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On P. J. Myrberg's approximation theorem for some Kleinian groups

Dedicated to Prof. Y. Kusunoki on his sixtieth birthday.

By

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Introduction.

The apprximation theorem about Fuchsian groups or about the geodesic flow on surfaces of constant negative curvature, obtained by P. J. Myrberg, is based only on some topological and (hyperbolic) geometrical facts. So its proof may be considered elementary.

If we try to extend Myrberg's result to a Kleinian group, we find his method works also efficient for purely loxodromic groups, but we shall face some difficulties for groups which contain parabolic transformations. Such difficulties can be overcome actually by ergodic method.

We shall give in this paper, however, an elementary proof, independent of ergodic theorems, of the approximation theorem for Kleinian groups which are geometrically finite and of the first kind. Moreover, by replacing the terms in our proof, we obtain another proof of the original theorem for Fuchian groups. It seems to the author interesting to find in this paper that parabolic elements, which are generally considered as troublesome existence, turn out to play an important role in the proof.

Further we shall show an analogy to the approximation theorem for classical Schottky groups.

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§ 1. Preliminaires.

1.1. An isometry on the hyperbolic or non-euclidean 3-space $\boldsymbol{B}^3=\{x=(x_1, x_2, x_3)\in\boldsymbol{R}^3;\ |x|<1\}$, with the Poincaré metric $ds=2|dx|/(1-|x|^2)$, is called a hyperbolic motion.

It is well known that a hyperbolic motion extends to a Möbius transformation on $\hat{\boldsymbol{R}}^3=\boldsymbol{R}^3\cup\{\infty\}$, which has at least one fixed point in $\hat{\boldsymbol{R}}^3$. For simplicity we consider here only orientation preserving motions. Then they are classified into three types:

A motion, whose Möbius extension has fixed points only on the unit sphere $S=\{|x|=1\}$, is called to be parabolic or loxodromic according as the number of its fixed points is one or two respectively. Another type of the motion, which is a conjugation of some $k\in SO(3)$, the group of special orthogonal matrices, is called to be elliptic.

1.2. A Kleinian group Γ is a group of the hyperbolic motions which acts discontinuously on $\boldsymbol{B}^3$. The limit set Λ of Γ is the set of points of accumulation of Γ-equivalents of $x\in\boldsymbol{B}^3$, which is independent of the choice of x. The limit set Λ is necessarily on S. Γ is called to be the first kind if Λ coincide with S; of the second kind, otherwise.

If, especially, a Kleinian group has a fundamental polygon (for the definition See 2.1.) with finitely many faces, then it is called to be geometrically finite. For further details, See [Ah], [G] etc.

1.3. A non-euclidean line (simply we denote as n.e. line) is a circular arc in $\boldsymbol{B}^3$ which is orthogonal to S.

Definition 1.1. Let Λ be the limit set of a Kleinian group Γ. Suppose $x\in\Lambda$ and R is an n.e. half line ended at x, then x is said to be transitive if, for any n.e. line L connecting two points in Λ, we can find a sequence of elements γ_ν in Γ such that

$$\gamma_\nu(R)\longrightarrow L \qquad (\nu\longrightarrow\infty) \tag{1.1}$$

However it is not difficult to see that, if (1.1) holds for one half line ended at x, then it holds for all such half lines.

One of our main results is:

Theorem 1.1. (P. J. Myrberg's approximation theorem) *If Γ is a geometrically finite Kleinian groups of the first kind then all points in $\Lambda=S$, except a subset of the Lebesgue measure zero, are transitive.*

For the cases that Γ consists of only loxodromic and elliptic elements, the method in Myrberg's paper, where he treats Fuchsian groups, leads to the result. So we give the proof of the theorem for Γ with parabolic elements.

From this theorem we can derive immediately that, for almost all $x\in S$, the projection of the n.e. half line ended at x on the quotient manifold $M=\boldsymbol{B}^3/\Gamma$ draws a everywhere dense orbit.

§2. A decomposition of S into two sets.

2.1. Let Γ be geometrically finite, of the first kind with parabolic elements. We may assume without loss of generality that no nontrivial element of Γ fixes the origin O, for otherwise we can take a conjugation of Γ by a proper hyperbolic motion to satisfy this condition. For convenience we write $\Gamma^*=\Gamma-\{\text{identity}\}$. Then we can construct the Dirichlet fundamental polygon of Γ

centered at O as

$$P_0=\{x\in \boldsymbol{B}^3;\ d(x, O)<d(x, \gamma(O)) \text{ for all } \gamma\in\Gamma^*\}$$

where $d(\ ,\ )$ is the n.e. distance. By our assumption P_0 has finitely many faces.

2.2. Let p be a parabolic fixed point which is on the boundary of P_0 ([G] Theorem 2.6.1.), and $M_p=\{\gamma\in\Gamma;\ \gamma(p)=p\}$ be the stabilizer of p. For the representatives $\{\gamma_\nu\}_{\nu=0}^{\infty}$ of all right cosets Γ/M_p we put $B_\nu^r=\gamma_\nu B_0^r$, where B_0^r is the horoball of euclidean radius r based at p, which we call here an r-horoball. We consider only sufficient small r for which the B_ν^r's are mutually disjoint.

The "shadow" on S of B_ν^r under the central projection is denoted by $\hat{B}_\nu^r$. Put $\Omega_r=\bigcup_{\nu=0}^{\infty}\hat{B}_\nu^r$, then obviously Ω_r is monotonously decreasing with decreasing r. We define the following sets:

$$\Omega_0=\lim_{r\to 0}\Omega_r=\bigcap_{r>0}\Omega_r \tag{2.1}$$
$$F_r=S-\bigcup_{p\in\partial P_0}\Omega_r \qquad (r\geqq 0)$$

$\Omega_0=\emptyset$ provided that Γ is purely loxodromic, but in our case it is nonvoid because it contains Γ-equivalents of p. On the other hand F_0 is always nonvoid since it contains all loxodromic fixed points.

2.3. First we show that F_0 has the Lebesgue measure zero. For this purpose we fix an r and denote by P_0^r the rest of P_0 from which removed all r-horoballs intersecting it. Let B_0, B_1 be closed balls centered at O such that

$$B_0\subset P_0^r\subset B_1 \qquad (\subset \boldsymbol{B}^3) \tag{2.2}$$

and let r_0 be the euclidean radius of B_0.

Lemma 2.1. *Let ϕ_r be the spherical radius of the shadow on S of γB_0 for $\gamma\in\Gamma$. If for an r-horoball B_ν^r the n.e. distance between γP_0 and B_ν^r holds $d(\gamma P_0^r, B_\nu^r)\leqq\delta$, then there is a constant $\rho_\delta(>0)$ depends only on δ such that*

$$\rho_\delta A(\phi_r)<m(\hat{B}_\nu^r)$$

where $A(\phi)$ is the area of the cap of sphereical radius ϕ and $m(\hat{B}_\nu^r)$ is that of $\hat{B}_\nu^r$.

