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The incompressible limit and the initial layer of the compressible Euler equation

By

Seiji UKAI

1. Introduction.

The Euler equation of compressible ideal fluid flow in $\boldsymbol{R}^n$ is written, in appropriate nondimensional form (cf. [5]), as

$$
\begin{aligned}
&\frac{1}{\gamma p}(p_t+v\cdot\nabla p)+\nabla\cdot v=0,\\
(1.1)\qquad &\rho(v_t+v\cdot\nabla v)+\lambda^2\nabla p=0,\\
&(p,\ v)|_{t=0}=(p_0,\ v_0).
\end{aligned}
$$

Here, the unknowns are the pressure $p=p(t, x)>0$ and velocity $v=v(t, x)\in\boldsymbol{R}^n$, $t\geqq 0$, $x\in\boldsymbol{R}^n$, while ρ is the density governed by the equation of state $p=\rho^\gamma$, $\gamma>1$, and λ, the parameter arising from nondimensionalization, is $M^{-1}\gamma^{-1/2}$, M being the Mach number.

In this paper we discuss the limit of solutions as $\lambda\to\infty$. Some fundamental facts on this limit have been established by Klainerman and Majda in [5], (see also [4] for the periodic case and [1], [2] for bounded domains). In particular, it is shown that unique solutions exist for all large λ on the time interval $[0, T]$ independent of λ, and that if the initial datum is incompressible datum, then the solutions converge as $\lambda\to\infty$ uniformly on $[0, T]$ to a solution of the incompressible Euler equation.

The aim of the present paper is to show that even if initial datum is not incompressible, the limit still exists and satisfies the incompressible Euler equation. However, the uniform convergence breaks near $t=0$, due to the development of initial layer.

To state our result more precisely, we put, as in [5],

$$p(t,\ x)=\bar{p}+\lambda^{-1}q(t,\ x)\,,\quad p_0(x)=\bar{p}+\lambda^{-1}q_0(x)\,,$$

where $\bar{p}$ is an arbitrarily fixed positive number. Set $u=(q, v)$ and $u_0=(q_0, v_0)$. Let H^s denote the Sobolev space $H^s(\boldsymbol{R}^n)$ with norm $\|\cdot\|_s$. Throughout the paper, we take $s\geqq s_0+1$, $s_0=[n/2]+1$. The following theorem is the part of results from [5] which is relevant to us.

Theorem 1.1 ([5]). (i) *For any C_0, $k_0>0$, there exist two positive numbers*

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T and C_1 such that for each $\lambda \geqq 1$ and $u_0 \in H^s$ with

$$\|u_0\|_s \leqq C_0, \quad p_0 \geqq k_0, \tag{1.2}$$

the equation (1.1) *has a unique classical solution $u=u^\lambda$ on $[0, T]\times \boldsymbol{R}^n$, satisfying*

$$u^\lambda \in C^0([0, T]; H^s) \cap C^1([0, T]; H^{s-1}) \tag{1.3}$$

and

$$\|u^\lambda(t)\|_s \leqq C_1, \qquad t \in [0, T]. \tag{1.4}$$

(ii) *Suppose the initial u_0 satisfies the additional condition*

$$\begin{aligned} &q_0 = \lambda^{-1} q_0^1, \\ &v_0 = v_0^0 + \lambda^{-1} v_0^1, \quad \nabla \cdot v_0^0 = 0, \\ &\|(q_0^1, v_0^1)\|_s \leqq C_0. \end{aligned} \tag{1.5}$$

Then the solution u^λ of (i) *has the additional estimate*

$$\|u_t^\lambda(t)\|_{s-1} \leqq C_1 \tag{1.6}$$

on the same time interval $[0, T]$. *Furthermore, u^λ converges;*

$$\begin{aligned} u^\lambda \to u^\infty = (0, v^\infty) \text{ weakly* in } L^\infty([0, T]; H^s), \text{ and} \\ \text{strongly in } C^0_{loc}([0, T]\times \boldsymbol{R}^n), \end{aligned} \tag{1.7}$$

as $\lambda \to \infty$, where v^∞ belongs to the class (1.3) *and is a unique solution, together with some $p^\infty \in C^0([0, T]; H^s)$, of the Cauchy problem for the incompressible Euler equation;*

$$\begin{aligned} &\nabla \cdot v^\infty = 0, \\ &\bar{\rho}(v_t^\infty + v^\infty \cdot \nabla v^\infty) + \nabla p^\infty = 0, \\ &v^\infty|_{t=0} = v_0^0, \end{aligned} \tag{1.8}$$

where $\bar{\rho} = (\bar{p})^{1/\gamma}$.

