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Holomorphic families of Riemann mapping functions

To Yukio Kusunoki on his 60th birthday

By

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1. Introduction.

Suppose a closed Jordan curve γ_0 in the plane is deformed into other closed Jordan curves γ_λ, where λ is a complex variable in the unit disk $\boldsymbol{D}$. Furthermore, suppose the deformations depend holomorphically on λ. In such a situation one is naturally led to investigate the dependence on λ of suitably normalized Riemann mapping functions f_λ of the unit disk onto G_λ, the inside region of γ_λ. For a summary of earlier results on this topic see Warschawski [8]. A result of Rodin [5] shows that, in general, f_λ depends real analytically on the real and imaginary parts of λ. It is of interest to know when this real analytic dependence is actually complex analytic (i.e., holomorphic in λ). In the present paper we give some methods and results which have been useful in our preliminary investigation of the question.

2. Definitions.

Throughout this paper $\boldsymbol{D}_r$, where $r>0$, denotes the disk $\{z\in \boldsymbol{C}: |z|<r\}$. The Riemann sphere will be denoted $\hat{\boldsymbol{C}}$. Let $E\subset\hat{\boldsymbol{C}}$. A map

$$F: \boldsymbol{D}_r\times E \longrightarrow \hat{\boldsymbol{C}} \tag{2.1}$$

is called a *holomorphic motion of E* if the following three conditions are satisfied:

(i) For all $\lambda\in\boldsymbol{D}_r$ the map $F(\lambda,\cdot): E\to\hat{\boldsymbol{C}}$ is injective.

(ii) For all $z\in E$ the map $F(\cdot, z): \boldsymbol{D}_r\to\hat{\boldsymbol{C}}$ is holomorphic.

(iii) For all $z\in E$, $F(0, z)=z$.

The "λ-lemma" of Mañé-Sad-Sullivan [4] states that the holomorphic motion (2.1) extends to a holomorphic motion

$$\hat{F}: \boldsymbol{D}_r\times Cl\,E \longrightarrow \hat{\boldsymbol{C}} \tag{2.2}$$

of the closure of E. Furthermore, for each $\lambda\in\boldsymbol{D}_r$ the map

$$\hat{F}(\lambda,\cdot): Cl\,E \longrightarrow \hat{F}(\lambda, Cl\,E) \tag{2.3}$$

is a homeomorphism and is quasiconformal on every open subset of $Cl\,E$.

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The "improved λ-lemma" refers to the following result: if F in (2.1) is a holomorphic motion of E then there is a holomorphic motion

$$\tilde{F}: \boldsymbol{D}_{r/3} \times \hat{\boldsymbol{C}} \longrightarrow \hat{\boldsymbol{C}} \tag{2.4}$$

of $\hat{\boldsymbol{C}}$ such that $\tilde{F}(\lambda, z)=F(\lambda, z)$ for all $(\lambda, z) \in \boldsymbol{D}_{r/3} \times E$. (This result was proved in Sullivan-Thurston [7] with "$\boldsymbol{D}_{r/3}$" replaced by "$\boldsymbol{D}_\rho$ for some $\rho \in (0, r]$", the form (2.4) of the result was proved in Bers-Royden [1]).

Throughout this paper γ_0 will denote a closed Jordan curve in $\boldsymbol{C}$. We will consider a holomorphic motion

$$F: \boldsymbol{D}_r \times \gamma_0 \longrightarrow \boldsymbol{C}$$

of γ_0. For each $\lambda \in \boldsymbol{D}_r$ the set

$$\gamma_\lambda = \{F(\lambda, z): z \in \gamma_0\}$$

is a closed Jordan curve in $\boldsymbol{C}$ since the λ-lemma asserts that (2.3) is a homeomorphism. We shall let G_λ denote the region inside of γ_λ $(\lambda \in \boldsymbol{D}_r)$.

3. Let us first observe that the indexed family $\{\gamma_\lambda: \lambda \in \boldsymbol{D}\}$, where γ_λ is considered as a subset of $\boldsymbol{C}$ rather than as a parametrized curve, uniquely determines the holomorphic motion $F: \boldsymbol{D} \times \gamma_0 \to \boldsymbol{C}$ that generated it.

