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On Whittaker vectors for generalized Gelfand-Graev representations of semisimple Lie groups

By

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§ 0. Introduction

0.1. Historical background. Early in the 1960's, I. M. Gelfand and M. I. Graev [3] attempted to construct and classify irreducible representations of Chevalley groups over a finite field through irreducible decompositions of the representations induced from characters of a maximal unipotent subgroup. Such an induced representation is called a Gelfand-Graev representation if the character is non-degenerate. They showed that this representation is multiplicity free.

The Gelfand-Graev representations for real semisimple Lie groups are defined in the same way. J. A. Shalika [12] extended the above multiplicity one theorem to quasi-split linear semisimple Lie groups (more generally, to such groups over a local field). For Chevalley groups, H. Jacquet [6] constructed intertwining operators from the principal series representations to the Gelfand-Graev representations through analytic continuation of an integral operator, so called Whittaker integral. G. Schiffmann [13] treated the problem of analytic continuation of Whittaker integral for linear semisimple Lie groups of real rank one. Using his results, M. Hashizume [4] dealt with it for reductive algebraic groups over $\boldsymbol{R}$ of higher rank for the spherical principal series representations.

Recently, N. Kawanaka [8] introduced, generalizing the idea of Gelfand-Graev, the generalized Gelfand-Graev representations of Chevalley groups over a finite field, and proved Ennola duality using their characters. As was suggested in [8], these representations seem to give us more precise informations on irreducible representations than those given only by Gelfand-Graev representations.

0.2. As a first step of our study of the generalized Gelfand-Graev representations of real semisimple Lie groups, we extend in this article the above results for Gelfand-Graev representations to the generalized ones. Standing at the same point of view as [4], we treat intertwining operators from the principal series representations to the contragredient representations of generalized Gelfand-Graev representations.

This article consists of two parts. In the first part §§ 2∼3, we deal with a uniqueness property of intertwining operators. The main result of Part I is

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Theorem 3.7, which gives a multiplicity theorem for the generalized Gelfand-Graev representations. In the second part §§ 4∼5, we construct explicitly some intertwining operators. For this purpose, we treat Whittaker integral and its analytic continuation. In the following subsections 0.3 and 0.4, we explain, in details, our results in Part I and Part II respectively.

0.3. Uniqueness of intertwining operators. For a precise description of our results, we first explain some notations and definitions on semisimple Lie groups and their representations. Let G be a connected real semisimple Lie group with finite center. Denote by $\mathfrak{g}$ its Lie algebra. Let $G=KAN$ be an Iwasawa decomposition of G, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ the corresponding decomposition of $\mathfrak{g}$. We denote by W the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Choose a positive system Λ^+ in the set Λ of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ so that $\mathfrak{n}=\sum_{\lambda\in\Lambda^+}\mathfrak{g}_\lambda$, where $\mathfrak{g}_\lambda$ denotes the root space of a root λ. Let U be the maximal unipotent subgroup of G opposite to N.

For a non-zero nilpotent element $A\in\mathfrak{g}$, there exists an $\mathfrak{sl}_2$-triplet $\{A, H, B\}\subseteq\mathfrak{g}$ containing A. Using the adjoint representation of $\boldsymbol{R}A\oplus\boldsymbol{R}H\oplus\boldsymbol{R}B\simeq\mathfrak{sl}(2, \boldsymbol{R})$ on $\mathfrak{g}$, we can associate to A a connected unipotent subgroup $U(1.5)_A$ and its unitary character η_A such that

$$\eta_A(\exp X)=\exp\sqrt{-1}\,Q(X, B) \qquad \text{for } X\in\mathfrak{u}(1.5)_A,$$

where $\mathfrak{u}(1.5)_A$ is the Lie algebra of $U(1.5)_A$ and Q is the Killing form of $\mathfrak{g}$. We call the smooth representation π_{η_A} of G induced from η_A a *generalized Gelfand-Graev representation* associated to A (for the precise definition, see 3.1).

Take a subset F of the set Π of simple roots in Λ^+. Let $P_F=M_FA_FN_F$ be a Langlands decomposition of the standard parabolic subgroup P_F corresponding to F. This correspondence is given so that the root system generated by F is the restricted root system of M_F. Let $\mathfrak{p}_F=\mathfrak{m}_F\oplus\mathfrak{a}_F\oplus\mathfrak{n}_F$ be the corresponding decomposition of the Lie algebra $\mathfrak{p}_F$ of P_F, and let U_F (resp. $\mathfrak{u}_F$) be the opposite of N_F (resp. $\mathfrak{n}_F$). For a smooth representation (σ, E_σ) of M_F and $\nu\in(\mathfrak{a}_F)_C{}^*$, we denote by $(\pi_{\sigma,\nu}, H_{\sigma,\nu})$ the smooth representation of G induced from $\sigma\otimes e^\nu\otimes(1_{N_F})$. For a character η of a closed subgroup U' of U, let $\pi_\eta^\vee$ denote the contragredient representation of $\pi_\eta=\mathrm{Ind}_{U'}^G(\eta)$ (smoothly).

The space $\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_\eta^\vee)$ of intertwining operators from $\pi_{\sigma,\nu}$ to $\pi_\eta^\vee$ is isomorphic to the space $\mathcal{I}_\pi(G)$ (see 2.1) of intertwining E_σ-distributions on G, and to the space $\mathrm{Wh}_\eta(H_{\sigma,\nu}^\vee)$ (see Definition 1.1) of *Whittaker vectors* of type (U', η).

In § 2, we treat the space $\mathcal{I}_\pi(G)$ when η is a character of $U_{F'}$ for some $F'\subseteq\Pi$. In case that the number of double cosets in $U_{F'}\backslash G/P_F$ is at most countable, one can apply a powerful formula by F. Bruhat which gives an estimate of $\dim\mathcal{I}_\pi(G)$. Unfortunately, this is far from to be countable in general, so his formula can not be applied directly. We modify his theory as follows. One has an estimate of $\dim\mathcal{I}_\pi(G)$ in Proposition 2.3 by Bruhat decomposition of G. Suggested by this proposition, we treat for every $s\in W$ the space $\mathcal{I}_{\pi,s}$ of intertwining distributions on an open subset $\Omega_s(\supseteqq UsP_F)$ of G with supports contained in UsP_F. UsP_F is divided into continuously many double cosets of $U_{F'}\backslash G/P_F$ in general, which differs from the case of Bruhat. On the supports of distributions in $\mathcal{T}_{\pi,s}$, we obtain Theorem 2.11, the main result of § 2, under the assumption that σ is finite dimensional.

In § 3, we apply Theorem 2.11 to the generalized Gelfand-Graev representations. Then we obtain one of our main results which asserts a uniqueness property of intertwining operators.

Theorem 1 (see Theorem 3.7). *Let G be a connected real semisimple Lie group of matrices. Let A be a non-zero nilpotent element of $\mathfrak{g}$ and $\{A, H, B\}\subseteqq\mathfrak{g}$ be an $\mathfrak{sl}_2$-triplet. Assume that there exists a subset $F''\subseteqq\Pi$ such that $U(1.5)_A$ can be taken as $U_{F''}$. Consider the generalized Gelfand-Graev representation π_{η_A} associated to A. Let $F\subseteqq\Pi$. For a finite dimensional representation (σ, E_σ) of M_F and $\nu\in(\mathfrak{a}_F)_C{}^*$, one has*

(1) $\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_A}^{\vee})=\{0\}$ *if* $\mathrm{Ad}\,(G)B\cap\mathfrak{n}_F=\emptyset$.

(2) $\dim \mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_A}^{\vee})\begin{cases} \leqq\dim E_\sigma & \textit{if } F=F'', \\ =0 & \textit{if } F\supsetneqq F''. \end{cases}$

The assumption "$U(1.5)_A=U_{F''}$ for some $F''\subseteqq\Pi$" of this theorem is essentially satisfied for even nilpotent elements A, and for all nilpotent elements if $\mathfrak{g}$ is a complex semisimple Lie algebra of type (A_l).

Theorem 1 is an extension in a certain sense of Theorem 2.2 of M. Hashizume [4] to the generalized Gelfand-Graev representations. We emphasize the next point for an original value of our theorem. Let $A\in\mathfrak{g}$ be a non-zero nilpotent element in Theorem 1. For an $F\subseteqq\Pi$, $\neq\emptyset$, let σ be a finite dimensional representation of M_F and $\nu\in(\mathfrak{a}_F)_C{}^*$. Then there are no non-zero intertwining operators from $\pi_{\sigma,\nu}$ to $\pi_\eta^{\vee}$ if η is a non-degenerate character of U. So the representation $\pi_{\sigma,\nu}$ never can be caught within the limit of original Gelfand-Graev representations. However, thanks to Theorem 1 (2), we can expect that irreducible $\pi_{\sigma,\nu}$ with $F=F''$ actually occurs in $\pi_{\eta_A}^{\vee}$ with finite multiplicity. Therefore it seems to be quite natural to associate with such $\pi_{\sigma,\nu}$ the nilpotent $\mathrm{Ad}\,(G)$-orbit of A.

0.4. Construction of intertwining operators. In §§ 4~5, we construct intertwining operators through analytic continuation of Whittaker integrals. This realizes our expectation in the last part of 0.3. Suggested by Casselman's subrepresentation theorem, we consider the representations $\pi_{\sigma,\nu}$ induced from the minimal parabolic subgroup $P=MAN$. Let σ be an irreducible finite dimensional representation of M and $\nu\in\mathfrak{a}_C{}^*$. For an $s\in W$, put $U_s=U\cap s^{-1}Ns$. For a unitary character η of U_s, we introduce a *Whittaker integral* after [4]:

$$W^{e^{\vee}}(\sigma, \nu, \eta)f(g)=\int_{U_s}\langle e^{\vee}, f(gu)\rangle\eta(u)^{-1}du \qquad (f\in H_{\sigma,\nu},\ g\in G) \tag{0.1}$$

for $e^{\vee}\in E_{\sigma}^{\vee}\backslash(0)$, where $E_{\sigma}^{\vee}$ is the dual space of E_{σ}. This integral is absolutely convergent if ν lies in an open convex domain $D_s\subseteq \mathfrak{a}_C{}^*$ given as

$$D_s=\{\nu\in \mathfrak{a}_C{}^*;\ \langle \mathrm{Re}\,\nu, \lambda\rangle>0 \quad \text{for all } \lambda\in\Delta^+ \text{ such that } s\lambda\in-\Delta^+\}.$$

And the map $f\to W^{e^{\vee}}(\sigma, \nu, \eta)f$ gives a non-zero intertwining operator from $\pi_{\sigma,\nu}$ to $\pi_{\eta}^{\vee}$. Moreover it is holomorphic with respect to $\nu\in D_s$.

In §5, we examine how far the Whittaker integrals can be extended as meromorphic functions of ν. For this purpose, we restrict G to complex semisimple Lie groups. H. Jacquet [6] showed that there exists a subgroup $W'_{s,\eta}$ of W such that the integral (0.1) can be extended to a holomorphic function on the convex hull $[W'_{s,\eta}D_s]$ of $W'_{s,\eta}D_s$ (see Theorem 5.2). Combining this with meromorphic continuation of intertwining operators between the principal series representations, we find out that the integral (0.1) extends meromorphically to a larger domain than $[W'_{s,\eta}D_s]$ in general (see 5.1). We apply this method to Whittaker integrals $W^{e^{\vee}}(\sigma, \nu, \eta_A)f(g)$ for the generalized Gelfand-Graev representations π_{η_A} in case that $U(1.5)_A=U_s$ for some $s\in W$.

In Proposition 5.11, we give a sufficient condition for meromorphic continuation of $W^{e^{\vee}}(\sigma, \nu, \eta_A)f(g)$ to the whole $\mathfrak{a}_C{}^*$ when A is a certain nilpotent element such that the Ad(G)-orbit through A intersects $\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$, the sum of root spaces $\mathfrak{g}_{-\lambda}$ for simple roots λ. If the above A is an even nilpotent element, the function $\nu\to W^{e^{\vee}}(\sigma, \nu, \eta_A)f(g)$ extends to a meromorphic function on $\mathfrak{a}_C{}^*$ (Corollary 5.12). We apply Proposition 5.11 to complex simple Lie groups in the subsections 5.4~5.7. The complex simple Lie algebra of type (A_l) is special in the point that every nilpotent Ad(G)-orbit intersects $\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$. We establish, in Theorem 5.13, meromorphic continuation of Whittaker integral $W^{e^{\vee}}(\sigma, \nu, \eta_A)f(g)$ to the whole $\mathfrak{a}_C{}^*$ for every nilpotent element A, when $\mathfrak{g}$ is of type (A_l). We regard these Corollary 5.12 and Theorem 5.13, especially the latter, as the results of great importance in this article. We summarize them in the following theorem.

