{ij}, i\cdot j=1, \cdots d, V_i, i=1, \cdots d, V, x)$. Assuming uniform ellipticity and some regularity conditions, they showed the existence of a unique classical solution V of the following parabolic (or elliptic) Bellman equation,

$$\begin{cases} \dfrac{\partial V}{\partial t}=\inf_u F^u\Big(\dfrac{\partial^2 V}{\partial x_i \partial x_j}, i, j=1 \cdots d, \dfrac{\partial V}{\partial x_i}, i=1 \cdots d, V, x\Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0, T)\times R^d \\ V(0, x)=\phi(x), \quad \text{on } R^d. \end{cases}$$

In this article we will discuss control problems associated with the following simple case of F^u,

$F^u(V_{ij}, i, j=1, \cdots d, V_i\ i=1, \cdots d, V, x)=A(u)V-c(x, u)V+f(x, V, u)$ where $A(u)$ is a second order elliptic operator which may be degenerate. Namely we deal with the following H-J-B eq. (1.7) by the probability method.

$$(1.7)\quad \begin{cases} \dfrac{\partial V}{\partial t}=\sup_{u\in\Gamma}(A(u)V-c(x, u)V+f(x, V, u)) \\ \qquad\qquad\qquad\qquad\qquad \text{in } (0, T)\times R^d \\ V(0, x)=\phi(x) \quad \text{on } R^d, \end{cases}$$

Recalling the relation between (1.3) and (1.6), a solution of (1.7) seems to turn out the value function defined by the integral equation (1.8).

$$(1.8)\quad V(t, x)=\sup_{u\in\Gamma} E\int_0^t e^{-\int_0^s c(\xi(\theta), U(\theta))d\theta} f(\xi(s), V(t-s, \xi(s), U(s))ds$$
$$+e^{-\int_0^t c(\xi(\theta), U(\theta))d\theta}\phi(\xi(t)).$$

In § 2, we will show the existence of a unique solution of (1.8). Using the routine of stochastic control, we can prove the Bellman principle and the solution V of (1.8) provides a semigroup with generator (1.7), [Theorems 2 & 3]. Moreover V becomes a viscosity solution of (1.7) [Theorem 4].

If $f(x, v, u)=\lambda(u)\sum_{k=0}^{\infty} p_k(u)v^k$ with $p_1(u)=0$, $p_k(u)\geqq 0$ and $\sum_k p_k(u)=1$ and $c(x, u)=\lambda(u)>0$, then $A(u)V-\lambda(u)V+f(x, V, u)$ is the generator of branching diffusion. Therefore we can construct a solution of (1.7), using a stochastic control for branching diffusions, besides the value function V of (1.8). Moreover we have a semigroup $\overline{W}(t)$ with generator of (1.7), by the routine of time discrete approximation. Then $\overline{W}(t)\hat{\phi}(x)$ and $V(t, x)$ coincide under mild conditions. In §5, we assume that λ and p_k do not depend on u. Applying the same method as [8], we can see that the viscosity solution of (1.7) belongs to $W_\infty^{1,2}$, under the conditions of complementary non-degeneracy and smoothness of coefficients [Theorem 7].

2. Stochastic control associated with (1.8).

Let Γ be a compact subset of R^k, called a control region. $B(t)$, $t\geqq 0$ denotes a d-dimensional Brownian motion, defined on a probability space (Ω, F, P). Put $F_t=\sigma$-field generated by $B(s)$, $s\leqq t$ $(=\sigma_t(B))$. By an admissible control we mean a Γ-valued F_t-progressively measurable process. $\mathfrak{A}$ denotes the totality of admissible controls.

Let α be a $d\times d$ symmetric matrix valued function on $R^d\times\Gamma$, and γ and c R^d and R^1-valued functions on $R^d\times\Gamma$ respectively. We assume the following conditions

(A1) $$g(x, u)\in \mathrm{BUC}(R^d\times\Gamma), \quad g=\alpha_{ij}, \gamma_i, c$$

say, $|g(x, u)|\leqq b$, $g=\alpha, \gamma$, and $c(x, u)\geqq\lambda>-\infty$.

(A2) $$\sup_{u\in\Gamma}|g(x, u)-g(y, u)|\leqq K|x-y|, \quad g=\alpha, \gamma, c,$$

Suppose that $f\in\mathrm{BUC}(R^d\times R^1\times\Gamma)$ satisfies (A3)

(A3) $$\begin{cases} \sup_{u\in\Gamma}|f(x, v, u)-f(y, w, u)|\leqq K|x-y|+h|u-w|, \\ \sup_{x,v,u}|f(x, v, u)|\leqq b. \end{cases}$$

By virtue of (A1) and (A2), CSDE(1.1) has a unique solution $\xi(t)=\xi(t, x, U)$ which is F_t-progressively measurable.

Theorem 1. *Eq.* (1.8) *has a unique solution* $V\in\mathrm{BUC}([0, T]\times R^d)$ *for any* $T>0$.

Proof. For simplicity we put

(2.1) $$\begin{aligned} &F(t, x, \phi, g, U) \\ &=E_x\int_0^t e^{-\int_0^s c(\xi(\theta), U(\theta))d\theta} f(\xi(s), g(t-s, \xi(s)), U(s))ds \\ &\quad + e^{-\int_0^t c(\xi(s), U(s))ds}\phi(\xi(t)) \end{aligned}$$

where $\xi(t)=\xi(t, x, U)$ and sub x of E_x means the starting point of ξ. Define V_k, $k=0, 1, 2\cdots$ as follows,

$$V_0(t, x)=\phi(x) \tag{2.2}$$

$$V_k(t, x)=\sup_{U\in\mathfrak{A}} F(t, x, \phi, V_{k-1}, U), \ k=1, 2, \cdots . \tag{2.3}$$

Putting $\|\phi\|$=supremum value of $|\phi|$, we get

Lemma

(i) $$|V_k(t, x)|\leqq\|f\|\frac{1-e^{-\lambda t}}{\lambda}+\|\phi\|e^{-\lambda t}$$

where $\frac{1-e^{-\lambda t}}{\lambda}$ *stands for* t *when* $\lambda=0$.

(ii) $$V_k\in \mathrm{BUC}([0, T]\times R^d) \quad \textit{for any} \quad T>0$$

Proof (i) is clear by (A1).

(ii). Put $\xi(t)=\xi(t, x, U)$ and $\eta(t)=\xi(t, y, U)$.
Recalling the following evaluations

$$E|\xi(t)-\eta(t)|^2\leqq|x-y|^2e^{3Kt} \quad (\text{say } |x-y|^2q^2(t)) \tag{2.4}$$

and

$$E|\xi(t+\theta)-\xi(t)|^2\leqq 2b^2(\theta+\theta^2) \tag{2.5}$$

we have

$$\begin{aligned} &|F(t+\theta, x, \phi, g, U)-F(t, y, \phi, g, U)| \\ &\leqq\int_t^{t+\theta} e^{-\lambda s}\|f\|ds+E\int_0^t e^{-\lambda s}K|\xi(s)-\eta(s)| \\ &\quad +h|g(t+\theta-s, \xi(s))-g(t-s, \eta(s))|ds \\ &\quad +E\int_0^t |e^{-\int_0^s c(\xi(\theta),U(\theta))d\theta}-e^{-\int_0^s c(\eta(\theta),U(\theta))d\theta}|ds\|f\| \\ &\quad +E|e^{-\int_0^{t+\theta} c(\xi(s),U(s))ds}-e^{-\int_0^t c(\eta(s),U(s))ds}|\,\|\phi\| \\ &\quad +e^{-\lambda t}E|\phi(\xi(t+\theta))-\phi(\eta(t))| \end{aligned} \tag{2.6}$$

Using (2.4) and (2.5) we can easily prove (ii) by induction.

Now we will prove Theorem. From the definition of V_k, we see

$$\begin{aligned} |V_{k+1}(t, x)-V_k(t, x)|\leqq\sup_{U\in\mathfrak{A}} E\int_0^t e^{-\lambda s}h|V_k(t-s, \xi(s, x, U)) \\ -V_{k-1}(t-s, \xi(s, x, U))|ds \end{aligned} \tag{2.7}$$

Putting $\rho_k(t)=\|V_k(t, \cdot)-V_{k-1}(t, \cdot)\|$, (2.7) turns out

$$\rho_{k+1}(t)\leqq\int_0^t e^{-\lambda s}h\rho_k(t-s)ds=h\int_0^t e^{-\lambda(t-s)}\rho_k(s)ds$$

i. e,

(2.8) $$e^{\lambda t}\rho_{k+1}(t)\leqq h\int_0^t e^{\lambda s}\rho_k(s)ds$$

Since $e^{\lambda t}\rho_k(t)\leqq\|f\|\dfrac{e^{\lambda t}-1}{\lambda}+\|\phi\|$, by Lemma (i), (2.8) implies

(2.9) $$e^{\lambda t}\rho_{k+1}(t)\leqq\frac{h^k t^k}{k!}\Big(\|f\|\frac{e^{\lambda t}-1}{\lambda}+\|\phi\|\Big).$$

Therefore $\sum_{k=1}^{\infty}(\sup_{t\leq T}\rho_k(t))<\infty$, for $T>0$. So, V_k converges uniformly on $[0, T]\times R^d$, as $k\to\infty$. Put $V=\lim_{k\to\infty}V_k$. Then $V\in\mathrm{BUC}([0, T]\times R^d)$ by Lemma (ii). Moreover

(2.10)
$$\begin{aligned}&|F(t, x, \phi, V_k, U)-F(t, x, \phi, V, U)|\\&\leqq h\int_0^t e^{-\lambda s}\|V_k(t-s, \cdot)-V(t-s, \cdot)\|ds\\&\leqq\sum_{j=k}^{\infty}\sup_{\theta\leq t}\rho_j(\theta)\int_0^t e^{-\lambda s}ds.\end{aligned}$$

Hence, as $k\to\infty$, we have

$$\sup_{U\in\mathfrak{A}}\|F(t, \cdot, \phi, V_k, U)-F(t, \cdot, \phi, V, U)\|\longrightarrow 0.$$

So, we see

$$\sup_{U\in\mathfrak{A}}F(t, x, \phi, V, U)=V(t, x)$$

i.e., V is a solution of (1.8).

Let $\bar{V}\in\mathrm{BUC}([0, T]\times R^d)$, for any $T>0$, be a solution of (1.8). Then, by the same argument as (2.9), we have

$$\|\bar{V}(t, \cdot)-V(t, \cdot)\|\leqq\frac{h^k t^k}{k!}\|\bar{V}(t, \cdot)-V(t, \cdot)\|,\qquad\text{for } k=1, 2, \cdots.$$

Hence $\bar{V}=V$. This completes the proof of Theorem.

