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A topological characterization of C^2/G

By

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Introduction

In [6], C. P. Ramanujam has given a topological characterization of C^2 as an affine variety. In [1] it was proved that if $C^2 \to V$ is a proper morphism onto an affine normal surface V, then V is topologically contractible, the fundamental group at infinity of V is finite and $V \approx C^2/G$ where G is a "small", finite subgroup of $GL(2, C)$ acting linearly on C^2 (G is "small" if it contains no non-trivial pseudo-reflections).

In this paper we prove a generalization of C. P. Ramanujam's result which is at the same time a converse of the above result in [1].

Theorem. *Let V be an affine, normal surface/C, which is topologically contractible. Assume further that the fundamental group at infinity of V, $\pi_1^\infty(V)$, is finite. Then $V \approx C^2/G$ where G is a "small" subgroup of $GL(2, C)$ isomorphic to $\pi_1^\infty(V)$.*

In particular, V has at most one singular point.

For the definition of $\pi_1^\infty(V)$, see §1. In §2, we will give some examples of affine normal surfaces which fail to satisfy one of the properties of contractibility or finiteness of $\pi_1^\infty(V)$.

§1.

Proof. Let $P = \{p_1 \cdots p_n\}$ be the singular locus of V. If P is empty then V is non-singular and the result is proved in [1]. In this case $V \approx C^2$.

Now assume that $n \geq 1$. Let Ψ: $Y \to V$ be a resolution of singularities such that $T_i = \Psi^{-1}(p_i)$ is a divisor all whose components are smooth curves with normal crossings (i.e., intersecting transversally and no three meeting in a point) and such that Y is a resolution which is minimal with all these properties. Let $Y \subset X$ where X is a smooth projective surface such that Y is a Zariski dense open subset with $T_\infty = X - Y$ a divisor with normal crossings and is minimal. Since V is affine, T_∞ is connected and T_i for $1 \leq i \leq n$ are connected because V is normal.

Let U_i be a small "tubular" neighbourhood of T_i for $i = 1, 2, \ldots, n, \infty$ such that U_i are pairwise disjoint. Let $Z = X - (U_1 \cup U_2 \cdots \cup U_n \cup U_\infty)$, so that Z is a compact

4 -dimensional (real) manifold with boundary components $\partial_i = \partial U_i$ for $i=1,\dots, n, \infty$. By definition, $\pi_1^\infty(V) = \pi_1(U_\infty - T_\infty) \approx \pi_1(\partial_\infty)$ and this is finite by hypothesis. By a generalization of the Lefschetz hyperplane section theorem (see [5]) it follows that if R is any Zariski closed proper subset of X, then $\pi_1(U_\infty - R) \to \pi_1(X-R)$ is surjective. In particular, taking $R = T_1 \cup \cdots \cup T_n \cup T_\infty$, we obtain $\pi_1(U_\infty - T_\infty) \to \pi_1(X - T_1 \cup \cdots \cup T_n \cup T_\infty)$ is surjective. But $X - (T_1 \cup \cdots \cup T_n \cup T_\infty) = Z$ and Z is a strong deformation retract of $V-P$. Thus $\pi_1(V-P)$ is finite (and actually a homomorphic image of $\pi_1^\infty(V)$). We will need the following lemmas:

Lemma 1. *Let φ: $W \to \Delta$ be a proper morphism where Δ is a smooth algebraic curve, W is smooth and most fibers of φ are isomorphic to $\boldsymbol{P}^1$. Suppose S is a singular fiber of φ and φ has a section A. Let S' be the union of certain components of S none of which meet A and none of which is an exceptional curve of the 1st kind. Then*

$\sharp(S) \geq \sharp(S') + 2$*($\sharp$ denotes the number of components).*

Proof. Since S must contain an exceptional curve of the 1st kind, we see that $\sharp(S) \geq \sharp(S') + 1$. Let E be the component of S for which $E.A = 1$. Then in the scheme-theoretic representation of S, E occurs with multiplicity 1. By a simple argument in $\boldsymbol{P}^1$-fibrations, S should contain an exceptional curve of the 1st kind E', $E' \neq E$. See, for example [4, p. 115]. The lemma is proved.