Let $\mathfrak{g}=\bigoplus_{1\leq i\leq p}\mathfrak{g}^{(i)}$ be the direct sum decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ into simple ideals $\mathfrak{g}^{(i)}$. For $X\in\mathfrak{g}$, write $X=\sum_{1\leq i\leq p}X^{(i)}$ with $X^{(i)}\in\mathfrak{g}^{(i)}$ for $1\leqq i\leqq p$.

Theorem 2 (see Corollary 5.12 and Theorem 5.13). *Let G be a connected complex semisimple Lie group. Let A_0 be a nilpotent element in $\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$ such that each $\mathfrak{g}^{(i)}$ component $A_0{}^{(i)}$ of A_0 is even unless $\mathfrak{g}^{(i)}$ is of type (A_l). Then there exists an* Ad(G)*-conjugate A of A_0 satisfying the following conditions* (1) *and* (2).

(1) *There exists a subset $F(A)$ of Π such that $U(1.5)_A$ can be chosen as $U_{F(A)}$.*

(2) *The function $\nu\to W^{e^{\vee}}(\sigma, \nu, \eta_A)f(g)$ extends to a meromorphic function on the whole $\mathfrak{a}_C{}^*$, for $\sigma\in\hat{M}$, $f\in(H_{\sigma,\nu})_K$ and $g\in G$.*

Here $\hat{M}$ denotes the set of characters of the torus M, and $(H_{\sigma,\nu})_K$ the space of K-finite vectors in $H_{\sigma,\nu}$.

For general $\mathfrak{g}$, it is no longer true that every nilpotent $\mathrm{Ad}(G)$-orbit intersects $\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$. Here is a technical limit of Proposition 5.11. Moreover, there exist nilpotent $\mathrm{Ad}(G)$-orbits such that the corresponding Whittaker integrals never can be extended meromorphically to $\mathfrak{a}_C{}^*$ within the limit of our present method. We give in 5.5 such an example for the complex simple Lie groups of type (G_2). In this case, the Whittaker integral can be extended meromorphically to a half space by our method. But we do not know whether it can be extended to a meromorphic function on a larger domain or not.

Our method is fairly successful in order to show that Whittaker integral $W^{e^\vee}(\sigma, \nu, \eta_A)f(g)$ extends meromorphically to the whole $\mathfrak{a}_C{}^*$. However, it would be necessary to consider not only our present method but also other ones for the general treatment of analytic continuation of Whittaker integrals.

In the last part of § 5, we list up, for complex simple Lie groups of rank 2, our results of analytic continuation of Whittaker integrals $W^{e^\vee}(\sigma, \nu, \eta_A)f(g)$.

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§ 1. Notations and preliminaries

We explain the notations used throughout this paper.

1.1. Smooth representations of Lie groups. Let G be a Lie group countable at infinity (i.e., G may be written as a countable union of compact subsets). We denote by $\mathfrak{g}$ the Lie algebra of G. We identify $\mathfrak{g}$ with the space of right G-invariant vector fields on G as follows:

$$Xf(g)=\frac{d}{dt}f(\exp(-tX)g)|_{t=0} \qquad (g\in G) \tag{1.1}$$

for $X\in\mathfrak{g}$ and a differentiable function f on G. Let $U(\mathfrak{g}_C)$ be the universal enveloping algebra of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. Then $U(\mathfrak{g}_C)$ is naturally identified with the algebra of all right G-invariant differential operators through (1.1).

Let us recall the notion of smooth representations of G after [4] and [16]. In this subsection, we denote by E a locally convex, complete, Hausdorff, topological vector space. Let $C^\infty(G, E)$ be the space of all E-valued smooth functions on G. We shall equip $C^\infty(G, E)$ with the topology of uniform convergence on compact subsets of a function and its derivatives. This topology is called the Schwartz topology for $C^\infty(G, E)$. Let $\mathcal{D}(G, E)$ be the space of all E-valued smooth functions on G with compact supports. For a compact subset Ω of G, $\mathcal{D}_\Omega(G, E)$ denotes the space of all functions in $\mathcal{D}(G, E)$ with supports contained in Ω. We define a topology of $\mathcal{D}_\Omega(G, E)$ by the topology inherited from the Schwartz topology for $C^\infty(G, E)$. Then the space $\mathcal{D}(G, E)$ is topologized as the strict inductive limit of the spaces $\mathcal{D}_\Omega(G, E)$, where Ω ranges all compact subsets of G. This topology is called the Schwartz topology for $\mathcal{D}(G, E)$. In case

$E=\boldsymbol{C}$, we drop the symbol E in the above notations.

Let π be a continuous representation of G on E. A vector $v\in E$ is called a *smooth vector* for π if E-valued function $\tilde{v}: g\to\pi(g)v$ on G is in $\mathcal{C}^\infty(G, E)$. We denote by E^∞ the space of all smooth vectors in E. Then E^∞ is a $\pi(G)$-stable dense subspace of E. The map $E^\infty\ni v\to\tilde{v}\in\mathcal{C}^\infty(G, E)$ is a linear injection and the image of E^∞ is a closed subspace of $\mathcal{C}^\infty(G, E)$. We identify E^∞ with this closed subspace of $\mathcal{C}^\infty(G, E)$ through this mapping. This induced topology on E^∞ is, in general, finer than that inherited from E. Henceforth, unless specifically stated to the contrary, we shall consider E^∞ as equipped with the finer topology. We call (π, E) a *smooth representation* if $E=E^\infty$ with coincidence of topologies.

For a continuous representation (π, E) of G, the operators $\pi(x)$ $(x\in G)$ restricted to E^∞ define a smooth representation π_∞ of G on E^∞. This representation (π_∞, E^∞) is called the *smooth representation associated to* π.

We explain the notion of smoothy induced representations. For a Lie group G, we denote by $d_G x$ (or simply dx) a left Haar measure on G and by δ_G (or simply δ) the modular function on G with respect to $d_G x$: $\delta_G(y)=d_G(xy)/d_G x$ $(y\in G)$. Let H be a closed subgroup of G. For a smooth representation σ of H on a Fréchet space F, we consider the space $\mathcal{D}_\sigma(G, F)$ of all F-valued smooth functions f on G satisfying the following conditions (1) and (2).

(1) $f(xh)=\sqrt{\dfrac{\delta_H(h)}{\delta_G(h)}}\sigma(h^{-1})f(x) \qquad (x\in G,\ h\in H)$,

(2) f has a compact support mod H, in other words, the canonical image in G/H of the support of f is compact.

We define a topology of $\mathcal{D}_\sigma(G, F)$ in the following way. For a compact subset Ω of G, denote by $\mathcal{D}_{\sigma,\Omega}(G, F)$ the space of all functions in $\mathcal{D}_\sigma(G, F)$ with supports contained in ΩH. We place on $\mathcal{D}_{\sigma,\Omega}(G, F)$ the relative topology inherited from $\mathcal{C}^\infty(G, F)$. Equip $\mathcal{D}_\sigma(G, F)$ with the strict inductive limit of the topologies of $\mathcal{D}_{\sigma,\Omega}(G, F)$, where Ω ranges all compact subsets of G. Then $\mathcal{D}_\sigma(G, F)$ is an LF-space, that is, a strict inductive limit of a sequence of Fréchet spaces. The left translation defines a smooth representation π_σ of G on $\mathcal{D}_\sigma(G, F)$, which is called the *smooth representation induced from* σ.

Let π be a continuous representation of G on E. We denote by $E^\vee$ the the space of all continuous linear functionals on E with strong dual topology. We consider the contragredient representation (in algebraic sense) $\pi^\vee$ on $E^\vee$ defined by $\langle\pi^\vee(x)T, v\rangle=\langle T, \pi(x^{-1})v\rangle$ $(x\in G, v\in E, T\in E^\vee)$. If the representation (π, E) is smooth and $E^\vee$ is complete with respect to the strong dual topology, $(\pi^\vee, E^\vee)$ defines a smooth representation.

1.2. Notations for semisimple Lie groups. We prepare some notations for real semisimple Lie groups and Lie algebras. Let G be a connected real semisimple Lie group with finite center. Denote by $\mathfrak{g}$ its Lie algebra. Let θ be a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{q}$ the corresponding Cartan decomposition. Here $\mathfrak{k}$ (resp. $\mathfrak{q}$) is the space of all $X\in\mathfrak{g}$ such that $\theta X=X$ (resp. $\theta X=-X$). Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{q}$. By Λ we denote the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and by Λ^+ a set of positive roots in Λ. Put $\mathfrak{n}=\sum_{\lambda\in\Lambda^+}\mathfrak{g}_\lambda$ and $\mathfrak{u}=\theta\mathfrak{n}$, where $\mathfrak{g}_\lambda$ is the root space corresponding to a root λ. Let K, A, N and U be the analytic subgroup of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{u}$ respectively. Then $G=KAN$ (resp. $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$) is an Iwasawa decomposition of G (resp. $\mathfrak{g}$). For a Lie subgroup H of G and Lie subalgebras $\mathfrak{h}$ and $\mathfrak{x}$ of $\mathfrak{g}$, we denote by $Z_H(\mathfrak{x})$ (resp. $\mathfrak{z}_\mathfrak{h}(\mathfrak{x})$) and $N_H(\mathfrak{x})$ (resp. $\mathfrak{n}_\mathfrak{h}(\mathfrak{x})$) the centralizer of $\mathfrak{x}$ in H (resp. $\mathfrak{h}$) and the normalizer of $\mathfrak{x}$ in H (resp. $\mathfrak{h}$) respectively. Put $M=Z_K(\mathfrak{a})$ (resp. $\mathfrak{m}=\mathfrak{z}_\mathfrak{k}(\mathfrak{a})$). Then $P=MAN$ (resp. $\mathfrak{p}=\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$) is a minimal parabolic subgroup of G (resp. a minimal parabolic subalgebra of $\mathfrak{g}$). We denote by W the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Then $W=N_K(\mathfrak{a})/M$. For an $s\in W$, we denote a fixed representative of s in $N_K(\mathfrak{a})$ again by s.

Let Π be the set of simple roots in Λ^+. For a subset F of Π, let $\langle F\rangle$ denote the set of positive roots written as linear combinations of the elements in F, and put $\langle F\rangle'=\Lambda^+\backslash\langle F\rangle$.

For a subset $\mathfrak{h}$ of $\mathfrak{g}$, we denote by $\mathfrak{h}^\perp$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form Q of $\mathfrak{g}$. For $\nu\in\mathfrak{a}^*$, the dual space of $\mathfrak{a}$, define $H_\nu\in\mathfrak{a}$ by $\nu(H)=Q(H, H_\nu)$ for all $H\in\mathfrak{a}$. We introduce some subalgebras of $\mathfrak{g}$ as follows:

$$\mathfrak{a}(F)=\sum_{\lambda\in F}\boldsymbol{R}H_\lambda\,,\qquad \mathfrak{a}_F=\mathfrak{a}\cap\mathfrak{a}(F)^\perp\,,$$

$$\mathfrak{n}(F)=\sum_{\lambda\in\langle F\rangle}\mathfrak{g}_\lambda\,,\qquad \mathfrak{n}_F=\sum_{\lambda\in\langle F\rangle'}\mathfrak{g}_\lambda\,,$$

$$\mathfrak{u}(F)=\theta\mathfrak{n}(F)\,,\qquad \mathfrak{u}_F=\theta\mathfrak{n}_F\,.$$

Moreover

$$\mathfrak{m}_F=\mathfrak{u}(F)\oplus\mathfrak{a}(F)\oplus\mathfrak{m}\oplus\mathfrak{n}(F)\,,\qquad \mathfrak{l}_F=\mathfrak{m}_F\oplus\mathfrak{a}_F\,,$$

$$\mathfrak{p}_F=\mathfrak{m}_F\oplus\mathfrak{a}_F\oplus\mathfrak{n}_F\,.$$

Then $\mathfrak{p}_F$ is self-normalizing and is called the standard parabolic subalgebra of $\mathfrak{g}$ corresponding to F. The subalgebra $\mathfrak{m}_F$ is θ-stable, hence reductive in $\mathfrak{g}$. By $A(F)$, A_F, $N(F)$, N_F, $U(F)$ and U_F we denote the analytic subgroup of G corresponding to $\mathfrak{a}(F)$, $\mathfrak{a}_F$, $\mathfrak{n}(F)$, $\mathfrak{n}_F$, $\mathfrak{u}(F)$ and $\mathfrak{u}_F$ respectively. Then one has decompositions $A=A(F)\times A_F$, $N=N(F)\ltimes N_F$ and $U=U(F)\ltimes U_F$.