Proof. Let r_γ be the euclidean radius of γB_0. Suppose $B_\nu^r=\eta B_0^r$ and put $\xi=\gamma^{-1}\eta$. Then γ^{-1} translates γP_0, B_ν^r to P_0, ξB_0^r resp. and it holds

$$d(P_0^r, \xi B_0^r)\leqq\delta \tag{2.3}$$

There are only finitely many elements ξ of Γ which satisfy the inequality (2.3). Let $\xi_1, \cdots, \xi_n$ be those solutions. Then $\xi=\gamma^{-1}\eta=\xi_j$ for some j. Let R_η, R_j be the radius of ηB_0^r, $\xi_j B_0^r$ resp.. Then

$$r_\gamma=\frac{(1-|\gamma(0)|^2)th(\rho/2)}{1-|\gamma(0)|^2 th^2(\rho/2)},\quad R_\eta=\frac{R_j(1-|\gamma(0)|^2)}{(1-R_j)|\gamma^{-1}(0)-\xi_j(p)|^2+R_j(1-|\gamma(0)|^2)},$$

where $\rho=\log(1+r)/(1-r)$ is the n.e. radius of B_0 and we denote tanh simply by th. Hence

$$\frac{R_\eta}{r_r}>\frac{R_j(1-th^2(\rho/2))}{4-3R_j}\geqq C_\delta=\min_{j=1,\cdots,n}\frac{R_j(1-th^2(\rho/2))}{4-3R_j}$$

Clearly $R_\eta<\Phi_\nu$=the spherical radius of $\hat{B}_\iota^r$. On the other hand if we denote the euclidean center of γB_0^r by b, then $r_\gamma=|b|\sin\phi_\gamma$. Therefore for γB_0^r apart far away from O, $r_\gamma\doteqdot\phi_\gamma$. So that we have $\phi_\gamma\leqq C'r_\gamma$ for some constant C'. Since for $\phi<\pi/2$ it holds

$$\frac{\omega_2}{2}\left(\frac{2}{\pi}\right)\phi^2\leqq A(\phi)\leqq\frac{\omega_2}{2}\phi^2$$

where $\omega_2=4\pi$, the full measure of S, we obatin except the case $\gamma=id.$ that

$$A(\Phi_\iota)/A(\phi_\gamma)\geqq(2\Phi_\iota/\pi\phi_\gamma)^2\geqq(2C_\delta/\pi C')^2>0.$$

Furthemore $A(\Phi_\iota)/A(\phi_\gamma)>\min_{j=1,\cdots,n}2\omega_2R_j/\pi^2>0$, for $\gamma=id.$ so that we can find a desired constant ρ_δ. Q.E.D.

A similar consideration leads to

Lemma 2.2. *Let ϕ_γ, Φ_γ be the spherical radius of the shadow of γB_0, γB_1 resp. for $\gamma\in\Gamma$. If $d(\gamma P_0^r, \eta P_0^r)=0$ then there is a constant $C_0(>0)$ independent of γ, η such that $C_0A(\Phi_\eta)<A(\phi_\gamma)$.*

We denote by V_a the n.e. cone with the vertex a which inscribes B_0 and V_a^* the unbounded component of V_a-B_0 with respect to the Poincaré metric. Then we can find a finite number of r-horoballs

$$B_{\iota_1}^r,\ B_{\iota_2}^r,\ \cdots,\ B_{\iota_q}^r \tag{2.4}$$

such that for all $a\in\boldsymbol{B}^3-B_0$ there exists a ball $B_{\iota_\mu}^r$ in (2.4) which is contained in V_a^*.

Lemma 2.3. *We fix a number $s(0<s<1)$. Let C be any sphericap on S and ϕ be its radius. If the cap C' of radius $s\phi$ concentric with C contains a point belongs to F_r, then there is a subcap $\hat{C}$ of C such that*

$$(1)\quad \hat{C}\subset\Omega_r,\qquad (2)\quad \rho m(C)<m(\hat{C})$$

where ρ is a positive constant independent of C.

Proof. We take some $b\in C'\cap F_r$ and denote by R the radius $\overrightarrow{Ob}$ and by V the cone with the vertex O and the base C. By definition R meets infinitely many copies ηP_0^r of P_0^r. Suppose ηP_0^r is the first of them contained totally in V and ξP_0^r is the one intersects R just before ηP_0^r when we enumerate these copies from O. Then since the shadow of ξB_1 is not wholly covered by C, for its spherical radius holds $(1-s)\phi<2\Phi_\xi$. Therefore if $\Phi_\xi<\pi/2$, then

$$A(\Phi_\xi)/A(\phi)>(2\Phi_\xi/\pi\phi)^2>\hat{\rho}=((1-s)/\pi)^2 \tag{2.5}.$$

Otherwise (i.e. for the case $\xi=id.$) $A(\Phi_\xi)/A(\phi)\geqq 1>\hat{\rho}$. Because of $d(\xi P_0^r, \eta P_0^r)=0$ previous lemma yields $A(\phi_\eta)/A(\Phi_\xi)>C_0$. Hence with (2.5) it leads

$$A(\phi_\eta)>\hat{\rho}C_0A(\phi)=\hat{\rho}C_0m(C) \qquad (2.6).$$

We put $V'=\eta^{-1}(V)$, $O'=\eta^{-1}(O)$ and denote by $V_{0'}$ the n.e. cone with the vertex O' which inscribes B_0. Since $\eta B_0\subset V$, we have $V_0\subset V'$. As we remarked, we can find one $B_{\nu_\mu}^r$ among (2.4) which is contained in $V_{0'}^*$. Then it holds

$$d(\eta P_0^r, \eta B_{\nu_\mu}^r)=d(P_0^r, B_{\nu_\mu}^r)<\delta=\max_{\mu=1,\cdots,q} d(P_0^r, B_{\nu_\mu}^r).$$

Hence by lemma 2.1 and (2.6)

$$m(\eta \hat{B}_{\nu_\mu}^r)>\rho_\delta A(\phi_\eta)>C_0\hat{\rho}\rho_\delta m(C).$$

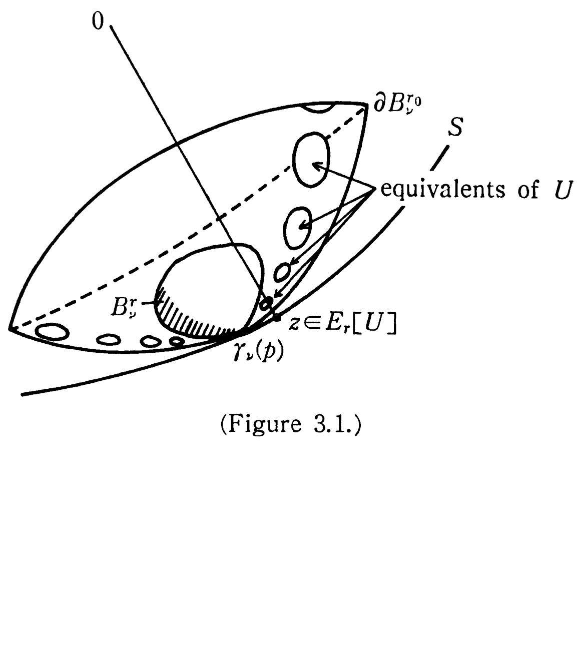
Now put $\hat{C}$=the shadow of $\eta B_{\nu_\mu}^r$ and $\rho=C_0\hat{\rho}\rho_\delta$, then these satisfy the statement of the lemma. Q.E.D.

