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On discontinuous subgroups with parabolic transformations of the Möbius groups

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

By

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Introduction

In the theory of Kleinian groups, the following result by H. Shimizu [10] (see also Kra [7, p. 68], Leutbecher [8, Lemma 2.1]) is one of the most fundamental fact.

Proposition. *Let G be a discrete subgroup of $PSL(2; \boldsymbol{C})=SL(2; \boldsymbol{C})/\{\pm E\}$, where E is the unit matrix. If $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \in G$, then for any $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in G$ we have $|c| \geq 1$ or $c=0$.*

It is known that one can regard a discrete subgroup of $PSL(2; \boldsymbol{C})$ as a group of Möbius transformations, acting discontinuously on the upper half 3-space $\boldsymbol{H}^3=\{(x_1, x_2, x_3) \in \boldsymbol{R}^3;\ x_3>0\}$, and that an element $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in PSL(2; \boldsymbol{C})$, $c \neq 0$, corresponds to a Möbius transformation with isometric sphere of radius $|c|^{-1}$. In this point of view the above Proposition informs us that if a group G of Möbius transformations, acting discontinuously on $\boldsymbol{H}^3$, contains the parabolic transformation: $(x_1, x_2, x_3) \mapsto (x_1+1, x_2, x_3)$, then for any $T \in G$, $T(\infty) \neq \infty$, the radius of the isometric sphere of T does not exceed one. Our aim is to study this property in the higher dimensional cases. In §2 we shall state the main results in this paper, after providing some definitions and basic facts in §1. The proofs of our results will appear in §§2 and 3.

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§1. Preliminaries.

Let $\boldsymbol{R}^n$ and $\overline{\boldsymbol{R}^n}=\boldsymbol{R}^n \cup \{\infty\}$ be the n-dimensional Euclidean space and its one-point compactification, respectively. To represent points in $\boldsymbol{R}^n$ we use $n\times 1$-matrices mostly but sometimes we use pairs of $(n-1)\times 1$-matrices and scalars. We denote the transpose of a matrix A by ${}^tA$ and the standard basis of $\boldsymbol{R}^n$ by $e_1, e_2,\dots, e_n$, e.g. ${}^te_1=(1, 0,\dots, 0)$.

A mapping of the form

$$(1.1)\qquad x \longmapsto kUx+a,$$

where $k>0$, $U\in O(n)$ and $a\in \boldsymbol{R}^n$, is called a *(Euclidean) similarity*. The reflection in the unit $(n-1)$-sphere S^{n-1} is denoted by

$$(1.2)\qquad x \longmapsto J(x)=x^*=x/|x|^2 \qquad (0^*=\infty,\ \infty^*=0).$$

The *(full) Möbius group* $\mathscr{M}_n$ is defined by the group of the Möbius transformations of $\overline{\boldsymbol{R}^n}$, which is generated by all similarities and the reflection J. All transformations considered in this paper are Möbius transformations only, hence we call them transformations for short from now on.

The transformations except the identity are classified into three kinds of mappings as follows. An element in $\mathscr{M}_n$ is said to be *parabolic* if it is conjugate (in $\mathscr{M}_n$) to a similarity of the form

$$(1.3)\qquad x \longmapsto Ux+a,$$

where $U\in O(n)$, $a\in \boldsymbol{R}^n\setminus\{0\}$ and $Ua=a$, and *loxodromic* if it is conjugate to a similarity of the form

$$(1.4)\qquad x \longmapsto kVx,$$

where $k>0$, $k\neq 1$ and $V\in O(n)$. An element $T\in\mathscr{M}_n$, $T\neq E$, (we use the same letter E for the identity transformation with the unit matrix,) which is neither parabolic nor loxodromic is said to be *elliptic*. A parabolic transformation such that $U=E$ in the above definition (1.3) is said to be *strictly parabolic*. A parabolic transformation which is not strictly parabolic is named to be *parabolic with torsion* in Apanasov [3].