Denote by W_F the subgroup of W generated by reflections corresponding to the elements of F, and put $P_F=PW_FP$. Then P_F is self-normalizing and is the normalizer of $\mathfrak{n}_F$ in G, which is called the standard parabolic subgroup of G corresponding to F. Put $L_F=Z_G(\mathfrak{a}_F)$, then $L_F=M_FA_F$ with $M_F=Z_K(\mathfrak{a}_F)(M_F)_0$, where $(M_F)_0$ is the analytic subgroup of G with Lie algebra $\mathfrak{m}_F$. An Iwasawa decomposition of M_F is given by $M_F=K(F)A(F)N(F)$ with $K(F)=K\cap M_F$. The group P_F admits a Langlands decomposition $P_F=M_FA_FN_F$.

For a smooth representation σ of M_F on a Fréchet space E_σ and $\nu\in(\mathfrak{a}_F)_C{}^*$, we denote by $(\pi_{\sigma,\nu}, H_{\sigma,\nu})$ the smooth representation of G induced from the representation $\sigma\otimes e^\nu\otimes 1_{N_F}$ of $P_F=M_FA_FN_F$. The space $H_{\sigma,\nu}$ consists of E_σ-valued smooth functions f on G such that $f(xman)=a^{-\nu-\rho}\sigma(m)^{-1}f(x)$ for $x\in G$, $m\in M_F$,

$a\in A_F$ and $n\in N_F$. Here $\rho=2^{-1}\sum_{\lambda\in\Delta^+}(\dim \mathfrak{g}_\lambda)\lambda$ and we put $a^\mu=\exp\langle\mu, H\rangle$ for $\mu\in\mathfrak{a}_C^*$ and $a=\exp H\in A$. Each element μ' in $\mathfrak{a}(F)_C^*$ (resp. $(\mathfrak{a}_F)_C^*$) is regarded as an element in $\mathfrak{a}_C^*$ with $\langle\mu', \mathfrak{a}_F\rangle=\{0\}$ (resp. $\langle\mu', \mathfrak{a}(F)\rangle=\{0\}$).

1.3. Definition of Whittaker vectors. Let U' be a closed subgroup of U. For a character (i.e., one dimensional representation) η of U', we put

$$C_\eta^\infty(G)=\{f\in C^\infty(G);\ f(gu)=\eta(u)f(g) \quad \text{for } g\in G \text{ and } u\in U'\},$$

and equip it with the topology inherited from the Schwartz topology for $C^\infty(G)$. G acts on $C_\eta^\infty(G)$ by the left translation which defines a smooth representation $\tilde{\pi}_\eta$.

We introduce the notion of Whittaker vector as a generalization of that of Kostant in [10].

Definition 1.1. For a smooth representation π of G on a Fréchet space E, put

$$\mathrm{Wh}_\eta(E^\vee)=\{T\in E^\vee;\ \pi^\vee(u)T=\eta(u)T \quad \text{for } u\in U'\}.$$

Each element in the space $\mathrm{Wh}_\eta(E^\vee)$ is called a *Whittaker vector of type* (U', η).

Let (π_i, E_i) $(i=1, 2)$ be continuous representations of G on locally convex, complete, Hausdorff, topological vector spaces E_i. We denote by $\mathrm{Hom}_G(\pi_1, \pi_2)$ the space of continuous intertwining operators from E_1 to E_2.

The next lemma makes clear the relation between Whittaker vectors and intertwining operators.

Lemma 1.2 ([4, §1]). *Under the notations of Definition* 1.1, *one has isomorphisms of vector spaces*

$$\mathrm{Wh}_\eta(E^\vee)\simeq\mathrm{Hom}_G(\pi, \tilde{\pi}_\eta)\simeq\mathrm{Hom}_G(\pi, \pi_\eta^\vee).$$

The correspondence is given as follows:

(1) $\mathrm{Wh}_\eta(E^\vee)\ni T \longrightarrow A_T\in\mathrm{Hom}_G(\pi, \tilde{\pi}_\eta)$,

$$A_T(v)(g)=\langle\pi^\vee(g)T, v\rangle \quad \textit{for } v\in E \textit{ and } g\in G.$$

(2) $\mathrm{Hom}_G(\pi, \tilde{\pi}_\eta)\ni A \longrightarrow A^\vee\in\mathrm{Hom}_G(\pi, \pi_\eta^\vee)$,

$$\langle A^\vee v, P_\eta\phi\rangle=\int_G(Av)(x)\phi(x)d_Gx \quad \textit{for } v\in E \textit{ and } \phi\in\mathcal{D}(G).$$

Here P_η *is an open continuous linear surjection from* $\mathcal{D}(G)$ *to* $\mathcal{D}_\eta(G)$ *given by*

$$(P_\eta\phi)(x)=\int_{U'}\phi(xu')\eta(u')du' \quad (\phi\in\mathcal{D}(G),\ x\in G),$$

and du' *denotes a Haar measure on* U'.

Part I. On uniqueness of Whittaker vectors for generalized Gelfand-Graev representations

§ 2. Distributions corresponding to Whittaker vectors

Let G be a connected real semisimple Lie group with finite center. In this section, we treat Whittaker vectors using Bruhat's method.

Let $F, F' \subseteq \Pi$. Let σ be a smooth representation of M_F on a Fréchet space E_σ, η a character of $U_{F'}$ and $\nu \in (\mathfrak{a}_F)_C{}^*$. By the canonical projection $\mathcal{D}(G, E_\sigma) \to H_{\sigma,\nu}$, each element in $H_{\sigma,\nu}^{\vee}$ can be viewed as an E_σ-distribution on G. Consider Whittaker vectors of type $(U_{F'}, \eta)$ in $H_{\sigma,\nu}^{\vee}$. We study the supports of distributions corresponding to these Whittaker vectors. Our main result of this section is Theorem 2.11 under the assumption that σ is finite dimensional.

2.1. Whittaker vectors and intertwining distributions. For $\phi \in \mathcal{D}(G, E_\sigma)$, put

$$(P_{\sigma,\nu}\phi)(x) = \int_{M_F \times A_F \times N_F} a^{\nu+\rho}\sigma(m)\phi(xman)\,dm\,da\,dn\,,$$

where dm, da and dn are Haar measures on M_F, A_F and N_F respectively. $P_{\sigma,\nu}$ gives an open continuous linear surjection from $\mathcal{D}(G, E_\sigma)$ to $H_{\sigma,\nu}$. Then $P_{\sigma,\nu}$ induces a linear injection ${}^tP_{\sigma,\nu}: H_{\sigma,\nu}^{\vee} \to \mathcal{D}(G, E_\sigma)^{\vee}$ by $\langle {}^tP_{\sigma,\nu}T_0, \phi\rangle = \langle T_0, P_{\sigma,\nu}\phi\rangle$ for $T_0 \in H_{\sigma,\nu}^{\vee}$ and $\phi \in \mathcal{D}(G, E_\sigma)$. The image ${}^tP_{\sigma,\nu}H_{\sigma,\nu}^{\vee}$ consists of all $T \in \mathcal{D}(G, E_\sigma)^{\vee}$ such that

$$\langle T, R_{p^{-1}}\phi\rangle = \langle T, a^{\nu-\rho}\sigma(m)\phi\rangle \qquad \text{for } p = man \in M_F A_F N_F \text{ and } \phi \in \mathcal{D}(G, E_\sigma),$$

where $R_y\phi(x) = \phi(xy)$ $(x, y \in G)$. Moreover one easily has

Lemma 2.1. *${}^tP_{\sigma,\nu}$ induces a linear bijection from* $\mathrm{Wh}_\eta(H_{\sigma,\nu}^{\vee})$ *to the space of all $T \in \mathcal{D}(G, E_\sigma)^{\vee}$ such that*

$$\langle T, L_{u^{-1}}R_{p^{-1}}\phi\rangle = \langle T, \eta(u)a^{\nu-\rho}\sigma(m)\phi\rangle \tag{2.1}$$

for $p = man \in M_F A_F N_F$, $u \in U_{F'}$ and $\phi \in \mathcal{D}(G, E_\sigma)$, where $L_y\phi(x) = \phi(y^{-1}x)$ $(x, y \in G)$.

We define an action of the product group $U_{F'} \times P_F$ on G by

$$(u, p)\cdot g = ugp^{-1} \qquad \text{for } (u, p) \in U_{F'} \times P_F \text{ and } g \in G\,.$$

Then $U_{F'} \times P_F$ acts on spaces of functions on G, and then on spaces of distributions on G by duality. For example, $\phi^x(g) = \phi(x^{-1}\cdot g)$ $(g \in G)$,

$$\langle T^x, \phi\rangle = \langle T, \phi^{x^{-1}}\rangle \qquad \text{for } x \in U_{F'} \times P_F,\ \phi \in \mathcal{D}(G, E_\sigma),\ T \in \mathcal{D}(G, E_\sigma)^{\vee}.$$

Define a smooth representation π of $U_{F'} \times P_F$ on E_σ by $\pi(y) = \eta(u)a^{\nu-\rho}\sigma(m)$ for $y = (u, man)$ with $u \in U_{F'}$ and $man \in M_F A_F N_F$. Now put

$$\mathcal{T}_\pi(G)=\{T\in\mathcal{D}(G, E_\sigma)^\vee;\ T^x={}^t\pi(x)T \qquad \text{for } x\in U_{F'}\times P_F\},$$

where ${}^t\pi(x)$ is the transpose of the operator $\pi(x)$. By Lemma 2.1, one has

$$\mathrm{Wh}_\eta(H_{\sigma,\nu}^\vee)\simeq\mathcal{T}_\pi(G)\,. \tag{2.2}$$

Now we give an explicit description of orbits of $U_{F'}\times P_F$ on G. First we have a decomposition by Bruhat

$$G=\bigcup_{s\in W/W_F} G_s \text{ (disjoint union)} \quad \text{with } G_s=UsP_F\,.$$

Then $G_s=(U\cap sU_Fs^{-1})sP_F$ and every element of G_s is expressed uniquely as a product of elements of $U\cap sU_Fs^{-1}$ and sP_F. And G_s becomes a normal submanifold of G diffeomorphic to the product $(U\cap sU_Fs^{-1})\times P_F$ in the canonical way. Using this this formula and the semidirect product decomposition $U=U(F')\ltimes U_{F'}$, we obtain the following lemma.

Lemma 2.2. *For an $s\in W$, $G_s=(U(F')\cap sU_Fs^{-1})U_{F'}sP_F$, and it has an expression $G_s=(U(F')\cap sU_Fs^{-1})(U_{F'}\cap sU_Fs^{-1})sP_F$ in such a way that it is diffeomorphic to the product of $U(F')\cap sU_Fs^{-1}$, $U_{F'}\cap sU_Fs^{-1}$ and P_F in the canonical way.*

Consequently, G admits a decomposition

$$G=\bigcup_{s\in W/W_F}\ \bigcup_{y\in U(F')\cap sU_Fs^{-1}} U_{F'}ysP_F \quad \text{(disjoint union)}. \tag{2.3}$$

In general, the number of orbits of $U_{F'}\times P_F$ in G is not countable, which is contrary to the case of Bruhat. This prevents us from direct application of Bruhat's theorem ([16, Theorem 5.3.2.3]). However, thanks to the decomposition (2.3), his technique itself is available also in our situation.