By this lemma we can verify that the measure of F_r is zero after a routine consideration. Since $F_0=\bigcup_n F_{1/n}$, thus we know at last the measure of F_0 is zero.

§ 3. Transitive points in Ω_0.

3.1. We fix a sufficiently small r_0. M_ρ acts on the horosphere $\Sigma_0^{r_0}=\partial B_0^{r_0}$ ([G] 2.6.2.) and $\Sigma_0^{r_0}/M_\rho$ is compact. We can construct a fundamental region Q of M_ρ on $\Sigma_0^{r_0}$ as follows ([G] 2.6.3.): Let $\gamma_0(p)=p, \gamma_1(p), \cdots, \gamma_N(p)$ be the equivalents of p on the boundary of P_0. If we put $\Pi_i=\gamma_i(\Sigma_0^{r_0})\cap P_0 (i=0, \cdots, N)$ then $Q=\bigcup_{i=0}^{N}\gamma_i^{-1}(\Pi_i)$.

3.2. Definition 3.1. For $r(0<r<r_0)$ and a subset U of Q, we define $E_r[U]$ as a set of all $x\in\Omega_0$ satisfy the following property; There is an r-horoball B_ν^r which meets the radius $\overrightarrow{Ox}$ such that, the radius toward x intersects $\gamma_\nu(\Sigma_0^{r_0})$ on the set $\gamma_\nu(M_P(U))$ after passing through B_ν^r (See Figure 3.1.).



(Figure 3.1.)

Lemma 3.1. *We denote the radius of B_ν^r, B_ν^{r0} by $r(B_\nu^r)$, $r(B_\nu^{r0})$ resp. then there is a constant $\tilde{C}_r$ independent of ν such that*

$$\begin{aligned} &(1)\quad r(B_\nu^r)/r(B_\nu^{r0}) \leqq \tilde{C}_r \\ &(2)\quad \min_{r\to 0} \tilde{C}_r = 0 \end{aligned} \tag{3.1}$$

Proof. Let ω be the base $\gamma_\nu(p)$ of B_ν^r, B_ν^{r0} and x_1, x_2 be the intersecting point of $O\omega$ and ∂B_ν^{r0}, ∂B_ν^r resp.. Then $d=d(x_1, x_2)$, the n. e. distance between x_1 and x_2, is independent of ν. We put $\rho_1=1-|x_1|$, $\rho_2=1-|x_2|$ the diameter of B_ν^{r0}, B_ν^r resp., then

$$d=\int_{1-\rho_1}^{1-\rho_2} \frac{2dt}{1-t^2} = \log \frac{2-\rho_2}{\rho_2} - \log \frac{2-\rho_1}{\rho_1}.$$

This leads to

$$\rho_1 = \frac{2\rho_2}{\rho_2+(2-\rho_2)e^{-d}} > \frac{2\rho_2}{\rho_2+2e^{-d}},$$

Hence

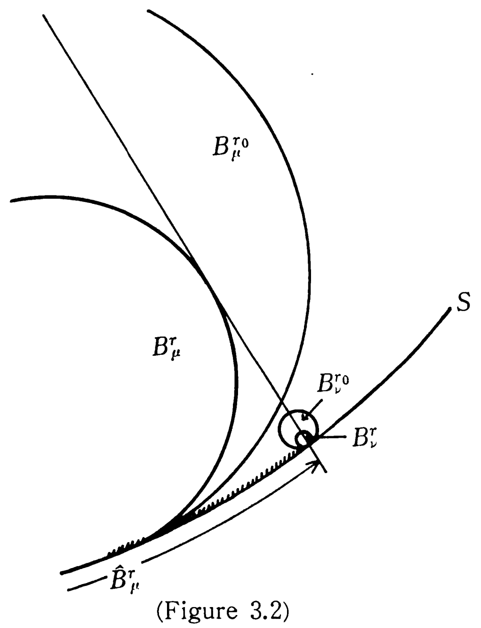
$$r(B_\nu^r)/r(B_\nu^{r0}) = \rho_2/\rho_1 < (\rho_2/2)+e^{-d}.$$

We may assume that B_0^r has the largest radius among its Γ-equivalents, so that $r(B_\nu^r)/r(B_\nu^{r0}) < \tilde{C}_r = r - e^{-d}$. If $r\to 0$, then $d\to\infty$. Therefore $\lim_{r\to 0} \tilde{C}_r = 0$. Q. E. D.

By this lemma and an elementary geometrical consideration, we obtain

Lemma 3.2. *For two r-horoballs B_ν^r, B_μ^r satisfying*

$$(1)\quad r(B_\nu^r) < r(B_\mu^r) \quad \textit{and} \quad (2)\quad \hat{B}_\nu^r \cap \hat{B}_\mu^r \neq \emptyset,$$



(Figure 3.2)

there is a constant C_τ independent of η, ν such that, if we denote by ϕ the spherical radius of $\hat{B}_\mu^\tau$, then $\hat{B}_\nu^\tau$ is contained in the concentric cap of $\tilde{B}_\mu^\tau$ of radius $C_\tau\phi$. Moreover $\lim C_\tau=1$. *(See Figure* 3.2. *In fact $C_\tau-1$ is of the order $O(\tilde{C}_\tau^2)$.)*

3.3. Let $\gamma_0(=id.)$, $\gamma_1, \cdots, \gamma_N$ be the elements of Γ referred in 3.1. We define $P=\bigcup_{i=0}^{N}\gamma_i^{-1}P_0$ and denote by D the union of $B_0^{\tau_0}$ and the interior domains to all spheres each of them is the extension of a face of P with p on its boundary. We can choose beforehand the representatives $\{\gamma_\nu\}$ of the cosets Γ/M_P so that $a_\nu=\gamma_\nu^{-1}(0)$ are outside of D.

Let V_ν be the n.e. cone with the vertex a_ν (where we agree that $a_0=0$) inscribes B_0^τ and denote by C_ν the component of $\overline{(\Sigma_0^{\tau_0}\cap V_\nu)}$ which contains p. C_ν is a spherical cap of $\Sigma_0^{\tau_0}$.