3. Semigroup $V(t)$.

Firstly we recall the following Bellman principle. Define W by

(3.1)
$$\begin{aligned}W(t, x)=\sup_{U\in\mathfrak{A}}E_x\int_0^t &e^{-\int_0^s c(\xi(\theta),U(\theta))d\theta}g(t-s, \xi(s), U(s))ds\\&+e^{-\int_0^t c(\xi(\theta),U(\theta))d\theta}\phi(\xi(t))\end{aligned}$$

where $g\in\mathrm{BUC}([0, T]\times R^d)$ for any $T>0$. Then the Bellman principle holds, i.e.

(3.2)
$$\begin{aligned}W(t+s, x)=\sup_{U\in\mathfrak{A}}E_x\int_0^t &e^{-\int_0^z c(\xi(\theta),U(\theta))d\theta}g(t+s-z, \xi(z), U(z))dz\\&+e^{-\int_0^t c(\xi(\theta),U(\theta))d\theta}W(s, \xi(t)).\end{aligned}$$

Putting $g(t, x, u)=f(x, V_k(t, x), u)$, we apply (3.1) and (3.2). Then we have

$$(3.3)\qquad V_{k+1}(t, s, x)=\sup_{U\in\mathfrak{A}} E_x\int_0^t e^{-\int_0^z c(\xi(\theta),U(\theta))d\theta} f(\xi(z), V_k(t+s-z, \xi(z), U(z))dz$$

$$+e^{-\int_0^t c(\xi(\theta),U(\theta))d\theta} V_{k+1}(s, \xi(t)).$$

Tending k to ∞, we can see Bellman principle for V, i.e.

$$(3.4)\qquad V(t+s, x)=\sup_{U\in\mathfrak{A}} E_x\int_0^t e^{-\int_0^t c(\xi(\theta),U(\theta))d\theta} f(\xi(z), V(t+s-z, \xi(z), U(z))dz$$

$$+e^{-\int_0^t c(\xi(\theta),U(\theta))d\theta} V(s, \xi(t)).$$

Stressing the dependency on the initial value ϕ, we will denote a unique solution $V(t, x)$ of (1.8) by $V(t, x, \phi)$. Put

$$(3.5)\qquad \tilde{V}(t, x, \phi)=V(t+s, x, \phi).$$

Then (3.4) turns out

$$(3.6)\qquad \tilde{V}(t, x, \phi)=\sup_{U\in\mathfrak{A}} F(t, x, V(s, \cdot), \tilde{V}(\cdot, \cdot, \phi), U).$$

This means that $\tilde{V}$ is a solution of (1.8) with the initial value $V(s, \cdot, \phi)$. Therefore, by the uniqueness of solution, we have

$$(3.7)\qquad V(t+s, x, \phi)=V(t, x, V(s, \cdot, \phi)).$$

Now define $V(t)$ by

$$(3.8)\qquad V(t)\phi=V(t, \cdot, \phi)$$

Then $V(t)$ is a transformation on BUC(R^d) by Theorem 1.

Theorem 2. *$V(t)$ has the following properties*
(i) $V(0)=$*identity*
(ii) $V(t+s)=V(t)V(s)$
(iii) $\|V(t)\phi-\phi\|\to 0$ *as* $t\to 0$
(iv) $\|V(t)\phi-V(t)\psi\|\leqq e^{(h-\lambda)t}\|\phi-\psi\|$
(v) *$V(t)\phi$ is Lipschitz continuous, if ϕ is so.*

Proof. (i) is clear and (ii) is nothing but (3.7). Since $V(t)\phi(x)$ is uniformly continuous on $[0, T]\times R^d$, (iii) holds.

$$(3.9)\qquad |V(t)\phi(x)-V(t)\psi(x)|$$

$$\leqq\sup_{U\in\mathfrak{A}}|F(t, x, \phi, V(\cdot, \phi), U)-F(t, x, \psi, V(\cdot, \psi), U)|$$

$$\leqq\sup_{U\in\mathfrak{A}}\int_0^t e^{-\lambda s}hE_x|V(t-s)\phi(\xi(s))-V(t-s)\psi(\xi(s))|ds+e^{-\lambda t}\|\phi-\psi\|$$

Put $\rho(t)=\|V(t)\phi-V(t)\psi\|$. Then ρ is continuous and (3.9) implies

$$\rho(t)\leqq h\int_0^t e^{-\lambda s}\rho(t-s)ds+e^{-\lambda t}\|\phi-\psi\| \tag{3.10}$$

So we have (iv).

Lip(A) denotes the totality of Lipschitz continuous functions on R^d with Lipschitz constant A. Let $g\in \mathrm{BUC}([0, T]\times R^d)$ for any $T>0$ and $g(t, \cdot)\in \mathrm{Lip}(A)$, $\phi\in \mathrm{Lip}(B)$. Using (2.4) and (2.6) we have

$$\begin{aligned}&|F(t, x, \phi, g, U)-F(t, y, \phi, g, U)|\\ &\leqq\Big[\int_0^t e^{-\lambda s}(K+hA)q(s)ds+\|f\|\int_0^t e^{-\lambda s}K\int_0^s q(\theta)d\theta ds\\ &\quad+Bq(t)e^{-\lambda t}\Big]|x-y|,\end{aligned} \tag{3.11}$$

where $q(t)=e^{3Kt/2}$. Taking θ such that

$$h\int_0^\theta e^{-\lambda s}q(s)ds=\frac{1}{2}, \tag{3.12}$$

we have, for $t\leqq\theta$,

$$|F(t, x, \phi, g, U)-F(t, y, \phi, g, U)|\leqq\Big(Bp+r+\frac{A}{2}\Big)|x-y| \tag{3.13}$$

where p and r do not depend on g and ϕ. Now putting $g(t, x)=\phi$ and $A=B$, we have

$$V_1(t, \cdot)\in \mathrm{Lip}\Big(Bp+r+\frac{B}{2}\Big) \qquad \text{for} \quad t\leqq\theta . \tag{3.14}$$

Suppose $V_k(t, \cdot)\in \mathrm{Lip}\Big(\sum_{j=0}^{k-1}\Big(\frac{1}{2}\Big)^j(Bp+r)+\Big(\frac{1}{2}\Big)^k B\Big)$, for $t\leqq\theta$. Then recalling (3.13), we see

$$\begin{aligned}&|F(t, x, \phi, V_k, U)-F(t, y, \phi, V_k, U)\\ &\leqq\Big[Bp+r+\frac{1}{2}\Big(\sum_{j=0}^{k-1}\Big(\frac{1}{2}\Big)^j(Bp+r)+\Big(\frac{1}{2}\Big)^k B\Big)\Big]|x-y|\end{aligned}$$

i. e.

$$V_{k+1}(t, \cdot)\in \mathrm{Lip}\Big(\sum_{j=0}^{k}\Big(\frac{1}{2}\Big)^j(Bp+r)+\Big(\frac{1}{2}\Big)^{k+1}B\Big), \qquad \text{for} \quad t\leqq\theta . \tag{3.15}$$

Since V_k converges to V uniformly on $[0, \theta]\times R^d$, we see

$$V(t)\phi\in \mathrm{Lip}(2(Bp+r)) \qquad \text{for} \quad t\leqq\theta . \tag{3.16}$$

Repeating the same computation for $V(t)(V(\theta)\phi)$, we have

$$V(t+\theta)\phi\in \mathrm{Lip}(2(B_1p+r)) \qquad \text{for} \quad t\leqq\theta \tag{3.17}$$

where $B_1=2(Bp+r)$. This means that $V(t)\phi$ is Lipschitz continuous whenever $t\leqq 2\theta$. Repeating this argument, we can conclude (V).

Theorem 3. *Let $\mathfrak{G}$ be the strong generator of $V(t)$. Then*

(3.18) $\mathscr{D}(\mathfrak{G}) \supset \left\{\phi \in \mathrm{BUC}(R^d)\,;\ \frac{\partial \phi}{\partial x_i},\ \frac{\partial^2 \phi}{\partial x_i \partial x_j} \in \mathrm{BUC}(R^d),\ i,\ j=1, \cdots, d\right\} (=\mathscr{D})$

and

(3.19) $\mathfrak{G}\phi = \sup_{u\in\Gamma}(A(u)\phi - c(x, u)\phi + f(x, \phi(x), u))$, for $\phi \in \mathscr{D}$.

Proof. We apply Ito's formula for $\phi \in \mathscr{D}$. Then

(3.20)
$$F(t, x, \phi, V(t)\phi, U) - \phi(x)$$
$$= E_x\int_0^t e^{-\int_0^s c(\xi(\theta), U(\theta))d\theta}(A(U(s))\phi(\xi(s)) - c(\xi(s), U(s))\phi(\xi(s))$$
$$+ f(\xi(s), V(t-s)\phi(\xi(s)), U(s))ds.$$

By the uniform continuity of α, γ, c, ϕ and V, we have

(3.21)
$$V(t)\phi(x) - \phi(x) = \sup_{U\in\mathfrak{A}} E\int_0^t A(U(s))\phi(x) - c(x, U(s))\phi(x)$$
$$+ f(x, \phi(x), U(s))ds + o(t)$$

where $o(t)$ is small uniformly in U and x. On the other hand

(3.22) the main term of rigth side of (3.21)
$$\leqq \int_0^t \sup_{u\in\Gamma} A(u)\phi(x) - c(x, u)\phi(x) + f(x, \phi(x), u)ds$$
$$= t\cdot\sup_{u\in\Gamma}(A(u)\phi(x) - c(x, u)\phi(x) + f(x, \phi(x), u))$$
$$\leqq \sup_{U\in\mathfrak{A}} E\int_0^t A(U(s))\phi(x) - c(x, U(s))\phi(x) + f(x, \phi(x), U(s))ds$$

Therefore $\leqq$ turns out $=$. Thus we can complete the proof.

Theorem 4. *$V(t)\phi(x)$ is a unique viscosity solution of* (1.6) *in* BUC([0, T] $\times R^d$) *for any $T>0$, if* $\sup\|\alpha_{ij}(\cdot, u)\|_{W^2(R^d)} < \infty$, $ij=1\cdots, d$.

Proof. Put $V(t, x) = V(t)\phi(x)$. Then $V \in \mathrm{BUC}([0, T]\times R^d)$ and we can approximate V by a smooth bounded function W_k, so that

(3.23) $|V(t, x) - W_k(t, x)| > 2^{-k}$ for any $(t, x) \in [0, T]\times R^d$.