Lemma 2. *Let φ: $W \to \boldsymbol{P}^1$ be a minimal elliptic fibration (W is projective, smooth and no fibre of φ contains an exceptional curve of the 1st kind). Let S_0 be a singular fibre of φ of type II (in the terminology of [3]) and let S_j, $1 \leq j \leq k$, $k=1$ or 2 be all the other singular fibres. Then for $k=2$, both S_j cannot be of type I_{b_j} (for notation see [3]). If $k=1$, then $S_1 = II^*$.*

Proof. The local monodromy for the fibres of types II and I_b are given respectively by the matrices $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. See [3]. Since $\boldsymbol{P}^1$ is simply-connected, it follows that the product of suitable conjugates of the monodromy matrices of all the singular fibres is the identity matrix in $SL(2, \boldsymbol{Z})$.

For $k=2$, let $S_1 = I_{b_1}$ and $S_2 = I_{b_2}$. Then $\exists L_j \in SL(2, \boldsymbol{Z})$ such that

$$L_1 \begin{pmatrix} 1 & b_1 \\ 1 & 1 \end{pmatrix} L_1^{-1}. \quad L_2 \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} L_2^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Comparing the traces on either side, we get $b_1, b_2 = 1$. Hence $b_1 = b_2 = 1$. But then for the topological Euler characteristic of W obtain $\chi(W) = \chi(S_1) + \chi(S_2) + \chi(S_0) = 2 + b_1 + b_2 = 4$. But Noether's formula implies that $12 \mid \chi(W)$ ($K^2 = 0$ since φ is a minimal fibration). This is a contradiction.

For $k=1$, the monodromy matrix for S_1 should be a conjugate of $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. From [3], $S_1 = II^*$.

Lemma 3. *Let φ: $W \to \boldsymbol{P}^1$ be an elliptic fibration (W smooth, projective) with* *a section A and a singular fibre S_0 of type II. Suppose S_1 is the only other singular fibre and E_1 a component of S_1 such that $E_1, A=1$. Assume that no other component of S_1 is an exceptional curve of the 1st kind. Let T_1 be the union of all the components of S_1 other than E_1. Then $W-(T_1 \cup S_0 \cup A)$ is not simply connected.*

Proof. We can choose tubular neighbourhoods $N(E_1)$, $N(T_1)$, $N(S_1)$ of E_1, T_1 and S_1 resp. such that $N(S_1)=N(E_1) \cup N(T_1)$. Clearly $W-S_0$ is a fibration over C with one singular fibre S_1 and a section $A'=A-S_0$. Hence $N(S_1)-A$ is a strong deformation retract of $W-(S_0 \cup A)$, in particular, $N(S_1)-(A \cup T_1)$ is a strong deformation retract of $W-(T_1 \cup S_0 \cup A)$. Since A' cuts E_1 transversally, it follows that $N(T_1)-T_1$ is a deformation retract of $N(S_1)-(A' \cup T_1)$. Thus we shall show that $\pi_1(N(T_1)-T_1)$ is non trivial.

First suppose that E_1 is not exceptional curve of the 1st kind. Then S_1 is of type II*. Since $E_1.A=1$, E_1 is the curve that occurs with multiplicity 1 in the scheme-theoretic fibre S_1 (see [3]). It follows that T_1 can be thought of as the resolution of the E_8 singularity and hence $\pi_1(N(T_1)-T_1)$ is isomorphic to the binary icosahedral group. (Note that this case has occurred in [1]).

Now suppose that E_1 is an exceptional curve of the 1st kind. Then E_1 can be blown down to a smooth point on a smooth surface W_1 with an elliptic fibration $\varphi_1\colon W_1 \to P^1$. The image of A will be a section of φ_1 which will meet the image of all the components in S_1 that meet E_1. Hence E_1 meets exactly one component in S_1. Repeating this argument, it follows that the dual graph of S_1 is

$$\begin{array}{ccccccccccccccccccc}
-1 & & -2 \quad\quad -2 & & -3 & & -2 & & -2 & & -2 & & -2 & & -2 & & -2 & & -2 \\
\times & \text{—} & \underbrace{\times \cdots\cdots \times}_{m \text{ vertices}} & \text{—} & \times & \text{—} & \times & \text{—} & \times & \text{—} & \times & \text{—} & \times & \text{—} & \times & \text{—} & \times & \text{—} & \times \\
E_1 & & & & & & & & & & & & & & | & & & & \\
 & & & & & & & & & & & & & & \times\ -2 & & & &
\end{array}$$

for some $m \geq 0$. Then it is easily seen that $\pi_1(N(T_1)-T_1)$ is non trivial.