It makes us easier to approach our problem if we consider it occasionary on the upper half space $\boldsymbol{H}^3=\{(x_1, x_2, x_3)\,;\, x_3>0\}$ of $\boldsymbol{R}^3$. For if we map $\boldsymbol{B}^3$ onto $\boldsymbol{H}^3$ by the conformal mapping h such that $h(O)=j=(0, 0, 1)$ and $h(p)=\infty$, then the image of $\Sigma_0^{\tau_0}$ is a parallel plane to (x_1, x_2)-plane, which we identify with the complex plane $\boldsymbol{C}=\{z=x_1+ix_2\}$, and M_P acts on $\Sigma_0^{\tau_0}$ as a group of euclidean motions. ([G] 2.6.2). For economy of notations we give the same letters $\Sigma_0^{\tau_0}$, M_P etc. to the h-images of those. Suppose

$$\Sigma_0^{\tau_0}=\{z+s_0j\,;\, z\in\boldsymbol{C}\}$$

$$\Sigma_0^{\tau}=\{z+sj\,;\, z\in\boldsymbol{C}\} \qquad (s>s_0).$$

We denote the compliment of D by K, then each generating line on the surface of V_ν corresponds to a circular arc orthogonal to $\boldsymbol{C}$ and of the hight s from a point in K. Since K is a compact set in $\boldsymbol{H}^3\cup\boldsymbol{C}$, we have easily

Lemma 3.3. *We identify $\Sigma_0^{\tau_0}=\{z+s_0j\}$ with the complex plane $\{z\in\boldsymbol{C}\}$. Then there is a constant d depends only on K such that when we put*

$$A_1=\{|z|>R_0-d\}, \qquad A_2=\{|z|>R_0+d\}$$

where R_0 is the radius of the disk C_0,

$$A_2\subset C_\nu\subset A_1 \qquad (\nu=0, 1, \cdots) \tag{3.2}$$

We define $N_1, N_2\subset M_P$ as

$$N_1=\{\eta\in M_P\,;\, \eta(\bar{Q})\cap A_1\neq\emptyset\}, \qquad N_2=\{\eta\in M_P\,;\, \eta(\bar{Q})\subset A_2\}.$$

Then by the previous lemma,

$$\bigcup_{\eta\in N_2}\eta(Q)\subset C_\nu\subset\bigcup_{\eta\in N_1}\eta(Q) \qquad (\nu=0, 1, \cdots) \tag{3.3}$$

We return to the consideration on $\boldsymbol{B}^3$, also in this case (3.3) holds. If we put $\tilde{V}_\nu=\gamma_\nu(V_\nu)$ then it is the n.e. cone with the vertex O inscribes B_ν^τ. Let $\tilde{C}_\nu=\gamma_\nu(C_\nu)$, which is the component of $\overline{(\tilde{V}_\nu\cap\partial B_\nu^{\tau_0})}$ which contains $\gamma_\nu(p)$. The area of

$\tilde{C}_\nu$ is estimated as

$$m(\tilde{C}_\nu)=\int_{C_\nu}|\gamma_\nu'(x)|^2 d\omega(x)\leqq\frac{(1-|a_\nu|^2)^2}{\inf\limits_{\substack{X\in\cup\eta(Q)\\ \eta\in N_1}}[x,\ a_\nu]^4}\sum_{\eta\in N_1}m(\eta(Q)).$$

where $d\omega$ is the area element on $\Sigma_0^{r_0}$ of total area $4\pi r_0^2$ and $[x, a_\nu]^2=1+|x|^2|a_\nu|^2-2x\cdot a_\nu$ ($x\cdot a_\nu$ is the scalar product of x and a_ν.).

If we denote by $\tilde{C}_\nu^U$ the all Γ-equivalents of U contained in $\tilde{C}_\nu$ then its area, provided U is measurable, is estimated as

$$m(\tilde{C}_\nu^U)\geqq\sum_{\eta\in N_2}\int_{\eta(U)}|\gamma_\nu'(x)|^2 d\omega(x)\geqq\frac{(1-|a_\nu|^2)^2}{\sup\limits_{\substack{X\in\cup\eta(\bar{Q})\\ \eta\in N_2}}[x,\ a_\nu]^4}\sum_{\eta\in N_2}m(\eta(U))$$

$$\geqq\frac{(1-|a_\nu|^2)^2}{\sup\limits_{\substack{X\in\cup\eta(\bar{Q})\\ \eta\in N_1}}[x,\ a_\nu]^4}\sum_{\eta\in N_2}m(\eta(U))$$

Hence

$$\frac{m(\tilde{C}_\nu^U)}{m(\tilde{C}_\nu)}\geqq\frac{\inf(x\in\bigcup_{\eta\in N_1}\eta(\bar{Q}))[x,\ a_\nu]^4\sum_{\eta\in N_2}m(\eta(U))}{\sup(x\in\bigcup_{\eta\in N_2}\eta(\bar{Q}))[x,\ a_\nu]^4\sum_{\eta\in N_1}m(\eta(Q))} \tag{3.4}$$

Since $a_\nu\notin D$, uniformly $[x, a_\nu]\to|a_\nu-p|\,(r\to 0)$. Hence

$$\lim_{r\to 0}\frac{m(\tilde{C}_\nu^U)}{m(\tilde{C}_\nu)}>\lim_{r\to 0}\frac{\sum_{\eta\in N_2}m(\eta(U))}{\sum_{\eta\in N_1}m(\eta(Q))} \tag{3.5}$$

The shadow of $\tilde{C}_\nu$ is just $\hat{B}_\nu^r$. We denote by $\hat{B}_\nu^r[U]$ the shadow of $\tilde{C}_\nu^U$ on S, then by definition $\hat{B}_\nu^r[U]\subset E_r[U]$. By letting $r\to 0$ with the fixed r_0, it holds

$$\lim_{r\to 0}\frac{m(\hat{B}_\nu^r[U])}{m(\hat{B}_\nu^r)}=\lim_{r\to 0}\frac{m(\tilde{C}_\nu^U)}{m(\tilde{C}_\nu)}. \tag{3.6}$$

Keeping the notations in lemma 3.3 but we denote again $\max(d, diamQ)$ by d we have

$$\frac{\sum_{\eta\in N_1-N_2}m(\eta(Q))}{\sum_{\eta\in N_2}m(\eta(Q))}\leqq\frac{\int_A|h'(x)|dx}{\int_{A_2}|h'(x)|dx}$$

$$=\frac{O((R_0-2d)^{-2}-(R_0+2d)^{-2})}{O((R_0+2d)^{-2})}\longrightarrow 0.$$

when $r\to 0$, where $A=\{R_0-2d<z<R_0+2d\}$.

Therefore we can modify (3.5) to have with (3.6)

$$\lim_{r\to 0}\frac{m(\hat{B}_\nu^r[U])}{m(\tilde{B}_\nu^r)}\geqq\lim_{r\to 0}\frac{\sum_{\eta\in N_2}m(\eta(U))}{\sum_{\eta\in N_2}m(\eta(Q))}. \tag{3.7}$$

The right hand side of (3.7) is estimated from below as

$$\frac{\sum_{\eta\in N_2} m(\eta(U))}{\sum_{\eta\in N_2} m(\eta(Q))} \geq \frac{\left(\sum_{\eta\in N_2} \frac{(1-|\eta^{-1}(O)|^2)^2}{\max_{X\in\bar{U}}[x,\ \eta^{-1}(O)]^4}\right)m(U)}{\left(\sum_{\eta\in N_2} \frac{(1-|\eta^{-1}(O)|^2)^2}{\min_{X\in\bar{Q}}[x,\ \eta^{-1}(O)]^4}\right)m(Q)} \tag{3.8}$$

If r is sufficiently small, then for $\eta\in M_P$ such that $\eta(\bar{Q})\subset A_2$ $\eta^{-1}(O)$ must be quite near to p. So that for $x\in\bar{Q}$, where $\bar{Q}$ is a compact set in $\boldsymbol{B}^3$, $[x,\ \eta^{-1}(O)]\rightarrow|x-p|$, when $r\rightarrow 0$.