For $s\in W$, put $\Omega_s=G_s\cup(\bigcup_{s'}G_{s'})$, where s' ranges through the elements in W such that $\dim G_{s'}>\dim G_s$. Each double coset G_w contained in the closure $\overline{G_s}$ of G_s has a dimension strictly smaller than that of G_s unless $G_w=G_s$. This implies that Ω_s is an open subset of G and that G_s is a closed submanifold of Ω_s ([12, Lemma 1.7]). We put

$$\mathcal{T}_{\pi,s}=\{T\in\mathcal{D}(\Omega_s, E_\sigma)^\vee;\ T^x={}^t\pi(x)T \qquad \text{for } x\in U_{F'}\times P_F,\ \mathrm{spt}(T)\subseteqq G_s\},$$

where $\mathrm{spt}(T)$ denotes the support of a distribution T.

The following proposition is well-known. But we give its proof to clarify the connection between $\mathcal{T}_\pi(G)$ and $\mathcal{T}_{\pi,s}$.

Proposition 2.3. *Let $\{s_j;\ 1\leqq j\leqq n\}\subseteqq W$ be a complete system of representatives of the coset space W/W_F. Then one has*

$$\dim\mathcal{T}_\pi(G)\leqq\textstyle\sum_{1\leq j\leq n}\dim\mathcal{T}_{\pi,s_j}\,.$$

Proof. We may assume that $\dim G_{s_j}\geqq\dim G_{s_{j+1}}$ for $1\leqq j\leqq n-1$. Put

$$\mathcal{T}^j=\{T\in\mathcal{T}_\pi(G);\ \mathrm{spt}(T)\subseteqq\bigcup_{j\leq k\leq n}G_{s_k}\} \quad (1\leqq j\leqq n),\quad \mathcal{T}^{n+1}=\{0\}.$$

For $1\leqq j\leqq n$, let $r_j: \mathcal{D}(G, E_\sigma)^\vee\to\mathcal{D}(\Omega_{s_j}, E_\sigma)^\vee$ be the map defined by restriction. Then we have immediately

$$r_j\mathcal{T}^j\subseteqq\mathcal{T}_{\pi,s_j} \quad\text{and}\quad \mathrm{Ker}\,(r_j|_{\mathcal{T}^j})=\mathcal{T}^{j+1} \qquad \text{for } 1\leqq j\leqq n\,.$$

Hence

$$\dim\mathcal{T}^j\leqq\dim\mathcal{T}_{\pi,s_j}+\dim\mathcal{T}^{j+1} \qquad \text{for } 1\leqq j\leqq n\,.$$

This proves the assertion. Q. E. D.

2.2. Distributions in $\mathcal{T}_{\pi,s}$ (I), from T ($\in\mathcal{T}_{\pi,s}$) to S (on $\mathcal{O}_1\subseteqq G_s$). For an $s\in W$, let $T\in\mathcal{T}_{\pi,s}\backslash\{0\}$ and $z_0\in\mathrm{spt}\,(T)$. We shall associate to T a quasi-invariant distribution on an open neighbourhood of z_0 in G_s under the action of $U_{F'}\times P_F$ ([16, 5.2]). We begin with the following lemma.

Lemma 2.4. *Let $s\in W$ and put $\mathfrak{l}_0=\mathfrak{u}\cap\mathrm{Ad}\,(s)\mathfrak{u}_F$. Then $\mathfrak{l}_0$ is transversal to G_s. In other words, $\mathfrak{g}=\mathfrak{l}_0\oplus T_e(G_sz^{-1})$ for every $z\in G_s$. Here $T_e(G_sz^{-1})$ denotes the tangent space of G_sz^{-1} at the unit element e of G.*

Proof. Let $z=u_1sp$ with $u_1\in U\cap sU_Fs^{-1}$ and $p\in P_F$. Then

$$T_e(G_sz^{-1})=\mathrm{Ad}\,(u_1)\{(\mathfrak{u}\cap\mathrm{Ad}\,(s)\mathfrak{u}_F)\oplus\mathrm{Ad}\,(s)\mathfrak{p}_F\},$$

because $G_sz^{-1}=u_1(U\cap sU_Fs^{-1})(sP_Fs^{-1})u_1^{-1}$. Hence it is sufficient to prove

$$\mathrm{Ad}\,(u_1)\{(\mathfrak{u}\cap\mathrm{Ad}\,(s)\mathfrak{u}_F)\oplus\mathrm{Ad}\,(s)\mathfrak{p}_F\}\cap\mathfrak{l}_0=\}0\}.$$

Since $\mathrm{Ad}\,(u_1)^{-1}\mathfrak{l}_0\subseteqq\mathrm{Ad}\,(s)\mathfrak{u}_F$, the left hand side of the above equality is contained in $\mathfrak{u}$. Clearly it is contained also in $\mathfrak{n}$, whence in $\mathfrak{u}\cap\mathfrak{n}=\{0\}$. Q. E. D.

Until the end of 2.3, we fix an element s in W. Let $X_1,\cdots,X_r$ (resp. $Y_1,\cdots,Y_{r'}$) be a basis of $\mathfrak{l}_0$ (resp. $T_e(G_ss^{-1})$). For $z=u_1sp$ with $u_1\in U\cap sU_Fs^{-1}$ and $p\in P_F$, put $Y_j^z=\mathrm{Ad}\,(u_1)Y_j$ for $1\leqq j\leqq r'$. Then $Y_1^z,\cdots,Y_{r'}^z$ forms a basis of $T_e(G_sz^{-1})$. For $\alpha=(\alpha_1,\cdots,\alpha_r)$ with non-negative integers α_i ($1\leqq i\leqq r$), we put $X^\alpha=X_1^{\alpha_1}\cdots X_r^{\alpha_r}\in U(\mathfrak{g}_C)$. Similarly $(Y^z)^\gamma=(Y_1^z)^{\gamma_1}\cdots(Y_{r'}^z)^{\gamma_{r'}}$ for $z\in G_s$ and $\gamma=(\gamma_1,\cdots,\gamma_{r'})$. By Poincaré-Birkhoff-Witt's theorem the set $\{(Y^z)^\gamma X^\alpha\}_{\gamma,\alpha}$ forms a basis of $U(\mathfrak{g}_C)$ for every $z\in G_s$.

Let k be a non-negative integer. For $x=(u,p)\in U_{F'}\times P_F$ and $z\in G_s$, we define a matrix $A_k(x,z)$ of degree $l={}_{r+k-1}C_{r-1}$ as follows. For $\alpha=(\alpha_1,\cdots,\alpha_r)$ such that $|\alpha|=\alpha_1+\cdots+\alpha_r=k$, we have an expansion

$$\mathrm{Ad}\,(u)X^\alpha=\sum_{|\beta|=k}a_{\beta\alpha}(x,z)X^\beta+\sum_{|\beta|<k,\,|\beta|+|\gamma|\leq k}c^\alpha_{\beta\gamma}(x,z)(Y^{x\cdot z})^\gamma X^\beta \tag{2.4}$$

with $a_{\beta\alpha}(x,z)$, $c^\alpha_{\beta\gamma}(x,z)\in\boldsymbol{C}$. Put

$$A_k(x,z)=(a_{\beta\alpha}(x,z))_{|\beta|=|\alpha|=k}\,.$$

By the definition of A_k, we easily have the following lemma.

Lemma 2.5. *The matrix valued function A_k on $(U_{F'}\times P_F)\times G_s$ is smooth and* *satisfies the following conditions* (1)~(3).

(1) $A_k((e, p), z)=I$ (I *the identity operator*) *for* $p\in P_F$ *and* $z\in G_s$.

(2) $A_k(xx', z)=A_k(x, x'\cdot z)A_k(x', z)$ *for* $x, x'\in U_{F'}\times P_F$ *and* $z\in G_s$.

(3) *The matrix* $A_k(x, z)$ *is unipotent if* $x\in(U_{F'}\times P_F)_z$, *where* $(U_{F'}\times P_F)_z$ *denotes the stabilizer of* z *in* $U_{F'}\times P_F$.

Now let $T\in\mathcal{T}_{\pi,s}\backslash\{0\}$ and $z_0\in\mathrm{spt}(T)$. In a sufficiently small relatively compact neighbourhood $\mathcal{O}$ of z_0 in Ω_s, there exist unique E_σ-distributions T_α on $\mathcal{O}_1=\mathcal{O}\cap G_s$ such that

$$T|_{\mathcal{O}}=\textstyle\sum_\alpha(-1)^{|\alpha|}X^\alpha T_\alpha \qquad \text{(finite sum)}, \tag{2.5}$$

where we regard each T_α as an E_σ-distribution on $\mathcal{O}$ by trivial extension ([16, Proposition A.2.1.2]). Let k be the largest integer such that $k=|\alpha|$ for some α with $T_\alpha\neq0$ and $\mathrm{spt}(T_\alpha)\ni z_0$. By replacing $\mathcal{O}$ with a smaller neighbourhood of z_0 if necessary, we may assume that $T_\alpha=0$ if $|\alpha|>k$.

Now we define an $(E_\sigma\otimes\boldsymbol{C}^l)$-distribution S on $\mathcal{O}_1$ as follows:

$$S(f)=\textstyle\sum_{|\alpha|=k}T_\alpha(f_\alpha) \qquad \text{for } f=(f_\alpha)_{|\alpha|=k}\in\mathcal{D}(\mathcal{O}_1, E_\sigma\otimes\boldsymbol{C}^l)$$

with $f_\alpha\in\mathcal{D}(\mathcal{O}_1, E_\sigma)$.

Lemma 2.6. *Let S be as above. Then S satisfies*

$$S^x(f)=S((\pi(x)\otimes{}^tA_k(x^{-1}, \cdot))f(\cdot))$$

for $x\in U_{F'}\times P_F$ *and* $f\in\mathcal{D}(\mathcal{O}_1, E_\sigma\otimes\boldsymbol{C}^l)$ *such that* $\mathrm{spt}(f)\subseteqq\mathcal{O}_1\cap x\cdot\mathcal{O}_1$

Proof. The proof is carried out just in the same way as in [16, 5.2.3]. To clarify the arguments succeeding to this lemma, we give the proof.

Let $x=(u, p)\in U_{F'}\times P_F$ and $h\in\mathcal{D}(\mathcal{O}, E_\sigma)$ such that $\mathrm{spt}(h)\subseteqq\mathcal{O}\cap x\cdot\mathcal{O}$. From the local expression (2.5) of T, one has

$$\begin{aligned}T^x(h)&=\textstyle\sum_\alpha(-1)^{|\alpha|}(X^\alpha T_\alpha)^x(h)=\sum_\alpha T_\alpha(X^\alpha(h^{x^{-1}}))\\ &=\textstyle\sum_\alpha T_\alpha(((\mathrm{Ad}(u)X^\alpha)h)^{x^{-1}}).\end{aligned}$$

Using the expansion (2.4), one has

$$\begin{aligned}T_\alpha(((\mathrm{Ad}(u)X^\alpha)h)^{x^{-1}})&=\textstyle\sum_{|\beta|=|\alpha|}(-1)^{|\beta|}X^\beta(a_{\beta\alpha}(x, \cdot)T_\alpha)^x(h)\\ &\quad+\textstyle\sum_{|\beta|<|\alpha|}(-1)^{|\beta|}(X^\beta T'_\beta)(h),\end{aligned}$$

where T'_β is an E_σ-distribution on $\mathcal{O}_1\cap x\cdot\mathcal{O}_1$ for each β such that $|\beta|<|\alpha|$. Hence

$$T^x(h)=\textstyle\sum_{|\beta|=k}(-1)^kX^\beta(\sum_{|\alpha|=k}a_{\beta\alpha}(x, \cdot)T_\alpha)^x(h)+\sum_{|\beta|<k}(-1)^{|\beta|}(X^\beta T''_\beta)(h),$$

where T''_β is an E_σ-distribution on $\mathcal{O}_1\cap x\cdot\mathcal{O}_1$ for each β such that $|\beta|<k$. On the other hand,

$$T^x(h)=\textstyle\sum_{|\beta|\leq k}(-1)^{|\beta|}X^\beta({}^t\pi(x)T_\beta)(h).$$

By the uniqueness of the local expression (2.5) of T, one has

$$T_\beta{}^x=\sum_{|\alpha|=k}a_{\beta\alpha}(x^{-1},\ \cdot)^t\pi(x)T_\alpha \qquad \text{on } \mathcal{O}_1\cap x\cdot\mathcal{O}_1$$

for each β such that $|\beta|=k$. This proves the lemma. Q. E. D.