A parabolic transformation has one fixed point and a loxodromic transformation has two fixed points. On the other hand an elliptic transformation does not necessarily have a fixed point; for example $-J$: $x\mapsto -x^*$, but if it has then the set of the fixed points is equal to the image of the j-sphere S^j, $0\leq j<n$, under some element in $\mathscr{M}_n$. (cf. Agard [1, §5.1])

It is known that a transformation, leaving ∞ fixed, is a similarity and that $T\in\mathscr{M}_n$, $T(\infty)\neq\infty$, has the form

$$(1.5)\qquad T(x)=kU(x-p)^*+q,$$

where $k>0$, $U\in O(n)$ and p, $q\in\boldsymbol{R}^n$. (cf. Ahlfors [2, pp. 21–22])

By $T'(x)$ we denote the Jacobian matrix of $T\in\mathscr{M}_n$ at $x\in\boldsymbol{R}^n$. It is easily seen that

(1.6) $$J'(x)=\frac{1}{|x|^2}\left(E-\frac{2x^t x}{|x|^2}\right).$$

Following Ahlfors, we set

(1.7) $$Q(x)=x^t x/|x|^2 \qquad (x\neq 0,\ \infty).$$

One can see that

(1.8) $$E-2Q(a)\in O(n),$$

in fact $E-2Q(a)$ is the reflection in the $(n-1)$-plane $\{x\in \boldsymbol{R}_n;\ {}^t ax=0\}$. The chain rule implies that $T'(x)$ can be written with a positive scalar λ and an orthogonal matrix V *as* $T'(x)=\lambda V$. (This means that T is conformal.) We denote by $|T'(x)|$ this positive scalar λ. From (1.5)–(1.7) it follows that for $T\in\mathscr{M}_n$, $T(\infty)\neq\infty$, the set $\{x\in\boldsymbol{R}^n;\ |T'(x)|=1\}$ is an $(n-1)$-sphere with center $p=T^{-1}(\infty)$. This sphere is called the *isometric sphere* of T. We denote its radius by $r(T)$. From the definition of $r(T)$ we obtain $k=r(T)^2$ in (1.5).

We can regard $\mathscr{M}_n$ as a subgroup of $\mathscr{M}_{n+1}$. If we represent points in $\boldsymbol{R}^{n+1}$ by means of the pairs (x, s) of $x\in\boldsymbol{R}^n$ and $s\in\boldsymbol{R}$ and identify $\boldsymbol{R}^n$ with the subset $\{(x, 0);\ x\in\boldsymbol{R}^n\}$ of $\boldsymbol{R}^{n+1}$. Then a similarity (1.1) can be extended to a similarity of the form

(1.9) $$(x, s)\longmapsto k(Ux, s)+(a, 0)$$

and the reflection (1.2) to the reflection in the unit n-sphere S^n

(1.10) $$(x, s)\longmapsto (x, s)^*=\frac{(x, s)}{|x|^2+s^2}.$$

The above extensions induce a mapping ι of $\mathscr{M}_n$ into $\mathscr{M}_{n+1}$ naturally. One can see that this induced mapping ι is well-defined and each transformation in its image $\iota(\mathscr{M}_n)$ keeps the $(n+1)$-dimensional upper half-space $\boldsymbol{H}^{n+1}=\{(x, s)\in\boldsymbol{R}^n\times\boldsymbol{R};\ s>0\}$ invariant. Moreover the followings can be seen. Set $\mathscr{M}(\boldsymbol{H}^{n+1})=\{T\in\mathscr{M}_{n+1};\ T(\boldsymbol{H}^{n+1})=\boldsymbol{H}^{n+1}\}$. Then the mapping ι is an isomorphism of $\mathscr{M}_n$ onto $\mathscr{M}(\boldsymbol{H}^{n+1})$ and preserves the radii of the isometric spheres. Therefore we identify the elements in $\mathscr{M}_n$ with the elements in $\mathscr{M}(\boldsymbol{H}^{n+1})$ and use the same letters.

Since $\mathscr{M}_n$ is a group of continuous self-mappings of $\overline{\boldsymbol{R}^n}$, which becomes a compact metric space with the help of the spherical metric, by uniform convergence a topology is induced on $\mathscr{M}_n$. We remark that this topology is equivalent to the topology defined in Ahlfors [2, p. 79].

The followings, well-known for the case of $n=2$, is valid for any dimensions.