As stated already, our aim is to establish a similar convergence result without assuming the condition (1.5). We will prove the

Theorem 1.2. *Let $n \geqq 3$ and let u^λ and u_0 be those of Theorem* 1.1 (i). *Suppose u_0 is constant in λ. Then, as $\lambda \to \infty$,*

$$\begin{aligned} u^\lambda \to u^\infty = (0, v^\infty) \text{ weakly* in } L^\infty([0, T]; H^s), \text{ and} \\ \text{strongly in } C^0_{loc}((0, T]\times \boldsymbol{R}^n), \end{aligned} \tag{1.9}$$

with a limit v^∞ belonging to the class (1.3) *and giving a unique solution to* (1.8) *with the initial condition replaced by*

$$v^\infty|_{t=0} = P v_0, \tag{1.10}$$

where P is the orthogonal projection of H^s onto the solenoidal subspace $H^s_\sigma = \{v \in H^s \mid \nabla \cdot v = 0\}$ (cf. [3]).

Remark 1.3. (i) Compare (1.9) with (1.7). In (1.9), $(0, T]$ cannot be replaced by $[0, T]$, that is, the convergence is not uniform near $t=0$. In fact, at $t=0$, $v^{\lambda}(0)=v_0$, and in general $v_0 \neq Pv_0=v^{\infty}(0)$. On the other hand, $v_0^0=Pv_0^0$ in (1.5). Thus, the initial layer develops if and only if v_0 is not solenoidal.

(ii) In case the initial $u_0=u_0^{\lambda}$ varies with λ, just as is the case in (1.5), then it suffices to assume that $u_0^{\lambda} \to u_0^{\infty}$ strongly in H^s: The conclusion of Theorem 1.2 remains valid, with v_0 replaced by v_0^{∞} in (1.10).

(iii) Theorem 1.1 is true also for the periodic (in x) case, (see [4]). However, Theorem 1.2 is not. This is because no decay estimates similar to that given in Proposition 2.1 below are possible for the periodic case.

The convergence (1.7) in Theorem 1.1 is a direct consequence of local compactness assured by the uniform estimates (1.4) and (1.6). In our situation, only (1.4) is available. As for u_t^{λ}, we merely have $\|u_t^{\lambda}\|_{s-1} \leqq C_1 \lambda$, see [5]. Thus there is not enough compactness in t.

In section 3, we will show that the solenoidal part Pv^{λ} still satisfies (1.6) while $(I-P)v^{\lambda} \to 0$ strongly for $t>0$. The main ingredient of the proof is the use of a nice asymptotic behavior as $\lambda \to \infty$ of solutions of the linearized equation about $u=(0, 0)$ of (1.1). This asymptotic behavior will be established in the next section. Our method of proof is similar to those developed in [7] and [8], and will have other applications.

2. Linearized Operator.

In terms of $u={}^t(q, v)$ and $u_0={}^t(q_0, v_0)$ (column vectors), the equation (1.1) takes the form

$$u_t + B_j^{\lambda}(u) u_{x_j} = 0, \quad u(0) = u_0, \tag{2.1}$$

where the summation convection is used, and B_j are $(n+1)\times(n+1)$ matrices given by

$$B_j^{\lambda}(u) = \begin{pmatrix} v_j & \lambda \gamma p e_j \\ \dfrac{\lambda}{\rho} {}^t e_j & v_j I_n \end{pmatrix},$$