2.3. Distributions in $\mathcal{T}_{\pi,s}$ (II), from S (*on* $\mathcal{O}_1\subseteq G_s$) to τ (on $\mathcal{R}\subseteq U(F')\cap sU_Fs^{-1}$). We shall associate to S an $(E_\sigma\otimes \boldsymbol{C}^l)$-distribution τ on an open subset $\mathcal{R}$ of $U(F')\cap sU_Fs^{-1}$ satisfying a certain condition with respect to an "action" of $U_{F'}\cap sP_Fs^{-1}$ (Proposition 2.10).

We proceed in more general situation. Let A and B be Lie groups so that B acts on A as a group of automorphisms of A: $B\times A\ni(y,\ x)\to x^y\in A$. For a closed subgroup H of A, put $M_2=B\times(A/H)$. Define an action of A on M_2 by

$$x\cdot(y,\ x_1H)=(y,\ x^{y^{-1}}x_1H) \qquad (x,\ x_1\in A,\ y\in B)\,.$$

For a Fréchet space E, let Φ be a differentiable E-multiplier on M_2 relative to A, i. e., Φ is a map from $A\times M_2$ to $\mathcal{L}(E)$, the space of continuous linear operators on E, such that

(1) $\Phi(e,\ z)=I$ (I the identity operator) for $z\in M_2$.
(2) $\Phi(xx',\ z)=\Phi(x',\ z)\Phi(x,\ x'\cdot z)$ $(x,\ x'\in A,\ z\in M_2)$.
(3) For every $\xi\in E$, $A\times M_2\ni(x,\ z)\to\Phi(x,\ z)\xi\in E$ is C^∞.
(4) The image of a compact subset of $A\times M_2$ by any fixed derivative of Φ is equicontinuous in $\mathcal{L}(E)$.

We consider a quasi-invariant E-distribution S on a neighbourhood $\mathcal{O}_1$ of a point $z_0\in M_2$ with multiplier Φ:

$$S^x(f)=S(\Phi(x^{-1},\ \cdot)f(\cdot)) \qquad \text{for } x\in A,\ f\in\mathcal{D}(\mathcal{O}_1,\ E) \tag{2.6}$$

such that $\operatorname{spt}(f)\subseteq\mathcal{O}_1\cap x\cdot\mathcal{O}_1$. Here S^x is defined as $S^x(f)=S(f^{x^{-1}})$ with $f^{x^{-1}}(z)=f(x\cdot z)$. Suppose $z_0\in\operatorname{spt}(S)$.

Remark 2.7. (1) Put $A=U_{F'}\times P_F$, $B=U(F')\cap sU_Fs^{-1}$, $H=\{(u,\ s^{-1}us);\ u\in U_{F'}\cap sP_Fs^{-1}\}$, $x^y=(yuy^{-1},\ p)$ for $x=(u,\ p)\in A$, $y\in B$. Then M_2 is identified with G_s in the canonical way. The action of A on M_2 conincides with that of $U_{F'}\times P_F$ on G_s.

(2) Set $E=E_\sigma\otimes\boldsymbol{C}^l$, $\Phi(x,\ z)=\pi(x^{-1})\otimes{}^tA_k(x,\ z)$ for $x\in A$, $z\in M_2$. Then S in Lemma 2.6 satisfies the above condition (2.6).

Put $M_1=B\times A$. Define an action of A on M_1 by

$$\bar{x}\cdot(y,\ x_1)=(y,\ xx_1) \qquad \text{for } x\in A,\ (y,\ x_1)\in M_1\,.$$

Let $j: M_1\to M_2$ be a submersion such that $j(y,\ x_1)=(y,\ x_1H)$ for $(y,\ x_1)\in M_1$. For $\alpha\in\mathcal{D}(M_1,\ E)$, put

$$f_\alpha(y,\ x_1H)=\int_H\alpha(y,\ x_1h)dh \qquad ((y,\ x_1H)\in M_2)\,,$$

where dh denotes a left Haar measure on H. As is well known, $\alpha\to f_\alpha$ gives an open continuous linear surjection from $\mathcal{D}(j^{-1}(\mathcal{O}_1), E)$ to $\mathcal{D}(\mathcal{O}_1, E)$. Moreover $\mathrm{spt}\,(f_\alpha)\subseteqq j(\mathrm{spt}\,\alpha)$ for all $\alpha\in\mathcal{D}(j^{-1}(\mathcal{O}_1), E)$.

Put $\mathcal{P}_1=\{(y, x)\in M_1;\ (y, x^{y^{-1}})\in j^{-1}(\mathcal{O}_1)\}$. In order to associate with S an A-invariant distribution on $\mathcal{P}_1$, we put

$$\alpha'(y, x)=\Phi((x^y)^{-1}, (y, xH))\alpha(y, x^y) \qquad \text{for } \alpha\in\mathcal{D}(\mathcal{P}_1, E).$$

Then the linear map $\alpha\to\alpha'$ gives a topological isomorphism from $\mathcal{D}(\mathcal{P}_1, E)$ to $\mathcal{D}(j^{-1}(\mathcal{O}_1), E)$. There exists a unique E-distribution S^* on $\mathcal{P}_1$ such that

$$S^*(\alpha)=S(g_\alpha) \qquad \text{with } g_\alpha=f_{\alpha'} \qquad \text{for } \alpha\in\mathcal{D}(\mathcal{P}_1, E).$$

We proceed as follows: S (on $\mathcal{O}_1$)$\to S^*$ (on $\mathcal{P}_1$)$\to\tau$ (on $\mathcal{R}$) (see infra).

Lemma 2.8. *The distribution S^* on $\mathcal{P}_1$ is invariant under A: $S^{*x'}(\alpha)=S^*(\alpha)$ for $x'\in A$ and $\alpha\in\mathcal{D}(\mathcal{P}_1, E)$ with* $\mathrm{spt}\,(\alpha)\subseteqq\mathcal{P}_1\cap\bar{x}'\cdot\mathcal{P}_1$.

Proof. Let x' and α be as above. For $\beta=\alpha^{x'^{-1}}$, one has

$$g_\beta(y, x_1H)=\int_H\Phi(((x_1h)^y)^{-1}, (y, x_1H))\alpha(y, (x'^{y^{-1}}x_1h)^y)dh\,.$$

On the other hand,

$$\begin{aligned}(g_\alpha)^{x'^{-1}}(y, x_1H)&=g_\alpha(y, x'^{y^{-1}}x_1H)\\&=\int_H\Phi(((x'^{y^{-1}}x_1h)^y)^{-1}, (y, x'^{y^{-1}}x_1H))\alpha(y, (x'^{y^{-1}}x_1h)^y)dh\\&=\Phi(x'^{-1}, x'\cdot(y, x_1H))\int_H\Phi(((x_1h)^y)^{-1}, (y, x_1H))\alpha(y, (x'^{y^{-1}}x_1h)^y)dh.\end{aligned}$$

Therefore one has $g_\beta=\Phi(x', \cdot)(g_\alpha)^{x'^{-1}}$. Consequently,

$$S^{*x'}(\alpha)=S(\Phi(x', \cdot)(g_\alpha)^{x'^{-1}})=S^{x'^{-1}}((g_\alpha)^{x'^{-1}})=S(g_\alpha)=S^*(\alpha)\,.$$

Q. E. D.

Write $z_0=j(y_0, x_0{}^{y_0})$ with $y_0\in B$, $x_0\in A$. Then $(y_0, x_0)\in\mathcal{P}_1$ and $(y_0, x_0)\in\in\mathrm{spt}\,(S^*)$. By replacing $\mathcal{P}_1$ with a smaller neighbourhood of (y_0, x_0) if necessary, we may assume that $\mathcal{P}_1=\mathcal{R}\times Q$ with an open neighbourhood $\mathcal{R}$ (resp. Q) of y_0 (resp. x_0) in B (resp. A). By Lemma 2.8 combined with the usual "pasting" arguments, S^* can be extended to a distribution on $\mathcal{R}\times A$ invariant under A. We denote it again by S^*.

For $\alpha\in\mathcal{D}(\mathcal{R}\times A, E)$, we put

$$\beta_\alpha(y)=\int_A\alpha(y, x)dx \qquad (y\in\mathcal{R}),$$

where dx is a left Haar measure on A. Then $\alpha\to\beta_\alpha$ gives a surjective linear map from $\mathcal{D}(\mathcal{R}\times A, E)$ to $\mathcal{D}(\mathcal{R}, E)$, and there exists a unique E-distribution τ on $\mathcal{R}$ such that

$$\tau(\beta_\alpha)=S^*(\alpha) \qquad \text{for } \alpha\in\mathcal{D}(\mathcal{R}\times A, E).$$

Obviously $y_0 \in \mathrm{spt}(\tau)$. Then τ satisfies the condition in the following proposition.

Proposition 2.9. *There exists an open neighbourhood $\mathcal{M}$ of the unit of H such that $\tau(\beta)=\tau({}^h\beta)$ for $h \in \mathcal{M}$ and $\beta \in \mathcal{D}(\mathcal{R}, E)$. Here we put*

$$({}^h\beta)(y)=\delta_A(h^{-1})\delta_A(h^y)\Phi(h^y, (y, H))\beta(y) \qquad (y \in \mathcal{R}),$$

with δ_A the modular function of A.

Proof. Let $\alpha \in \mathcal{D}(M_1, E)$. For $h' \in H$, we put $\alpha^{h'}*(y, x)=\alpha(y, xh')$. Let $\mathcal{M}$ be a symmetric open neighbourhood of the unit of H, and Q' an open neighbourhood of x_0 in A satisfying $Q'[\mathcal{R}, \mathcal{M}]^{-1} \subseteqq Q$ and $Q'\mathcal{M} \subseteqq Q$. Here we put $[y, h]=h^y h^{-1}$ for $h \in H$, $y \in B$.

Now let $\alpha \in \mathcal{D}(\mathcal{R} \times Q', E)$ and $h' \in \mathcal{M}$, then $\alpha^{h'}* \in \mathcal{D}(\mathcal{P}_1, E)$. Obviously one has $\beta_{\alpha^{h'}*}=\delta_A(h'^{-1})\beta_\alpha$. On the other hand,

$$g_{\alpha^{h'}*}(y, xH)=\int_H \Phi(((xh)^y)^{-1}, (y, xH))\alpha(y, (xhh')^y[y, h'^{-1}])dh$$

$$=\delta_A(h'^{-1})\int_H \Phi(((xh)^y)^{-1}, (y, xH))\Phi(h'^y, (y, H))$$

$$\cdot\alpha(y, (xh)^y[y, h'^{-1}])dh.$$

Therefore, if we define $\alpha_{h'} \in \mathcal{D}(M_1, E)$ by

$$\alpha_{h'}(y, x)=\Phi(h'^y, (y, H))\alpha(y, x[y, h'^{-1}]),$$

we thus obtain $g_{\alpha^{h'}*}=\delta_A(h'^{-1})g_{\alpha_{h'}}$. Note that $\mathrm{spt}(\alpha_{h'}) \subseteqq \mathcal{P}_1$ by the definition of $\mathcal{M}$. Clearly one has $\beta_{\alpha_{h'}}={}^{h'}(\beta_\alpha)$, and consequently

$$\tau(\beta_\alpha)=\delta_A(h')\tau(\beta_{\alpha^{h'}*})=S(g_{\alpha_{h'}})=\tau({}^{h'}(\beta_\alpha)).$$

Since the map $\mathcal{D}(\mathcal{P} \times Q', E) \ni \alpha \to \beta_\alpha \in \mathcal{D}(\mathcal{R}, E)$ is surjective, we complete the proof. Q. E. D.

We return to our original situation in 2.2. By Remark 2.7 and Proposition 2.9, we conclude

Proposition 2.10. *Let S be as in Lemma 2.6. Then there exists an $(E_\sigma \otimes \boldsymbol{C}^l)$-distribution τ on an open neighbourhood $\mathcal{R} \subseteqq U(F') \cap sU_F s^{-1}$ of y_0 satisfying the following conditions.*

(1) $\mathrm{spt}(\tau) \ni y_0$.

(2) *There exists a neighbourhood $\mathcal{M}$ of the unit of $U_{F'} \cap sP_F s^{-1}$ such that $\tau(\beta)=\tau({}^m\beta)$ for $m \in \mathcal{M}$ and $\beta \in \mathcal{D}(\mathcal{R}, E_\sigma \otimes \boldsymbol{C}^l)$. Here we put*

$$({}^m\beta)(y)=\{\pi(ymy^{-1}, s^{-1}ms)^{-1} \otimes {}^tA_k(ymy^{-1}, s^{-1}ms)\}\beta(y).$$

2.4. Supports of distributions in $\mathcal{T}_{\pi,s}$. Now we state the main result of this section which is crucial in the next section.