Lemma 4. *Let V be an affine, normal, irreducible surface. If $\pi_1^\infty(V)=(1)$ then V can be embedded in a normal projective surface $\bar{V}$ such that $\bar{V}$ is smooth along $\bar{V}-V$ and the dual graph corresponding to $\bar{V}-V$ is* $\begin{array}{ccc} 0 & & 0 \\ \times & \text{—} & \times \end{array}$.

If $\pi_1^\infty(V)$ is isomorphic to the binary icosahedral group, then $\bar{V}$ as above can be chosen so that the dual graph of $\bar{V}-V$ is

$$\begin{array}{ccccc}
-2 & & -1 & & -3 \\
\times & \text{—} & \times & \text{—} & \times \\
 & & | & & \\
 & & \times\ -5 & &
\end{array}$$

Proof. See [1].

Now we continue with the proof of the Theorem.

Step I: Assume that $\pi_1(V-P)=(1)$. Then we will show that $P=\emptyset$ and $V \approx C^2$. Since V is contractible, $H_i(V, P)=(0)$ for $i \geq 2$ (all the homology groups which occur are with Z-coefficients). Also $H_1(V, P) \approx Z^{n-1}$ Let $U=\cup U_i$, $\partial=\bigcup_{i=1}^{n} \partial_i$. Then $H_i(Z, \partial) \approx H_i(V, U) \approx H_i(V, P)$. By assumption $\pi_1(Z) \simeq \pi_1(V-P)=(1)$. Hence $H_1(Z)=(0)$. From the exact sequence

$$H_2(Z, \partial) \longrightarrow H_1(\partial) \longrightarrow H_1(Z) \text{ we see that}$$

$H_1(\partial)=(0)$. Hence for $i=1,\dots, n$. $H_1(\partial_i)=(0)$. By Poincare duality $H_2(Z, \partial_\infty) \simeq H^2(Z, \partial)$. But $H^2(Z, \partial)\approx \mathrm{Hom}\,(H_2(Z, \partial), \boldsymbol{Z})\oplus \mathrm{Ext}^1\,(H_1(Z, \partial), \boldsymbol{Z})$. Since $H_1(Z, \partial)$ is torsion free, we see that $H^2(Z, \partial)=(0)$. Again from the exact sequence $H_2(Z, \partial_\infty)\to H_1(\partial_\infty)\to H_1(Z)$ we get $H_1(\partial_\infty)=(0)$. Then $G=\pi_1^\infty(V)$ is a finite perfect group and thus $G=(1)$ or the binary icosahedral group (see [1]). By Lemma 4 the dual graph corresponding to T_∞ can be assumed to be $\overset{0}{\times}\!-\!\overset{0}{\times}$ (when $G=(1)$) or $\begin{array}{ccccc} \overset{-2}{\times} & - & \overset{-1}{\times} & - & \overset{-3}{\times} \\ & & | & & \\ & & \times\, -5 & & \end{array}$

We will first show that $G=(1)$ and hence $T_\infty = \overset{0}{\times}\!-\!\overset{0}{\times}$. If not, we have T_∞ has dual graph $\begin{array}{ccccc} \overset{-2}{\times} & - & \overset{-1}{\times} & - & \overset{-3}{\times}. \\ & & | & & \\ & & \times\, -5 & & \end{array}$

As in [1], by three successive blowing down, we get a rational curve C on a smooth surface W with $C^2=1$ and C having a unique singular point which is an ordinary cusp of multiplicity 2. It is easily seen that the linear system C has a unique base point (at which C is smooth) and after blowing-up at this point and taking a 2- dimensional sub-system of C we get an elliptic fibration $\varphi\colon \tilde{X}\to \boldsymbol{P}^1$ with the proper transform C' of C being a singular fibre of φ of type II and a section A. Also now $T_\infty = A\cup C'$. Then $\tilde{X}-(A\cup C')=Y$. Let $S_1,\dots, S_k$, $S_{k+1},\dots, S_{k+l}$ be the other singular fibres, S_i, $i\le k$ being simply connected and S_i, $i>k$ non simplyconnected. Clearly there are no multiple fibres.

