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On a characterization of the Sobolev spaces over an abstract Wiener space

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Introduction

Let us recall the classical Sobolev spaces in a finite-dimensional case. Consider $H^1(\boldsymbol{R}^d)$ for example. Usually it is defined by means of the Schwartz distribution, that is,

(A) $H^1(\boldsymbol{R}^d) \equiv \{f \in L^2(\boldsymbol{R}^d)$; For each $i=1,\dots, d$, the distribution derivative $\frac{\partial}{\partial x_i} f$ belongs to $L^2(\boldsymbol{R}^d)\}$.

$H^1(\boldsymbol{R}^d)$ is a Hilbert space with norm $\|f\|_{H^1} \equiv \left(\|f\|_{L^2}^2 + \sum_{i=1}^{d} \left\|\frac{\partial}{\partial x_i} f\right\|_{L^2}^2\right)^{1/2}$.

However, if we have to define it without the notion of distribution, we may adopt the following definition.

(B) $H^1(\boldsymbol{R}^d) \equiv$ the completion of the space $C_0^1(\boldsymbol{R}^d)$ with respect to $\| \ \|_{H^1}$.

Or, due to Nikodym, we can take the next one.

(C) $H^1(\boldsymbol{R}^d) \equiv \{f \in L^2(\boldsymbol{R}^d)$; For each $i=1,\dots, d$, there exists a version $\tilde{f}_i$ of f such that $\tilde{f}_i$ is absolutely continuous along almost all lines parallel to the x_i-axis, and its Radon-Nikodym derivative $\frac{\partial}{\partial x_i}\tilde{f}_i$ belongs to $L^2(\boldsymbol{R}^d)$.$\}$

Now, talking about the Sobolev spaces over an abstract Wiener space, two typical definitions are known; one is due to Shigekawa [3] (cf. [5]), and the other is due to Kusuoka-Stroock [2]. In short words, we can say that the former definition is an infinite-dimensional analogue of type (B), and the latter one is that of type (C). In this paper, we first present a theorem in which Shigekawa's Sobolev spaces are characterized, in an analogous way to (A), by means of so-called generalized Wiener functionals. Then, as its application, we will prove that those two definitions of Shigekawa and Kusuoka-Stroock in fact determine the same spaces.

1. Shigekawa's Sobolev spaces and Kusuoka-Stroock's Sobolev spaces

First we shall introduce several notions and notations.

(W, H, μ) is an abstract Wiener space, whose Borel structure is given by $\overline{\mathscr{B}(W)^{\mu}}$, i.e., the completion of the topological σ-field $\mathscr{B}(W)$ with respect to μ. E is a separable Hilbert space and its inner product and norm are denoted by $\langle\ ,\ \rangle_E$ and $|\ |_E$ respectively. We consider it as a measurable space with the topological σ-field $\mathscr{B}(E)$. For $1 \leqq p < +\infty$, we define E-valued L^p-spaces as follows; $L^p(E) = L^p(W; E) \equiv \{f: W \to E;\ \overline{\mathscr{B}(W)^{\mu}}|\mathscr{B}(E)$-measurable and $\int_W |f(w)|_E^p \mu(dw) < +\infty\}$. As usual, if $f, g \in L^p(E)$ coincide μ-a.e., we identify them in $L^p(E)$. Hence,

$$\|f\|_{p;E} \equiv \left(\int_W |f(w)|_E^p \mu(dw)\right)^{1/p}, \quad f \in L^p(E)$$

is a norm of $L^p(E)$, and $L^p(E)$ is a Banach space with this norm.

We set $\mathscr{H}(E) = \mathscr{H}(H; E) \equiv \{V: H \to E;\ V$ is a linear operator of Hilbert-Schmidt type.$\}$. $\mathscr{H}(E)$ is a Hilbert space whose norm is the Hilbert-Schmidt norm. Inductively we set $\mathscr{H}^n(E) \equiv \mathscr{H}(\mathscr{H}^{n-1}(E))$, $n = 2, 3, \ldots$, where $\mathscr{H}^1(E) \equiv \mathscr{H}(E)$.