Theorem 1. *Let*

$$F_1: \boldsymbol{D} \times \gamma_0 \longrightarrow \boldsymbol{C} \quad \text{and} \quad F_2: \boldsymbol{D} \times \gamma_0 \longrightarrow \boldsymbol{C}$$

be holomorphic motions of γ_0. Suppose that for each $\lambda \in \boldsymbol{D}$ we have

$$\{F_1(\lambda, z): z \in \gamma_0\} = \{F_2(\lambda, z): z \in \gamma_0\}. \tag{3.1}$$

Then $F_1(\lambda, z)=F_2(\lambda, z)$ for all $(\lambda, z) \in \boldsymbol{D} \times \gamma_0$.

Proof. By the improved λ-lemma the restriction of F_1 to $\boldsymbol{D}_{1/3} \times \gamma_0$ can be extended to a holomorphic motion

$$\hat{F}_1: \boldsymbol{D}_{1/3} \times (G_0 \cup \gamma_0) \longrightarrow \boldsymbol{C}.$$

Consider the function

$$\hat{F}_2: \boldsymbol{D}_{1/3} \times (G_0 \cup \gamma_0) \longrightarrow \boldsymbol{C}$$

defined by

$$\hat{F}_2(\lambda, z) = \begin{cases} \hat{F}_1(\lambda, z) & \text{for} \quad (\lambda, z) \in \boldsymbol{D}_{1/3} \times G_0 \\ F_2(\lambda, z) & \text{for} \quad (\lambda, z) \in \boldsymbol{D}_{1/3} \times \gamma_0 . \end{cases}$$

It is easily seen using (3.1) that $\hat{F}_2$ is a holomorphic motion of $G_0 \cup \gamma_0$. For each fixed $\lambda \in \boldsymbol{D}_{1/3}$ the functions

$$z \longmapsto \hat{F}_1(\lambda, z) \quad \text{and} \quad z \longmapsto \hat{F}_2(\lambda, z)$$

are continuous on $G_0 \cup \gamma_0$ (by the λ-lemma). Since they agree on G_0, they agree on γ_0. Therefore $F_1(\lambda, z) = F_2(\lambda, z)$ for $(\lambda, z) \in \boldsymbol{D} \times \gamma_0$.

We continue to consider the family $\{\gamma_\lambda: \lambda \in \boldsymbol{D}\}$ generated by a holomorphic motion $F: \boldsymbol{D} \times \gamma_0 \to \boldsymbol{C}$ of γ_0. If γ_0 is a quasicircle then each γ_λ is also a quasicircle. In Gehring-Pommerenke [3], e.g., this is proved by considering the crossratio

$$\frac{F(\lambda, z_1) - F(\lambda, z_4)}{F(\lambda, z_1) - F(\lambda, z_3)} \cdot \frac{F(\lambda, z_2) - F(\lambda, z_3)}{F(\lambda, z_2) - F(\lambda, z_4)} \tag{3.2}$$

where z_1, z_2, z_3, z_4 are four distinct ordered points on γ_0. The crossratio (3.2) is a holomorphic function of λ with values in $\hat{\boldsymbol{C}} - \{0, 1, \infty\}$. For $\lambda = 0$ its values are bounded independently of the four points (this is a characterization of quasicircles); by Schottky's theorem the same is true for any λ with $|\lambda| < 1$. The following generalization, however, requires a completely different proof.

Theorem 2. *Let $\{\gamma_\lambda: \lambda \in \boldsymbol{D}\}$ be generated by holomorphic motion of γ_0. For each $\lambda \in \boldsymbol{D}$ there exists a quasiconformal homeomorphism of $\hat{\boldsymbol{C}}$ which maps γ_0 onto γ_λ.*

Proof. Fix $\lambda \in \boldsymbol{D}$. Choose $0 = \lambda_0, \lambda_1, \ldots, \lambda_n = \lambda$ so that $|\lambda_{j+1} - \lambda_j| < (1 - |\lambda_j|)/3$ for $j = 0, 1, \ldots, n-1$. By the improved λ-lemma each map $f(\lambda_j, z) \mapsto f(\lambda_{j+1}, z)$, for $z \in \gamma_0$, of γ_{λ_j} onto $\gamma_{\lambda_{j+1}}$ extends to a quasiconformal homeomorphism G_j of $\hat{\boldsymbol{C}}$. Thus $G = G_{n-1} \circ \cdots \circ G_1 \circ G_0$ is the desired quasiconformal homeomorphism of $\hat{\boldsymbol{C}}$ which maps γ_0 onto γ_λ.