Define $\overline{W_k}$ by

(3.24)
$$\overline{W_k}(t, x) = \sup_{U\in\mathfrak{A}} E_x\int_0^t e^{-\int_0^s c(\xi(\theta), U(\theta))d\theta} f(\xi(s), W_k(t-s, \xi(s)), U(s))ds$$
$$+ e^{-\int_0^t c(\xi(\theta), U(\theta))d\theta}\phi(\xi(t)).$$

Then, as $k\to\infty$, $\overline{W_k}$ converges to V uniformly in $[0, T]\times R^d$ by (3.23). Moreover $\overline{W_k}$ is a unique viscosity solution of (3.25),

(3.25)
$$\begin{cases} \dfrac{\partial w}{\partial t} = \sup_{u\in\Gamma} A(u)W - c(x, u)W + f(x, W_k(t, x), u), & \text{in } (0, T)\times R^d \\ W(0, \cdot) = \phi. \end{cases}$$

Since $f(x, W_k(t, x), u)$ converges to $f(x, V(t, x), u)$ uniformly in $[0, T]\times R^d \times \Gamma$, as $k\to\infty$, V is a viscosity solution of (1.6), by virtue of stability of viscosity solution.

Let $W\in \mathrm{BUC}([0, T]\times R^d)$ be a viscosity solution of (1.7). Putting $g(t, x, u) = f(x, W(t, x), u)$, $g\in\mathrm{BUC}([0, T]\times R^d\times\Gamma)$ and W is a unique viscosity solution of (3.26)

$$\begin{cases} \dfrac{\partial w}{\partial t}=\sup_{u\in\Gamma} A(u)W-c(x, u)W+g(t, x, u) \quad \text{in} \quad (0, T)\times R^d \\ W(0, \cdot)=\phi. \end{cases} \tag{3.26}$$

By the uniqueness of viscosity solution of (3.26), W is expressed by value function of stochastic control, that is,

$$W(t, x)=\sup_{U\in\mathfrak{A}} E_x\int_0^t e^{-\int_0^s c(\xi(\theta), U(\theta))d\theta} g(t-s, \xi(s), U(s))ds + e^{-\int_0^t c(\xi(\theta), U(\theta))d\theta}\phi(\xi(t)). \tag{3.27}$$

This turns out the following equality,

$$W(t, x)=\sup_{U\in\mathfrak{A}} F(t, x, \phi, W, U). \tag{3.28}$$

Namely, W is a solution of (1.8). Therefore $W=V$. This completes the proof.

Remark. $V(t)$, $t\geqq 0$ satisfies the following condition, for ϕ and $\psi\in\mathfrak{D}$,

$$\lim_{t\to 0}\frac{1}{t}(V(t)(\phi+t\psi)-\phi)(x)=\psi(x)+\mathfrak{G}\phi(x), \qquad \text{for} \quad x\in R^d, \tag{3.29}$$

Proof.

(3.30)

$$E_x\Big[\int_0^t e^{-\int_0^s c(\xi(\theta), U(\theta))d\theta} f(\xi(s), V(t-s)(\phi+t\psi)(\xi(s)), U(s))ds + e^{-\int_0^t c(\xi(\theta), U(\theta))d\theta}(\phi(\xi(t))+t\psi(\xi(t)))-\phi(x)\Big]$$

$$=E_x\Big[\int_0^t e^{-\int_0^t c(\xi(\theta), U(\theta))d\theta}\{f(\xi(s), V(t-s)\phi(\xi(t)), U(s))+A(U(s))\phi(\xi(s), U(s))\phi(\xi(s))\}ds + te^{-\int_0^t c(\xi(\theta), U(\theta))d\theta}\psi(\xi(t))\Big]$$

$$+E_x\Big[\int_0^t e^{-\int_0^s c(\xi(\theta), U(\theta))d\theta} f(\xi(s), V(t-s)(\phi+t\psi)(\xi(s)), U(s)) - f(\xi(s), V(t-s)\phi(\xi(s)), U(s))ds\Big]$$

By Theorem 2(iv), we have

$$|\text{the 3rd term of right side}|\leqq hte^{ht}\|\psi\|. \tag{3.31}$$

So, using the same argument as (3.19), we can derive (3.26).

Besides (A1)$\sim$(A3) we assume

(A4) $$v\leqq w \Rightarrow f(x, v, u)\leqq f(x, w, u) \quad \text{for any} \quad x, u.$$

The condition (A4) clearly implies the monotonicity of $V(t)$, i.e.,

(3.32) $$V(t)\phi\leqq V(t)\psi \quad \text{if} \quad \phi\leqq\psi.$$

The following can be shown in the same way as [12].

Proposition. *Suppose that* (A1)$\sim$(A4) *hold. Let* $\sup_{u\in\Gamma}\|\alpha_{ij}(\cdot, u)\|_{W^2(R^d)}<\infty$ *and* $S(t)$, $t\geqq 0$, *be a strongly continuous semigroup on* BUS(R^d), *whose generater satisfies* (3.18) *and* (3.19). *If* $S(t)$ *has the properties* (3.29) *and* (3.32), *then*

$$S(t)=V(t), \quad \text{for any} \quad t\geqq 0.$$

Concerning a classical solution of H-J-B equation corresponding to a family of quasilinear operators, we can find neater results in [7], [15].

4. Controlled branching diffusion.

This section is concerned with stochastic control for branching diffusions. We assume the following conditions besides (A1) and (A2).

(A5) $$c(x, u)=\lambda>0$$

(A6) $$f(x, v, u)=\lambda\sum_{k=0}^{\infty} p_k v^k$$

where $p_k\geqq 0$, $p_1=0$ and $\sum_{k=0}^{\infty} p_k=1$.

(A7) $$M=\sum_{k=0}^{\infty} k^2 p_k<\infty, \quad \text{and put} \quad m=\sum_{k=0}^{\infty} k p_k.$$

Let $\bar{B}$ be a d-dimensional branching Brownian motion on $[0, T]$, [1, 5, 14]. $Z(t)$ denotes the number of Brownian particles at t. Let τ_1 be the 1st branching time with the following distribution,

(4.1) $$P(\tau_1>t/Z(0)=1)=e^{-\lambda t}$$

and

(4.2) $$P(Z(\tau_1)=k/Z(0)=1)=p_k, \quad k=0, 1, 2, \cdots.$$

Namely each Brownian particle has an exponentially distributed branching time and creates at that time independent $(k-1)$ Brownian new particles with probability p_k, $k=2, 3, \cdots$ and disappears with probability p_0. Hereafter we assume $Z(0)=1$. Since each Brownian particle has its ancestor, we connect each new born particle with its ancestor and get Brownian motion up to its life time. Let δ be a trap. When a Brownian particle B disappears at τ, we put $B(t)=\delta$ for $t\geqq\tau$. Namely a particle moves on $R^d\cup\{\delta\}$. Let Z^* be the number of

Brownian particles on $R^d\cup\{\delta\}$ of $\bar{B}(T)$. Then $(B^1(t, \omega), \cdots, B_{Z^*(\omega)}(t, \omega))$, $0\leqq t\leqq T$, can express $\bar{B}$. Moreover, under the condition $Z^*\geqq i$, B_i is expressed as follows, before its life time ζ_i,

$$B_i(t)=\xi_1(t)\chi_{[0,\tau_1)}(t)+(\xi_1(\tau_1)+\xi_2(t-\tau_1))\chi_{[\tau_1,\tau_2)}(t)+\cdots$$
$$+\Big(\sum_{k=1}^{n-1}\xi_k(\tau_k-\tau_{k-1})+\xi_n(t-\tau_{n-1})\Big)\chi_{[\tau_{n-1},\tau_n)}(t), \qquad \text{if} \quad \zeta_i=\tau_n$$

where τ_k is the kth branching time of $\bar{B}$ and $\xi_1, \xi_2 \cdots$ are suitable independent Brownian motions (may depends on i). Hence

$$P(B_i(t_k)\in A_k,\ k=1, \cdots, l/\tau_1,\ Z(\tau_1), \cdots, \tau_j,\ Z(\tau_j), \cdots, \zeta_i)$$
$$=P(B_i(t_k)\in A_k,\ k=1, \cdots, l/\zeta_i)$$

Put $L(i, t, w)=\min\{j\,;\, B_j(t, w)=B_i(t, w)$ for ${}^\forall s\leqq t\}$. Namely $B_i(\cdot, w)$ and $B_j(\cdot, w)$ have the same ancestor before t. Define z by

$$z(t, w)=\{L(i, t, w)\,;\,\zeta_i>t\}.$$

Then, the number of elements of $z(t)=Z(t)$. For example, if $t<\tau_1$, then $B_k(t)=B_1(t)$ and $\zeta_k\geqq\tau_1$ for any k. So $L(i, t, w)=1$ and $z(t)=\{1\}$. $Z(t)=0$ for $t\geqq s$ whenever $Z(s)=0$. Moreover $Z(t)$ is a Galton-Watson process and

$$EZ(t)=e^{(m-1)\lambda t}, \tag{4.3}$$

$$\operatorname{Var}(Z(t))=\begin{cases} \dfrac{M+1}{m-1}(e^{2(m-1)\lambda t}-e^{(m-1)\lambda t} & \text{for} \quad m\neq 1 \\ (M+1)\lambda t & \text{for} \quad m=1\,. \end{cases} \tag{4.4}$$

So there exists a constant Λ such that

$$EZ^2(t)\leqq e^{\Lambda t}.$$

Finally we can easily see that, for any i, $B_i(t)$, $0\leqq t<\zeta_i$, is a Brownian motion under the condition $z(t)=\{j_1, \cdots, j_k\}$.

Now we consider controlled branching diffusion starting at $x\in R^d$, in the following way. Let $U\,;\,[0, T)\times C([0, T)\to R^d)\to\Gamma$ be progressively measurable with respect to the σ-field on $C([0, T)\to R^d)$. U is called an admissible control and $\mathfrak{A}$ denotes the totality of admissible controls.

Consider the following CSDE, for $U\in\mathfrak{A}$

$$\begin{cases} dX_k(t)=\alpha(X_k(t), U(t, B_k))dB_k+\gamma(X_k(t), U(t, B_k))dt \\ X_k(0)=x \end{cases} \tag{4.5}$$

under the condition $Z^*\geqq k$.