(1) Shigekawa's Sobolev spaces.

Definition 1.1. (i) A mapping $f: W \to \boldsymbol{R}^1$ is said to be a *polynomial*, if $\exists n \in \boldsymbol{N}$, $\exists l_1, \ldots, l_n \in W^*$, and $\exists \tilde{f}: \boldsymbol{R}^n \to \boldsymbol{R}^1$, polynomial in n variables such that

$$f(w) = \tilde{f}((l_1, w), \ldots, (l_n, w)), \quad w \in W. \tag{1.1}$$

The totality of polynomials is denoted by $\boldsymbol{P}$.

(ii) A mapping $f: W \to E$ is said to be an *E-valued polynomial*, if $\exists m \in \boldsymbol{N}$, $\exists f_1, \ldots, f_m \in \boldsymbol{P}$, and $\exists e_1, \ldots, e_m \in E$ such that

$$f(w) = \textstyle\sum_{i=1}^m f_i(w) e_i, \quad w \in W. \tag{1.2}$$

The totality of E-valued polynomials is denoted by $\boldsymbol{P}(E)$.

It should be noted that $\boldsymbol{P}(E)$ is a dense subspace of $L^p(E)$ for every $1 \leqq p < +\infty$. Let S be a linear operator mapping $\boldsymbol{P}$ into itslef. Then it naturally induces a linear operator mapping $\boldsymbol{P}(E)$ into itself in the following way; for $f \in \boldsymbol{P}(E)$ with an expression (1.2), we define an operator $\bar{S}: \boldsymbol{P}(E) \to \boldsymbol{P}(E)$, by putting $\bar{S}f \equiv \sum_{i=1}^m (Sf_i) e_i$ (this definition doesn't depend on the expression of f). We will denote $\bar{S}$ by the same notaion S.

Definition 1.2. A linear mapping $D: \boldsymbol{P}(E) \to \boldsymbol{P}(\mathscr{H}(E))$ is defined by

$$Df(w)[h] \equiv \lim_{t \to 0} \frac{1}{t}(f(w+th) - f(w)), \quad w \in W, \quad h \in H, \ f \in \boldsymbol{P}(E). \tag{1.3}$$

D is known as *the Fréchet derivative operator*. The n-th iteration D^n is a linear operator mapping $\boldsymbol{P}(E)$ into $\boldsymbol{P}(\mathscr{H}^n(E))$. Shigekawa's Sobolev spaces are defined in terms of D, that is;

Definition 1.3. Let $1<p<+\infty$ and $n\in\boldsymbol{N}$. We endow $\boldsymbol{P}(E)$ with a norm

$$\|f\|_{p,n;E}\equiv\|f\|_{p;E}+\|D^n f\|_{p;\mathscr{H}^n(E)},\quad f\in\boldsymbol{P}(E), \tag{1.4}$$

and define the Sobolev space $\boldsymbol{D}_{p,n}(E)$ as the completion of $\boldsymbol{P}(E)$ with respace to this norm.

(2) Kusuoka-Stroock's Sobolev spaces.

Now, we will introduce Kusuoka-Stroock's Sobolev spaces. The difference from those of Shigekawa's stands, of course, in the differential operation.

Definition 1.4. (Kusuoka [1]). (i) A measurable (i.e., $\overline{\mathscr{B}(W)^\mu}|\mathscr{B}(E)$-measurable) mapping f: $W\to E$ is said to be *ray absolutely continuous* (abbr. RAC), if for every $h\in H$, there exists a measurable mapping $\tilde{f}_h$: $W\to E$ such that

$$f(w)=\tilde{f}_h(w),\quad \mu\text{-}a.e. w\in W$$

and for any $w\in W$,

$$\tilde{f}_h(w+th),\quad t\in\boldsymbol{R},\quad \text{is absolutely continuous in } t.$$

(ii) A measurable mapping f: $W\to E$ is said to be *stochastically Gateaux differentiable* (abbr. SGD), if there exists a measurable mapping F: $W\to\mathscr{H}(E)$ such that for any $h\in H$,

$\frac{1}{t}(f(w+th)-f(w))$ converges to $F(w)[h]$ in probability with respect to μ as $t\to 0$.