Let s_j=number of irreducible components of $S_j\cap(T_1\cup\cdots\cup T_n)$ and $\sharp(S_j)=$ number of irreducible components of S_j. Since T_i do not intersect $C'\cup A$, for $1\le i\le n$, each T_i is contained in a singular fibre S_j for $j\ge 1$. Hence $\sum_{j=1}^{k+l} s_j = \sum_{i=1}^{n} \sharp(T_i)$. Also $\sharp(S_j)\ge s_j+1$ because A is a section. By the contractibility of V, it is easily seen that the second betti number

$$b_2(\tilde{X})=\sum_{i=1}^{n}\sharp(T_i)+\sharp(T_\infty)=\sum_{j=1}^{k+l}s_j+2$$

Since S_j is simply- connected for $j\le k$,

$$\chi_{top}(S_j)=\sharp(S_j)+1 \qquad \text{and} \quad \text{for} \quad j\ge k+1,\ \chi(S_j)=\sharp(S_j)$$

Now

$$\chi(\tilde{X})=\sum s_j+4=\sum_{j=0}^{k+l}\chi(S_j)=2+\sum_{j=1}^{k}(\sharp(S_j)+1)+\sum_{j=k+1}^{k+l}(\sharp S_j)$$

$\ge 2+\sum_{j=1}^{k}(s_j+2)+\sum_{j=k+1}^{k+l}(s_j+1)$ (here we are using the fact that the only curves in X which do not intersect T_∞ are contained in $\bigcup_{i=1}^{n} T_i$ and a singular fibre of φ cannot consists entirely of the T_i as the quadratic form on T_i is negative definite). Thus $\chi(\tilde{X})\ge 2+2k+l+\sum_{j=1}^{k+l}s_j$. Thus $2k+l\le 2$. By lemma 2 the possibilities $(k, l)=(0, 1)$ or $(0, 2)$ are ruled out. Thus $(k, l)=(1, 0)$. Then it follows that $\sharp(S_1)=$

s_1+1. By Lemma 3 it follows that $n=1$ and $\tilde{X}-(T_1 \cup T_\infty) \approx V-P$ is not simply connected.

This is a contradiction. Thus $G=(1)$ and T_∞ can be assuned to be $\underset{C_1}{\overset{0}{\times}} - \underset{C_2}{\overset{0}{\times}}$.
Then the linear system $|C_1|$ defines a P^1-fibration φ: $X \to P^1$ for which C_2 is a section. If $S_1, \ldots, S_r$ are the singular fibres of φ, as before each T_i for $1 \le i \le n$, is contained in some S_j and $\sum_{i=1}^{n} \#(T_i) = \sum_{j=1}^{r} s_j$. By Lemma 1 $\#(S_j) \ge s_j+2$ and hence we have

$$\sum s_j+4=4+\sum_{j=1}^{r}(\chi(S_j)-2) \ge 4+\sum_{j=1}^{r}(s_j+1)=4+\sum_{j=1}^{r} s_j+r$$

Hence $r=0$. Thus V is smooth as claimed.

Step II. In this step we reduce the situation to the step I

We know that $\pi_1(V-P)$ is finite and by step I we can assume it to be non-trivial. Let π': $\tilde{V}' \to V-P$ be the universal covering projection.

Then we can embed $\tilde{V}' \subset \tilde{V}$ where $\tilde{V}$ is a normal affine surface, $\tilde{V}-\tilde{V}'$ is a finite set of points and π' extends to a proper morphism π: $\tilde{V} \to V$. The fundamental group at infinity for $\tilde{V}$ is a subgroup of $\pi_1^\infty(V)$. We will show, that $\tilde{V}$ is contractible. Once this is done, by step I, $\tilde{V} \approx C^2$ as an affine variety and $\tilde{V}/G \approx V$, where G is a finite group of automorphisms of $\tilde{V}$ (the covering transformations of $\tilde{V}'$ extended to $\tilde{V}$ giving automorphisms of $\tilde{V}$).

To show that $\tilde{V}$ is contractible, we proceed as follows. Let m_i be the number of points in $\pi^{-1}(p_i)$ and $m=\sum_{i=1}^{n} m_i$. Since $\tilde{V}$ is a Galois extension of V, $m_i|d$ for $i=1, \ldots, n$ where $d=\deg \pi$. We define $\tilde{\partial}_\infty$, $\tilde{\partial}$, $\tilde{Z}$ for $\tilde{V}$ just as for V. Then $\tilde{\partial}_\infty$ is connected. Since $\tilde{V}'$ is simply connected, so is $\tilde{Z}$ and hence $H_1(\tilde{Z})=(0)$. Also $\pi_1(\tilde{\partial}_\infty)$ is finite. It follows from the cohomology exact sequence of the pair $(\tilde{Z}, \tilde{\partial}_\infty)$ that $H^1(\tilde{Z}, \tilde{\partial}_\infty)=(0)$. By duality $H_3(\tilde{Z}, \tilde{\partial})=(0)$. Clearly $H_i(\tilde{Z}, \tilde{\partial})=(0)$ for $i \ge 4$. In particular it follows that $H_3(\tilde{\partial}) \approx H_3(\tilde{Z})$. Hence the third betti number $b_3(\tilde{Z})=m$. We shall show that $H_2(\tilde{Z}, \tilde{\partial})=(0)$. Assuming this for a moment, it follows that $H_i(\tilde{V}, \pi^{-1}(P)) \approx H_i(\tilde{Z}, \tilde{\partial})=0$ for $i \ge 2$. By the homology exaxt sequence of $(\tilde{V}, \pi^{-1}(P))$, it follows that $H_i(\tilde{V})=(0)$ for $i \ge 1$($\tilde{V}$ is simply-connected). Thus $\tilde{V}$ is contractible.