For an $s \in W$, put

$$D_{\eta}^{s}=\{y\in U(F')\cap sU_{F}s^{-1};\ \eta|_{U_{F'}\cap ysP_{F}(ys)^{-1}}\equiv 1\}.$$

Then $D_{\eta}^{s}U_{F'}sP_{F}$ is a closed subset of $G_{s}=UsP_{F}$, in general, of lower dimension.

Theorem 2.11. *Let the notations be as above. Assume that σ is a finite dimensional representation of M_{F}. For every $s\in W$, a distribution in $\mathcal{T}_{\pi,s}$ has always the support contained in $D_{\eta}^{s}U_{F'}sP_{F}$.*

Proof. Let $T\in\mathcal{T}_{\pi,s}$ and $z^{0}=y_{0}(x_{0}\cdot s)\in\mathrm{spt}(T)$ with $y_{0}\in U(F')\cap sU_{F}s^{-1}$ and $x_{0}\in U_{F'}\times P_{F}$. We use the notations in 2.3 in this situation. Assume that $y_{0}\notin D_{\eta}^{s}$. Then there exists an $m\in\mathcal{M}$ (Proposition 2.10) such that $\eta(y_{0}my_{0}^{-1})\neq 1$. By Proposition 2.10, we see

$$\tau_{y}(\{I-\eta(ym^{-1}y^{-1})(\sigma\otimes e^{\nu-\rho}\otimes 1)(s^{-1}m^{-1}s)\otimes{}^{t}A_{k}((ymy^{-1},\ s^{-1}ms),\ ys)\}\beta(y))=0$$

for all $\beta\in\mathcal{D}(\mathcal{R},\ E_{\sigma}\otimes\boldsymbol{C}^{l})$, where τ_{y} means that τ is applied on the function in y. On the other hand, the operator $(\sigma\otimes e^{\nu-\rho}\otimes 1)(s^{-1}m^{-1}s)$ is unipotent by the highest weight theory of irreducible finite dimensional representations of $([\mathfrak{m}_{F},\ \mathfrak{m}_{F}])_{C}$. By Lemma 2.5 (3), the matrices ${}^{t}A_{k}((ymy^{-1},\ s^{-1}ms),\ ys)$ are unipotent for all $y\in\mathcal{R}$. This implies that the operators

$$I-\eta(ym^{-1}y^{-1})(\sigma\otimes e^{\nu-\rho}\otimes 1)(s^{-1}m^{-1}s)\otimes{}^{t}A_{k}((ymy^{-1},\ s^{-1}ms),\ ys)$$

are invertible when y ranges a sufficiently small neighbourhood of y_{0} in $\mathcal{R}$. Hence $y_{0}\notin\mathrm{spt}(\tau)$. This is a contradiction. Q. E. D.

We apply our result to the case $F=F'=\emptyset$. A character η of U is called *non-degenerate* if the restriction of η to $U\cap s^{-1}Ns$ is non-trivial for every $s\in W\backslash\{e\}$. The following well-known theorem (e. g., [4]) is a direct consequence of Theorem 2.11.

Corollary 2.12. *Let η be a non-degenerate character of U. For a finite dimensional representation $(\sigma,\ E_{\sigma})$ of M and $\nu\in\mathfrak{a}_{C}^{*}$, one has*

$$\dim\mathrm{Wh}_{\eta}(H_{\sigma,\nu}^{\vee})\leqq\dim E_{\sigma}.$$

Proof. This corollary follows from Lemma 2.1, Proposition 2.3, Theorem 2.11 and the fact that $D_{\eta}^{s}=\emptyset$ if $s\neq e$. Q. E. D.

Note. After I had proved Theorem 2.11, I learned the following. For reductive groups over a non-archimedean local field, M. L. Karel treated in [7] Whittaker vectors in the similar situation as ours in 2.1 except that σ is supposed to be one dimensional in his case. And he obtained a uniqueness property ([7, Theorem 3.2]) of such Whittaker vectors when the character η is "generic". His method is based on the vanishing of certain integrals over compact subgroups of a unipotent subgroup. So it can not be extended immediately to archimedean cases. Nevertheless, our result (Theorem 2.11) shows, as its direct consequence, that a fact parallel to non-archimedean cases holds also in archimedean cases.

§ 3. Multiplicity theorem for generalized Gelfand-Graev representations

In this section, we define the genepalized Gelfand-Graev representations of semisimple Lie groups just as in [8]. Using the results in § 2, we prove a multiplicity theorem (Theorem 3.7) for some of such representations.

3.1. Let G be a connected real semisimple Lie group with finite center. Retain the notations in 1.2.

Let A be a non-zero nilpotent element of $\mathfrak{g}$. By Jacobson-Morozov, there exists an $\mathfrak{sl}_2$-triplet $\{A, H, B\} \subseteq \mathfrak{g}$ containing A:

$$[H, A]=2A, \qquad [H, B]=-2B, \qquad [A, B]=H.$$

We put

$$\mathfrak{g}(i)_A=\{X\in\mathfrak{g}\,;\,[H, X]=iX\},$$

$$\mathfrak{u}(i)_A=\oplus_{k\geq i}\mathfrak{g}(k)_A, \qquad \mathfrak{n}(i)_A=\oplus_{k\geq i}\mathfrak{g}(-k)_A$$

for each integer i. For $i\geq 1$, denote by $U(i)_A$ (resp. $N(i)_A$) the analytic subgroup of G corresponding to $\mathfrak{u}(i)_A$ (resp. $\mathfrak{n}(i)_A$). By the representation theory of $\mathfrak{sl}_2$, one has $\mathfrak{g}=\oplus_{i\in Z}\mathfrak{g}(i)_A$. If one notices that the spaces $\mathfrak{g}(i)_A$ and $\mathfrak{g}(j)_A$ are mutually orthogonal with respect to Q (Killing form) unless $i+j=0$, the dual space $\mathfrak{u}(1)_A{}^*$ of $\mathfrak{u}(1)_A$ is identified with $\mathfrak{n}(1)_A$ by $\mathfrak{n}(1)_A\ni Y\to Y^*\in\mathfrak{u}(1)_A{}^*$, $\langle Y^*, X\rangle=Q(X, Y)$ for $X\in\mathfrak{u}(1)_A$.

By taking a suitable $\mathrm{Ad}(G)$-conjugate of A instead of A, we may assume that $H\in\mathfrak{a}$ and $\langle\lambda, H\rangle\leq 0$ for all $\lambda\in\Lambda^+$. Put $F_A'=\{\lambda\in\Pi\,;\,\langle\lambda, H\rangle=0\}$, then $\mathfrak{l}_{F'_A}=\mathfrak{g}(0)_A$ and $\mathfrak{u}_{F'_A}=\mathfrak{u}(1)_A$.

Let $\mathfrak{u}(1.5)_A$ be a subalgebra of $\mathfrak{u}(1)_A$ satisfying the following conditions.

(1) $\mathfrak{u}(2)_A\subseteq\mathfrak{u}(1.5)_A\subseteq\mathfrak{u}(1)_A$,

(2) $\mathfrak{u}(1.5)_A$ is a subalgebra of $\mathfrak{u}(1)_A$ subordinate to $B^*\in\mathfrak{u}(1)_A{}^*$, i. e., $B^*([\mathfrak{u}(1.5)_A, \mathfrak{u}(1.5)_A])=(0)$, and of maximal dimension among such subalgebras.

Such a subalgebra $\mathfrak{u}(1.5)_A$ actually exists, because $\mathfrak{u}(2)_A$ is subordinate to B^*. Since the alternating bilinear form on $\mathfrak{g}(1)_A\times\mathfrak{g}(1)_A$ defined by $(X_1, X_2)\to B^*([X_1, X_2])$ is non-degenerate, $\mathfrak{u}(1.5)_A$ can be written as $\mathfrak{u}(1.5)_A=\mathfrak{u}(2)_A\oplus\mathfrak{u}'$ with a vector subspace $\mathfrak{u}'$ of $\mathfrak{g}(1)_A$ such that $2\dim\mathfrak{u}'=\dim\mathfrak{g}(1)_A$ and $B^*([\mathfrak{u}', \mathfrak{u}'])=\{0\}$. Let $U(1.5)_A$ be the analytic subgroup of G corresponding to $\mathfrak{u}(1.5)_A$. We define a unitary character η_A of $U(1.5)_A$ by

$$\eta_A(\exp X)=\exp\{\sqrt{-1}\langle B^*, X\rangle\} \qquad \text{for } X\in\mathfrak{u}(1.5)_A.$$

Definition 3.1. Let $A\in\mathfrak{g}$ be a non-zero nilpotent element. The smooth representation $(\pi_{\eta_A}, \mathcal{D}_{\eta_A}(G))$ induced from the character η_A of $U(1.5)_A$ is called a *generalized Gelfand-Graev representation* associated to A.

In general, $U(1.5)_A$ is not uniquely determined by A. Nevertheless, we call every representation defined as above a generalized Gelfand-Graev representation associated to A. We note the following: Let (ξ_A, E_A) be the smooth representation of $U(1)_A$ associated to the unitary representation of $U(1)_A$ unitarily induced from η_A. By Kirillov's theory on representations of nilpotent Lie groups, (ξ_A, E_A) is irreducible and independent (up to equivalence) of a choice of $U(1.5)_A$. Moreover $\mathcal{D}_{\eta_A}(G)$ is embedded continuously into $\mathcal{D}_{\xi_A}(G, E_A)$ as a G-module.

In case that A is a regular nilpotent element of $\mathfrak{g}$, $U(1)_A=U(2)_A=U$. And η_A gives a non-degenerate character of U if $\mathfrak{g}$ is, at least, quasi-split. In this case, the representation π_{η_A} has been called the *Gelfand-Graev representation*. This is the reason why π_{η_A} is called a *generalized* Gelfand-Graev representation for each nilpotent element A.

When G is a connected complex semisimple Lie group, we regard G as a real group in the following way: Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{a}$ the real form of $\mathfrak{h}$ consisting of elements on which every root with respect to $\mathfrak{h}$ takes a real value. Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}$ which contains $\sqrt{-1}\,\mathfrak{a}$ as a maximal abelian subalgebra. Denote by θ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{k}$. These notations $\mathfrak{k}$, $\mathfrak{a}$ and θ are compatible with those defined in 1.2. So we use the notations in 1.2.

3.2. We introduce some additional notations and make some preparations for the main result (Theorem 3.7) of this section.

For a real vector space $\mathfrak{v}$, $\mathfrak{v}_C$ denotes the complexification of $\mathfrak{v}$. Let $\mathfrak{h}^-$ be a maximal abelian subalgebra of $\mathfrak{m}$. Then $\mathfrak{h}=\mathfrak{h}^-\oplus\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Denote by $\tilde{\Lambda}$ the set of roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$. Choose a positive system $\tilde{\Lambda}^+$ of $\tilde{\Lambda}$ such that $\tilde{\Lambda}^+|_{\mathfrak{a}}\supseteqq\Lambda^+$. For $F\subseteqq\Pi$, let Λ_F be the set of non-zero (ad $\mathfrak{a}_F$)-weights in $\mathfrak{g}$. In general, the set Λ_F is not a root system for $\mathfrak{a}_F^*$, nevertheless, it is often called the set of $\mathfrak{a}_F$-roots by abuse of the lauguage. Put $\Lambda_F^+=\Lambda_F\cap(\Lambda^+|_{\mathfrak{a}_F})$, $\Pi_F=\Lambda_F\cap(\Pi|_{\mathfrak{a}_F})$. Let $r_F\colon \tilde{\Lambda}\to\Lambda_F\cup\{0\}$ be the map defined by the restriction. Put $\tilde{F}=r_F^{-1}(\{0\})\cap\tilde{\Pi}$, where $\tilde{\Pi}$ denotes the set of simple roots in $\tilde{\Lambda}^+$. Regarding $\tilde{\mathfrak{g}}=\mathfrak{g}_C$ as a real Lie algebra, we use the notations in 1.2 with the symbol "~" on the head. For example, $\tilde{\mathfrak{a}}=\sqrt{-1}\,\mathfrak{h}^-\oplus\mathfrak{a}$, $\tilde{\mathfrak{a}}(\tilde{F})=\sqrt{-1}\,\mathfrak{h}^-\oplus\mathfrak{a}(F)$, $\tilde{\mathfrak{n}}=\sum_{\alpha\in\tilde{\Lambda}^+}\tilde{\mathfrak{g}}_\alpha$.