Such F, if it exists, is unique in μ-a.e. sense, and is denoted by $\tilde{D}f$. If $\tilde{D}f$ is SGD again, we define $\tilde{D}^2 f\equiv\tilde{D}(\tilde{D}f)$, and inductively, if $\tilde{D}^{n-1}f$ is SGD, $\tilde{D}^n f$ is defined by $\tilde{D}^n f\equiv\tilde{D}(\tilde{D}^{n-1}f)$.

Definition 1.5. (Kusuoka-Stroock [2]). Let $1<p<+\infty$. First we define the space $\tilde{\boldsymbol{D}}_{p,1}(E)$ by

$$\tilde{\boldsymbol{D}}_{p,1}(E)\equiv\{f\in L^p(E); f \text{ is } RAC \text{ and } SGD,\ \tilde{D}f\in L^p(\mathscr{H}(E))\}, \tag{1.5}$$

and endow it with a norm

$$\|f\|_{\tilde{p},1;E}\equiv\|f\|_{p;E}+\|\tilde{D}f\|_{p;\mathscr{H}(E)},\quad f\in\tilde{\boldsymbol{D}}_{p,1}(E). \tag{1.6}$$

Then for $n=2, 3,\dots$, we define the spaces $\tilde{\boldsymbol{D}}_{p,n}(E)$ inductively by

$$\tilde{\boldsymbol{D}}_{p,n}(E)\equiv\{f\in\tilde{\boldsymbol{D}}_{p,n-1}(E);\ \tilde{D}f\in\tilde{\boldsymbol{D}}_{p,n-1}(\mathscr{H}(E))\}, \tag{1.7}$$

and endow them with the following norms respectively.

$$\|f\|_{\tilde{p},n;E}\equiv\|f\|_{p;E}+\|\tilde{D}^n f\|_{p;\mathscr{H}^n(E)},\quad f\in\tilde{\boldsymbol{D}}_{p,n}(E). \tag{1.8}$$

It is known that the normed spaces $(\tilde{\boldsymbol{D}}_{p,n}(E),\ \|\ \|_{\tilde{p},n;E})$, $n=1, 2,\dots$, are complete, i.e., they are Banach spaces. For the proof, see Lemma 1.1 in Kusuoka [1].

2. A characterization theorem of Shigekawa's Sobolev spaces by means of generalized Wiener functionals.

In order to prove the theorem at the title of this section, let us summarize some results obtained by Sugita [5].

Let L be *the Ornstein-Uhlenbeck operator.* It is known that for $f \in \boldsymbol{P}$ with its Wiener-Itô decomposition $f = \sum_n f_n$ (finite sum), $Lf = \sum_n (-n) f_n$ holds. Then we can define an operator $(I-L)^{r/2}$ on $\boldsymbol{P}$, where I denotes the identity mapping and r is an arbitrary real number, by putting $(I-L)^{r/2} f \equiv \sum_n (1+n)^{r/2} f_n$. As mentioned before, we regard it as an operator mapping $\boldsymbol{P}(E)$ into itself, and finally we define a system of norms

$$\|f\|_{p,r;E} \equiv \|(I-L)^{r/2} f\|_{p;E}, \quad 1<p<+\infty, \quad r \in \boldsymbol{R},$$

on $\boldsymbol{P}(E)$, and denote the completion of $\boldsymbol{P}(E)$ with respect to $\| \ \|_{p,r;E}$ by $\boldsymbol{D}_{p,r}(E)$. Since for $n \in \boldsymbol{N}$, the two norms $\|\cdot\|_{p;E} + \|D^n \cdot\|_{p;\mathscr{H}^n(E)}$ and $\|(I-L)^{n/2} \cdot\|_{p;E}$ induce the same topology on $\boldsymbol{P}(E)$, the above definition is consistent with Definition 1.3 (see Sugita [5]).

Once the Sobolev spaces $\boldsymbol{D}_{p,r}(E)$ are defined for all real numbers r (especially for negative r), more profound arguments are possible. For instance, the following theorem holds.