Our next theorem shows that, in general, the Riemann mapping function cannot be expected to depend holomorphically on λ. Let us say that a holomorphic motion $F: \boldsymbol{D} \times E \to \hat{\boldsymbol{C}}$ is *trivial* if $F(\lambda, z)$ is of the form

$$F(\lambda, z) = \frac{a(\lambda)z + b(\lambda)}{c(\lambda)z + d(\lambda)} \tag{3.3}$$

where a, b, c, d are holomorphic in $\boldsymbol{D}$.

Theorem 3. *Let $\{\gamma_\lambda: \lambda \in \boldsymbol{D}\}$ be generated by a holomorphic motion of γ_0. Let $f_\lambda: \boldsymbol{D} \to G_\lambda$ be a Riemann mapping function of $\boldsymbol{D}$ onto the inside of γ_λ, and let $f_\lambda^*: \boldsymbol{D}^* \to G_\lambda^*$ be a Riemann mapping function of the exterior of $\boldsymbol{D}$ onto the exterior of γ_λ. If f_λ and f_λ^* depend holomorphically on λ and if γ_0 has measure zero then the holomorphic motion of γ_0 is trivial.*

Proof. Let $\{\gamma_\lambda: \lambda \in \boldsymbol{D}\}$ be generated by the holomorphic motion $F: \boldsymbol{D} \times \gamma_0 \to \boldsymbol{C}$ of γ_0. Define

$$\hat{F}: \boldsymbol{D} \times \hat{\boldsymbol{C}} \longrightarrow \hat{\boldsymbol{C}}$$

by

$$\hat{F}(\lambda, z) = \begin{cases} f_\lambda(f_0^{-1}(z)) & z \text{ inside } \gamma_0 \\ f_\lambda^*(f_0^{*-1}(z)) & z \text{ outside } \gamma_0 . \\ F(\lambda, z) & z \text{ on } \gamma_0 \end{cases}$$

If f_λ, f_λ^* depend holomorphically on λ for $\lambda \in \boldsymbol{D}$ then $\hat{F}$ is a holomorphic motion of $\hat{\boldsymbol{C}}$. For fixed $\lambda \in \boldsymbol{D}$ the function

$$z \longmapsto H(z) \equiv \hat{F}(\lambda, z)$$

is quasiconformal (by the λ-lemma) and it satisfies $H_{\bar{z}} = 0$ almost everywhere. Hence H is a linear fractional transformation. Therefore $\hat{F}$ has the form (3.3); the functions a, b, c, d are holomorphic since $\hat{F}(\lambda, z)$ is holomorphic in λ for any fixed $z \in \hat{\boldsymbol{C}}$.

The next result gives a means for testing whether the Riemann mapping functions can be normalized to depend holomorphically on λ.

Theorem 4. *Let $F: \boldsymbol{D} \times \gamma_0 \to \boldsymbol{C}$ be a holomorphic motion of γ_0. A necessary and sufficient condition that the Riemann mapping functions*

$$f_\lambda : \boldsymbol{D} \longrightarrow G_\lambda \tag{3.4}$$

can be normalized to depend holomorphically on λ is that for each $\lambda \in \boldsymbol{D}$ the function

$$z \longmapsto F(\lambda, z) : \gamma_0 \longrightarrow \gamma_\lambda \tag{3.5}$$

is the boundary value function of a function holomorphic in G_0.

Proof. Suppose that for each $\lambda \in \boldsymbol{D}$ (3.5) is the boundary value function of a function holomorphic in G_0. Since (3.5) is continuous (by the λ-lemma), there is a function $\hat{F}: \boldsymbol{D} \times (G_0 \cup \gamma_0) \to \boldsymbol{C}$ such that each function $z \mapsto \hat{F}(\lambda, z)$ is holomorphic for $z \in G_0$, continuous for $z \in G_0 \cup \gamma_0$, and is equal to $z \mapsto F(\lambda, z)$ for $z \in \gamma_0$.