$e_j=(0, \cdots, 1, \cdots, 0)$ being unit vectors and I_n the unit matrix, in $\boldsymbol{R}^n$. We write (2.1) as

$$\begin{aligned} &u_t + B_j^{\lambda}(0) u_{x_j} = F^{\lambda}(u, \nabla u) \equiv -(B_j^{\lambda}(u) - B_j^{\lambda}(0)) u_{x_j}, \\ &u(0) = u_0. \end{aligned} \tag{2.2}$$

In this section we study the group $U^{\lambda}(t)$ generated by the linearized operator $-B_j^{\lambda}(0)\partial/\partial x_j$. Using the Fourier transform

$$(\mathcal{F}u)(\xi) = \tilde{u}(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-ix\cdot\xi} u(x) dx,$$

we readily find the group $U^{\lambda}(t)$ in the form

$$U^{\lambda}(t)=\mathcal{F}^{-1}e^{-tB(\lambda,\xi)}\mathcal{F},$$

where B is the $(n+1)\times(n+1)$ matrix defined by

$$B(\lambda, \xi)=iB_j^{\lambda}(0)\xi_j=\begin{pmatrix} 0 & i\lambda\gamma\xi \\ i\lambda{}^t\xi & 0 \end{pmatrix},$$

with $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ (law vector). Here we have put $\bar{p}=\bar{\rho}=1$, which does not lose generality because (1.1) is already in nondimensional form.

The matrix $B(\lambda, \xi)$ has eigenvalues 0 and $\pm i\lambda\sqrt{\gamma}|\xi|$. The eigenvalue 0 is of multiplicity $n-1$ with the eigenvectors

$$e_j(\xi)={}^t(0, \tilde{e}_j(\xi)), \qquad j=1, 2, \cdots, n-1,$$

where $\tilde{e}_j(\xi)\in \boldsymbol{R}^n$ are such that

$$\xi\cdot\tilde{e}_j(\xi)=0, \quad \tilde{e}_j(\xi)\cdot\tilde{e}_k(\xi)=\delta_{jk}, \tag{2.3}$$

for all $\xi\in\boldsymbol{R}^n$. Obviously, the adjoint eigenvectors $e_j^*(\xi)$ are ${}^te_j(\xi)$. The eigenvalues $\pm i\lambda\sqrt{\gamma}|\xi|$ are both simple, with the eigenvectors

$$e_{\pm}(\xi)=c_0{}^t(\pm\sqrt{\gamma}, \omega), \quad c_0=(2\sqrt{\gamma})^{-1/2}, \quad \omega=\xi/|\xi|, \tag{2.4}$$

and the adjoint ones

$$e_{\pm}^*(\xi)=c_0(\pm 1, \sqrt{\gamma}\omega).$$

$\{e_j^*(\xi), e_{\pm}^*(\xi)\}$ is the adjoint basis to $\{e_j(\xi), e_{\pm}(\xi)\}$.

Accordingly, the group $U^{\lambda}(t)$ has the following orthogonal decomposition.

$$U^{\lambda}(t)=U_1+U_2^{\lambda}(t), \quad U_1U_2^{\lambda}(t)=U_2^{\lambda}(t)U_1=0, \tag{2.5}$$

$$U_1u_0={}^t(0, Pv_0), \quad Pv_0=\mathcal{F}^{-1}(\tilde{e}_j^*(\xi)\cdot\hat{v}_0(\xi)){}^t\tilde{e}_j(\xi), \tag{2.6}$$

$$U_2^{\lambda}(t)u_0=\mathcal{F}^{-1}e^{\mp i\lambda\sqrt{\gamma}|\xi|}(e_{\pm}^*(\xi)\cdot\hat{u}_0(\xi))e_{\pm}(\xi), \tag{2.7}$$

(The summation convention is used in an obvious manner for $\pm$.) U_1 is the part of $U^{\lambda}(t)$ corresponding to the eigenvalue 0, and $U_2^{\lambda}(t)$ that to $\pm i\lambda\sqrt{\gamma}|\xi|$. From (2.3) it is seen that the operator P in (2.6), the v-component of U_1, is just the projection P introduced in § 1. By virtue of the Parseval theorem, we obtain

$$\|Pv_0\|_l\leqq\|v_0\|_l, \quad \|U_2^{\lambda}(t)u_0\|_l\leqq\sqrt{\gamma}\|u_0\|_l, \qquad l\in\boldsymbol{R}. \tag{2.8}$$

Also we see that in general $U_2^{\lambda}(t)u_0$ does not tend to zero as λ, $t\to\infty$ in H^l. However, it has an L^{∞} decay. Denote the norm of $L^p(\boldsymbol{R}^n)$ by $|\cdot|_p$. The following is the main result of this section.