Theorem 2.1. (Sugita [5]). (i) *If* $1<p \leqq q<+\infty$ *and* $r \leqq s$, *then* $\boldsymbol{D}_{q,s}(E) \subsetneqq \boldsymbol{D}_{p,r}(E)$ *holds. Here "$\subsetneqq$" stands for the continuous imbedding. Consequently, we have the following diagram; for* $1<p \leqq q<+\infty$ *and* $0 \leqq r \leqq s<+\infty$,

$$\begin{array}{ccccccccc}
\boldsymbol{D}_{p,s}(E) & \subsetneqq & \boldsymbol{D}_{p,r}(E) & \subsetneqq & \boldsymbol{D}_{p,0}(E)=L^p(E) & \subsetneqq & \boldsymbol{D}_{p,-r}(E) & \subsetneqq & \boldsymbol{D}_{p,-s}(E) \\
\cup\uparrow & & \cup\uparrow & & \cup\uparrow & & \cup\uparrow & & \cup\uparrow \\
\boldsymbol{D}_{q,s}(E) & \subsetneqq & \boldsymbol{D}_{q,r}(E) & \subsetneqq & \boldsymbol{D}_{q,0}(E)=L^q(E) & \subsetneqq & \boldsymbol{D}_{q,-r}(E) & \subsetneqq & \boldsymbol{D}_{q,-s}(E)
\end{array}$$

(ii) *Under the standard identification of* $L^2(E)^* = L^2(E)$, *we have*

$$\boldsymbol{D}_{p,r}(E)^* = \boldsymbol{D}_{q,-r}(E), \quad \text{where } 1<p,\ q<+\infty, \frac{1}{p}+\frac{1}{q}=1 \quad \text{and } r \in \boldsymbol{R}.$$

(iii) *The linear operators* D *and* L *uniquely extend to bounded operators respectively as follows; for* $1<p<+\infty$ *and* $r \in \boldsymbol{R}$,

$$D\colon \boldsymbol{D}_{p,r}(E) \longrightarrow \boldsymbol{D}_{p,r-1}(\mathscr{H}(E))$$

$$L\colon \boldsymbol{D}_{p,r}(E) \longrightarrow \boldsymbol{D}_{p,r-2}(E)$$

Therefore the dual operator D^* *of* D *is bounded as a linear operator,*

$$D^*\colon \boldsymbol{D}_{p,r}(\mathscr{H}(E)) \longrightarrow \boldsymbol{D}_{p,r-1}(E),$$

and it satisfies the condition $D^*D = L$.

Let R be a linear operator on $\boldsymbol{P}(E)$ defined by $R \equiv \int_0^{+\infty} (e^{Lt} - J_0)\, dt$, where

$J_0 f \equiv \int_W f(w)\mu(dw)$, $f \in \boldsymbol{P}(E)$. Then following lemma can be easily proved in the same way as Theorem 3.2 in Sugita [5].

Lemma 2.1. *R and J_0 uniquely extend to bounded linear operators as*

$$R: \boldsymbol{D}_{p,r}(E) \longrightarrow \boldsymbol{D}_{p,r+2}(E), \quad 1<p<+\infty, \quad r \in \boldsymbol{R}$$

and

$$J_0: \boldsymbol{D}_{p,r}(E) \longrightarrow \boldsymbol{D}_{q,s}(E), \quad 1<p,\, q<+\infty, \quad r,\, s \in \boldsymbol{R}$$

respectively. Since $J_0^2 = J_0$ on $\boldsymbol{P}(E)$, $J_0 f$ is an E-valued constant functional for every $f \in \boldsymbol{D}_{p,r}(E)$.