Hence the right hand side of (3.8) tends to $m(U)/m(Q)$ when r tends to 0. Therefore we obtain finally

$$\lim_{r\rightarrow 0}\frac{m(\tilde{B}_\nu^r[U])}{m(\hat{B}_\nu^r)}\geq\frac{m(U)}{m(Q)}$$

and we can conclude

Lemma 3.4. *Let $U(\subset Q)$ be a measurable set, then there is a constant b_r depends only on r and U such that*

(1) $m(\tilde{B}_\nu^r[U])\geqq b_r m(\hat{B}_\nu^r)$, *where* $\hat{B}_\nu^r[U]\subset E_r[U]$

and

(2) $\lim b_r=m(U)/m(Q)$.

3.4. Theorem 3.5. *For a set $U(\subset Q)$ of positive measure, all points in Ω_0, except a subset of null measure, are contained in $E_r[U]$ for sufficiently small r.*

Proof. We denote by $\tilde{B}_\nu^r$ the concentric cap of $\hat{B}_\nu^r$ whose radius is C_r times the length of that of $\tilde{B}_0$, where C_r is the constant appeared in lemma 3.2. Then for sufficiently small r, $m(\hat{B}_\nu^r)/m(\tilde{B}_\nu^r)=C_r^{-2}$ and this tends to 1 with r tends to 0. On the other hand by lemma 3.4, $\lim_{r\rightarrow 0} m(\hat{B}_\nu^r[U])/m(\hat{B}_\nu^r)\geqq m(U)/m(Q)>0$. Hence

$$\lim_{r\rightarrow 0}(m(\tilde{B}_\nu^r)-m(\hat{B}_\nu^r[U]))<(1-m(U)/m(Q))m(\hat{B}_\nu^r).$$

Therefore there is a number $r(U)(>0)$ such that for $r<r(U)$

$$m(\tilde{B}_\nu^r)-m(\hat{B}_\nu^r[U])<d_r m(\hat{B}_\nu^r) \tag{3.9}$$

where $d_r<1$.

Suppose $r<r(U)$. For any sufficient small $\varepsilon(>0)$ we take an open set $G_1(\supset\Omega_0)$ on S such that $m(G_1-\Omega_0)<\varepsilon$. Here we may neglect for our purpose the set of all Γ-equivalent points of p, so that we can assume for all points x in Ω_0, the radius $\overrightarrow{Ox}$ meets infinitely many r-horoballs. Since the shadows of these balls are contained in G_1 with finitely many exceptions, there is a plentiful supply of r-horoballs whose shadows are contained in G_1. Among B_ν^r's such that $\hat{B}_\nu^r\subset G_1$ we choose one of the largest radius and denote it by $B_{\nu_1}^r$. Next we choose one of the largest radius, which we denote by $B_{\nu_2}^r$, among B_ν^r's such that $\hat{B}_\nu^r\subset G_1-Cl(B_{\nu_1}^r)$($Cl(B)$ means the closure of B) and repeat this procedure, that is, we choose an r-horoball of the largest radius which is denoted by

$B^r_{\nu_n}$, among B^r_ν's such that $\hat{B}^r_\nu \subset G_1 - \bigcup_{k=1}^{n-1} Cl(\hat{B}^r_{\nu_k})(n=1, 2, \cdots)$.

Then $\hat{B}^r_{\nu_n}$'s are disjoint caps in G_1. We don't know if $\hat{B}^r_{\nu_n}$'s cover Ω_0, but we can say about $\tilde{B}^r_{\nu_n}$'s that

Lemma 3.6. $\{\tilde{B}^r_{\nu_n}\}_{n=1}^\infty$ *is a covering of* Ω_0.

Proof. For any $x \in \Omega_0$, let B^r_ν be the one of the largest radius among r-horoballs intersect Ox. If $B^r_\nu = B^r_{\nu_n}$ for some n then $x \in \hat{B}^r_{\nu_n} \subset \tilde{B}^r_{\nu_n}$. Otherwise, by the choice of $B^r_{\nu_n}$'s, there exists a $B^r_{\nu_n}$ such that $r(B^r_\nu) < r(B^r_{\nu_n})$ and $\hat{B}^r_\nu \cap \hat{B}^r_{\nu_n} \neq \emptyset$. Hence by lemma 3.2, $x \in \hat{B}^r_\nu \subset \tilde{B}^r_{\nu_n}$. Q.E.D.

Note that $\hat{B}^r_\nu[U] \subset E_r[U]$. If we set $\Omega_1 = \Omega_0 - \bigcup_{n=1}^\infty \tilde{B}^r_{\nu_n}[U]$, then by (3.9) and the above lemma,

$$m(\Omega_1) \leqq \sum_{n=1}^\infty \{m(\tilde{B}^r_{\nu_n}) - m(\hat{B}^r_{\nu_n}[U])\} \leqq d_r \sum_{n=1}^\infty m(\hat{B}^r_{\nu_n})$$
$$\leqq d_r m(G_1) \leqq d_r(m(\Omega_0)+\varepsilon) \tag{3.10}$$

Again we take an open set $G_2(\supset \Omega_1)$ on S such that $m(G_2 - \Omega_1) < \varepsilon$. Similarly we choose an r-horoball of the largest radius, which we denote by $B^r_{\nu_1}$, among B^r_ν's such that $\hat{B}^r_\nu \subset G_2$, and inductively we choose one of the largest radius, which is denoted by $\hat{B}^r_{\nu_n}$, among B^r_ν's such that $\hat{B}^r_\nu \subset G_2 - \bigcup_{k=1}^{n-1} Cl(\hat{B}^r_{\nu_k})$ for $n=1, 2, \cdots$. By the same consideration in lemma 3.6 we know $\Omega_1 \subset \bigcup_{n=1}^\infty \tilde{B}^r_{\nu_n}$.