(1) *If G is a discrete subgroup of $\mathscr{M}_n$, then G is discontinuous on $\boldsymbol{H}^{n+1}$.*

(2) *The converse of* (1) *holds.*

(3) *If G is discontinuous in $\overline{\boldsymbol{R}^n}$, that is, discontinuous on an open subset of $\overline{\boldsymbol{R}^n}$, then so is G on $\boldsymbol{H}^{n+1}$.*

It is easy to see (2). The statements (1) and (3) are shown implicitly in Ahlfors [2, pp. 79–82].

§2. Statements and proofs

A parabolic transformation in $\mathscr{M}_n$ $(=\mathscr{M}(H^{n+1}))$ is cojugate to a similarity of the form

$$x \longmapsto Ux+e_n, \quad x\in \boldsymbol{R}^n \qquad (Ue_n=e_n), \tag{2.1}$$

where $U\in O(n)$. We remark that torsion U in (2.1) is not uniquely determined under this "normalization" but ord $U=\sharp\langle U\rangle$, the order of U, is.

For a subgroup G of $\mathscr{M}_n$ we set $G_\infty=\{T\in G;\ T(\infty)=\infty\}$.

Theorem 1. *Suppose that a subgroup G of $\mathscr{M}_n$ acts discontinuously on $\boldsymbol{H}^{n+1}$ and contains a parabolic transformation X of the form* (2.1). *If* ord $U<+\infty$, *then for every transformation Y in $G\smallsetminus G_\infty$ the radius $r(Y)$ of the isometric sphere of Y satisfies*

$$r(Y)\leq \mathrm{ord}\, U. \tag{2.2}$$

The estimate (2.2) in Theorem 1 is sharp in the following sense.

Theorem 2. *Let $X\in\mathscr{M}_n$ be a parabolic transformation of the form* (2.1). *If* ord $U<+\infty$, *then there exists a transformation $Y\in\mathscr{M}_n$ with isometric sphere of radius* ord U *for which $\langle X, Y\rangle$ is discontinuous on $\boldsymbol{H}^{n+1}$ (, and also in $\overline{\boldsymbol{R}^n}$ if $n\geq 2$).*

Corollary 1. *Let G and X be as in Theorem* 1. *Then the horoball $B=\{x\in\boldsymbol{H}^{n+1};$ ${}^te_{n+1}x>$*ord $U\}$ *at ∞ is precisely invariant under G_∞, i.e. $Y(B)=B$ if $Y\in G_\infty$ and $Y(B)\cap B=\emptyset$ if $Y\in G\smallsetminus G_\infty$.*

Proof. Suppose that G_∞ contains a similarity $Y(x)=kVx+a$ such that $k\neq 1$. By taking Y^{-1}, if necessary, we may assume that $k<1$. It is easily seen that Y has a fixed point $b=(E-kV)^{-1}a$. Let $T(x)=x+b$ and set $\tilde{X}=T^{-1}XT$ and $\tilde{Y}=T^{-1}YT$. Then we have $\tilde{X}(x)=Ux+c$, where $c=e_n+(U-E)b$, and $\tilde{Y}(x)=kVx$. By (1.9) we have $\tilde{Y}^m\tilde{X}\tilde{Y}^{-m}(e_{n+1})=e_{n+1}+k^mV^mc$. Since $c\neq 0$, $\tilde{Y}^m\tilde{X}\tilde{Y}^{-m}$ are all distinct. Letting $m\to+\infty$ we see that the orbits $\tilde{Y}^m\tilde{X}\tilde{Y}^{-m}(e_{n+1})$ converge to e_{n+1}, which contradicts the discontinuity of $\tilde{G}=T^{-1}GT$. Therefore G_∞ consists of only similarities $x\to Wx+a$, $W\in O(n)$. These transformations keep B invariant. On the other hand we see from Theorem 1 that $r(Y)\leq$ ord U for any $Y\in G\smallsetminus G_\infty$. Hence it follows that $Y(B)\subset\{x\in\boldsymbol{H}^{n+1};\ {}^te_{n+1}x<\mathrm{ord}\, U\}$ for any $Y\in G\smallsetminus G_\infty$. This completes the proof,

q. e. d.