Let $f_0: \boldsymbol{D} \to G_0$ be a fixed Riemann mapping function. Let $\tilde{f}_0: \boldsymbol{D} \cup \partial \boldsymbol{D} \to G_0 \cup \gamma_0$ be its continuous extension. For fixed λ the function

$$\zeta \longmapsto f_\lambda(\xi) \equiv \hat{F}(\lambda, \tilde{f}_0(\xi)) : \boldsymbol{D} \cup \partial \boldsymbol{D} \longrightarrow G_\lambda \cup \gamma_\lambda$$

is, by the argument principle, the continuous extension of a Riemann mapping function of $\boldsymbol{D}$ onto G_λ. The Cauchy integral formula shows that f_λ depends holomorphically on λ.

Conversely, suppose that, for each $\lambda \in \boldsymbol{D}$, $g_\lambda: \boldsymbol{D} \to G_\lambda$ is a Riemann mapping function which depends holomorphically on λ. Let $\tilde{g}_\lambda: \boldsymbol{D} \cup \partial \boldsymbol{D} \to G_\lambda \cup \gamma_\lambda$ be the continuous extension of g_λ. Define

$$\hat{F}(\lambda, z) = \begin{cases} g_\lambda(g_0^{-1}(z)) & z \in G_0 \\ F(\lambda, z) & z \in \gamma_0 . \end{cases}$$

Then $\hat{F}: \boldsymbol{D} \times (G_0 \cup \gamma_0) \to \boldsymbol{C}$ is a holomorphic motion of $G_0 \cup \gamma_0$. Therefore $z \mapsto \hat{F}(\lambda, z)$ is continuous. It follows that $F(\lambda, z) = \tilde{g}_\lambda(\tilde{g}_0^{-1}(z))$ for $z \in \gamma_0$. This shows that $z \to F(\lambda, z)$ for $z \in \gamma_0$ is the boundary value function of the holomorphic function $z \mapsto g_\lambda(g_0^{-1}(z))$ for $z \in G_0$.

Theorem 5. *Let $f(\lambda, z)$ be holomorphic in $z \in \boldsymbol{D}$ and measurable in $\lambda \in \boldsymbol{D}$. Furthermore, let*

$$|f(\lambda, z)| < M \qquad \text{for} \quad \lambda \in \boldsymbol{D}, \quad z \in \boldsymbol{D}. \tag{3.6}$$

We assume that $E \subset \partial \boldsymbol{D}$ has the property that for all $\zeta \in E$, $\lambda \in \boldsymbol{D}$ the limit

$$f(\lambda, \zeta) = \lim_{r \to 1-0} f(\lambda, r\zeta) \tag{3.7}$$

exists, $f(\lambda, \zeta)$ is holomorphic in $\lambda \in \boldsymbol{D}$, and mes $E > 0$. *Then $f(\lambda, z)$ is holomorphic for $(\lambda, z) \in \boldsymbol{D} \times \boldsymbol{D}$.*

Proof. Let

$$f(\lambda, z) = \sum_{n=0}^{\infty} a_n(\lambda) z^n \quad (z \in \boldsymbol{D}). \tag{3.8}$$

Let $\zeta \in E$. Let C be any piecewise smooth closed curve in $\boldsymbol{D}$. We consider

$$g(z) \equiv \int_C f(\lambda, z) d\lambda = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \boldsymbol{D}) \tag{3.9}$$

where

$$b_n = \int_C a_n(\lambda) \quad n = 0, 1, \ldots. \tag{3.10}$$

We obtain from (3.6) and (3.7), by Lebesgue's bounded convergence theorem, that if $r_k \to 1-0$ as $k \to \infty$ then

$$\lim_{k \to \infty} g(r_k \zeta) = \int_C f(\lambda, \zeta) d\lambda.$$

The above integral vanishes because $f(\lambda, \zeta)$ is holomorphic in $\lambda \in \boldsymbol{D}$. Hence we conclude that

$$g(r\zeta) \longrightarrow 0 \quad \text{as} \quad r \longrightarrow 1-0 \tag{3.11}$$

for each $\zeta \in E$.

Let $L(C)$ be the length of C. Since

$$|g(z)| < M \cdot L(C) \quad (z \in \boldsymbol{D})$$

by (3.6) and (3.9), we obtain from the Riesz uniqueness theorem and from (3.11) that $g(z) \equiv 0$.