Using the successive approximation, we can easily see the existence of a unique strong solution $X_k=X_k(\cdot, x, U)$. Moreover X_k has the same law as X_1, up to its life time ζ_k and $\bar{X}(t, w)=(X_j(t, w),\ j\in z(t, w))$ has the same branching law as $\bar{B}$. Let $\xi=\xi(\cdot, x, U)$ be a non-branching part of $\bar{X}$, that is

$$(4.6)\qquad \begin{cases} d\xi(t)=\alpha(\xi(t),\ U(t,\ B))dB+\gamma(\xi(t),\ U(t,\ B))dt \\ \xi(0)=x\,. \end{cases}$$

When U is a constant function, say $u\in\Gamma$, $\bar{X}$ is a branching diffusion with generator $\mathfrak{G}(u)$,

$$(4.7)\qquad \mathfrak{G}(u)\phi=\sum_{ij} a_{ij}(x,\ u)\frac{\partial^2\phi}{\partial x_i\partial x^j}\sum_i \gamma_i(x,\ u)\frac{\partial\phi}{\partial x_i}-\lambda\phi+\lambda\sum_{k=0}^{\infty}p_k\phi^k\,.$$

Put $C=\{\phi\in\mathrm{BUC}(R^d)\,;\,0\leqq\phi\leqq 1 \text{ on } R^d\}$. Then C is a convex closed subset of $\mathrm{BUC}(R^d)$. For $\phi\in C$ we define a pay-off function W as follows

$$(4.8)\qquad W(t,\ x,\ \phi,\ U)=E\prod_{i\in z(t)}\phi(X_i(t,\ x,\ U))$$

where $\prod_{i\in\Phi}\phi=1$ for $z(t)=\Phi$ (=empty set)

$$(4.9)\qquad W(t,\ x,\ \phi)=\sup_{U\in\mathfrak{A}}W(t,\ x,\ \phi,\ U)\,.$$

According to [5], we denote $\prod_{i\in z(t)}\phi(X_i(t,\ x,\ U))$ by $\hat{\phi}(\bar{X}(t,\ x,\ u))$. Using (2.4) and (2.5) we see the following proposition.

Proposition 4.1. *Put* $p=(m-1)\lambda$ *and* $q(t)=e^{3kt/2}$.

(i) $W(t,\ \cdot\phi,\ U)\in C\cap\mathrm{Lip}(ae^{pt}q(t))$ *if* $\phi\in C\cap\mathrm{Lip}(a)$.

(ii) $W(t,\ \cdot\phi)\in C\cap\mathrm{Lip}(ae^{pt}q(t))$, *if* $\phi\in C\cap\mathrm{Lip}(a)$

(iii) $W(t,\ x,\ \phi,\ U)$ *is uniformly continuous in* x, *uniformly in* $U\in\mathfrak{A}$, *whenever* $\phi\in C$.

(iv) $W(t,\ \cdot\phi)\in C$, *if* $\phi\in C$

(v) $\|W(t,\ \cdot\phi)-W(t,\ \cdot\psi)\|\leqq\|\phi-\psi\|e^{pt}$

(vi) $\sup_{U\in\mathfrak{A}}\|W(t,\ \cdot\phi,\ U)-W(s,\ \cdot\phi,\ U)\|\to 0$ *as* $t\to s$,

(vii) $\|W(t,\ \cdot\phi)-W(s,\ \cdot\phi)\|\to 0$, *as* $t\to s$.

Proof. It is enough to show (i) (iii) (v) and (vi).

(i)
$$\begin{aligned} &|W(t,\ x,\ \phi,\ U)-W(t,\ y,\ \phi,\ U)| \\ &\leqq E\sum_{i\in z(t)}|\phi(X_i(t,\ x,\ U))-\phi(X_i(t,\ y,\ U))| \\ &=EE\Big(\sum_{i\in z(t)}|\phi(X_i(t,\ x,\ U)-\phi(X_i(t,\ y,\ U))|/z(t)\Big) \\ &=\sum_{k=1}^{\infty}kE|\phi(\xi(t,\ x,\ U))-\phi(\xi(t,\ y,\ U))|P(Z(t)=k) \\ &\leqq\sum_{k=1}^{\infty}ka|x-y|q(t)P(Z(t)=k)=aq(t)e^{pt}|x-y|. \end{aligned}$$

(iii) and (v) are proved in the same way.

(vi)
$$|W(t,\ x,\ \phi,\ U)-W(s,\ x,\ \phi,\ U)|$$

(4.10) $\leqq E|\prod_{i\in z(t)}\phi(X_i(t, x, U))-\prod_{i\in z(s)}\phi(X_i(s, x, U))|$

$$\leqq E\Big(|\prod_{i\in z(s)}\phi(X_i(t, x, U)-\prod_{i\in z(s)}\phi(X_i(s, x, U))|\ ;$$

$$\text{no branching time}\in[s, t]\Big)+P(\exists\text{branching time}\in[s, t])$$

$$\leqq E\Big(\sum_{i\in z(s)}|\phi(X_i(t, x, U))-\phi(X_i(s, x, U))|\Big)+P(\exists\text{branching time}\in[s, t]).$$

Since, for $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that

$$|\phi(x)-\phi(y)|<\varepsilon \quad \text{if} \quad |x-y|<\delta, \quad \text{we see}$$

(4.11) $\quad$ 1st term of right side $\leqq\varepsilon EZ(s)+E\sum_{i\in z(s)}\chi_{(\delta\infty)}(|X_i(t, x, U)$

$$-X_i(s, x, U)|)$$

(4.12) $\quad$ 2nd term of right side of (4.11)

$$=EE\Big(\sum_{i\in z(s)}\chi_{(\delta\infty)}(|X_i(t, x, U)-X_i(s, x, U)|)/\sigma_s(\bar{B})\Big)$$

$$=E\sum_{i\in z(s)}P(|X_i(t, x, U)-X_i(s, x, U)|>\delta/\sigma_s(\bar{B}))$$

$$\leqq\sum_{k=1}^{\infty}P(Z(s)=k)k\frac{2b^2(t-s+(t-s)^2)}{\delta^2}$$

Combining (4.11) and (4.12) with (4.10), we can complete the proof of (vi).

Next we will prove the Bellman principle for W.

Proposition 4.2.

(4.13)
$$W(t+s, x, \phi)=W(t, x, W(s, \cdot\phi))$$

Proof.

(4.14)
$$W(t+s, x, \phi, U)=E\hat{\phi}(\bar{X}(t+s, x, U))$$
$$=EE(\hat{\phi}(\bar{X}(t+s, x, U))/\sigma_t(\bar{B}))$$

Since, under the conditional probability $P(\cdot/\sigma_t(\bar{B}))$, $\bar{B}(\theta+t)$, $\theta\geqq0$, becomes $Z(t)$ independent branching Brownian motions, say $\bar{B}_l\,;\,l\in z(t)$, we see

(4.15)
$$E(\hat{\phi}(\bar{X}(t+s, x, U))/\sigma_t(\bar{B}))\leqq\prod_{i\in z(t)}W(s, X_i(t, x, U)), \phi)$$
$$=\hat{W}(s, \bar{X}(t, x, U), \phi)$$

Hence we have, from (4.14) and (4.15),

(4.16)
$$W(t+s, x, \phi)\leqq W(t, x, W(s, \cdot, \phi))$$

For the converse inequality, we recall the regularity of W in Proposition 4.1

and take ε-optimal control as follows. For $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)$, such that if $|x-y|<\delta$ then

$$|W(s, x, \phi, U)-W(s, y, \phi, U)|<\varepsilon \quad \text{for any} \quad U\in\mathfrak{A} \tag{4.17}$$

and

$$|W(s, x, \phi)-W(s, y, \phi)|<\varepsilon.$$

Let D_i, $i=1, 2, \cdots$ be a Borel partition of R^d such that diameter of $D_i<\delta$, $i=1, 2, \cdots$. Fix $x_i\in D_i$ arbitrarily and take $U_i\in\mathfrak{A}$ so that

$$W(s, x_i, \phi)-\varepsilon<W(s, x_i, \phi, U_i). \tag{4.18}$$

Then we have, for $y\in D_i$

$$\begin{aligned} W(s, y, \phi)&<W(s, x_i, \phi)+\varepsilon<W(s, x_i, \phi, U_i)+2\varepsilon \\ &<W(s, y, \phi, U_i)+3\varepsilon. \end{aligned} \tag{4.19}$$

Namely U_i is a 3ε-optimal for any $y\in D_i$.

Define an admissible control $\tilde{U}: [0, T)\times C([0, T)\to R^d)\to\Gamma$ as follows,

$$\tilde{U}(\theta, w)=\begin{cases} U(\theta, w) & \theta<t \\ \sum U_i(\theta-t, w(\cdot+t)-w(t))\chi_{D_i}(\Phi(w)) & \theta\geqq t \end{cases} \tag{4.20}$$

where $\Phi=\Phi_{t,x,U}: C([0, t]\to R^d)\to R^d$, so that,

$$\Phi(w)=\xi(t, w)=\text{solution of (4.6) for } B(\cdot, w) \tag{4.21}$$

Putting $\tilde{U}_i(\theta, w)=U_i(\theta-t, w(\cdot+t)-w(t))$, we have

$$W(t+s, x, \phi, \tilde{U})=EE(\hat{\phi}(\bar{X}(t+s, x, \tilde{U}))/\sigma_t(\bar{B})) \tag{4.22}$$

and

$$\begin{aligned} &E(\hat{\phi}(\bar{X}(t+s, x, \tilde{U}))/\sigma_t(\bar{B})) \\ &=\prod_{l\in z(t)}\sum_{i=1}^{\infty}E\hat{\phi}(\bar{X}_l(s, X_l(t, x, U), U_i)\chi_{D_i}(X_l(t, x, U)) \\ &\geqq\prod_{l\in z(t)}(W(s, X_l(t, x, U), \phi)-3\varepsilon)\vee 0 \end{aligned} \tag{4.23}$$

where $a\vee b=\max(a, b)$. Taking the expectation of both sides, we have

$$W(t+s, x, \phi, \tilde{U})\geqq W(t, x, (W(s, \cdot, \phi)-3\varepsilon)\vee 0, U)$$

Hence

$$W(t+s, x, \phi)\geqq W(t, x, W(s, \cdot, \phi)-3\varepsilon)\vee 0, U) \tag{4.24}$$

As $\varepsilon\downarrow 0$, we see, from Proposition 4.1 (v),

$$W(t+s, x, \phi)\geqq W(t, x, W(s, \cdot, \phi), U). \tag{4.25}$$

Since U is arbitrary, (4.25) derives the required one and completes the proof.

Define $W(t)\,;C\to C$ by

$$W(t)\phi(x)=W(t, x, \phi) \tag{4.26}$$

Then we have the following theorem, from Propositions 4.1 and 4.2.

Theorem 5.
(i) $W(0)=identity\quad W(t+s)=W(t)W(s)$
(ii) $W(t)\phi\leqq W(t)\psi \quad if \quad \phi\leqq\psi$
(iii) $\|W(t)\phi-W(t)\psi\|\leqq e^{(m-1)\lambda t}\|\phi-\psi\|$
(iv) *Let* $\mathfrak{G}$ *be the strong generator of* $W(t)$. *Then* $\mathcal{D}(\mathfrak{G})\supset\mathcal{D}\cap C$

and

$$\mathfrak{G}\phi=\sup_{u\in\Gamma}A(u)\phi-\lambda\phi+\lambda\sum_{k=0}^{\infty}p_k\phi^k, \qquad for \quad \phi\in\mathcal{D}\cap C. \tag{4.27}$$

Proof of (iv).