Proposition 2.1. *Let $n\geqq 3$ and $l>n/2$. Then, there is a constant $C\geqq 0$ such that*

(2.9) $$|U_2^\lambda(t)u_0|_\infty \leqq C|\lambda t|^{-\delta}|u_0|_1^\delta \|u_0\|_l^{1-\delta}$$

holds for all λ, $t\in \boldsymbol{R}$ *and* $u_0\in H^l\cap L^1$, *with* $\delta=1-(n-1)/(l+n/2-1)$.

Proof. Referring to (2.7), it suffices to evaluate the integral of the type

(2.10) $$\psi(\mu, x)=\int_{R^n} e^{i\mu|\xi|+ix\cdot\xi}\alpha(\omega)\hat{\phi}(\xi)d\xi,$$

for $\mu\in \boldsymbol{R}$, $\alpha\in C^\infty(S^{n-1})$ and $\phi\in H^l$. Split the integral over $|\xi|\geqq R$ and $|\xi|\leqq R$, with $R>0$ to be determined later, and write the respective integrals as ψ_1 and ψ_2. By Schwarz' inequality, we get

(2.11) $$\begin{aligned}|\psi_1(\mu, x)| &\leqq \alpha_0\int_{|\xi|\geqq R}|\hat{\phi}(\xi)|d\xi \\ &\leqq \alpha_0\Big(\int_{|\xi|\geqq R}(1+|\xi|)^{-2l}d\xi\Big)^{1/2}\|\phi\|_l \leqq CR^{-(l-n/2)}\|\phi\|_l,\end{aligned}$$

for $l>n/2$, where $\alpha_0=\sup|\alpha(\omega)|$. To evaluate ψ_2, we put $r=|\xi|$ and substitute $\hat{\phi}=\mathcal{F}\phi$ to deduce

(2.12) $$\begin{aligned}\psi_2(\mu, x)&=\int_0^R e^{i\mu r}g(r, x)dr, \\ g(r, x)&=(2\pi)^{-n/2}r^{n-1}\int_{R^n}h(r, x-y)\phi(y)dy, \\ h(r, x)&=\int_{S^{n-1}}e^{irx\cdot\omega}\alpha(\omega)d\omega.\end{aligned}$$

By integration by parts, we find

$$\psi_2(\mu, x)=\frac{1}{i\mu}\Big\{[e^{i\mu r}g(r, x)]_0^R-\int_0^R e^{i\mu r}g'(r, x)dr\Big\},$$

where $g'=\partial g/\partial r$. Hence,

$$|\psi_2(\mu, x)|\leqq|\mu|^{-1}\Big(|g(R, \cdot)|_\infty+|g(0, \cdot)|_\infty+\int_0^R|g'(r, \cdot)|_\infty dr\Big).$$

We shall evaluate the right hand side. In [6], it is shown that

(2.13) $$\Big|\int_{S^{n-1}}e^{ix\cdot\omega}\beta(\omega)d\omega\Big|\leqq C|x|^{-l}$$

holds for any $l\in[0, (n-1)/2]$ and $\beta\in C^\infty(S^{n-1})$, with a constant $C\geqq 0$ independent of x. By this with $l=0$, we get $|h(r, x)|\leqq C$ and so,

(2.14) $$|g(r, \cdot)|_\infty\leqq Cr^{n-1}|\phi|_1.$$

Differentiation of g under integral sign yields

$$g'(r, x)=(n-1)r^{-1}g(r, x)+(2\pi)^{-n/2}r^{n-1}\int_{R^n}h'(r, x-y)\phi(y)dy,$$

$$h'(r, x)=i\int_{S^{n-1}} x\cdot\omega e^{irx\cdot\omega}\alpha(\omega)d\omega .$$