In the case of ord $U=+\infty$, by contraries, there does not exist such a uniform estimate as in Theroem 1. We can show

Theorem 3. *Let X be a parabolic transformation of the form* (2.1). *If* ord U $=+\infty$, *then for any $r>0$ there exists a transformation $Y\in\mathscr{M}_n$, $Y(\infty)\neq\infty$, with isometric sphere of radius $r(Y)\geq r$, for which $\langle X, Y\rangle$ is discontinuous in $\overline{\boldsymbol{R}^n}$.*

As a corollary of Theorem 3 we immediately obtain

Corollary 2. *Let X be as in Theorem 3. Then there is no horoball B at ∞ such that for all $Y \in \mathcal{M}_n$ for which $\langle X, Y\rangle$ is discontinuous on $\boldsymbol{H}^{n+1}$, B is precisely invariant under $\langle X, Y\rangle$.*

Moreover we can show

Theorem 4. *Let X be a parabolic transformation of the form* (2.1). *If* ord $U = +\infty$, *then there exists an infinitely generated subgroup G of $\mathcal{M}_n$ such that i) $X \in G$, ii) G is discontinuous in $\overline{\boldsymbol{R}^n}$ and iii) $\{r(Y);\ Y \in G \setminus G_\infty\}$ is not bounded.*

Corollary 3. *Let G be as in Theorem 4. Then there is no horoball at ∞ which is precisely invariant under G_∞.*

Theorems 2, 3 and 4 will be shown in §3 by constructing examples.

We begin by proving Theorem 1 under the assumption $U=E$ in (2.1). Under this assumption Theorem 1 is proved in Apanasov [4] and Wielenberg [11, Proposition 4]. (See also Apanasov [5, Theorem 3.3], which says that in the estimate (2.2), even if ord $U=+\infty$, ord U could be replaced by 1, but, as we shall see in §3, it is not valid. He actually proves for the case $U=E$ only.) They make use of the representation of $\mathcal{M}_n$ by means of the Lorentz group $O(n+1, 1)$ and derive the result from the discreteness of G. We give here a more direct proof, in which the assertion from the discontinuity on $\boldsymbol{H}^{n+1}$ of G.

Lemma 1. *If T has the form*

$$T\colon x \longmapsto r^2 V(x-p)^* + q, \tag{2.3}$$

then we have

$$q = T(\infty), \quad p = T^{-1}(\infty), \tag{2.4}$$

$$V = \frac{|x-p|^2}{r^2}\, T'(x)\{E - 2Q(x-p)\}. \tag{2.5}$$

Lemma 1 is easily obtained by elementary calculus.

Lemma 2. *Let T be a transformation such that $T(\infty) \neq \infty$. Then*

$$|T(x)-T(y)| = \frac{r(T)^2|x-y|}{|x-T^{-1}(\infty)|\,|y-T^{-1}(\infty)|} \tag{2.6}$$

and

$$|T(x)-T(\infty)|\,|x-T^{-1}(\infty)| = r(T)^2. \tag{2.7}$$

Since T is of the form (2.3), one can see (2.6) by direct computation or elementary geometrical consideration. (In fact (2.3) means that T is a composite mapping of the reflection in the sphere with center $T^{-1}(\infty)$ and radius $r(T)$, and a Euclidean isometry.) Letting $y \to \infty$ we get (2.7) from (2.6).

Proof of Theorem 1. (The case $U=E$) Let $Y \in G \setminus G_\infty$, then Y is written as

$Y(x)=r^2V(x-p)^*+q$. We define

$$Y_0=Y, \quad Y_{j+1}=Y_jXY_j^{-1}, \tag{2.8}$$
$$Y_j(x)=r_j^2V_j(x-p_j)^*+q_j.$$

Setting $T=Y_{j+1}$, $x=Y_j(\infty)$ in (2.7) and then $T=Y_j$, $x=X^{\pm1}Y_j^{-1}(\infty)$, we have

$$r_{j+1}^2=|Y_j(\infty)-Y_jXY_j^{-1}(\infty)|\,|Y_j(\infty)-Y_jX^{-1}Y_j^{-1}(\infty)|$$
$$=\frac{r_j^2}{|XY_j^{-1}(\infty)-Y_j^{-1}(\infty)|}\frac{r_j^2}{|X^{-1}Y_j^{-1}(\infty)-Y_j^{-1}(\infty)|}=r_j^4,$$