$W(t, x, \phi, U)-\phi(x)$

$=E(\phi(X_1(t, x, U))-\phi(x)\,;$ no branching time in $[0, t])$

$+E(\hat{\phi}(\bar{X}(t, x, U))-\phi(x)\,;\ ^{\exists}$branching time $\in[0, t])$

$=E(\phi(X_1(t, x, U))-\phi(x))-E(\phi(X_1(t, x, U)\,;\ ^{\exists}$branching time $\in[0, t])$

$+E(\hat{\phi}(\bar{X}(t, x, U)\,;\ ^{\exists}$branching time $\in[0, t])$

Again using the regularity of α, γ, ϕ and W, we can see in the same way as Theorem 3

$$W(t, x, \phi)-\phi(x) \tag{4.28}$$
$$=t\Big(\sup_{u\in\Gamma}A(u)\phi-\lambda\phi+\lambda\sum_{k=0}^{\infty}p_k\phi^k\Big)+o(t).$$

This completes the proof.

Theorem 6. *$V(t, x)=W(t)\phi(x)$ is a viscosity solution of* (4.29),

$$\begin{cases} \dfrac{\partial V}{\partial t}-\sup_{u\in\Gamma}A(u)V+\lambda V-\lambda\sum_{k=0}^{\infty}p_kV^k=0, & in \quad (0, T)\times R^d \\ V(0, x)=\phi(x), \quad on \quad R^d. \end{cases} \tag{4.29}$$

Moreover, if $W\in$BUC$([0, T]\times R^d)$ *is a viscosity solution of* (4.29) *and* $|W(t, x)|\leqq 1$ *then* $V=W$, *under the condition* $\sup_{u\in\Gamma}\|\alpha_{ij}(\cdot, u)\|_{W^2(R^d)}<\infty$, $i, j=1, \cdots, d$.

Proof.. Let $\psi\in$BUC$((0, T)\times R^d)$ be a smooth function such that $\dfrac{\partial\psi}{\partial t}$, $\dfrac{\partial\psi}{\partial x_i}$ and $\dfrac{\partial^2\psi}{\partial x_i\partial x_j}$ belong to BUC$((0, T)\times R^d)$. Suppose that $V-\psi$ has a strict maximum at $(t_0, x_0)\in(0, T)\times R^d$. Now we will show

$$\frac{\partial\phi}{\partial t}(t_0, x_0)-\sup_{u\in\Gamma}A(u)\phi(t_0, x_0)+\lambda V(t_0, x_0)-\lambda\sum_{k=0}^{\infty}p_kV^k(t_0, x_0)\leqq 0 \tag{4.30}$$

For the proof of (4.30) we may assume

$$\phi(t_0, x_0)=V(t_0, x_0) \tag{4.31}$$

Therefore

$$0\leqq V(t, x)\leqq(\phi\wedge 1)(t, x), \quad \text{in} \quad (0, T)\times R^d. \tag{4.32}$$

We apply a similar argument as (4.27)

$$\begin{aligned}
V(t_0, x_0)&=W(t_0)\phi(x_0)=W(\theta)V(t_0-\theta, \cdot)(x_0) \\
&=\sup_{U\in\mathfrak{A}}E\hat{V}(t_0-\theta, \bar{X}(\theta, x_0, U)\leqq\sup_{U\in\mathfrak{A}}E(\phi\wedge 1)(t_0-\theta, X(\theta, x_0, U)) \\
&=\sup_{U\in\mathfrak{A}}E(\phi\wedge 1)(t_0-\theta, X_1(\theta, x_0, U)) \\
&\quad -E\{(\phi\wedge 1)(t_0-\theta, X_1(\theta, x_0, U)\,;\,{}^{\exists}\text{branching time}\in[0, \theta]\} \\
&\quad +E\{(\widehat{\phi\wedge 1})(t_0-\theta, \bar{X}(\theta, x_0, U)\,;\,{}^{\exists}\text{branching time}\in[0, \theta]\}
\end{aligned} \tag{4.33}$$

(4.34) 2nd term of right side $=-(\phi\wedge 1)(t_0, x_0)\lambda\theta+o(\theta)$

3rd term of right side $=\sum_{k=0}^{\infty}p_k(\phi\wedge 1)^k(t_0, x_0)\lambda\theta+o(\theta)$

where $o(\theta)$ is small uniformly in $U\in\mathfrak{A}$. Recalling (4.31) we see $(\phi\wedge 1)(t_0, x_0)=V(t_0, x_0)$. Moreover Ito's formula tells us

$$\begin{aligned}
&E(\phi\wedge 1)(t_0-\theta, X_1(\theta, x_0, U))-\phi(t_0, x_0) \\
&\leqq E(\phi(t_0-\theta, X_1(\theta, x_0, U))-\phi(t_0, x_0) \\
&=E\int_0^{\theta}-\frac{\partial\phi}{\partial t}(t_0-t, X_1(t, x_0U))+A(U(t))\phi(t_0-t, X_1(t, x, U))dt
\end{aligned} \tag{4.35}$$

Thus, combining (4.34) and (4.35) with (4.33), we have

$$\begin{aligned}
0&\leqq\sup E\int_0^{\theta}-\frac{\partial\phi}{\partial t}(t_0-t, X_1(t, x_0, U))+A(U(t))\phi(t_0-t, X_1(t, x_0, U))dt \\
&\quad +\Big(-V(t_0, x_0)+\sum_{k=0}^{\infty}p_kV^k(t_0, x_0)\Big)\lambda\theta+o(\theta) \\
&=\Big(-\frac{\partial\phi}{\partial t}(t_0, x_0)+\sup_{u\in\Gamma}A(u)\phi(t_0, x_0)-\lambda V(t_0, x_0)+\lambda\sum_{k=0}^{\infty}p_kV^k(t_0, x_0)\Big)\theta+o(\theta)
\end{aligned}$$

This derives (4.30).

In the same way we can show that, if $V-\phi$ has a strict minimum at (t_0, x_0), then

$$\frac{\partial\phi}{\partial t}(t_0, x_0)-\sup_{u\in\Gamma}A(u)\phi(t_0, x_0)+\lambda V(t_0, x_0)-\lambda\sum_{k=0}^{\infty}p_kV^k(t_0, x_0)\geqq 0.$$

This concludes the former half of Theorem.

Put $g(v)=\lambda \sum_{k=0}^{\infty} p_k v^k$. Then $|g'(v)| \leqq \lambda m$ whenever $|v| \leqq 1$. Therefore, putting $f(v)=g((-1 \vee v) \wedge 1)$, we see

$$|f(v)-f(v')| \leqq \lambda m |v-v'|$$

Now Theorem 4 implies the uniqueness of viscosity solution of

$$(4.36)\qquad \begin{cases} \dfrac{\partial V}{\partial t}=\sup_{u\in\Gamma} A(u)V-\lambda V+f(V), & \text{in } (0, T)\times R^d \\ V(0, x)=\phi(x) \quad \text{on } R^d. \end{cases}$$

This concludes the later half of Theorem.

§5. Regularity of $W(t)\phi(x)$.

In this section we assume (A8) besides (A5)~(A7)

$$(A8)\qquad g(\cdot, u)\in \mathcal{D} \quad \text{for any } u\in\Gamma \text{ and } \sup\|g(\cdot, u)\|_{C^2(R^d)}<\infty$$

$$\text{where } g=\alpha_{ij}, \gamma_i, i, j=1, \cdots, d.$$

This condition implies that the solution ξ of (4.6) depends on its starting point x smoothly, that is, there exist B-adapted square integrable processes Y_{ij} and Z_{ijk} such that

$$(5.1)\qquad E\Big(\frac{\xi_i(t, x+\theta e_j, U)-\xi_i(t, x, U)}{\theta}-Y_{ij}(t, x, U)\Big)^2 \longrightarrow 0$$

as $\theta\to 0$, where e_j is the unit vector $(0, \cdots 0, 1, \cdots, 0)$

$$(5.2)\qquad E\Big(\frac{Y_{ij}(t, x, \theta e_k, , U)-Y_{ij}(t, x, U)}{\theta}-Z_{ijk}(t, x, U)\Big)^2 \longrightarrow 0$$

as $\theta\to 0$.

Namely $Y_{ij}(t, x, U)=\dfrac{\partial\xi_i(t, x, U)}{\partial x_j}$ and $Z_{ijk}(t, x, U)=\dfrac{\partial\xi_i(t, x, U)}{\partial x_j \partial x_k}$ in the sense of L^2-derivatives.

Proposition 5.1. *If $\phi\in C\wedge\mathcal{D}$, then $W(t, \cdot, \phi, U)\in\mathcal{D}\wedge C$. Moreover*

$$(5.3)\qquad \sup_{U\in\mathfrak{A}}\|W(t, \cdot, \phi, U)\|_{C^2(R^d)}<\infty$$

Proof. We apply the routine arguments. By (5.1) we have

$$(5.4)\qquad \frac{\partial W}{\partial x_j}(t, x, \phi, U)$$

$$=\sum_{q=1}^{d} E \sum_{l\in z(t)} \prod_{\substack{i\in z(t)\\ i\neq l}} \phi(X_i(t, x, U))\frac{\partial\phi}{\partial x_q}(X_l(t, x, U))\frac{\partial X_{l,q}}{\partial x_j}(t, x, U)$$

Hence

(5.5) $$M_1(T)=\max_{j=1\cdots d}\sup_{t\leq T,U\in\mathfrak{A}}\left\|\frac{\partial W}{\partial x_j}(t,\cdot,\phi,U)\right\|<\infty.$$

In the same way we have

(5.6) $$M_2(T)=\max_{j,k=1,\cdots d}\sup_{t\leq T,U\in\mathfrak{A}}\left\|\frac{\partial^2 W}{\partial x_j\partial x_j}(t,\cdot,\phi,U)\right\|<\infty.$$

This completes the proof.