Definition 2.1. By Theorem 2.1(i), the following definitions make sense.

$$\boldsymbol{D}_{+\infty}(E) \equiv \cap \{\boldsymbol{D}_{p,r}(E);\, 1<p<+\infty, \quad r \in \boldsymbol{R}\}$$

$$\boldsymbol{D}_{-\infty}(E) \equiv \cup \{\boldsymbol{D}_{p,r}(E);\, 1<p<+\infty, \quad r \in \boldsymbol{R}\}$$

$\boldsymbol{D}_{+\infty}(E)$ is a complete countably normed space and called the space of *test functionals.* By Theorem 2.1(ii), we see that $\boldsymbol{D}_{+\infty}(E)^* = \boldsymbol{D}_{-\infty}(E)$ so that $\boldsymbol{D}_{-\infty}(E)$ is called the space of *generalized Wiener functionals.*

Since the extensions of the operator D mentioned in Theorem 2.1(iii) are consistent, i.e., the diagram

$$\begin{tikzcd} \boldsymbol{D}_{q,s}(E) \arrow[d, "D"] \arrow[r, phantom, "\subsetneqq"] & \boldsymbol{D}_{p,r}(E) \arrow[d, "D"] \\ \boldsymbol{D}_{q,s-1}(\mathscr{H}(E)) \arrow[r, phantom, "\subsetneqq"] & \boldsymbol{D}_{p,r-1}(\mathscr{H}(E)) \end{tikzcd}$$

is commutative for any $1<p \leqq q<+\infty$ and $r \leqq s$, D is in fact well-defined on the whole $\boldsymbol{D}_{-\infty}(E)$ taking value in $\boldsymbol{D}_{-\infty}(\mathscr{H}(E))$. Similarly, the operators $L, R; \boldsymbol{D}_{-\infty}(E) \to \boldsymbol{D}_{-\infty}(E)$ and $D^*: \boldsymbol{D}_{-\infty}(\mathscr{H}(E)) \to \boldsymbol{D}_{-\infty}(E)$ are well-defined.

Now, we proceed to the main theorem of the section, whose original form is seen in Shigekawa [4]. After these preparations, its proof is quite easy.

Theorem 2.2. *Let $f \in \boldsymbol{D}_{-\infty}(E)$, $1<p<+\infty$, $r \in \boldsymbol{R}$ and $k \in \boldsymbol{N}$. If $D^k f \in \boldsymbol{D}_{p,r}(\mathscr{H}^k(E))$, then $f \in \boldsymbol{D}_{p,r+k}(E)$.*

Proof. It is sufficient to prove the theorem for the case $k=1$. Let us note that $-RD^*Df = (I-J_0)f$. By Theorem 2.1(iii) and Lemma 2.1, the compound mapping $-RD^*$ carries $\boldsymbol{D}_{p,r}(\mathscr{H}(E))$ into $\boldsymbol{D}_{p,r+1}(E)$. Therefore, if $Df \in \boldsymbol{D}_{p,r}(\mathscr{H}(E))$, $(I-J_0)f$ belongs to $\boldsymbol{D}_{p,r+1}(E)$. But since $J_0 f$ is a constant vector, f itself belongs to $\boldsymbol{D}_{p,r+1}(E)$. Q.E.D.

It is useful to regard the opertor D on $\boldsymbol{D}_{-\infty}(E)$ as the differentiation in *the distribution sense.* Namely, for $f \in \boldsymbol{D}_{-\infty}(E)$, if there exists $F \in \boldsymbol{D}_{-\infty}(\mathscr{H}(E))$ such that

$$(2.1) \qquad {}_{\boldsymbol{D}_{-\infty}(E)}(f,\, D^* g)_{\boldsymbol{D}_{+\infty}(E)} = {}_{\boldsymbol{D}_{-\infty}(\mathscr{H}(E))}(F,\, g)_{\boldsymbol{D}_{+\infty}(\mathscr{H}(E))}$$

for each $g \in \boldsymbol{D}_{+\infty}(\mathscr{H}(E))$, it clealy holds that $F = Df$. In this context, we obtain the following corollary to Theorem 2.2.