If $n\geqq 3$, we can take $l=1$ in (2.13), to deduce $|h'|\leqq C|x||rx|^{-1}=Cr^{-1}$. This and (2.14) give $|g'(r, \cdot)|_\infty\leqq Cr^{n-2}|\phi|_1$, and then,

$$|\phi_2(\mu, \cdot)|_\infty\leqq C|\mu|^{-1}R^{n-1}|\phi|_1 . \tag{2.15}$$

Finally, choose $R=(|\mu|\|\phi\|_l/|\phi|_1)^{1/(l+n/2-1)}$ in (2.11) and (2.15), and find

$$|\phi(\mu, \cdot)|_\infty\leqq C|\mu|^{-\delta}|\phi|_1^\delta\|\phi\|_l^{1-\delta} .$$

This is the desired estimate.

As a corollary, we have a decay for u_0 in H^l but not in $H^l\cap L^1$.

Proposition 2.2. *Let $n\geqq 3$ and $u_0\in H$ with $l>n/2$. Then, for any $\tau>0$,*

$$\sup_{\tau\leq t}|U_2^\lambda(t)u_0|_\infty\to 0 \qquad \textit{as} \quad \lambda\to\infty . \tag{2.16}$$

Proof. The proposition is true if $u_0\in C_0^\infty(\boldsymbol{R}^n)$. Now, it suffices to note that $C_0^\infty(\boldsymbol{R}^n)$ is dense in H^l and that

$$|U_2^\lambda(t)u_0|_\infty\leqq C\|u_0\|_l ,$$

which comes from (2.11) by putting $R=0$. Thus, we are done.

3. Proof of Theorem 1.2.

Let $u^\lambda=(q^\lambda, v^\lambda)$ be the solution of Theorem 1.1 (i). By virtue of the uniform estimate (1.4), there exists a subsequence such that

$$u^\lambda\to u^\infty \quad \text{weakly* in } L^\infty([0, T]; H^s), \tag{3.1}$$

as $\lambda\to\infty$, with a limit $u^\infty=(q^\infty, v^\infty)\in L^\infty([0, T]; H^s)$. Now we will establish a stronger convergence. To this end, we first note that since u^λ is a unique solution to (2.2), it can be expressed as

$$u^\lambda(t)=U^\lambda(t)u_0+\int_0^t U^\lambda(t-\tau)F^\lambda(\tau)d\tau , \tag{3.2}$$

where $F^\lambda(\tau)=F^\lambda(u^\lambda(\tau), \nabla u^\lambda(\tau))$ was defined in (2.2). We shall evaluate F^λ. Recall $s\geqq s_0+1$, $s_0=[n/2]+1$ in Theorem 1.1.

Lemma 3.1. *There is a constant $C\geqq 0$ independent of $\lambda\geqq 1$ and $\tau\in[0, T]$, and there holds*

$$|F^\lambda(\tau)|_1+\|F^\lambda(\tau)\|_{s-1}\leqq C . \tag{3.3}$$

Proof. Write F^λ explicitly as

$$F^\lambda=-{}^t(v^\lambda\cdot\nabla q^\lambda+\lambda\gamma(p^\lambda-1)\nabla\cdot v^\lambda,\ v^\lambda\cdot\nabla v^\lambda+\lambda(1/\rho^\lambda-1)\nabla q^\lambda) . \tag{3.4}$$

Recall that we have put $\bar{p}=\bar{\rho}=1$. By Schwarz' inequality,

$$|F^{\lambda}|_{1}\leqq(|v^{\lambda}|_{2}+|\lambda\gamma(p^{\lambda}-1)|_{2}+|\lambda(1/\rho^{\lambda}-1)|_{2})|\nabla u^{\lambda}|_{2}. \tag{3.5}$$