where we use the fact $X(\infty)=\infty$ in the first equality and $X^{\pm1}(x)=x\pm e_n$ in the third, hence $r_{j+1}=r_j^2$. This implies

$$r_j=r^{2^j}. \tag{2.9}$$

From Lemma 1 and (2.8) we get

$$q_{j+1}=q_j+r_j^2V_je_n, \quad p_{j+1}=q_j-r_j^2V_je_n, \tag{2.10}$$

in particular,

$$|q_{j+1}-p_{j+1}|=2r_j^2=2r_{j+1}. \tag{2.11}$$

Differentiation of (2.8) yields $Y'_{j+1}(q_{j+1})=Y'_{j+1}(Y_{j+1}(\infty))=Y'_j(X^2(p_j))Y'_j(X(p_j))^{-1}$. Since we have from Lemma 1 $Y'_j(X^k(p_j))=(r_j/k)^2V_j\{E-2Q(ke_n)\}$, we obtain $Y'_{j+1}(q_{j+1})=\frac{1}{4}E$, where we use $Q(2e_n)=Q(e_n)$. Therefore again from Lemma 1 we get

$$V_{j+1}=E-2Q(V_je_n). \tag{2.12}$$

Now we suppose that $r>1$. Then from (2.10) and (2.9)

$$||q_j|-r_j|=||q_j|-|q_j-q_{j-1}||\le|q_{j-1}|$$
$$\le\sum_{k=1}^{j-1}|q_k-q_{k-1}|+|q_0|=\sum_{k=1}^{j-1}r_k+|q|\sim r^{2^{j-1}}=r_{j-1}.$$

Thus our assumption $r>1$ implies

$$\lim_{j\to\infty}\frac{|q_j|}{r_j}=1, \tag{2.13}$$

and also from (2.10)

$$\lim_{j\to\infty}\frac{|p_j|}{r_j}=1. \tag{2.14}$$

Let us show that the orbits of e_{n+1} under Y_j have an accumulation point in $\boldsymbol{H}^{n+1}$, which contradicts the discontinuity of G. First, from (2.8), (1.9), (1.10) and (2.14), we have

(2.15) $$^{t}e_{n+1}Y_j(e_{n+1})=r_j^2/(|p_j|^2+1)\longrightarrow 1.$$

Next, from (2.6)

(2.16) $$|Y_j(0)-Y_j(e_{n+1})|=r_j^2/|p_j|\,|p_j-e_{n+1}|\longrightarrow 1.$$

On the other hand, from (2.10), (2.13) and (2.12)

$$\begin{aligned}|p_{j+1}|^2q_{j+1}&=q_{j+1}{}^{t}p_{j+1}p_{j+1}\\&=r_j^6V_je_n+r_j^4\{E-2Q(V_je_n)\}q_j+O(r_j^4)\\&=r_j^4\{E-2Q(V_je_n)\}(q_j-r_j^2V_je_n)+O(r_j^4)\\&=r_{j+1}^2V_{j+1}p_{j+1}+O(r_{j+1}^2).\end{aligned}$$

We note that $\{E-2Q(a)\}a=-a$. Hence we have

(2.17) $$\begin{aligned}|p_j|^2|Y_j(0)|&=|p_j|^2|q_j-r_j^2V_jp_j^*|\\&=|\,|p_j|^2q_j-r_j^2V_jp_j|=O(r_j^2).\end{aligned}$$

It follows from (2.14) that the orbits $\{Y_j(0)\}$ are bounded and therefore, because of (2.16), so are the orbits $\{Y_j(e_{n+1})\}$ in $\boldsymbol{R}^{n+1}$. This and (2.15) implies that the orbits $\{Y_j(e_{n+1})\}$ have an accumulation point in $\boldsymbol{H}^{n+1}$.

In the general case we consider the subgroup $\langle X^m, Y\rangle$ of G, where $m=\operatorname{ord}U$, instead of $\langle X, Y\rangle$. Since $X^m(x)=x+me_n$, the above argument can be applied to the conjugate group $T_m^{-1}\langle X^m, Y\rangle T_m=\langle T_m^{-1}X^mT_m, T_m^{-1}YT_m\rangle$, where $T_m(x)=mx$. Because we have $r(T_m^{-1}YT_m)=r(Y)/m$, thus we obtain (2.2), q.e.d.