Corollary 2.1. *Let $\varepsilon>0$, $1<p<+\infty$, and $n\in \boldsymbol{N}$. Suppose $f\in L^{1+\varepsilon}(E)$ and that there exists $F\in L^p(\mathscr{H}^n(E))$ such that*

$$(2.2)\qquad \int_W \langle f, (D^*)^n(g\otimes h_1\otimes\cdots\otimes h_n)\rangle_E d\mu=\int_W \langle F[h_1,\dots, h_n], g\rangle_E d\mu$$

for every $g\in \boldsymbol{P}(E)$ and $h_1,\dots, h_n\in W^*$ $(\subset H^*=H)$, where we set

$$(2.3)\qquad (g\otimes h_1\otimes\cdots\otimes h_n)[\cdot,\cdot\cdot,\cdot]\equiv g\langle h_1,\cdot\rangle_H\times\cdots\times\langle h_n,\cdot\rangle_H.$$

Then, it holds that $f\in \boldsymbol{D}_{p,n}(E)$ and $F(w)=D^nf(w)$ μ-a.e.$w\in W$.

Proof. It is enough to note that $L^{1+\varepsilon}(E)\subset \boldsymbol{D}_{-\infty}(E)$ and that the totality of the finite sums of functionals of type (2.3) is dense in $\boldsymbol{D}_{+\infty}(\mathscr{H}^n(E))$. Q.E.D.

Remark 1. $D^*(g\otimes h)$ can be calculated as follows. Here (,) stands for the pairing of W^* and W, and $\{h_i\}_{i=1}^\infty$ is a complete orthonormal system of H.

$$\begin{aligned} D^*(g(w)\otimes h) &= -\operatorname{trace} D(g(w)\otimes h)+g(w)(h, w)\\ &\equiv -\sum_{i=1}^\infty Dg(w)[h_i]\times\langle h, h_i\rangle_H+g(w)(h, w)\end{aligned}$$

The last term of the equality does not depend on the choice of $\{h_i\}_{i=1}^\infty$.

3. The proof of $\boldsymbol{D}_{p,n}(E)=\tilde{\boldsymbol{D}}_{p,n}(E)$.

Finally, as an application of Theorem 2.2 and Corollary 2.1, we prove that Kusuoka-Stroock's Sobolev spaces coincide with Shigekawa's. Namely, the following theorem holds.

Theorem 3.1. *For $1<p<+\infty$ and $n\in \boldsymbol{N}$, we have $\boldsymbol{D}_{p,n}(E)=\tilde{\boldsymbol{D}}_{p,n}(E)$, furthermore, if f is its element, then $Df(w)=\tilde{D}f(w)$ μ-a.e.$w\in W$.*

Proof. It is easy to show that $\boldsymbol{D}_{p,n}(E)\subset\tilde{\boldsymbol{D}}_{p,n}(E)$. Indeed, for $f\in \boldsymbol{P}(E)$, it clearly holds that $\tilde{D}^nf(w)=D^nf(w)$ μ-a.e.w. Hence, by the completeness of $\tilde{\boldsymbol{D}}_{p,n}(E)$, we have

$$\begin{aligned}\boldsymbol{D}_{p,n}(E)&=\overline{\boldsymbol{P}(E)}^{\|\ \|_{p,n;E}}=\overline{\boldsymbol{P}(E)}^{\|\ \|_{\tilde{p},n;E}}\\ &\subset\overline{\tilde{\boldsymbol{D}}_{p,n}(E)}^{\|\ \|_{\tilde{p},n;E}}=\tilde{\boldsymbol{D}}_{p,n}(E).\end{aligned}$$

Now, let us prove the inverse inclusion $\tilde{\boldsymbol{D}}_{p,n}(E)\subset\boldsymbol{D}_{p,n}(E)$. First, we prove it for $n=1$, in three steps. Take an arbitrary $f\in\tilde{\boldsymbol{D}}_{p,1}(E)$. Then, there is a version $\tilde{f}_h$ of f mentioned in Definition 1.4(i), for every $h\in H$. Hereafter we take an arbitrary $h\in W^*(\subset H^*=H)$ and fix it.

Step 1; *It holds that*

$$\frac{d}{dt}\tilde{f}_h(w+th)=\tilde{D}f(w+th)[h]$$

for almost every $(w, t)\in W\times \boldsymbol{R}$ with respect to the product measure $\mu(dw)dt$.