By definition $\lambda(p^{\lambda}-1)=q^{\lambda}$ and $\rho^{\lambda}=(p^{\lambda})^{1/\gamma}$, and so,

$$\lambda(1/\rho^{\lambda}-1)=\gamma^{-1}q^{\lambda}\int_{0}^{1}(1+\theta\lambda^{-1}q^{\lambda})^{-(\gamma+1)/\gamma}d\theta. \tag{3.6}$$

In [5], the solution u^{λ} is constructed in such a manner that $p^{\lambda}\geqq k_{0}/2$ (k_{0} is as in (1.2)). Hence, from (3.5),

$$|F^{\lambda}|_{1}\leqq C|u^{\lambda}|_{2}|\nabla u^{\lambda}|_{2}.$$

Further, by standard calculus inequalities (see Lemmas 2 and 3 of [5]), we have,

$$\|F^{\lambda}\|_{s-1}\leqq C(1+\|u^{\lambda}\|_{s_{0}+1})^{s-1}\|u^{\lambda}\|_{s}.$$

Combining these estimates with (1.4) yields (3.3).

Returning to u^{λ} of (3.2), we decompose it according to the decomposition (2.5):

$$u^{\lambda}(t)=u_{1}^{\lambda}(t)+u_{2}^{\lambda}(t), \tag{3.7}$$

$$u_{1}^{\lambda}(t)={}^{t}(0,\ v_{1}^{\lambda}(t)),\quad v_{1}^{\lambda}(t)=Pv_{0}+\int_{0}^{t}PG^{\lambda}(\tau)d\tau, \tag{3.8}$$

where $G^{\lambda}(\tau)$ denotes the v-component of $F^{\lambda}(\tau)$ (see (3.4)), and

$$u_{2}^{\lambda}(t)=U_{2}^{\lambda}(t)u_{0}+\int_{0}^{t}U_{2}^{\lambda}(t-\tau)F^{\lambda}(\tau)d\tau. \tag{3.9}$$

We shall show that v_{1}^{λ} satisfies (1.6) while $u_{2}^{\lambda}\to 0$ strongly.

Lemma 3.2. $\|v_{1}^{\lambda}(t)\|_{s}+\|v_{1t}^{\lambda}(t)\|_{s-1}\leqq C.$

Proof. Since the decomposition (3.7) is orthogonal, we have $\|u^{\lambda}\|_{s}^{2}=\|u_{1}^{\lambda}\|_{s}^{2}+\|u_{2}^{\lambda}\|_{s}^{2}$. Then by (1.4), $\|u_{1}^{\lambda}\|_{s}\leqq C$. Differentiate (3.8) and obtain $v_{1t}^{\lambda}=PG^{\lambda}(t)$. But P is a bounded operator on H^{s-1} (see (2.8)), and G^{λ}, the v-component of F^{λ}, enjoys (3.3). This completes the proof of the lemma.

Lemma 3.3. $\sup_{\tau\leq t}|u_{2}^{\lambda}(t)|_{\infty}\to 0$ $(\lambda\to\infty)$, *for any* $\tau>0$, *if* $n\geqq 3$.

Proof. Since $s-1\geqq s_{0}>n/2$, Proposition 2.1 can be applied to (3.9), with $l=s-1$. We have,

$$|u_{2}^{\lambda}(t)|_{\infty}\leqq|U_{2}^{\lambda}(t)u_{0}|_{\infty}+\frac{C}{|\lambda|^{\delta}}\int_{0}^{t}\frac{1}{|t-\tau|^{\delta}}|F^{\lambda}(\tau)|_{1}^{\delta}\|F^{\lambda}(\tau)\|_{s-1}^{1-\delta}d\tau,$$

where $\delta=1-(n-1)/(s+n/2-2)$ is δ of Proposition 2.1 for $s-1$. By Lemma 3.1 and since $\delta<1$, the last term in the above is majorized by

$$C|\lambda|^{-\delta}\int_0^t |t-\tau|^{-\delta}d\tau \leqq C'|\lambda|^{-\delta},$$

for all $t \in [0, T]$. This and Proposition 2.2 complete the proof of the lemma.