Put $\Omega_2 = \Omega_1 - \bigcup_{n=1}^\infty \hat{B}^r_{\nu_n}[U]$, then

$$m(\Omega_2) < \sum_{n=1}^\infty \{m(\tilde{B}^r_{\nu_n}) - m(\hat{B}^r_{\nu_n}[U])\} \leqq d_r m(G_2)$$
$$< d_r^2 m(\Omega_0) + (d_r + d_r^2)\varepsilon.$$

We repeat this procedure to obtain

$$m(\Omega_k) < d_r^k m(\Omega_0) + (d_r + d_r^2 + \cdots + d_r^k)\varepsilon.$$

Hence

$$\lim_{k \to \infty} m(\Omega_k) < d_r \varepsilon/(1-d_r).$$

Since ε is an arbitrary small number, we conclude

$$m(\Omega_0 - E_r[U]) = 0. \qquad \text{Q.E.D.}$$

3.5. Let $\{a_n\}_{n=1}^\infty$ be the countable dense set in Q. We denote by U^s_n the intersection of Q and the spherical cap on $\Sigma^{r_0}_0$ centered at a_n of radius $1/s(s= N, N+1, \cdots$, where N is sufficiently large.). We put $E_{1/m}[U^s_n] = E(m, n, s)$ for integer m ranges $>1/r(U^s_n)$. By theorem 3.5 almost all x in Ω_0 is contained in $\bigcap_{m,n,s} E(m, n, s)$. Let's see what we can say about this points set. First we fix n and s and make m tend to ∞. Then the radius $R = \overrightarrow{Ox}(x \in \bigcup_{m,n,s} E(m, n, s))$

meets a horoshere $\partial B_\nu^{\gamma_0}$ at a point, say y, equivalent to a point in U_n^s after it passed through $1/m$-horoball $B_\nu^{1/m}$. We take $\eta(\in\Gamma)$ so that $\eta(y)\in U_n^s$, then $R'=\eta(R)$ intersects $\Sigma_0^{\gamma_0}$ on U_n^s after it passed through $B_0^{1/m}$. Since $\eta(O)$ converge to p when $m\to\infty$, there is a sequence in Γ-equivalents of R which converge to a n.e. line initiated at p which intersects $\Sigma_0^{\gamma_0}$ on U_n^s.

Next, we fix n and make s tend to ∞, then, for the n.e. line initiated at p which intersects $\Sigma_0^{\gamma_0}$ at $a_n(n=1, 2, \cdots)$ there is a convergent sequence of Γ-equivalents of R to this line. Since a_n are dense in Q, for any n.e. line initiated at p which intersects $\Sigma_0^{\gamma_0}$ on Q, there is a convergent sequence of Γ-equivalents of R to the line. Since $\Sigma_0^{\gamma_0}$ is the tesselation of the copies of Q under M_P, which keeps p invariant, finally we know: For any n.e. line initiated at p, there is a sequence of elements γ_ν in Γ such that $\gamma_\nu R$ converges to it.

$\Gamma(p)=\{\gamma(p)\,;\,\gamma\in\Gamma\}$ is everywhere dense on S, so that for any n.e. line L, we can choose a convergent sequence to L in the set of all n.e. line initiated in $\Gamma(p)$. Hence we have the final result:

Theorem 3.7. *All points in Ω_0, except a subset of null measure, are transitive.*

So with the result in sec. 2, we complete the proof of Theorem 1.1.

§4. An analoguous theorem to the approximation theorem for classical Schottky groups.

4.1. Let $S_1, S_2, \cdots, S_{2g}$ be $2g$ orthogonal sheres to $S(g\geqq 2)$ whose interior balls are pairwise disjoint. Suppose for each $k(1\leqq k\leqq g)$ we are given a Möbius transformation γ_k which keeps $\boldsymbol{B}^3$ invariant (So it is a hyperbolic motion on $\boldsymbol{B}^3$) and maps the domain exterior to S_{k+g} onto the interior ball to S_k. Then a Kleinian group Γ freely generated by the γ_k's is called a classical Schottky group.

The domain P in $\bar{\boldsymbol{B}}^3=\boldsymbol{B}^3\cup S$ bounded by S_k's is a fundamental domain Γ. Γ acts discontinuously on $\Omega=\bigcup_{\gamma\in\Gamma}\gamma(\bar{P})$ and the quotient space Ω/Γ is a compact 3-manifold homeomorphic to a ball with g handlebodies and its interior has a hyperbolic structure induced by that of $\boldsymbol{B}^3$. The complement Λ of Ω is the limit set of Γ.

4.2. The set of all nontrivial elements of Γ is denoted by Γ^*. Then each element of Γ^* is expressed uniquely as a word in generators $\Sigma=\{\gamma_1, \cdots, \gamma_g, \gamma_{g+1}, \cdots, \gamma_{2g}\}$, where we put $\gamma_{g+k}=\gamma_k^{-1}$,

$$\gamma=\gamma_{j1}\gamma_{j2}\cdots\gamma_{jn} \qquad (\gamma_{jk}\in\Sigma\ \gamma_{jk}\cdot\gamma_{jk+1}\neq 1)$$

viz. $\gamma(x)=\gamma_{ji}(\gamma_{j2}(\cdots(\gamma_{jn}(x)\cdots)$.

Here we make an agreement on a notation about the composition: We write the composition of $\gamma=\gamma_{i1}\cdots\gamma_{jm}$ and $\eta=\gamma_{j1}\cdots\gamma_{jn}$ as $\gamma\cdot\eta$ with a dot $(\cdot)$ only when $\gamma_{im}\neq\gamma_{j1}^{-1}$. To express the ordinary composition, we write as $\gamma\eta$.

We denote the ball interior to S_k by $B_{\gamma_k}(k=1, \cdots, 2g)$ and put $\gamma(B_{\gamma k})=B_{\gamma\cdot\gamma_k}$ when the composition is defined in the above sense. Then $B_\eta \subset B_\gamma$ if and only if $\eta=\gamma\cdot\xi$ for some $\xi\in\Gamma$. Note that $\gamma(P)\subset B_\gamma$.

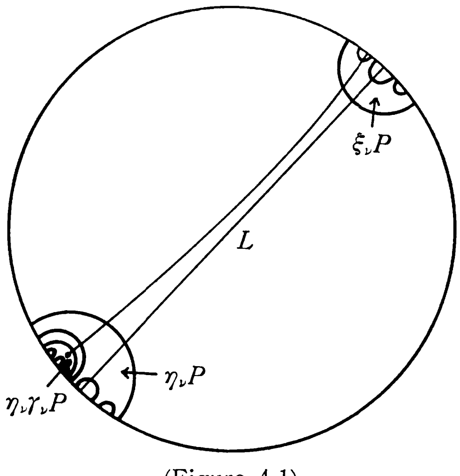
Our second purpose in this paper is to prove

Theorem 4.1. *If the limit set Λ of a Schottky group Γ has the Hausdorff dimension μ, then all points in Λ, except a subset of the μ-Hausdorff meassure zero, are transitive.*

4.3. To begin with, we give a characterization of the transitive point which is due to the observation by Myrberg [*M*]. We may assume that the fundamental domain P contains the origin O. Let L be any n. e. line with end points $a, b\in\Lambda$. If we run along L toward a or b, we meet infinitely many copies of P. So we can choose two sequences η_ν, ξ_ν of elements in Γ so that

$$\lim_{\nu\to\infty}\eta_\nu(P)=a \quad \text{and} \quad \lim_{\nu\to\infty}\xi_\nu(P)=b. \tag{4.1}$$