Proof; Since $\tilde{f}_h$ is absolutely continuous in t for all $w\in W$, we have

$$\frac{1}{s}\{\tilde{f}_h(w+th+sh)-\tilde{f}_h(w+th)\} \longrightarrow \frac{d}{dt}\tilde{f}_h(w+th), \quad \text{as} \quad s \longrightarrow 0,$$

for a.e. t and all $w \in W$. On the other hand, $\mu(\cdot - th)$ is absolutely continuous relative to $\mu(\cdot)$, and f is *SGD*, so we have

$$\frac{1}{s}\{f(w+th+sh)-f(w+th)\} \longrightarrow \tilde{D}f(w+th)[h], \quad \text{as} \quad s \longrightarrow 0,$$

in probability with respect to μ, for each $t \in \boldsymbol{R}$. The assertion follows from these facts. Q.E.D.

Step 2; *For each* $g \in \boldsymbol{P}(E)$, $\int_W \langle f(w+th), g(w)\rangle_E \mu(dw)$ *is differentiable in* $t \in \boldsymbol{R}$, *and we have*

$$\frac{d}{dt}\int_W \langle f(w+th), g(w)\rangle_E \mu(dw) = \int_W \langle f(w+th), D^*(g(w)\otimes h)\rangle_E \mu(dw).$$

Proof; By the Cameron-Martin formula,

$$\frac{1}{s}\left\{\int_W \langle f(w+th+sh), g(w)\rangle_E \mu(dw) - \int_W \langle f(w+th), g(w)\rangle_E \mu(dw)\right\}$$

$$= \int_W \langle f(w+th), \frac{1}{s}\left[g(w-sh)\exp\left\{s(h, w)-\frac{1}{2}s^2|h|_H^2\right\} - g(w)\right]\rangle_E \mu(dw).$$

Here, we easily calculate the following limit,

$$\lim_{s\to 0}\frac{1}{s}\left[g(w-sh)\exp\left\{s(h, w)-\frac{1}{2}s^2|h|_H^2\right\} - g(w)\right]$$

$$= \frac{d}{ds}g(w-sh)\exp\left\{s(h, w)-\frac{1}{2}s^2|h|_H^2\right\}\Big|_{s=0} = D^*(g(w)\otimes h).$$

On the other hand, since $\left|\frac{d}{ds}g(w-sh)\exp\left\{s(h, w)-\frac{1}{2}s^2|h|_H^2\right\}\right|_E^q$ is uniformly μ-integrable for any $1<q<+\infty$, on a bounded interval $-\delta<s<\delta(\delta>0)$, which is shown by the Cameron-Martin formula again, we are allowed to commute the integration and the limit, i.e., the assertion holds. Q.E.D.

Remark 2. According to the above proof, we see that Step 2 holds if only $f \in L^{1+\varepsilon}(E)$ for some $\varepsilon>0$. The condition $f \in \tilde{\boldsymbol{D}}_{p,1}(E)$ is not necessary.

Step 3; *Let* $g \in \boldsymbol{P}(E)$. *Then we have that for all* $t \in \boldsymbol{R}$,

$$\text{(3.1)} \qquad \frac{d}{dt}\int_W \langle \tilde{f}_h(w+th), g(w)\rangle_E \mu(dw) = \int_W \langle \frac{d}{dt}\tilde{f}_h(w+th), g(w)\rangle_E \mu(dw).$$

Proof; By Step 1, the condition $\tilde{D}f \in L^p(\mathscr{H}(E))$ and the Cameron-Martin formula, $\left|\frac{d}{dt}\tilde{f}_h(w+th)\right|_E^p$ is uniformly μ-integrable on an arbitrary bounded interval $a<t<b$. Consequently, $\langle \frac{d}{dt}\tilde{f}_h(w+th), g(w)\rangle_E$ is $\mu(dw)dt$-integrable on $W\times(a, b)$. This

enables us to apply Fubini's theorem, in integrating the both sides of the next equality in w, $\langle \tilde{f}_h(w+ch),\ g(w)\rangle_E - \langle \tilde{f}_h(w+ah),\ g(w)\rangle_E = \int_a^c \langle \frac{d}{dt}\tilde{f}_h(w+th),\ g(w)\rangle dt$, where $a \leqq c \leqq b$. That is, we have the following.