In order to show that $H_2(\tilde{Z}, \tilde{\partial})=(0)$, we first show that the second betti number $b_2(\tilde{Z})=(0)$. For the Euler-characteristics of $\tilde{Z}$ and Z, we have $d \cdot \chi(Z)=d(1-n)$ $=1+b_2(\tilde{Z})-m$. Let t be such that $0 \le t \le n$ and $m_i<d$ for $i \le t$ and $m_{t+1}=\cdots=$ $m_n=d$. For $i \le t$, $m_i \le \frac{d}{2}$ since $m_i|d$. Hence $0<\frac{1+b_2(\tilde{Z})}{d}=1-n+\frac{m}{d}=1-n+$ $\sum_{i=1}^{t} \frac{m_i}{d}+\sum_{t+1}^{n} \frac{m_i}{d} \le 1-n+\frac{t}{2}+(n-t)=1-\frac{t}{2}$

This forces $t \le 1$. If $t=0$, then $\tilde{V} \to V$ will be unramified, contrary to $\pi_1(V)=(1)$. Hence $t=1$. Hence $\pi_1(\partial_i) \to \pi_1(Z)$ are trivial maps for $i>1$ and $\pi_1(\partial_1) \to \pi_1(Z)$ has image H. The index of H in $\pi_1(Z)$ is m_1. Now consider the unramified covering $\tilde{W}' \to Z$, with $\pi_1(\tilde{W}')=H$. Then each ∂_i has exactly m_1 lifts in $\tilde{W}'$. This in turn means that $\tilde{W}'$ extends to an unramified covering $\tilde{W} \to V$, of degree m_1. By the simply connectivity of V, it follows that $m_1=1$. Now $1+b_2(\tilde{Z})=d(1-n)+m=d(1-n)+1+(n-1)d$, implies that $b_2(\tilde{Z})=0$.

Again use the exact sequence $(0)=H_3(\tilde{Z}, \tilde{\partial})\to H_2(\tilde{\partial})\to H_2(\tilde{Z})\to H_2(\tilde{Z}, \tilde{\partial})\to H_1(\tilde{\partial})\to H_1(\tilde{Z})$. $H_2(\tilde{Z})$ is a finite group and by duality $H_2(\tilde{\partial})\approx H^1(\tilde{\partial})$ is torsion -free. Hence $H_2(\tilde{\partial})=(0)$ and therefore $H_1(\tilde{\partial})$ is finite and $H_2(\tilde{Z}\ \tilde{\partial})$ is finite. Again $H_2(\tilde{Z}, \tilde{\partial})\approx H^2(\tilde{Z}, \tilde{\partial}_\infty)$ is torsion free (being isomorphic to Hom $(H_2(\tilde{Z}, \tilde{\partial}_\infty), \boldsymbol{Z})\oplus$Ext$(H_1(\tilde{Z}, \tilde{\partial}_\infty)\boldsymbol{Z}))$. Hence $H_2(\tilde{Z}, \tilde{\partial})=(0)$ as desired.

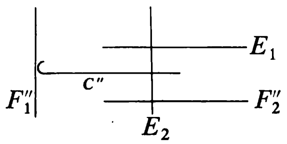
It is known that any finite subgroup of Aut$(\boldsymbol{C}^2)$ is conjugate to a subgroup of $GL(2, \boldsymbol{C})$. See [2] for a proof. From our proof it follows that $\pi_1(V-P)=\pi_1^\infty(V)$ and $\tilde{V}\approx \boldsymbol{C}^2$. It follows that $G=\pi_1(V-P)$ has no non-trivial pseudo- reflections i.e. G is a small subgroup of $GL(2, \boldsymbol{C})$. Since the action of elements of G on $\boldsymbol{C}^2$ is homogeneous linear, it follows that V has only one singular point. This completes the proof of the theorem.