Lemma 3.2. *Let $F'\subseteqq\Pi$. Then the nilpotent subalgebra $\mathfrak{u}_{F'}$ of $\mathfrak{g}$ has a structure given as $\mathfrak{u}_{F'}=\sum_{\lambda\in\Pi_{F'}}\mathfrak{g}_{-\lambda}\oplus[\mathfrak{u}_{F'}, \mathfrak{u}_{F'}]$, where $\mathfrak{g}_\mu$ ($\mu\in\Lambda_{F'}$) denotes the $\mathfrak{a}_{F'}$-root space corresponding to μ.*

Proof. It is sufficient to show that the complex Lie algebra $(\mathfrak{u}_{F'})_C$ is generated by $(\sum_{\lambda\in\Pi_{F'}}\mathfrak{g}_{-\lambda})_C$. We see easily

$$(\mathfrak{u}_{F'})_C=\textstyle\sum_{\alpha\in\langle\tilde{F}'\rangle'}\tilde{\mathfrak{g}}_{-\alpha}\,, \qquad (\sum_{\lambda\in\Pi_{F'}}\mathfrak{g}_{-\lambda})_C=\sum_\beta\tilde{\mathfrak{g}}_{-\beta}\,,$$

where the last sum $\sum_\beta$ runs through $\beta\in\langle\tilde{F}'\rangle'$ such that $\beta|_{\mathfrak{a}_{F'}}\in\Pi_{F'}$. Let $\mathfrak{u}'$ be the complex Lie subalgebra generated by $\sum_\beta\tilde{\mathfrak{g}}_{-\beta}$. Note that $\sum_\beta\tilde{\mathfrak{g}}_{-\beta}$ is $(\mathfrak{l}_{F'})_C$-

stable, whence so is $\mathfrak{u}'$. Put $\tilde{\Pi}=\{\alpha_1, \cdots, \alpha_q\}$. Assume that $\mathfrak{u}'\neq(\mathfrak{u}_{F'})_C$. Then there exists $\gamma=\sum_{1\leq i\leq q}n_i\alpha_i\in\langle\tilde{F}'\rangle'$ with $n_i\geqq 0$ such that $\tilde{\mathfrak{g}}_{-\gamma}\subsetneqq\mathfrak{u}'$. Take a such γ so that the sum $\sum n_i$ is minimal. As is well known, $\gamma=\gamma'+\gamma_j$ for some $\gamma'\in\tilde{\Lambda}^+$ and $\alpha_j\in\tilde{\Pi}$. Necessarily $\gamma'\in\langle\tilde{F}'\rangle'$. By the assumption for γ, one has $\tilde{\mathfrak{g}}_{-\gamma'}\subseteqq\mathfrak{u}'$. In case $\alpha_j\in\langle\tilde{F}'\rangle$, $\tilde{\mathfrak{g}}_{-\gamma}=[\tilde{\mathfrak{g}}_{-\gamma}, \tilde{\mathfrak{g}}_{-\alpha_j}]\subseteqq[\mathfrak{u}', (\mathfrak{l}_{F'})_C]\subseteqq\mathfrak{u}'$. In case $\alpha_j\in\langle\tilde{F}'\rangle'$, $\tilde{\mathfrak{g}}_{-\gamma}\subseteqq[\mathfrak{u}', \mathfrak{u}']\subseteqq\mathfrak{u}'$. This is a contradiction. Q. E. D.

Let η be a unitary character of $U_{F'}$. By the above lemma, there exists a unique element $B_\eta\in\sum_{\lambda\in\Pi_{F'}}\mathfrak{g}_\lambda$ such that $\eta(\exp X)=\exp\{\sqrt{-1}\,Q(B_\eta, X)\}$ for all $X\in\mathfrak{u}_{F'}$. The map $\eta\to B_\eta$ gives a bijective correspondence between the set of unitary characters of $U_{F'}$ and $\sum_{\lambda\in\Pi_{F'}}\mathfrak{g}_\lambda$.

Lemma 3.3. *Let $F\subseteqq\Pi$. For $s\in W$ and $y\in U(F')\cap sU_Fs^{-1}$, the restriction of η to $U_{F'}\cap ysP_F(ys)^{-1}$ is trivial if and only if* $\mathrm{Ad}(ys)^{-1}B_\eta\in\mathfrak{u}_F$.

Proof. We see easily that η restricted to $U_{F'}\cap ysP_F(ys)^{-1}$ is trivial if and only if $\mathrm{Ad}(y)^{-1}B_\eta\in\mathrm{Ad}(s)\mathfrak{p}_F{}^\perp$, since $\mathrm{Ad}(y)^{-1}B_\eta\in\mathfrak{u}_{F'}$ and $\mathfrak{u}_{F'}=\mathfrak{p}_{F'}{}^\perp$. This means that $\mathrm{Ad}(ys)^{-1}B_\eta\in\mathfrak{p}_F{}^\perp=\mathfrak{u}_F$. Q. E. D.

Until the end of this section, we assume that G is a connected real semisimple Lie group of matrices. Then G is contained in the complexified connected matrix group $\tilde{G}$ having $\tilde{\mathfrak{g}}$ $(=\mathfrak{g}_C)$ as Lie algebra. Also for $\tilde{G}$, we use the notations in 1.2 with the symbol "~" on the head. For a Lie subgroup L of G with Lie algebra $\mathfrak{l}$, denote by L_C the analytic subgroup of $\tilde{G}$ corresponding to $\mathfrak{l}_C$.

Proposition 3.4. *Let B be an element in $\mathfrak{u}_{F'}$ such that $[B, \mathfrak{p}_{F'}]=\mathfrak{u}_{F'}$ and $Z_{\tilde{G}}(B)\subseteqq(P_{F'})_C$. Then one has $\{g\in G\,;\ \mathrm{Ad}(g)B\in\mathfrak{u}_{F'}\}=P_{F'}$.*

Proof. First we note that $\mathfrak{z}_\mathfrak{g}(B)\subseteqq\mathfrak{p}_{F'}$. In fact, $Q(X, \mathfrak{u}_{F'})=Q(X, [B, \mathfrak{p}_{F'}])=Q([X, B], \mathfrak{p}_{F'})$ for $X\in\mathfrak{g}$. Hence $X\in\mathfrak{u}_{F'}{}^\perp=\mathfrak{p}_{F'}$ if $X\in\mathfrak{z}_\mathfrak{g}(B)$. Since $[B, \mathfrak{p}_{F'}]=\mathfrak{u}_{F'}$, $\mathcal{O}=\{Y\in(\mathfrak{u}_{F'})_C\,;\ [Y, (\mathfrak{p}_{F'})_C]=(\mathfrak{u}_{F'})_C\}$ is an open, dense and connected subset in $(\mathfrak{u}_{F'})_C$ containing B. Moreover $\mathcal{O}$ is a single $\mathrm{Ad}((P_{F'})_C)$-orbit in $(\mathfrak{u}_{F'})_C$.

Let $g\in G$ such that $X=\mathrm{Ad}(g)B\in\mathfrak{u}_{F'}$. Then

$$\begin{aligned}\dim[X, \mathfrak{p}_{F'}]&=\dim\mathfrak{p}_{F'}-\dim\mathfrak{z}_{\mathfrak{p}_{F'}}(X)\geqq\dim\mathfrak{p}_{F'}-\dim\mathfrak{z}_\mathfrak{g}(X)\\&=\dim\mathfrak{p}_{F'}-\dim\mathfrak{z}_\mathfrak{g}(B)=\dim\mathfrak{p}_{F'}-\dim\mathfrak{z}_{\mathfrak{p}_{F'}}(B)\\&=\dim[B, \mathfrak{p}_{F'}]=\dim\mathfrak{u}_{F'}.\end{aligned}$$

This means that $X\in\mathcal{O}$. Hence there exists $p\in(P_{F'})_C$ such that $\mathrm{Ad}(g)B=\mathrm{Ad}(p)B$. Thus $p^{-1}g\in Z_{\tilde{G}}(B)\subseteqq(P_{F'})_C$, $g\in(P_{F'})_C\cap G$. Since $(P_{F'})_C=N_{\tilde{G}}((\mathfrak{u}_{F'})_C)$, one has $(P_{F'})_C\cap G=N_G((\mathfrak{u}_{F'})_C)=N_G(\mathfrak{u}_{F'})=P_{F'}$. This completes the proof. Q.E.D.

In the next lemma, we give an example which satisfies the assumption in Proposition 3.4.

Lemma 3.5. *Keep to the notations in* 3.1. *Let A be a non-zero nilpotent element of $\mathfrak{g}$ such that $\mathfrak{u}(1.5)_A$ can be chosen as $\mathfrak{u}_{F''}$ for some $F''\subseteqq \Pi$. Then one has*

(1) $Z_{\tilde{G}}(B)\subseteqq(P_{F'_A})_C\subseteqq(P_{F''})_C$, (2) $[B, \mathfrak{p}_{F''}]=\mathfrak{u}_{F''}$.

Proof. (1) Let $g\in Z_{\tilde{G}}(B)$. We have two $\mathfrak{sl}_2$-triplets $\{A, H, B\}$ and $\{\mathrm{Ad}\,(g)A, \mathrm{Ad}\,(g)H, B\}$ in $\tilde{\mathfrak{g}}$ containing B. By Kostant ([9, Theorem 3.6]), there exists an element x in the unipotent radical of $Z_{\tilde{G}}(B)$ such that $\mathrm{Ad}\,(g)H=\mathrm{Ad}\,(x)H$. Then $x^{-1}g\in Z_{\tilde{G}}(H)\subseteqq N_{\tilde{G}}((\mathfrak{u}_{F'_A})_C)=(P_{F'_A})_C$. Since the unipotent radical of $Z_{\tilde{G}}(B)$ is contained in $(P_{F'_A})_C$ by the representation theory of $\mathfrak{sl}_2$, one has $g\in(P_{F'_A})_C$.

(2) Note that $\mathfrak{p}_{F''}=\mathfrak{p}_{F'_A}\oplus\mathfrak{u}''$ with a subspace $\mathfrak{u}''\subseteqq\mathfrak{g}(1)_A$ such that $2\dim\mathfrak{u}''=\dim\mathfrak{g}(1)_A$. Then one has

$$\mathfrak{u}_{F''}\supseteqq[B, \mathfrak{p}_{F''}]=\mathfrak{u}(2)_A\oplus[B, \mathfrak{u}''] .$$

Since $\mathfrak{z}_{\mathfrak{g}}(B)\subseteqq\mathfrak{g}_{F'_A}$, one has $2\dim[B, \mathfrak{u}'']=\dim\mathfrak{g}(1)_A$. Hence $\dim\mathfrak{u}_{F''}=\dim[B, \mathfrak{p}_{F''}]$, $\mathfrak{u}_{F''}=[B, \mathfrak{p}_{F''}]$. Q. E. D.

Remark 3.6. $U(1.5)_A$ can be chosen as $U_{F''}$ for some $F''\subseteqq\Pi$ if and only if there exists an abelian subspace $\mathfrak{u}''$ of $\mathfrak{g}(1)_A$ such that

(1) $[\mathfrak{g}(0)_A, \mathfrak{u}'']\subseteqq\mathfrak{u}''$ and (2) $2\dim\mathfrak{u}''=\dim\mathfrak{g}(1)_A$.

This is a sufficient condition for the existence of a weak polarization of the nilpotent element B ([11, Proposition 5.2]). In case $G=SL(n, \boldsymbol{C})$, such a $\mathfrak{u}''$ always exists (see 3.4).

3.3. A multiplicity theorem. We state the main theorem of this section.

Theorem 3.7. *Let G be a connected real semisimple Lie group of matrices. Let A be a non-zero nilpotent element of $\mathfrak{g}$ such that $U(1.5)_A$ can be chosen as $U_{F''}$ for some $F''\subseteqq\Pi$. We consider a generalized Gelfand-Graev representation $(\pi_{\eta_A}, \mathcal{D}_{\eta_A}(G))$ associated to A. Let $F\subseteqq\Pi$. For a finite dimensional representation (σ, E_σ) of M_F and $\nu\in(\mathfrak{a}_F)_C^*$, one has*

(1) $\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_A}^\vee)=\{0\}$ *if* $\mathrm{Ad}\,(ys)^{-1}B\notin\mathfrak{u}_F$ *for all* $s\in W$ *and* $y\in U(F'')\cap sU_Fs^{-1}$.