From (5.3) we have, for any unit vector $\chi\in R^d$

(5.7) $$\begin{aligned}&W(t,x+\theta\chi,\phi)+W(t,x-\theta\chi,\phi)-2W(t,x,\phi)\\ &\geqq\sup_{U\in\mathfrak{A}}W(t,x+\theta\chi,\phi,U)+W(t,x-\theta\chi,\phi,U)-2\sup_{U\in\mathfrak{A}}W(t,x,\phi,U)\\ &\geqq\inf_{U\in\mathfrak{A}}W(t,x+\theta\chi,\phi,U)+W(t,x-\theta\chi,\phi,U)-2W(t,x,\phi,U)\\ &\geqq-\theta^2M_2(t).\end{aligned}$$

Consequently

(5.8) $$\frac{\partial^2 W}{\partial\chi^2}(t,x,\phi)\geqq-M_2(t)\text{ in distribution sense.}$$

Proposition 5.2. *For $\phi\in\mathfrak{D}\cap C$, there exists $M_3(t)$ such that*

(5.9) $$\sup_{U\in\mathfrak{A}}\|W(t+\theta,\cdot,\phi,U)-W(t,\cdot,\phi,U)\|\leqq M_3(t)\theta.$$

Proof. Fix x and U arbitrarily and put $X_i(t)=X_i(t,x,U)$

(5.10) $$\begin{aligned}&W(t-\theta,x,\phi,U)-W(t,x,\phi,U)\\ &=E(\prod_{i\in z(t)}\phi(X_i(t+\theta))-\prod_{i\in z(t)}\phi(X_i(t)))\\ &\quad-E(\prod_{i\in z(t)}\phi(X_i(t+\theta))-\prod_{i\in z(t)}\phi(X_i(t)))\,;\,{}^\exists\text{branching time}\in[t,t+\theta])\\ &\quad+E(\hat{\phi}(\bar{X}(t+\theta))-\hat{\phi}(\bar{X}(t))\,;\,{}^\exists\text{branching time}\in[t,t+\theta])\end{aligned}$$

(5.11) $$\begin{aligned}|\text{2nd term}|+|\text{3rd term}|&\leqq 2P({}^\exists\text{branching time}\in[t,t+\theta])\\ &\leqq 2\theta\lambda e^{(m-1)\lambda t}.\end{aligned}$$

On the other hand, using Ito's formula we see

(5.12) $$|\text{1st term}|\leqq\sup_{u\in\Gamma}\|A(u)\phi\|\theta e^{(m-1)\lambda t}$$

Putting $M_3(t)=(\sup_{u\in\Gamma}\|A(u)\phi\|+2\lambda)e^{(m-1)\lambda t}$, we can conclude the proof.

Now we can show the following regularity according to [8].

Theorcm 7. *For $\phi\in\mathfrak{D}\cap C$, we have*

(i) $$W(\cdot\,\cdot,\phi)\in W^1_\infty((0,T)\times R^d)$$

Moreover $A(u)W(t, \cdot, \phi)\in L_\infty(R^d)$ *and*

(5.13) $$\sup_{t\le U}\sup_{u\in\Gamma}\|A(u)W(t, \cdot, \phi)\|<\infty .$$

(ii) *Suppose the following complementary non-degeneracy,*
(A.9) $\exists\nu>0$ *such that, for any* $x\in R^d$ *there exist* $n, u_1, \cdots, u_n\in\Gamma$ *and* $\theta_i\in(0, 1)$ $i=1, \cdots, n$ *such that* $\sum_{i=1}^n \theta_i=1$ *and*

(5.14) $$\sum_{i j=1}^d \sum_{k=1}^n \theta_k a_{ij}(x, u_k)\xi_i\xi_j\ge\nu|\xi|^2, \qquad \textit{for any} \quad \xi\in R^d.$$

Then $W(\cdot\ \cdot, \phi)\in W_\infty^{1,2}((0, T)\times R^d)$.

Since W is a viscosity solution of (4.29), Theorem 7 (ii) means

(5.15) $$\begin{cases} \dfrac{\partial W}{\partial t}=\sup\limits_{u\in\Gamma}A(u)W-\lambda W+\sum\limits_{k=1}^\infty p_k W^k & \text{a.e in} \quad (0, T)\times R^d \\ W(0, x, \phi)=\phi(x) & \text{on} \quad R^d. \end{cases}$$

6. Controlled branching semigroup.

Put $S=R^d$ and $\boldsymbol{S}=\bigcup_{n=1}^\infty S^n$. We endow an usual topology on $\boldsymbol{S}$.

Let $\overline{Y}(t, \bar{x}, u)$, $t\ge0$, be a branching diffusion on $\boldsymbol{S}$ starting at $\bar{x}\in\boldsymbol{S}$. Suppose that its branching system is $\{p_k(u),\ k=0, 1, 2, \cdots\}$, i.e.

(6.1) $$p_1(u)=0, \quad p_k(u)\ge0, \quad \text{and} \quad \sum_{k=0}^\infty p_k(u)=1.$$

and its non-branching part is a diffusion with the following generator $\overline{A}(u)$,

(6.2) $$\overline{A}(u)\phi=\sum_{i j=1}^d a_{ij}(x, u)\frac{\partial^2\phi}{\partial x_i\partial x_j}+\sum_{i=1}^d \gamma_i(x, u)\frac{\partial\phi}{\partial x_i}-\lambda(u)\phi$$

for a smooth function ϕ.

Besides (A1) and (A2) we assume two conditions,

(A10) $$0<\inf_{u\in\Gamma}\lambda(u)\le\sup_{u\in\Gamma}\lambda(u)=C<\infty,$$

and

(A11) $$p_0(u)=0 \quad \text{for} \quad \forall u\in\Gamma \quad \text{and} \quad \sup_{u\in\Gamma}\sum k p_k(u)<\infty.$$

Namely $\overline{Y}(\cdot, x, u)$, $x\in S$ has an exponentially distributed branching time and at that time independent $(k-1)$ new diffusions are created with probability $p_k(u)$. For $\bar{x}=(x_1, \cdots, x_n)\in S^n$, we have $\overline{Y}(t, \bar{x}, u)=(\overline{Y}_1(t, x_1, u), \cdots, \overline{Y}_n(t, x_n, u))$ with independent branching diffusions $\overline{Y}_i$ starting at $x_i\in S$, $i=1, \cdots, n$. By (A11), the number of diffusion particles is an increasing Galton-Watson process and no explosion occurs.

In this section we will construct a non-linear semigroup on a suitable Banach space of continuous functions defined on S, which has a branching property and gives a viscosity solution of H-J-B eq. for (6.1) and (6.2).

Set $\rho_n(\bar{x}, \bar{y}) = \sum_{i=1}^{n} |x_i - y_i|$ for $\bar{x} = (x_1, \cdots, x_n)$ and $\bar{y} = (y_1, \cdots, y_n) \in S^n$ and $\|\Phi\|_n = \sup_{x \in S^n} |\Phi(\bar{x})|$ for a real valued function Φ defined on S^n. Put

$\mathrm{Lip}(a) = \{\Phi; S \to R^1$ such that (i) $\Phi/_{S^n}$ is a symmetric bounded function, $n = 1, 2, \cdots$, (ii) $\lim_{n\to\infty} \|\phi/_{S^n}\| = 0$, (iii) $|\Phi(\bar{x}) - \Phi(\bar{y})| \leqq a\, \rho_n(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in S^n$, $n = 1, 2, \cdots\}$ and $\mathcal{L} = \bigcup_{a>0} \mathrm{Lip}(a)$. We endow the supremum norm on $\mathcal{L}$ and denote its completion by $\bar{C}$. Then $\bar{C}$ is a Banach lattice with supremum norm and usual order. Put $\bar{D} = \{\Phi; S \to R^1$ satisfies the following condition; $^\exists N$ such that $\|\Phi\|_n = 0$ for $n \geqq N$ and $\Phi/_{S^n} \in \mathrm{BUC}(S^n)$, symmetric, and derivative $\in \mathrm{BUC}(S^n)$ for $n < N\}$. Then $\bar{D} \subset \mathcal{L}$ and $\bar{D}$ is dense in $\bar{C}$.

For $\Phi \in \bar{C}$ we define $T(t, u_1, \cdots, u_n)\Phi$ by

$$(6.3) \qquad T(t, u_1, \cdots, u_n)\Phi(\bar{x}) = E\Phi(\bar{Y}_1(t, x_1, u), \cdots, \bar{Y}_n(t, x_n, u_n))$$

for $\bar{x} = (x_1, \cdots, x_n) \in S^n$ where $\bar{Y}_1, \cdots, \bar{Y}_n$ are independent branching diffusions and $\bar{Y}_i(t, x_i, u_i)$ is a copy of $\bar{Y}(t, x_i, u_i)$, $x_i \in S$. Put $\Delta = 2^{-N}$ and define $J = J_N$ by

$$(6.4) \qquad J\Phi(\bar{x}) = \sup_{u_1, \cdots, u_n \in \Gamma} T(\Delta, u_1, \cdots, u_n)\Phi(\bar{x}) \qquad \text{for} \quad \bar{x} \in S^n.$$

Proposition 6.1. *Put* $m(u) = \sum_{k=1}^{\infty} k p_k(u)$ *and* $\mu = \sup \lambda(u)(m(u) - 1) + \frac{3}{2}K$.

(i) $|T(t, u_1, \cdots, u_n)\Phi(\bar{x}) - T(t, u_1, \cdots, u_n)\Phi(\bar{y})| \leqq a e^{\mu t} \rho_n(\bar{x}, \bar{y})$ *whenever* $\Phi \in \mathrm{Lip}(a)$.

(ii) $\|T(t, u_1, \cdots, u_n)\Phi\|_n \leqq \sup_{l \geqq n} \|\Phi\|_l$

(iii) $J\Phi \in \mathrm{Lip}(a e^{\mu \Delta})$, *if* $\Phi \in \mathrm{Lip}(a)$

(iv) $J\Phi \in \bar{C}$, *if* $\Phi \in \bar{C}$

(v) $\|J\Phi - J\Psi\| \leqq \|\Phi - \Psi\|$ *and* $\|J\Phi - J\Psi\|_n \leqq \sup_{l \geqq n} \|\Phi - \Psi\|_l$

(vi) $J\Phi_n \nearrow J\Phi$ *at each point, if* $\Phi_n \nearrow \Phi$ *at each point.*

Proof. For $\bar{x} = (x_1, \cdots, x_n)$ and $\bar{y} = (y_1, \cdots, y_n)$

$$(6.5) \quad |T(t, u_1, \cdots, u_n)\Phi(x_1, \cdots, x_n) - T(t, u_1, \cdots, u_n)\Phi(y_1, \cdots, y_n)|$$
$$\leqq E|\phi(\bar{Y}_1(t, x_1, u_1), \cdots, \bar{Y}_n(t, x_n, y_n)) - \Phi(\bar{Y}_1(t, y_1, u_1), \cdots, \bar{Y}_n(t, y_n, u_n)|$$
$$\leqq a \sum_{i=1}^{n} E|\bar{Y}_i(t, x_i, u_i) - \bar{Y}_i(t, y_i, u_i)|$$
$$\leqq a \sum_{i=1}^{n} |x_i - y_i| e^{\mu t} = a e^{\mu t} \rho_n(\bar{x}, \bar{y})$$

(ii) For $\bar{x} \in S^n$, $(\bar{Y}_1(t, x_1, u_1), \cdots, \bar{Y}_n(t, x_n, u_n)) \in S^l$ for some $l \geqq n$. This derives (ii).

(iii) is clear by (i) and (ii).

(iv) For $\Phi\in\bar{C}$ we can take $\Psi\in\mathcal{L}$ so that $\|\Phi-\Psi\|<\varepsilon$.

(6.6) $$|J\phi(\bar{x})-J\Psi(\bar{x})|\leqq\sup_{u\cdots u_n}|T(\Delta, u, \cdots, u_n)(\Phi-\Psi)(\bar{x})|\leqq\|\Phi-\Psi\|$$

So $J\Phi$ can be approximated by $J\Psi(\in\mathcal{L})$.