§2. In this section we will give some examples of affine normal surfaces for which the contractibility or the finiteness of the fundamental group at infinity does not hold. In [6] C. P. Ramanujam has given an example of smooth, affine contractible surface which is not isomorphic to $\boldsymbol{C}^2$. By our theorem (or by a direct argument) we know that the fundamental group at infinity for this surface is infinite.

Example 1. We will construct an affine, normal, contractible surface V, with two singular points.

On $\boldsymbol{P}^1\times\boldsymbol{P}^1$, there is a smooth rational curve C, with $C^2=4$. Let φ_0: $\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^1$ be the projection to one of the factors. Then C can be taken to be a bisection of φ and hence by Riemann- Hurwitz formula, there are two fibres F_1 and F_2 of φ_0 which meet C tangentially.

Blow-up $\boldsymbol{P}^1\times\boldsymbol{P}^1$ at the point $C\cap F_2$ to obtain a surface X_1. Let E_1 be the exceptional curve. Denote by L' the proper transform of a curve L. Then we see that C', F_2' and E_1 meet in a single point p_1, transversally. Blow up X_1 at P_1 to obtain a surface a surface X_2 with the following configuration of lines:



where E_2 is the exceptional curve, $(F_1'')^2=0$, $(C'')^2=2$, $(E_1')^2=-2$ and $(F_2'')^2=-2$ Blow down E_1' and F_2'' to normal points on a projective surface X, say $\Psi=X_2\to X$. Let $V=X-\Psi(F_1''\cup C'')$. Then V is the required example.

That V is affine, normal with two singular points is easily seen. To see that V is contractible, note that φ_0 definies a $\boldsymbol{P}^1$ fibration φ_2: $X_2\to\boldsymbol{P}^1$ with a singular fibre $E_1'\cup E_2\cup F_2''$. Hence φ_2 defines a $\boldsymbol{P}^1$-fibration φ: $X\to\boldsymbol{P}^1$ with a singular fibre $S=\Psi(E_2)$ which is a topological sphere. Also $\Psi(F_1'')$ is a good fibre of φ and hence $X-\Psi(F_1'')$ is a $\boldsymbol{P}^1$ fibration over $\boldsymbol{C}$. Since $\Psi(C'')$ and $S=\Psi(E_2)$ intersect transversally in a single point q, it follows that $V=X-\Psi(F_1''\cup C'')$ deforms to $S-q$

which is topologically a 2 disc. Hence V is contractible.

Ofcourse, $\pi_1^\infty(V)$ is infinite. Indeed $\pi_1^\infty(V) \simeq (Z/(2)) * (Z/(2))$.

Example 2. Let $W = P^1 \times P^1 - \Delta$, where Δ is the diagonal. On W $Z/(2)$ acts fixed point freely by interchanging the coordinates. Let V be the quotient space. Then V is a smooth affine variety with $\pi_1(V) \simeq Z/(2)$. The topological Euler characteristic of W is 2 and so for V it is 1. Thus $b_2(V) = 0$ and since $H_2(V, Z)$ is torsion free, we see that $H_2(V, Z) = (0)$. Since $\pi_1^\infty(W) \simeq Z/(2)$, $\pi_1^\infty(V)$ is a group of order 4 (it is actually cyclic of order 4)

Example 3. (Due to N. Mohan Kumar and V. Srinivas). Consider the minimal model $F_n (n \neq 0)$ of the rational function field $C(X, Y)$. (see [7]. There is a minimal P^1-fibration $\varphi: F_n \to P^1$ with a unique section S_0 such that $S_0^2 = -n$. Let S be any section with $S^2 = n$ and let F be a fibre of φ. Blow up F_n at a point on F other than $S \cap F$ and $S_0 \cap F$ to obtain a smooth surface X_1 and E_1 be the exceptional curve that occurs. As before denoting by L' the proper transform of a curve L, let $p_1 = E_1 \cap F'$. Blow up X_1 at p_1 to obtain a surface X_2 and let E_2 be the exceptional curve. Now blow down E_1', $\Psi: X_2 \to X$, to obtain a projective, normal surface with one singular point. Let $V = X - \Psi(S'' \cup F'')$. Then V is affine. Also $V - \Psi(E_2)$ is isomorphic with $F_n - (S \cup F) \simeq C^2$. Thus V is simply connected. One can easily show that $\pi_1^\infty(V)$ is finite and $H_2(V; Z) \simeq Z$. In particular V is not contractible.

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