By (3.10)

$$\int_C a_n(\lambda) d\lambda = 0, \quad n = 0, 1, \ldots. \tag{3.12}$$

Assume for the moment that Morera's theorem holds in the generalized form: if

$a(\lambda)$ is a bounded measurable function of $\lambda \in \boldsymbol{D}$ and if $\int_C a(\lambda)d\lambda = 0$ for every piecewise smooth closed curve C in $\boldsymbol{D}$ then $a(\lambda)$ is equal a.e. to a function holomorphic for $\lambda \in \boldsymbol{D}$. Then we could conclude from (3.12) that each $a_n(\lambda)$, and hence $f(\lambda, z)$, is holomorphic in λ.

For the convenience of the reader we shall give a proof of the generalized form of Morera's theorem. Let

$$h(z) = \int_0^z a(\lambda)d\lambda, \quad z \in \boldsymbol{D}. \tag{3.13}$$

Then h is well defined, independent of the path of integration. The function h is absolutely continuous and

$$h_x(z) = -ih_y(z) = a(z) \tag{3.14}$$

for almost all $z \in \boldsymbol{D}$. Thus we have a generalized Stokes' formula for any closed rectangle $R \subset \boldsymbol{D}$:

$$0 = \iint_R (ih_x - h_y)dxdy = \int_{\partial R} h(z)dz \tag{3.15}$$

where the first equality is due to (3.14) and the second is derived by writing the double integral as an iterated integral and using the absolute continuity to apply the fundamental theorem of calculus. From (3.15) we conclude that h is holomorphic in $\boldsymbol{D}$. Hence, by (3.14) ,$a(\lambda)$ is equal a.e. to the holomorphic function $h'(\lambda)$.

4. Examples.

We consider examples of holomorphically moving Jordan curves and discuss the dependence on λ of the associated Riemann mapping functions.

Example 1. *Let δ be a quasicircle. There is a holomorphic motion $\{\gamma_\lambda: \lambda \in \boldsymbol{D}\}$ of the unit circle $\gamma_0 = \partial \boldsymbol{D}$ such that $\gamma_k = \delta$ for some $k \in \boldsymbol{D}$ and the associated Riemann mapping functions f_λ: $\boldsymbol{D} \to G_\lambda$ can be normalized to depend holomorphically on $\lambda \in \boldsymbol{D}$.*

The construction is well known. Let f be a quasiconformal mapping of $\hat{\boldsymbol{C}}$ which maps $\boldsymbol{D}$ conformally onto $\triangle$, the inside of δ. We may assume f leaves 0, 1, ∞ fixed. Let $\mu = f_{\bar{z}}/f_z$, $k = \|\mu\|$. Let $\mu(\lambda) = (\lambda/k)\mu$ if $k \neq 0$. Let f_λ be the $\mu(\lambda)$-conformal homeomorphism of $\hat{\boldsymbol{C}}$ which leaves 0, 1, ∞ fixed. Then f_λ is conformal in $\boldsymbol{D}$, holomorphic in λ, and $f_k = f$. The holomorphic motion

$$F: \boldsymbol{D} \times \gamma_0 \longrightarrow \boldsymbol{C}$$

defined by $F(\lambda, z) = f_\lambda(z)$ has the required properties.

Example 2. (*Hadamard variation*). *Assume γ_0 is a closed Jordan curve with continuously turning tangent. Let $\vec{n}(z)$ be the unit normal vector at $z \in \gamma_0$. The holomorphic motion of γ_0*

$$(\lambda, z) \longmapsto F(\lambda, z) = z + \lambda \vec{n}(z) : \boldsymbol{D} \times \gamma_0 \longrightarrow \boldsymbol{C} \tag{4.1}$$

is the complexified Hadamard variation (Courant [2]). *The associated Riemann mapping functions can be normalized to depend holomorphically on λ if and only if γ_0 is a circle.*

Let $\phi\colon \boldsymbol{D} \cup \partial\boldsymbol{D} \to G_0 \cup \gamma_0$ be the continuous extension of a Riemann mapping function of $\boldsymbol{D}$ onto G_0. By Lindelöf's theorem $\arg \phi'$ extends continuously to $\partial\boldsymbol{D}$; we denote this continuous extension by $\arg \phi'(\xi)$, $\xi \in \partial\boldsymbol{D}$. From (4.1) we obtain

$$F(\lambda, \phi(\xi)) = \phi(\xi) + \lambda \xi e^{i \arg \phi'(\xi)} \quad (\xi \in \partial\boldsymbol{D}). \tag{4.2}$$