(v) is clear.

(vi) Since $\Phi_n\leqq\Phi_{n+1}\leqq\Phi$, $J\Phi_n$ is increasing and

$$\lim_{n\to\infty}J\Phi_n(\bar{x})\leqq J\Phi(\bar{x}).$$

On the other hand, the convergence theorem tells us

$$T(\Delta, u_1, \cdots, u_n)\Phi(\bar{x})=\lim_{k\to\infty}T(\Delta, u_1, \cdots, u_n)\Phi_k(\bar{x})$$
$$\leqq\lim_{k\to\infty}J\Phi_k(\bar{x}).$$

Taking the supremum with respect to $u_1, \cdots, u_n\in\Gamma$, we have

$$J\Phi(\bar{x})\leqq\lim_{k\to\infty}J\Phi_k(\bar{x}).$$

This derives (vi).

Now we will successively define $J^k; \bar{C}\to\bar{C}$, by

$$J^{k+1}\Phi=J(J^k\Phi).$$

Proposition 6.2. *J^k has the following properties*

(i) $J^k\Phi\leqq J^k\Psi$, *if* $\Phi\leqq\Psi$

(ii) $J^k\Phi_n\nearrow J^k\Phi$ *at each point, if* $\Phi_n\nearrow\Phi$ *at each point*

(iii) $\|J^k\Phi-J^k\Psi\|\leqq\|\Phi-\Psi\|$ *and* $\|J^k\Phi-J^k\Psi\|_n\leqq\sup_{l\geq n}\|\Phi-\Psi\|_l$

(iv) $J^k\Phi\in\mathrm{Lip}(ae^{\mu k\Delta})$, *if* $\Phi\in\mathrm{Lip}(a)$

(v) $\|J^k\Phi-\Phi|\leqq k\Delta A(\Phi)$ *for* $\Phi\in\bar{D}$

where $A(\Phi)=\sup_{n\leq l}\sup_{u_1\cdots u_n\in\Gamma}\|A(u_1, \cdots, u_n)\Phi\|_n+2lC\|\Phi\|$

if $\|\Phi\|_m=0$ *for* $m\geqq l$, *putting*

$$A(u_1, \cdots, u_n)\Phi(x_1, \cdots, x_n)=\sum_{i=1}^{n}A(u_i)\Phi(x_1, \cdots, x_{i-1}, \cdot, x_{i+1}, \cdots, x_n(x_i).$$

Proof. (i)$\sim$(iv) are clear from Proposition 6.1.

(6.7) $$\|J^k\Phi-\Phi\|\leqq\sum_{p=1}^{k}J^p\Phi-J^{p-1}\Phi\|\leqq k\|J\Phi-\Phi\|$$

For $\bar{x}=(x_1, \cdots, x_n)\in S^n$ we have

(6.8) $$T(\Delta, u_1, \cdots, u_n)\Phi(\bar{x})-\Phi(\bar{x})$$
$$=E\Phi(Y_1(\Delta, x_1, u_1), \cdots, Y_n(\Delta, x_n, u_n))-\Phi(x_1, \cdots, x_n)$$
$$-E(\Phi(Y_1(\Delta, x_1, u_1), \cdots, Y_n(\Delta, x_n, u_n)); \text{branching times}\leqq\Delta)$$
$$+E(\Phi(\bar{Y}_1(\Delta, x_1, u_1), \cdots, \bar{Y}_n(\Delta, x_n, u_n); \text{branching times}\leqq\Delta)$$

where Y_i is the non-branching part of $\bar{Y}_i$. By Ito's formula we have

(6.9)

$$\text{1st term of right side}=\int_0^\Delta A(u_1, \cdots, u_n)\Phi(Y_1(t, x_1, u_1), \cdots, Y_n(t, x_n, u_n))dt$$

Moreover

$$|\text{2nd term}|+|\text{3rd term}|\leqq 2\|\Phi\|\left(1-\prod_{i=1}^{n} e^{-\lambda(u_i)\Delta}\right) \leqq \|\Phi\| n\Delta C \tag{6.10}$$

So combining these computations with (6.7) and (6.8) we can get (v).

Define $W_N(t)\,;\,\bar{C}\to\bar{C}$ by

$$W_N(t)\Phi=J_N^k\Phi \qquad \text{for} \quad t=k2^{-N} \tag{6.11}$$

Then

$$W_N(t+s)=W_N(t)W_N(s) \qquad \text{for} \quad t=k2^{-N},\ s=j2^{-N} \tag{6.12}$$

and

$$W_{N-1}(t)\Phi\leqq W_N(t)\Phi \qquad \text{for} \quad t=k2^{-N+1} \tag{6.13}$$

Since $W_N(t)\Phi$ is increasing, as $N\to\infty$, we can define $\bar{W}(t)$ by

$$\bar{W}(t)\Phi(\bar{x})=\lim_{N\to\infty} W_N(t)\Phi(\bar{x}) \qquad \text{for binary } t. \tag{6.14}$$

Then we can easily see

$$|\bar{W}(t)\Phi(\bar{x})-\bar{W}(t)\Psi(\bar{x})|\leqq\|\Phi-\Psi\|$$

$$\|\bar{W}(t)\Phi\|_n\leqq\sup_{l\geq n}\|\Phi\|_l$$

$$\bar{W}(t)\Phi\in\mathrm{Lip}(ae^{\mu t}), \quad \text{if} \quad \Phi\in\mathrm{Lip}(a).$$

Therefore, $\bar{W}(t)\Phi\in\bar{C}$ if $\Phi\in\bar{C}$.

Proposition 6.3. *For binary t, $\bar{W}(t)\,;\,\bar{C}\to\bar{C}$ has the following properties*

(i) $\bar{W}(t)\Phi\leqq\bar{W}(t)\Psi$, *if* $\Phi\leqq\Psi$

(ii) $\|\bar{W}(t)\Phi-\Phi\|\leqq tA(\Phi)$ *for* $\Phi\in\bar{D}$

(iii) $\|\bar{W}(t)\Phi-\bar{W}(t)\Psi\|\leqq\|\Phi-\Psi\|$ *and* $\|\bar{W}(t)\Phi-\bar{W}(t)\Psi\|_n\leqq\sup_{l\geq n}\|\Phi-\Psi\|_l$

(iv) $\bar{W}(t+s)=\bar{W}(t)\bar{W}(s)$, $\bar{W}(0)=$*identity*

(v) $\|\bar{W}(t)\Phi-\Phi\|\to 0$ *as* $t\to 0$.

Proof. We prove only (iv) and (v).

$$\bar{W}(t+s)\Phi=\lim_{N\to\infty} W_N(t+s)\Phi=\lim_{N\to\infty} W_N(t)W_N(s)\Phi \leqq\lim_{N\to\infty} W_N(t)\bar{W}(s)\Phi\leqq\bar{W}(t)\bar{W}(s)\Phi. \tag{6.15}$$

For $t=2^{-p}k$, $s=2^{-p}k$ and $p\leqq n\leqq N$,

$$W_n(t)W_N(s)\Phi \leqq W_N(t)W_N(s)\Phi = W_N(t+s)\Phi .$$

As $N\to\infty$, we see, from Proposition 6.2 (ii),

$$W_n(t)\overline{W}(s)\Phi \leqq \overline{W}(t+s)\Phi .$$

Tending n to ∞, we get the converse inequality of (6.15). This completes the proof of (iv).

Since $\bar{D}$ is dense in $\bar{C}$, we can take $\Psi\in\bar{D}$ so that $\|\Phi-\Psi\|<\varepsilon$. Hence, by (ii),

$$\|\overline{W}(t)\Phi-\Phi\| \leqq \|\overline{W}(t)\Phi-W(t)\Psi\|+\|W(t)\Psi-\Psi\|+\|\Psi-\Phi\| \leqq 2\varepsilon+tA(\Psi)$$

This concludes (v).

Using (iv) and (v) we can extend $\overline{W}(t)$ on $t\in[0, \infty)$, that is, $\overline{W}(t)\,;\,\bar{C}\to\bar{C}$ is defined by

$$\overline{W}(t)\Phi=\lim_{n\to\infty}\overline{W}(t_n)\Phi \text{ whenever binary } t_n\to t. \tag{6.16}$$

Then we have

Proposition 6.4. (i)$\sim$(v) *still hold for* $W(t)$, $t\geqq 0$.

Next we will show the branching property. Let $\phi\in\mathrm{BUC}(S)$ and $0\leqq\phi\leqq 1-\varepsilon$. Then $\hat{\phi}\in\bar{C}$, where $\hat{\phi}$ is defined by

$$\hat{\phi}(x_1, \cdots, x_n)=\sum_{i=1}^{n}\phi(x_i) \qquad \text{on} \quad S^n .$$

$$T(\Delta, u_1, \cdots, u_n)\hat{\phi}(x_1, \cdots, x_n)=\sum_{i=1}^{n}E\hat{\phi}(\bar{Y}_i(\Delta, x_i, u_i)) \tag{6.17}$$

Hence, taking the supremum with respect to $u_1, \cdots, u_n$, we see

$$J\hat{\phi}(x_1, \cdots, x_n)=\sum_{i=1}^{n}J\hat{\phi}(x_i)=\widehat{(J\hat{\phi})_{/S}}(x_1, \cdots, x_n).$$

Repeating this argument, we have

$$J^2\hat{\phi}=J(J\hat{\phi})=J(\widehat{(J\hat{\phi})_{/S}})=\widehat{(J(\widehat{J\hat{\phi})_{/S}})_{/S}}=\widehat{(J^2\hat{\phi})_{/S}}.$$

Thus we have

$$W_N(t)\hat{\phi}=\widehat{(W_N(t)\hat{\phi})_{/S}}$$

i.e.

$$W_N(t)\hat{\phi}(x_1, \cdots, x_n)=\prod_{i=1}^{n}W_N(t)\hat{\phi}(x_i)$$

As $N\to\infty$, we see, for binary t

$$\overline{W}(t)\hat{\phi}(x_1, \cdots, x_n)=\prod_{i=1}^{n}\overline{W}(t)\hat{\phi}(x_i) \tag{6.18}$$

Since $\overline{W}(t)$ is continuous in t, (6.18) holds for any $t\geqq 0$. This means the branching property.