If x belongs to $B_{\gamma_\nu\cdot\eta_\nu^{-1}\xi_\nu}$ or to $B_{\gamma_\nu\cdot\xi_\nu^{-1}\eta_\nu}$ with some $\gamma_\nu\in\Gamma$ then the radius $R=\overrightarrow{Ox}$ meets P and $\gamma_\nu\cdot\eta_\nu^{-1}\xi_\nu(P)$ or $\gamma_\nu\cdot\xi_\nu^{-1}\eta_\nu(P)$ respectively. Now translate R by $\eta_\nu\gamma_\nu^{-1}$ or $\xi_\nu\gamma_\nu^{-1}$ to see the image passes through both $\eta_\nu(P)$ and $\xi_\nu(P)$. Therefore if x belongs to $B_{\gamma_\nu\cdot\eta_\nu^{-1}\xi_\nu}$ or to $B_{\gamma_\nu\cdot\xi_\nu^{-1}\eta_\nu}$ for all ν and some $\gamma_\nu\in\Gamma$ we can find Γ-images of R passing through $\eta_\nu(P)$ and $\xi_\nu(P)$ which converge to L. (See Figure 4.1)



(Figure 4.1)

Especially if, for all $\gamma\in\Gamma$, x belongs to $B_{\eta\cdot\gamma}$ or to $B_{\eta\cdot\gamma^{-1}}(\eta\in\Gamma)$, then Γ-orbits of $R=\overrightarrow{Ox}$ contain a convergent sequence to L. So that we have

Proposition 4.2. *A point x in Λ is transitive if, for all $\gamma\in\Gamma$, x belongs to $B_{\eta\cdot\gamma}$ or to $B_{\eta\cdot\gamma^{-1}}$ for some $\eta\in\Gamma$.*

But we can modify this slightly as follows:

Proposition 4.3. *A point x in Λ is transitive if, all $\gamma \in \Gamma$ such that $\gamma_{i1} \neq \gamma_{in}^{-1}$ when we express it by word as $\gamma_{i1} \cdots \gamma_{in}$, x belongs to $B_{\eta \cdot \gamma}$ or to $B_{\eta \cdot \gamma^{-1}}$ for some $\eta \in \Gamma$.*

Proof. Express, say, $\eta_\nu^{-1}\xi_\nu$ as $\gamma_{i1} \cdots \gamma_{in}$ where η_ν, ξ_ν are those appeared in above consideration, and suppose $\gamma_{i1} = \gamma_{in}^{-1}$, then we replace ξ_ν by $\xi_\nu \cdot \hat{\gamma}_\nu (\hat{\gamma}_\nu \in \Sigma$, $\hat{\gamma}_\nu \neq \gamma_{i1}^{-1})$. Then the polygon $\xi_\nu \cdot \hat{\gamma}_\nu(P)$ is adjacent to $\xi_\nu(P)$ and even in this case $\lim \xi_\nu \cdot \hat{\gamma}_\nu(P) = b$. Q.E.D.

We define a subset $[\gamma]$ of Λ for $\gamma \in \Gamma^*$ as the set of points x which belong to $B_{\eta \cdot \gamma}$ or to $B_{\eta \cdot \gamma^{-1}}$ for some $\eta \in \Gamma$.

Since Γ is countable, for the proof of the theorem we are sufficient to show.

Theorem 4.2. *$M_\mu(\Lambda - [\gamma]) = 0$ for all γ such that $\gamma_{i1} \neq \gamma_{in}^{-1}$ if $\gamma = \gamma_{i1} \cdots \gamma_{in} (\gamma_{ik} \in \Sigma)$, where we denote by $M_\mu(E)$ the μ-Hausdorff measure of E.*

§ 5. The proof of theorem 4.2.

Our proof of the theorem much owe to Akaza's results in [*Ak*−1] and [*Ak*−2].

5.1. Since we assume that $\gamma_{i1} \neq \gamma_{in}^{-1}$ for $\gamma = \gamma_{i1} \cdots \gamma_{in}$, at least one of $\eta \cdot \gamma$ or $\eta \cdot \gamma^{-1}$ is defined for any $\eta \in \Gamma$ in the sense mentioned in 4.2.

Lemma 5.1. *There is a positive constant ρ independent of $\eta \in \Gamma$ such that*

$$\rho M_\mu(B_\eta \cap \Lambda) \leqq M_\mu(B_{\eta \cdot \gamma} \cap \Lambda) \quad \text{or} \quad M_\mu(B_{\eta \cdot \gamma^{-1}} \cap \Lambda).$$

Proof. Let B be a closed ball of radius r contained in B_η. Put $\hat{B} = \xi^{-1}(B)$ where $\eta = \xi^{-1} \cdot \gamma_0 (\gamma_0 \in \Sigma)$. Then $\hat{B} \subset B_{\gamma_0}$ and

$$r(\hat{B})^2 = \frac{1}{4\pi} \int_{\partial B} |(\xi^{-1})'(x)|^2 d\omega(x) = \frac{1}{4\pi} \int_{\partial B} \frac{R_\xi^4}{|x - \xi(\infty)|^4} d\omega(x)$$

where $r(B)$, R_ξ are the radii of B and the isometric sphere of ξ respectively. Therefore

$$\frac{R_\xi^2 r}{\max |x - \xi(\infty)|^2} \leqq r(\hat{B}) \leqq \frac{R_\xi^2 r}{\min |x - \xi(\infty)|^2} \tag{5.1}$$

For simplicity, we set $M_\eta = \max_{x \in B_\eta} |x - \xi(\infty)|^2$, $m_\eta = \min_{x \in B_\eta} |x - \xi(\infty)|^2$. For a subset E of $\boldsymbol{R}^3$, we use the notation $C(E\,;\, r)$ to denote the family of the coverings of E by closed balls of radii smaller than r.

For sufficiently small r, we may assume that $B_\nu \subset B_\eta$ for all $\{B_\nu\} \subset C(B_\eta \cap \Lambda\,;\, r)$. Then by (5.1) if we set $\hat{B}_\nu = \xi^{-1}(B_\nu)$, $\{B_\nu\} \subset C(B_{r_0} \cap \Lambda\,;\, R_\xi^2 r / m_\eta)$. Conversely for sufficient small r, if $\{\hat{B}_\nu\} \subset C(B_{r_0} \cap \Lambda\,;\, r)$, we may assume that $\hat{B}_\nu \subset B_{r_0}$, and by setting $B_\nu = \xi(\hat{B}_\nu)$ we have $\{B_\nu\} \subset C(B_\eta \cap \Lambda\,;\, M_\eta r / R_\xi^2)$.