§3. Examples

In the case of $n=1$, that is, in the case of Fuchsian groups, the Modular group $PSL(2; \boldsymbol{Z})=\langle X: z\mapsto z+1, Y: z\mapsto -1/z\rangle$ is our desired example for Theorem 2. We remark that if $n=1$ then every parabolic transformation is strict, i.e. $U=E$. In addition, also for $n\geq 2$, if $U=E$ then a natural extension of $PSL(2; \boldsymbol{Z})$ to $\boldsymbol{H}^{n+1}$ turns out to be an example, too. Hence in order to show Theorems 2, 3 and 4 it is sufficient to construct examples only for the case of $U\neq E$, i.e. $\operatorname{ord}U\geq 2$ and $n\geq 2$. All our examples, for $n\geq 2$, act discontinuously not only on $\boldsymbol{H}^{n+1}$ but also in $\overline{\boldsymbol{R}^n}$. Therefore we consider the actions of transformations on $\boldsymbol{R}^n$ only. From now on to the end of this section we express points in $\boldsymbol{R}^n$ by means of the pairs (x, s) of $x\in\boldsymbol{R}^{n-1}$ and $s\in\boldsymbol{R}$. Then the action of the parabolic transformation X of the form (2.1) is represented as

(3.1) $$X: (x, s)\longmapsto (Ux, s+1),$$

where $U\in O(n-1)$. We remark that, strictly speaking, U in (3.1) is the restriction of U in (2.1) to the $(n-1)$-plane $\{(x, 0); x\in\boldsymbol{R}^{n-1}\}$. But, by abuse of language, we use the same letter.

Let m be an arbitrary integer such that $2\leq m\leq\operatorname{ord}U$. It is easy to see from the fact that $\dim\{S^{n-2}\cap\ker(U^j-E)\}\leq n-3$ for $1\leq j<m$, where S^{n-2} is the unit $(n-2)$-sphere $\{(x, 0); |x|=1\}$, that there exists a point ζ in S^{n-2} such that

$$\zeta\neq U^j\zeta \quad \text{for} \quad j=1, 2,\dots, m-1. \tag{3.2}$$

Hence we can find an open neighbourhood N (in S^{n-2}) of ζ such that

$$N\cap U^j(N)=\emptyset \quad \text{for} \quad j=1, 2,\dots, m-1. \tag{3.3}$$

Set

$$N_m=\{(x, s)\in(\boldsymbol{R}^{n-1}\setminus\{0\})\times\boldsymbol{R};\ x/|x|\in N,\ 0\leq s<m\}. \tag{3.4}$$

Let us suppose that $N_m\cap X^j(N_m)\neq\emptyset$ for some $j\in\boldsymbol{Z}\setminus\{0\}$. We may assume $j>0$. Let (x, s) be a point in $N_m\cap X^j(N_m)$, then from the definition (3.4) of N_m we see that $x/|x|\in N$, $0\leq s<m$, $U^{-j}(x/|x|)=U^{-j}x/|U^{-j}x|\in N$ and $0\leq s-j<m$. Hence we have $N\cap U^j(N)\ni x/|x|$ and $0<j<m$. This contradicts (3.3). Therefore we obtain

$$N_m\cap X^j(N_m)=\emptyset \quad \text{for any} \quad j\in\boldsymbol{Z}\setminus\{0\}. \tag{3.5}$$

Now we state a result, due to B. Maskit, which we use to construct examples. We begin with definitions. For a subgroup G of $\mathscr{M}_n$ we denote by $\Omega(G)$ $(\subset\overline{\boldsymbol{R}^n})$ the *set of discontinuity* of G. A non-empty subset D of $\Omega(G)$ is called a *partial fundamental set* (*PFS*) for G if no two points in D are equivalent under G. And in addition, if every point in $\Omega(G)$ is equivalent under G to a point in D, then D is said to be a *fundamental set* (*FS*).