Now we will show that $\overline{W}(t)$ provides a semigroup which turns out a viscosity solution of H-J-B eq. Fix $p\in(0, 1)$ arbitrarily and put

$$\tilde{C}=\{\phi\in \mathrm{BUC}(R^{d})\,;\,0\leqq\phi\leqq p\}.$$

Then $\tilde{C}$ is a convex closed subset of $\mathrm{BUC}(R^{d})$. Define $\tilde{W}(t)\,;\,\tilde{C}\rightarrow\tilde{C}$ by

$$\tilde{W}(t)\phi(x)=\overline{W}(t)\hat{\phi}(x)\quad \text{for}\quad \phi\in\tilde{C}. \tag{6.19}$$

Theorem 8. *Under the conditions* (A1) (A2) (A11) *and* (A12), $\tilde{W}(t)$, $t\geqq 0$, *is a non-linear semigroup on* $\tilde{C}$ *with the generator* $\mathfrak{G}$;

$$\mathfrak{G}\phi=\sup_{u\in\Gamma}\mathfrak{G}(u)\phi\qquad for\quad \phi\in\tilde{C}\cap\mathfrak{D} \tag{6.20}$$

where $\mathfrak{G}(u)\phi=A(u)\phi-\lambda(u)\phi+\lambda(u)\sum_{k=2}^{\infty}p_{k}(u)\phi^{k}$. *Moreover* $\tilde{W}(t)\phi$ *is a viscosity solution of* (6.21).

$$\begin{cases}\dfrac{\partial W}{\partial t}=\sup_{u\in\Gamma}\mathfrak{G}(u)W & in\quad (0, T)\times R^{d}\\ W(0, \cdot)=\phi & on\quad R^{d}.\end{cases} \tag{6.21}$$

If $\sup\|a_{ij}(\cdot, u)\|_{W^{2}(R^{d})}<\infty$, *then its viscosity solution is unique.*

Proof. The semigroup property is derived from the branching property of $\overline{W}(t)$. That is

$$\begin{aligned}\tilde{W}(t)(\tilde{W}(s)\phi(x)&=\overline{W}(t)(\widehat{\overline{W}(s)\hat{\phi})_{/S}(x)}\\ &=\overline{W}(t)\overline{W}(s)\hat{\phi}(x)=\overline{W}(t+s)\hat{\phi}(x)\\ &=\tilde{W}(t+s)\phi(x).\end{aligned} \tag{6.22}$$

By the routine we can show (6.20).

Next we will prove that $W(t, x)=\tilde{W}(t)\phi(x)$ is a viscosity solution of (6.21). Let $\psi\in C_{b}^{\infty}((0, T)\times R^{d})$(=bounded smooth functions with any bounded derivatives). Suppose that $W-\psi$ has a strict maximum at (t_{0}, x_{0}). We may assume $W(t_{0}, x_{0})=\psi(t_{0}, x_{0})$. Hence $W\leqq\psi$.

First we assume that $z\leqq\phi\leqq p-z$ with some $z>0$. For $h\in(0, t_{0})$, we have $\varepsilon(h)$ such that

$$\psi(t_{0}-h, x)\leqq\psi(t, x)-h\frac{\partial\psi}{\partial t}(t_{0}, x)+h\varepsilon(h)\qquad \text{on}\quad R^{d}, \tag{6.23}$$

and $\varepsilon(h)\rightarrow 0$ as $h\rightarrow 0$.

Denote the right side of (6.23) by $\psi(x)$ $(=\psi(x\,;\,h))$. On the other hand, for $\varepsilon\in(0, z/2)$, there exists $v\in C_{b}^{\infty}(R^{d})$ such that

$$W(t_{0}-h, \cdot)\leqq v(\cdot)\leqq W(t_{0}-h, \cdot)+\varepsilon. \tag{6.24}$$

Put $V=\psi$ on S, $=\hat{v}$ on S^{k}, $k\geqq 2$. Define $\Phi\,;\,S\rightarrow R^{1}$ by

$$(6.25)\qquad \Phi(x_1, \cdots, x_n)=\begin{cases} \sup\limits_{u_1\cdots u_n} \sum\limits_{k=1}^{n} \prod\limits_{i\neq k} v(x_i)\mathfrak{G}(u_k)v(x_k), & \text{on } S^n,\ n\geqq 2 \\ \sup\limits_{u\in\Gamma}\mathfrak{G}(u)\psi(x) & \text{on } S. \end{cases}$$

Then $\Phi\in\bar{C}$. Let $T(t, u)$ be the transition semigroup of $\bar{Y}(t, u)$. Then the generator $\mathfrak{G}(u_1)\times\cdots\times\mathfrak{G}(u_n)$ of $T(t, u_1)\times\cdots\times T(t, u_n)$ satisfies

$$(6.26)\qquad \mathfrak{G}(u_1)\times\cdots\times\mathfrak{G}(u_n)\hat{v}(\bar{x}_1, \cdots, \bar{x}_n)\leqq\Phi(\bar{x})$$

where $(\bar{x}_1, \cdots, \bar{x}_n)\in \boldsymbol{S}^k$ and $\bar{x}=(\bar{x}_1, \cdots, \bar{x}_n)=\boldsymbol{S}$. Hence we have

$$\begin{aligned} &T(t, u_1, \cdots, u_n)\hat{v}(\bar{x})-\hat{v}(\bar{x}) \\ &\quad=T(t, u_1)\times\cdots\times T(t, u_n)\hat{v}(x_1, \cdots, x_n)-\hat{v}(x_1, \cdots, x_n) \\ &\quad=\int_0^t T(s, u_1)\times\cdots\times T(s, u_n)(\mathfrak{G}(u_1)\times\cdots\times\mathfrak{G}(u_n))\hat{v}(x_1, \cdots, x_n)ds \\ &\quad=\int_0^t T(s, u_1, \cdots, u_n)\Phi(\bar{x})ds\leqq\int_0^t \overline{W}(s)\Phi(\bar{x})ds \end{aligned}$$

Taking the supremum w.r. to $u_1, \cdots, u_n$, we get, on $\boldsymbol{S}\setminus S$,

$$(6.27)\qquad \hat{J}v-\hat{v}\leqq\int_0^\Delta \overline{W}(s)\Phi ds$$

where $\Delta=2^{-N}$ and $J=J_N$. Hence, recalling the definition of Φ on $\boldsymbol{S}$, we see

$$(6.28)\qquad JV-V\leqq\int_0^\Delta \overline{W}(s)\Phi ds \qquad \text{on } \boldsymbol{S}.$$

Putting $V_1=JV-V$, we have

$$(6.29)\qquad J^2V-JV\leqq J(JV-V)=JV_1.$$

$$\begin{aligned} (6.30)\qquad T(\Delta, u_1, \cdots, u_n)V_1&\leqq T(\Delta, u_1, \cdots, u_n)\int_0^\Delta \overline{W}(s)\Phi ds \\ &=\int_0^\Delta T(\Delta, u_1, \cdots, u_n)\overline{W}(s)\Phi ds\leqq\int_0^\Delta \overline{W}(\Delta+s)\Phi ds \\ &=\int_\Delta^{2\Delta} \overline{W}(s)\Phi ds, \qquad \text{on } \boldsymbol{S}. \end{aligned}$$

Therefore we have

$$J^2V-JV\leqq\int_\Delta^{2\Delta}\overline{W}(s)\Phi ds.$$

Repeating the same evaluations, we get

$$J^kV-J^{k-1}V\leqq\int_{(k-1)\Delta}^{k\Delta}\overline{W}(s)\Phi ds.$$

Consequently we get

$$(6.31)\qquad \overline{W}(t)V-V\leqq\int_0^t\overline{W}(s)\Phi ds.$$

So, we have

(6.32)
$$\psi(t_0, x_0)=W(t_0, x_0)=W(h)\hat{W}(t_0-h, \cdot)(x_0)$$
$$=W(h)V(x_0)\leqq V(x_0)+\int_0^h \overline{W}(s)\Phi(x_0)ds$$

This implies

$$\frac{\partial\psi}{\partial t}(t_0, x_0)\leqq\varepsilon(h)+\frac{1}{h}\int_0^h \overline{W}(s)\Phi(x_0)ds.$$

Tending h to 0, we conclude that W is a subsolution of (6.21).

Let (t_0, x_0) be a strict minimum point of $W-\psi$. We may assume $W(t_0, x_0)=\psi(t_0, x_0)$. So $W\geqq\psi$. For $h\in(0, t_0)$ we can choose $\varepsilon(h)$, such that

(6.33)
$$\psi(t_0-h, x)\geqq\psi(t_0, x)-h\frac{\partial\psi}{\partial t}(t_0, x)+h\varepsilon(h), \qquad \text{on } \boldsymbol{S}$$

and $\varepsilon(h)\to 0$ as $h\to 0$. Denoting the right side of (6.33) by $\psi(x)$ $(=\psi(x\,;h))$, we apply the similar evaluations. Choose $v\in C_b^\infty(R^d)$ so that

(6.34)
$$0\leqq W(t_0-h, \cdot)-v(\cdot)<\varepsilon$$

and put

$$V(\bar{x})=\begin{cases} \psi & \text{on } S \\ \hat{v}(\bar{x}) & \text{on } \boldsymbol{S}\setminus S. \end{cases}$$

Then $V\in\mathcal{D}(\mathfrak{G}(u))$. Moreover, from (6.33) and (6.34), we see

(6.35)
$$\psi(t_0, x_0)=W(t_0, x_0)=W(h)\hat{W}(t_0-h, \cdot)(x_0)$$
$$\geqq W(h)V(x_0).$$

and

$$W(h)V(x_0)-V(x_0)\geqq T(h, u)V(x_0)-V(x_0)$$
$$=\int_0^h T(s, u)\mathfrak{G}(u)V(x_0)ds.$$

Therefore, recalling (6.33), we have

$$\frac{\partial\psi}{\partial t}(t_0, x_0)\geqq\mathfrak{G}(u)\psi(t_0, x_0).$$

Taking the supremum w.r. to $u\in\Gamma$, we can show that W is a supersolution of (6.21). Hence W is a viscosity solution of (6.21).

For a general ϕ, we can choose an approximate ϕ_n, so that

$$\frac{1}{n}\leqq\phi_n\leqq p-\frac{1}{n} \quad \text{and} \quad \|\phi_n-\phi\|<\frac{1}{n}.$$

Put $W_n(t, x)=\widetilde{W}(t)\phi_n(x)$ and $W(t, x)=\widetilde{W}(t)\phi(x)$. Then W_n tends to W uniformly on $[0, T]\times R^d$. Let (t_0, x_0) be a strict maximum point of $W-\psi$. Then there exists a maximum point (t_n, x_n) of $W_n-\psi$, which converges to (t_0, x_0), as $n\to\infty$. Since W_n is a viscosity solution of (6.21) with the initial value ϕ_n, W is a viscosity solution of (6.21) with the initial value ϕ, by the stability of viscosity solution.

Since the uniqueness part is derived from Theorem 4, this completes the proof of Theorem 8.

Remark. If λ and p_k are independent of u and $p_0=0$, then we have two viscosity solutions $V(t)\phi$ and $\widetilde{W}(t)\phi$ in §4 and §5 respectively. Using the time discrete approximation of an admissible control, we can show that $V(t)\phi=\widetilde{W}(t)\phi$.

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