Lemma 3.2 implies that $v_1^\lambda(t, x)$ is uniformly bounded and equicontinuous both in t and x. Hence, passing to a subsequence,

$$v_1^\lambda \to v_1^\infty \text{ strongly in } C_{loc}^0([0, T]\times \boldsymbol{R}^n),$$

with some $v_1^\infty \in C^0([0, T]\times \boldsymbol{R}^n)$. Taking account of Lemma 3.3, we then conclude (note that u_2^λ also belongs to the class (1.3)) that

$$u^\lambda = u_1^\lambda + u_2^\lambda \to {}^t(0, v_1^\infty) \text{ strongly in } C_{loc}^0((0, T]\times \boldsymbol{R}^n). \tag{3.10}$$

And by (3.1), we may put $q^\infty = 0$ and $v^\infty = v_1^\infty$. Thus $v^\infty \in L^\infty([0, T]; H^s) \cap C^0([0, T]\times \boldsymbol{R}^n)$. Furthermore, the orthogonality stated in (3.7) implies that $Pv^\lambda = v_1^\lambda$, and thereby, in the limit, $Pv^\infty = v_1^\infty = v^\infty$. This proves that $\nabla\cdot v^\infty = 0$.

It remains to prove that the whole sequence is convergent and that v^∞ is a solution of the incompressible Euler equation, belonging to the class (1.3). In view of the strong convergence (3.10), combined with (3.1), it follows from (3.4) that

$$G^\lambda \to -v^\infty\cdot\nabla v^\infty \text{ weakly* in } L^\infty([0, T]; H^0).$$

The space H^0 is then replaced by H^{s-1}, by using (3.3). Now, we can go to the limit on both sides of (3.8). The result is,

$$v^\infty(t) = Pv_0 - \int_0^t Pv^\infty(\tau)\cdot\nabla v^\infty(\tau)d\tau. \tag{3.11}$$

Since $v^\infty\cdot\nabla v^\infty \in L^\infty([0, T]; H^{s-1})$, the integral on the right hand side is in $\mathrm{Lip}([0, T]; H^{s-1})$. By assumption of Theorem 1.2, $v_0 \in H^s$. Hence, (3.11) implies

$$v^\infty \in L^\infty([0, T]; H^s) \cap \mathrm{Lip}([0, T]; H^{s-1}), \tag{3.12}$$

and

$$\begin{aligned} &v_t^\infty = -P(v^\infty\cdot\nabla v^\infty), \\ &v^\infty|_{t=0} = Pv_0. \end{aligned} \tag{3.13}$$

The equation (3.13) is just the incompressible Euler equation. In [3], it is proved that the solution to (3.13) is unique within the class (3.12). A particular consequence of this uniqueness is that all the convergence stated so far is true for the whole sequence, not only by passing to a subsequence. It is also proved in [3] that the unique solution to (3.13) necessarily belongs to the class (1.3). Now the proof of Theorem 1.2 is complete.

Remark 3.4. From (3.1) it follows that

$$\nabla\cdot v^\lambda \to \nabla\cdot v^\infty = 0 \text{ weakly* in } L^\infty([0, T]; H^{s-1}).$$

If $s>n/2+2$, we can infer that for any $\tau>0$,

$$\sup_{\tau \leq t} |\nabla \cdot v^{\lambda}(t)|_{\infty} \to 0. \tag{3.14}$$

Indeed, by Proposition 2.1 we find for $l>n/2$,

$$|\nabla U_2^{\lambda}(t) u_0|_{\infty} \leqq C |\lambda t|^{-\delta} |\nabla u_0|^{\delta} \|\nabla u_0\|_l^{1-\delta},$$

and proceeding as in Lemma 3.1, we have

$$|\nabla F^{\lambda}(\tau)|_1 + \|\nabla F^{\lambda}(\tau)\|_{s-2} \leqq C,$$

if $s-1>n/2$. Now, (3.14) can be proved with $\nabla \cdot v^{\lambda}$ replaced by ∇u_2^{λ}, using the argument of Lemma 3.3. Since $\nabla \cdot v_1^{\lambda}=0$, this verifies (3.14).

DEPARTMENT OF APPLIED PHYSICS
OSAKA CITY UNIVERSITY

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