According to Theorem 5 we investigate when

$$F(\lambda, \phi(\cdot))\colon \partial\boldsymbol{D} \longrightarrow \boldsymbol{C} \tag{4.3}$$

is the continuous extension to $\partial\boldsymbol{D}$ of a function holomorphic in $\boldsymbol{D}$. Evidently, this will be the case if and only if the function

$$g(\xi) \equiv \xi\, e^{i \arg \phi'(\xi)} \quad (\xi \in \partial\boldsymbol{D}) \tag{4.4}$$

is the continuous extension to $\partial\boldsymbol{D}$ of a function holomorphic in $\boldsymbol{D}$.

The function g in (4.4) maps $\partial\boldsymbol{D}$ into $\partial\boldsymbol{D}$ and the image curve has winding number one about the origin. Therefore, if g is the continuous extension to $\partial\boldsymbol{D}$ of a function holomorphic in $\boldsymbol{D}$, that function would have to be a Möbius transformation and so there would exist $k \in \partial\boldsymbol{D}$, $\alpha \in \boldsymbol{D}$ such that

$$\xi e^{i \arg \phi'(\xi)} = k \frac{\xi - \alpha}{\bar{\alpha}\xi - 1} \tag{4.5}$$

for all $\xi \in \partial\boldsymbol{D}$. The function

$$G(z) = \frac{k(z-\alpha)}{z(\bar{\alpha}z - 1)\phi'(z)} \tag{4.6}$$

is meromorphic in $\boldsymbol{D}$. In an annulus $|\alpha| < |z| < 1$ there is a single-valued branch of $\arg G(z)$ and by (4.5)

$$\lim_{z \to \partial\boldsymbol{D}} \arg G(z) = 0.$$

The reflection principle shows that G is a rational function which exhibits the symmetry

$$\bar{G}(z) = G(1/\bar{z}). \tag{4.7}$$

Therefore ϕ' is rational. It has no zeros or poles in $\boldsymbol{D}$. Also, it has no zeros or poles on $\partial\boldsymbol{D}$ since ∂G_0 has a continuously turning tangent.

If $\alpha = 0$ then ϕ' is constant. Assume $\alpha \neq 0$. Then we see from (4.6) and (4.7) that G has one pole inside of $\boldsymbol{D}$, namely $z = 0$, and therefore only one pole outside of $\boldsymbol{D}$ at $z = \infty$. G has only one zero inside of $\boldsymbol{D}$, at $z = \alpha$, and therefore only one zero outside of $\boldsymbol{D}$, at $z = 1/\bar{\alpha}$. We conclude that G has the form

$$G(z)=\frac{c(z-\alpha)(\bar{\alpha}z-1)}{z} \tag{4.8}$$

If we compare (4.6) and (4.8) we find that

$$\phi'(z)=\frac{\phi'(0)}{(\bar{\alpha}z-1)^2}$$

$$\phi(z)=\frac{\phi'(0)}{\bar{\alpha}(1-\bar{\alpha}z)}+\text{const.}$$

We conclude that $G_0=\phi(\boldsymbol{D})$ is a disk if (4.3) is the boundary value function of a function holomorphic in $\boldsymbol{D}$.

Example 3. (Siegel disks). Up to now the domains G_λ that we considered were Jordan domains. In many cases this was only for simplicity; those results hold as well for non-Jordan domains. In this example we explicitly refrain from assuming G_λ is Jordan.

This example concerns iterations of rational functions (degree $\geqslant 2$). It is a fact that the boundary of a Siegel disk of a rational function is contained in the closure of the forward orbits of the critical points. Let us call a subset of the critical points *ample* if the forward orbits are infinite and disjoint, and if the subset is maximal with respect to these two properties. If the set of normality (of the iterates) of a rational function contains a Siegel disk then the closure of the forward orbits of an ample set of critical points will contain the boundary of the Siegel disk.