The results in [Ak-2] (Lemma 5 and Theorem 4) show $0 < M(B_\gamma \cap \Lambda) < \infty$ for all $\gamma \in \Gamma^*$. Therefore

$$k=\min\left\{\frac{M_\mu(B_{\gamma_i\cdot\gamma^\varepsilon}\cap\Lambda)}{M_\mu(B_{\gamma_i}\cap\Lambda)}\ ;\ \gamma_i\in\Sigma(1\leqq i\leqq 2g)\varepsilon=\pm 1\right\}$$

is a finite positive number. From these we obtain

$$M_\mu(B_{\eta\cdot\gamma^\varepsilon}\cap\Lambda)\geqq(m_\eta/R_\xi^2)^\mu M_\mu(B_{\gamma_0\cdot\gamma^\varepsilon}\cap\Lambda)\geqq k(m_\eta/R_\xi^2)^\mu M_\mu(B_{\gamma_0}\cap\Lambda). \tag{5.2}$$

Again we recall the following Akaza's result [Ak-2]: Generally we set for $\gamma\in\Gamma^*$, $F(n,\gamma)=\{B_{\gamma\cdot\xi}\ ;\ \xi\in\Gamma,\ l(\gamma\cdot\xi)\geqq n\}$ where $l(\gamma\cdot\xi)$ is the word length of $\gamma\cdot\xi$ in Σ. Clearly $F(n;\gamma)$ is a covering of $B_\gamma\cap\Lambda$ and the radius of any ball in $F(n;\gamma)$ is less than given $\delta(>0)$ for a sufficiently large integer n. Let $F^{\delta/k_0}(n;\gamma)$ be a covering of $B_\gamma\cap\Lambda$ by balls in $F(n;\gamma)$ whose radii are not greater than $\delta/2k_0$ (k_0 is a positive constant depending only on Γ.). Then it holds

$$\begin{aligned} L_\mu(B_\gamma\cap\Lambda)&=\lim_{\delta\to 0}\ \inf_{\{F^{\delta/k_0}_{(n;\gamma)}\}}\ \sum_{B\in F^{\delta/k_0}_{(n;\gamma)}}(2r(B))^\mu \\ &\leqq C(k_0/2)^{-\mu}M_\mu(B\gamma\cap\Lambda). \end{aligned} \tag{5.3}$$

where $r(B)$ is the radius of B and C is an absolute constant. We apply these results to prove the lemma. By (5.1)

$$\{B=\xi(\hat{B})\ ;\ \hat{B}\in F^{\delta/k_0}(n\ ;\ \gamma_0)\}\subset F^{\delta M_\eta/k_0R_\xi^2}(n+n_0\ ;\ \eta)$$

where $n_0=l(\xi)$ is the word length of ξ. Then it holds by (5.2)

$$\begin{aligned} M_\mu(B_{\gamma_0}\cap\Lambda)&\geqq(k_0/2)^\mu C^{-1}L_\mu(B_{\gamma_0}\cap\Lambda) \\ &\geqq(k_0/2)^\mu C^{-1}(R_\xi^2/M_\eta)^\mu\lim_{\delta\to 0}\ \inf_{\{F^{\delta M_\eta/k_0R_\xi^2}_{(n+n_0;\eta)}\}}\sum(2r(B))^\mu \\ &\geqq(k_0/2)^\mu C^{-1}(R_\xi^2/M_\eta)^\mu M_\mu(B_\eta\cap\Lambda). \end{aligned} \tag{5.4}$$

From (5.2) and (5.4)

$$M_\mu(B_{\eta\cdot\gamma^\varepsilon}\cap\Lambda)\geqq k(k_0/2)^\mu C^{-1}(m_\eta/M_\eta)^\mu M_\mu(B_\eta\cap\Lambda) \tag{5.5}$$

Then to prove the lemma, it suffices to replace m_η/M_η in (5.5) by a constant independent of η.

$$\frac{m_\eta}{M_\eta}=\frac{\min_{x\in B\eta}|x-\xi(\infty)|^2}{\max_{x\in B\eta}|x-\xi(\infty)|^2}=\frac{|\xi(x_1)-\xi(\infty)|^2}{|\xi(x_2)-\xi(\infty)|^2} \tag{5.6}$$

for some $x_1,\ x_2\in B_{\gamma_0}$. Since the right hand side in (5.6) holds

$$\frac{|\xi(x_1)-\xi(\infty)|^2}{|\xi(x_2)-\xi(\infty)|^2}\geqq\left(1+\frac{|\xi(x_1)-\xi(x_2)|}{|\xi(x_1)-\xi(\infty)|}\right)^{-2}$$

and

$$\frac{|\xi(x_1)-\xi(x_2)|}{|\xi(x_1)-\xi(\infty)|}=\frac{|x_1-x_2|}{|x_2-\xi^{-1}(\infty)|}\leqq\frac{\operatorname{diam}B_{\gamma_0}}{\min_{x\in B_{\gamma_0}}|x-\xi^{-1}(\infty)|},$$

therefore $m_\eta/M_\eta>\text{const.}>0$. Q.E.D.

5.2. Since Λ is the union of the disjoint compact sets $B_{\gamma_i}\cap\Lambda(i=1, \cdots, 2g)$, where $\gamma_{g+i}=\gamma_i^{-1}$, $M_\mu(\Lambda)=\sum_{i=1}^{2g} M_\mu(B_{\gamma_i}\cap\Lambda)$. Put $E_1=\Lambda-\bigcup_{i=1}^{2g}(B_{\gamma_i\cdot\gamma^\varepsilon}\cap\Lambda)$, where $\bigcup_{i=1}^{2g}(B_{\gamma_i\cdot\gamma^\varepsilon}\cap\Lambda)\subset[\gamma](\varepsilon=\pm 1)$. Since E_1 and $\bigcup_{i=1}^{2g}(B_{\gamma_i\cdot\gamma^\varepsilon}\cap\Lambda)$ are disjoint compact sets,

$$M_\mu(E_1)=\sum_{i=1}^{2g}(M_\mu(B_{\gamma_i}\cap\Lambda)-M_\mu(B_{\gamma_i\cdot\gamma^\varepsilon}\cap\Lambda))$$

$$\leqq(1-\rho)\sum_{i=1}^{2g} M_\mu(B_{\gamma_i}\cap\Lambda)=(1-\rho)M_\mu(\Lambda)$$

by the previous lemma.

We put inductively $\Gamma^\nu=\{\eta\cdot\xi\,;\ \eta\in\Gamma\ l(\xi)=l(\gamma)=n_0,\ \xi\neq\gamma^{\pm 1}\}$ for integer $\nu(>0)$ and $\Gamma^0=\Gamma$. Then for $E_{\nu+1}=E_\nu-\bigcup_{\eta\in\Gamma^\nu}(B_{\eta\cdot\gamma^\varepsilon}\cap\Lambda)$ where $B_{\eta\cdot\gamma^\varepsilon}\cap\Lambda\subset[\gamma]$, similarly we have

$$M_\mu(E_{\nu+1})=\sum_{\eta\in\Gamma^\nu}(M_\mu(B_\eta\cap\Lambda)-M_\mu(B_{\eta\cdot\gamma^\varepsilon}\cap\Lambda))$$

$$\leqq(1-\rho)^{\nu+1}M_\mu(\Lambda).$$

So that $M_\mu(E_\nu)\to 0(\nu\to 0)$, and this leads to the result.

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Added information: S. Agard gives in $[A_g]$ a geometric proof of the approximation theorem for kleinian groups Γ of divergence type based on the fact that Γ acts ergodically on S.

$[A_g]$ S. Agard, A geometric proof of Mostow's rigidity theorem for groups of divergence type, Acta Math., **151** (1983), 231-252.

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