Theorem. (Maskit [8, Theorem 4]) *Let G_j $(j=1, 2.)$ be a subgroup of $\mathscr{M}_n$ such that $\Omega(G_j)\neq\emptyset$ and let D_j be a PFS for G_j. Let H be a common subgroup of G_1 and G_2, and let Δ be a PFS for H. Set $E_j=\cup T(D_j)$ $(j=1, 2.)$, where the union is taken over all T in H. If $E_1\cup E_2\supset\Omega(G_1)\cup\Omega(G_2)$ and $D'=\operatorname{int}(D)\neq\emptyset$, where $D=E_1\cap E_2\cap\Delta$. Then $G=\langle G_1, G_2\rangle$ is discontinuous, moreover D' is a PFS for G.*

Let us continue the construction of examples. First we consider the case of $\operatorname{ord} U=+\infty$. Let $B_j=B(j, m)$ $(j=1, 2.)$ be two mutually disjoint open balls with radii $m/2-1/4$ which are relatively compact in N_m, and let Y_m be a loxodromic transformation which maps $\operatorname{ext}(B_1)$ onto B_2. Then $r(Y_m)=m/2-1/4$, in fact, the isometric sphere of Y_m is ∂B_1. Since $\overline{\operatorname{ext}(B_1)}\cap\operatorname{ext}(B_2)$ is a *FS* for $\langle Y_m\rangle$, and N_m, containing $B_1\cap B_2$, is a *PFS* for $\langle X\rangle$, from the above Masakit's theorem (or Koebe's combination theorem) we see that $\langle X, Y_m\rangle$ is discontinuous. Because m is arbitrary, the group $\langle X, Y_m\rangle$ is an example for Theorem 3.

In the above argument, if we take balls $B(j, m)$ $(j=1, 2.\ m=1, 2,\dots)$ which satisfy the additional condition

$$\overline{B(j, m)}\cap\overline{B(j', m')}=\emptyset \quad \text{if} \quad (j, m)\neq(j', m'),$$

then the infinitely generated group $\langle X, Y_1, Y_2,\dots\rangle$ is found to be an example for Theorem 4.

Next we consider the case of ord $U < +\infty$, and set $m = \operatorname{ord} U$. Choose and fix a positive number λ such that if $|x-\lambda\zeta| \leq m$, $x \in \boldsymbol{R}^{n-1}$, then $x/|x| \in N$, where ζ and N are as in (3.2) and (3.3), respectively. Define a transformation Y of $\overline{\boldsymbol{R}^n}$ by

$$Y(x, s) = m^2\{E - 2Q(0, 1))\}\{(x, s) - (\lambda\zeta, 0)\}^* + (\lambda\zeta, 0).$$

We shall show that $\langle X, Y\rangle$ is a desired example for Theorem 2. It is obvious that $\langle X, Y\rangle$ satisfies the statement of Theorem 2 except that it is discontinuous.

Set $G_1 = \langle X^m, Y\rangle$, $G_2 = \langle X\rangle$ and $H = \langle X^m\rangle$. Let Π be an arbitrary (Euclidean) 2-plane which contains the line $\{(\lambda\zeta, s); s \in \boldsymbol{R}\}$. It is easy to see that G_1 leaves $\overline{\Pi} = \Pi \cup \{\infty\}$ fixed. In addition, because the restriction of G_1 to $\overline{\Pi}$ is conjugate to the Modular group $PSL(2; \boldsymbol{Z})$, we find that

$$D_1 = \{(x, s); |x-\lambda\zeta| > m, 0 \leq s < m\}$$

can be taken as a *PFS* for G_1. On the other hand G_2 and H have *PFS*'s N_m and $\Delta = \{(x, s); 0 \leq s < m\}$, respectively. Since we have

$$E_1 = \bigcup_{T \in H} T(D_1) = \{(x, s); |x - \lambda\zeta| > m\}$$

$$E_2 = \bigcup_{T \in H} T(N_m) = \{(x, s); \frac{x}{|x|} \in N\} \supset \boldsymbol{R}^n \setminus E_1,$$

we get

$$E_1 \cup E_2 = \boldsymbol{R}^n \supset \Omega(G_1) \cup \Omega(G_2).$$

And we have $\operatorname{int}(D_1 \cap N_m \cap \Delta) \neq \emptyset$. Hence from Maskit's Theorem it follows that $G = \langle G_1, G_2\rangle = \langle X, Y\rangle$ is discontinuous, q. e. d.

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