$$\int_W \langle \tilde{f}_h(w+ch),\ g(w)\rangle_E \mu(dw) - \int_W \langle \tilde{f}_h(w+ah),\ g(w)\rangle_E \mu(dw)$$

$$= \int_a^c \int_W \langle \frac{d}{dt}\tilde{f}_h(w+th),\ g(w)\rangle_E \mu(dw) dt$$

This implies (3.1) for almost all $t \in \boldsymbol{R}$. But, since the both sides of (3.1) are continuous in t (even more, Step 2 claims that they are differentiable), (3.1) holds for all $t \in \boldsymbol{R}$. Q.E.D.

Now that these steps are shown, the claim $f \in \boldsymbol{D}_{p,1}(E)$ is an easy consequence of Corollary 2.1. Indeed, from them, it follows that

$$\int_W \langle \tilde{D}f(w+th)[h],\ g(w)\rangle_E \mu(dw) = \int_W \langle f(w+th),\ D^*(g(w)\otimes h)\rangle_E \mu(dw),$$

for any $g \in \boldsymbol{P}(E)$ and $t \in \boldsymbol{R}$. In order to see $f \in \boldsymbol{D}_{p,1}(E)$, we have only to set $t=0$, and apply Corollary 2.1 (for $n=1$).

Finally we will prove the theorem for $n=2, 3, \ldots$. In these cases, since $\tilde{\boldsymbol{D}}_{p,n}(E) \subset \tilde{\boldsymbol{D}}_{p,1}(E)$ holds by definition, we have $\tilde{D}f(w) = Df(w)$ μ-a.e.w for $f \in \tilde{\boldsymbol{D}}_{p,n}(E) \subset \boldsymbol{D}_{p,1}(E)$. Hence, $\tilde{\boldsymbol{D}}_{p,n}(E) \subset \boldsymbol{D}_{p,n}(E)$ is clear by applying Theorem 2.2 (for $k=1$) repeatedly.

Q.E.D.

Remark 3. The above proof tells us that some condition for definition of $\tilde{\boldsymbol{D}}_{p,n}(E)$ can be loosened. Namely, we may define $\tilde{\boldsymbol{D}}_{p,1}(E)$ as follows.

$$\tilde{\boldsymbol{D}}_{p,1}(E) \equiv \{f \in L^{1+\varepsilon}(E) \quad \text{for } \exists\varepsilon>0;\ f \text{ is } RAC,\ SGD, \text{ and } \tilde{D}f \in L^p(\mathscr{H}(E))\}$$

Then its element automatically belongs to $L^p(E)$. Similarly, we may set $\tilde{\boldsymbol{D}}_{p,n}(E) \equiv \{f \in L^{1+\varepsilon}(E)$ for $\exists\varepsilon>0;\ f$ is RAC, SGD, and $\tilde{D}f \in \tilde{\boldsymbol{D}}_{p,n-1}(\mathscr{H}(E))\}$.

Remark 4. Of course, both Shigekawa's and Kusuoka-Stroock's Sobolev spaces can be defined for $p=1$. However, we cannot apply our method to examine whether $\boldsymbol{D}_{1,n}(E) = \tilde{\boldsymbol{D}}_{1,n}(E)$ holds or not.

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References

[1] S. Kusuoka, Dirichlet forms and diffusion processes on Banach spaces, J. F. Sci., Tokyo Univ. Sec. IA, **29**-1 (1982).

[2] S. Kusuoka-D. Stroock, Application of the Malliavin calculus, Part I, Proceedings of the Taniguchi International Symposium on Stochastic Analysis, Katata and Kyoto, 1982, edited

by K. Itô, North–Holland.

[3] I. Shigekawa, Derivatives of Wiener functionals and absolute continuity of induced measures, J. Math. Kyoto Univ., **20**–2 (1980).

[4] I. Shigekawa, Malliavin's calculus on L^p-spaces and de Rham-Hodge-Kodaira's decomposition theorem. (preprint in Japanese).

[5] H. Sugita, Sobolev spaces of Wiener functionals and Malliavin's calculus, J. Math. Kyoto Univ., **25**–1 (1985)