Theorem 6. (D. Sullivan) *Let $\Omega\subset\boldsymbol{C}$ be simply connected. For each $\lambda\in\Omega$ let R_λ be a rational function with a Siegel disk G_λ centered at $a(\lambda)$. Let $c_1(\lambda),\dots,$ $c_N(\lambda)$ be an ample set of critical points of R_λ. Assume that $a, c_1,\dots, c_N$ depend holomorphically on λ. Then for $\lambda\in\Omega$, ∂G_λ moves holomorphically and the Riemann mapping functions f: $\boldsymbol{D}\to G_\lambda$ can be normalized to depend holomorphically on λ.*

The proof is due to Dennis Sullivan [6]. Let E_λ be the union of the forward orbist of $c_1(\lambda),\dots, c_N(\lambda)$ under R_λ. As observed above, $\partial G_\lambda\subset Cl\,E_\lambda$. We shall first show that in a neighborhood of each point $\lambda_0\in\Omega$ the Riemann maps f_λ can be normalized to depend holomorphically on λ. The general result will then be seen to follow by the monodromy theorem.

For the local result we may assume $\lambda_0=0\in\boldsymbol{D}\subset\Omega$. The construction of E_λ determines a holomorphic motion of E_0 (note that we have assumed the orbits to be disjoint). By the improved λ-lemma, the restriction of this motion to $|\lambda|<1/3$ can be extended to a holomorphic motion F: $\boldsymbol{D}_{1/3}\times\hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ of $\hat{\boldsymbol{C}}$. We have

$$F(\lambda,\, R_0^m(c_n(0)))=R_\lambda^m(c_n(\lambda)) \quad (n=1,\dots,N;\ m=1,2,\dots)$$

and we can assume $F(\lambda, a(0))=a(\lambda)$.

Note that $F(\lambda, G_0)=G_\lambda$ for $\lambda\in\boldsymbol{D}_{1/3}$. Indeed, if $z_0\in G_0$ but $F(\lambda, z_0)\notin G_\lambda$ then there would be a point $z_1\in G_0$ with $F(\lambda, z_1)\in\partial G_\lambda$ (consider the motion of an arc in G_0 from $a(0)$ to z_0). From $F(\lambda, z_1)\in Cl\,E_\lambda$ it would follow that $z_1\in Cl\,E_0$, a contradiction. Thus $F(\lambda, G_0) \subset G_\lambda$. A similar argument gives the opposite inclusion.

Let $\{z_n\}$ be a sequence in G_0 which is fundamental for a prime end of G_0 (i.e., $g(z_n)$ converges to a point on $\partial \boldsymbol{D}$ whenever g is a conformal map of G_0 onto $\boldsymbol{D}$). For $\lambda \in \boldsymbol{D}_{1/3}$ set $z_n(\lambda) = F(\lambda, z_n)$. Let $C_n(\lambda)$ be the closure of the orbit of $z_n(\lambda)$ under R_λ; $C_n(\lambda)$ is an analytic closed Jordan curve. Let $\tilde{C}_n(\lambda)$ be the component of the complement of $C_n(\lambda)$ which contains $a(\lambda)$.

Let $\phi_n(\lambda, \cdot)$ be the conformal mapping of $\hat{C}_n(0)$ onto $\tilde{C}_n(\lambda)$, normalized so that $a(0) \mapsto a(\lambda)$ and so that $z_n(0)$ corresponds to $z_n(\lambda)$ under the continuous (in fact, analytic) extension of $\phi_n(\lambda, \cdot)$ to the boundary of $\tilde{C}_n(0)$; this extension will again be denoted by $\phi_n(\lambda, \cdot)$. Let $\phi(\lambda, \cdot)$ be the conformal map of G_0 onto G_λ, normalized so that $a(0) \mapsto a(\lambda)$ and so that the prime end $\{z_n(0)\}$ corresponds to the prime end of $\{z_n(\lambda)\}$ (the sequence $\{z_n(\lambda)\}$ is fundamental because it is the image of a fundamental sequence $\{z_n(0)\}$ under a map $F(\lambda, \cdot)$ which is continuous on $Cl\, G_0$).

For each fixed $\lambda \in \boldsymbol{D}_{1/3}$, $\phi_n(\dot{\lambda}, \cdot)$ converges to $\phi(\lambda, \cdot)$ uniformly on compact subsets of G_0. This can be seen most easily, perhaps, by mapping G_0, G_λ conformally onto unit disks so that $a(0)$, $a(\lambda)$ correspond to the origin and so that the prime ends $\{z_n(0)\}$, $\{z_n(\lambda)\}$ correspond to 1. The curves $\{C_n(0)\}$, $\{C_n(\lambda)\}$ will be mapped to cir cles. The mapping $\phi_n(\lambda, \cdot)$ is transformed to a map of the form $\zeta \mapsto b_n \zeta$ where $b_n \in \boldsymbol{C}$, $b_n \to 1$.

The equation

$$\phi_n(\lambda, \underline{R}_0^k z_n(0)) = R_\lambda^k(z_n(\lambda))$$

shows that on a dense subset of $C_n(0)$ the boundary values of $\phi_n(\lambda, \cdot)$ depend holomorphically on λ. By the λ-lemma it follws that all the boundary values of $\phi_n(\lambda, \cdot)$ on $C_n(0)$ depend holomorphically on λ. It follows, using the Cauchy integral formula for example, that $\phi_n(\lambda, \cdot)$ depends holomorphically on λ in $\tilde{C}_n(0)$.

Since $\phi_n(\lambda, \cdot)$ is holomorphic in λ, the same is true of the limit function $\phi(\lambda, \cdot)$. Let f be a fixed Riemann mapping function of $\boldsymbol{D}$ onto G_0. Then

$$f_\lambda(\cdot) = \phi(\lambda, f(\cdot))$$

is a Riemann mapping function of $\boldsymbol{D}$ onto G_λ, and f_λ depends holomorphically on λ.

This local result means that Ω can be covered by open disks U_i such that there are Riemann mapping functions

$$f_{\lambda,i} \colon \boldsymbol{D} \longrightarrow G_\lambda \colon 0 \longmapsto a(\lambda)$$

which are defined for $\lambda \in U_i$ and which depend holomorphically on $\lambda \in U_i$. If $U_i \cap U_j \neq \emptyset$ then

$$f_{\lambda,i}^{-1} \circ f_{\lambda,j}(z) = k_{ij} z \quad (z \in U_i \cap U_j) \tag{4.9}$$

where k_{ij} has modulus 1 and therefore, being holomorphic in λ, is independent of λ. Since Ω is simply connected there are constants k_i, $|k_i| = 1$, such that $k_{ij} = k_i / k_j$. Thus if we define

$$f_\lambda(z) \equiv f_{\lambda,i}(k_i z)) \quad (z \in U_i)$$

we obtain a well defined family of Riemann mapping functions $\boldsymbol{D} \to G_\lambda$ which depend holomorphically on $\lambda \in \Omega$. (The existence of the $\{k_i\}$, a topological fact, can be put into function theory context by using (4.9) to see that $\log f'_{\lambda,i}(0)$ is locally holomorphic in λ and has a globally single-valued real part. Therefore there is a single-valued holomorphic function $g(\lambda)$ in Ω with the same real part. Now $g(\lambda) - \log f'_{\lambda,i}(0)$ is an imaginary constant in U_i and so the constant

$$k_i = \frac{e^{g(\lambda)}}{f'_{\lambda,i}(0)} \quad (\lambda \in U_i)$$

has modulus 1; it satisfies $k_i/k_j = k_{ij}$ as required.)

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References

[1] L. Bers and H. Royden, Holomorphic families of injections, To appear.

[2] R. Courant, Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces, Interscience Publ. Inc., New York, 1950, 330 pp.

[3] F. W. Gehring and Ch. Pommerenke, On the Nehari univalence criterion and quasicircles, Comment. Math. Helv., **59** (1984), 226–242.

[4] R. Mañé, P. Sad, D. Sullivan, On the dynamics of rational maps, Ann. École Norm. Sup., **16** (1983), 193–217.

[5] B. Rodin, Behavior of the Riemann mapping function under complex analytic deformations of the domain, Complex Variables: Theory and Applications., To appear.

[6] D. Sullivan, Quasiconformal homeomorphisms of analytic endomorphisms., I.H.E.S. preprint.

[7] D. Sullivan and W. P. Thurston, Improved λ-lemma., To appear.

[8] S. E. Warschawski, Recent results in numerical methods of conformal mapping, Proc. Amer. Math. Soc. of Symposia in Applied Math., vol. 6 (1956).