(2) $\dim\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_A}^\vee)\begin{cases}\leqq\dim E_\sigma & \text{if } F=F'' ,\\ =0 & \text{if } F\supsetneqq F'' .\end{cases}$

Proof. We adapt the results in § 2 putting $F'=F''$ and $\eta=\eta_A$. By Theorem 2.11 combined with Lemma 3.3, the assertion (1) is clear. Now we assume that $F\supseteqq F''$. Let $s\in W$. By Lemma 3.3, we have

$$\begin{aligned} D_{\eta_A}^s&=\{y\in U(F'')\cap sU_Fs^{-1}\,;\ \mathrm{Ad}\,(ys)^{-1}B\in\mathfrak{u}_F\}\\ &\subseteqq\{y\in U(F'')\cap sU_Fs^{-1}\,;\ \mathrm{Ad}\,(ys)^{-1}B\in\mathfrak{u}_{F''}\}.\end{aligned}$$

By Proposition 3.4 and Lemma 3.5, we have $D_{\eta_A}^s=\emptyset$ unless $s\in W_{F''}$. Hence $\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_A}^\vee)=\{0\}$ if $F\supsetneqq F''$. In case $F=F''$, we have $D_{\eta_A}^s=\{e\}$ for $s\in W_F$. If one notices that $\Omega_s=G_s=U_FP_F$ for $s\in W_F$ and that U_FP_F is diffeomorphic to the product $U_F \times P_F$ in the canonical way, one has

$$\dim \operatorname{Hom}_G(\pi_{\sigma,\nu}, \pi^{\vee}_{\eta_A}) \leqq \dim \mathcal{T}_{\pi,e} = \dim E_\sigma . \qquad \text{Q. E. D.}$$

Remark 3.8. The cases (1) and (2) in the above theorem do not exhaust all the cases for F'' and F. For example, $\dim \operatorname{Hom}_G(\pi_{\sigma,\nu}, \pi^{\vee}_{\eta_A})$ can be equal to infinity in case $F \subsetneqq F''$ (see § 4).

3.4. Case of $G=SL(n, \boldsymbol{C})$. As an example, we give $(U(1.5)_A, \eta_A)$ explicitly in case $G=SL(n, \boldsymbol{C})$. Put $\mathfrak{h}=\{h=\operatorname{diag}(h_1, \cdots, h_n); \sum_{1\leq i\leq n} h_i=0, h_i \in \boldsymbol{C}\}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The real form $\mathfrak{a}$ of $\mathfrak{h}$ is given by $\mathfrak{a}=\{h=\operatorname{diag}(h_1, \cdots, h_n) \in \mathfrak{h}; h_i \in \boldsymbol{R}\}$. As a compact real form of $\mathfrak{g}$ we take $\mathfrak{k}=\mathfrak{u}(n)$, the Lie algebra of skew-Hermitian matrices of degree n. If we define $e_i \in \mathfrak{a}^*$ by $\langle e_i, h\rangle = h_i$ for $h=\operatorname{diag}(h_1, \cdots, h_n) \in \mathfrak{a}$, the root system Λ of $(\mathfrak{g}, \mathfrak{a})$ is given by $\Lambda=\{e_i-e_j; 1\leqq i, j \leqq n, i\neq j\}$. Choose a set of positive roots as $\Lambda^+=\{e_i-e_j; i>j\}$. Then $\Pi=\{\lambda_i=e_{i+1}-e_i; 1\leqq i \leqq n-1\}$ is the set of simple roots in Λ^+. The Weyl group W is identified with the symmetric group $\mathfrak{S}_n$ of degree n which acts on $\mathfrak{a}$ by permutation of diagonal entries.

We introduce a set of partitions of n,

$$P_n=\left\{\gamma=(n_1, n_2, \cdots, n_s); \begin{array}{l} n_1 \geqq n_2 \geqq \cdots \geqq n_s \geqq 1 \quad (n_i \in \boldsymbol{Z}) \\ \sum_{1\leq i\leq s} n_i = n \end{array}\right\}.$$

Associating its Jordan type to each nilpotent element of $\mathfrak{g}$, we can parametrize by P_n the nilpotent $\operatorname{Ad}(G)$-orbits in $\mathfrak{g}$.

For a positive integer m, we define three matrices of degree m as follows:

$$A_m=\begin{bmatrix} 0 & 1 & & & & 0 \\ 0 & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 1 \\ 0 & & & & 0 & 0 \end{bmatrix}, \qquad H_m=\begin{bmatrix} m-1 & & & & 0 \\ & m-3 & & & \\ & & \ddots & & \\ & & & 3-m & \\ 0 & & & & 1-m \end{bmatrix},$$

$$B_m=\begin{bmatrix} 0 & 0 & & & & 0 \\ b_1 & 0 & & & & \\ & b_2 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & b_{m-2} & 0 & 0 \\ 0 & & & & b_{m-1} & 0 \end{bmatrix} \quad \text{with } b_i=i(m-i).$$

Let $\gamma=(n_1, \cdots, n_s)\in P_n$. Arranging numbers in γ, we obtain $\gamma_1=(m_1, m_2, \cdots, m_r; k_1, k_2, \cdots, k_q)$ such that

$$m_1\geqq m_2\geqq \cdots \geqq m_r\geqq 2 \qquad \text{with } m_i \text{ even for } 1\leqq i\leqq r\,,$$

$$1\leqq k_1\leqq k_2\leqq \cdots \leqq k_q \qquad \text{with } k_j \text{ odd for } 1\leqq j\leqq q\,,$$

and $k+m=n$, $m=\sum_{1\leq i\leq r}m_i$, $k=\sum_{1\leq j\leq q}k_j$. As a representative of the nilpotent $\mathrm{Ad}(G)$-orbit corresponding to γ, take

$$A_\gamma^0=A_{m_1}\oplus A_{m_2}\oplus \cdots \oplus A_{m_r}\oplus A_{k_1}\oplus A_{k_2}\oplus \cdots \oplus A_{k_q}\,.$$

If we put

$$H_\gamma^0=H_{m_1}\oplus H_{m_2}\oplus \cdots \oplus H_{m_r}\oplus H_{k_1}\oplus H_{k_2}\oplus \cdots \oplus H_{k_q}\,,$$

$$B_\gamma^0=B_{m_1}\oplus B_{m_2}\oplus \cdots \oplus B_{m_r}\oplus B_{k_1}\oplus B_{k_2}\oplus \cdots \oplus B_{k_q}\,,$$

$\{A_\gamma^0, H_\gamma^0, B_\gamma^0\}$ is an $\mathfrak{sl}_2$-triplet containing A_γ^0. For a positive integer i, we denote by t_i the multiplicity of i in γ. For an integer j, let l_j be the number of diagonal entries in H_γ^0 equal to j. Then one has $l_i=l_{-i}=t_{i+1}+t_{i+3}+\cdots$ for $i\geqq 0$. Put $\tilde{l}_i=l_i+l_{i-1}$ for $i\geqq 1$. We define $H_\gamma\in\mathfrak{a}$ by

$$H_\gamma=((n_1-1)\times J_{n_1-1})\oplus \cdots \oplus(j\times J_j)\oplus \cdots \oplus((-n_1+1)\times J_{-n_1+1})\,,$$

where $J_j=I_{l_j}$ denotes the identity matrix of degree l_j. Then H_γ is in the closure of the negative Weyl chamber of $\mathfrak{a}$. There exists a unique $w_\gamma\in W$ which satisfies the following conditions (1) and (2).

(1) $w_\gamma H_\gamma^0=H_\gamma$.

(2) Let $H_\gamma^0=\mathrm{diag}(h_1, \cdots, h_n)$, then $w_\gamma(i)>w_\gamma(j)$ for each pair (i, j) such that $i>j$ and $h_i=h_j$.

Taking a representative of w_γ in $N_K(\mathfrak{a})$, we denote it again by w_γ. Put $A_\gamma=\mathrm{Ad}(w_\gamma)A_\gamma^0$ and $B_\gamma=\mathrm{Ad}(w_\gamma)B_\gamma^0$. We consider the $\mathfrak{sl}_2$-triplet $\{A_\gamma, H_\gamma, B_\gamma\}$ conjugate to $\{A_\gamma^0, H_\gamma^0, B_\gamma^0\}$. Then it is easy to see that $U(1.5)_A$ can be chosen as $U(1.5)_A=U_\gamma$ with

$$U_\gamma=\left\{\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & & \\ \ddots & L_3 & * & * & * & \\ \ddots & 0 & L_1 & * & * & \ddots \\ & 0 & 0 & L_2 & * & \ddots \\ & 0 & 0 & 0 & L_4 & \ddots \\ & & & \ddots & \ddots & \ddots \end{bmatrix} ;\ L_j=I_{\tilde{l}_j}\right\}.$$

The group U_γ is the unipotent radical of a parabolic subgroup of G.

Remark 3.9. The elements A_γ^0 and w_γ above play an important role when we consider analytic continuation of Whittaker integrals later in § 5.

Now we shall write down the condition in Theorem 3.7 (1) for the non-existence of intertwining operators explicitly in case $G=SL(n, \boldsymbol{C})$.

Let $\tilde{P}_n \supseteqq P_n$ be as

$$\tilde{P}_n=\{\beta=(l_1, l_2, \cdots, l_t);\ \textstyle\sum_{1\leq i\leq t} l_i=n,\ l_i\in \boldsymbol{N}\}.$$

For $\beta=(l_1, \cdots, l_t)\in\tilde{P}_n$, we put $F(\beta)=\Pi\backslash\{\lambda_{l_1}, \lambda_{l_1+l_2}, \cdots, \lambda_{l_1+l_2+\cdots+l_{t-1}}\}$. The map $\beta\to F(\beta)$ gives a bijective correspondence between $\tilde{P}_n$ and the set of all subset of Π.

Let $\gamma=(n_1, n_2, \cdots, n_s)\in P_n$ and $\beta=(l_1, l_2, \cdots, l_t)\in\tilde{P}_n$. For a positive integer j, denote by y_j the multiplicity of j in β, and put $k_j=y_j+y_{j+1}+\cdots\ (j\geqq 1)$, $n_{s+1}=n_{s+2}=\cdots=0$.

Proposition 3.10. *Let γ and β be as above. Then the condition* $\mathrm{Ad}\,(ys)^{-1}B_\gamma \in\mathfrak{n}_{F(\beta)}$ *for some $s\in W$ and some $y\in U(F(\tilde{\gamma}))\cap sU_{F(\beta)}s^{-1}$ is equivalent to*

$$n_1+\cdots+n_j\leqq k_1+\cdots+k_j \qquad \textit{for all } j\geqq 1\,.$$

Here we put $\tilde{\gamma}=(\cdots, \tilde{l}_3, \tilde{l}_1, \tilde{l}_2, \tilde{l}_4, \cdots)\in\tilde{P}_n$.

Proof. Let $s\in W$ and $y\in U(F(\tilde{\gamma}))\cap sU_{F(\beta)}s^{-1}$. First we show that $\mathrm{Ad}\,(s)^{-1}B_\gamma \in\mathfrak{n}_{F(\beta)}$ if $\mathrm{Ad}\,(ys)^{-1}B_\gamma\in\mathfrak{n}_{F(\beta)}$. Indeed, as in the proof of Lemma 3.3, one has $\mathrm{Ad}\,(ys)^{-1}B_\gamma\in\mathfrak{n}_{F(\beta)}$ if and only if $B_\gamma\in\{\mathrm{Ad}\,(y)(\mathfrak{u}_{F(\tilde{\gamma})}\cap\mathrm{Ad}\,(s)\mathfrak{p}_{F(\beta)})\}^\perp$. Now we assume that $B_\gamma\notin(\mathfrak{u}_{F(\tilde{\gamma})}\cap\mathrm{Ad}\,(s)\mathfrak{p}_{F(\beta)})^\perp$. Let μ be a positive root such that $\mathfrak{g}_{-\mu}\subseteqq \mathfrak{u}_{F(\tilde{\gamma})}\cap\mathrm{Ad}\,(s)\mathfrak{p}_{F(\beta)}$ and $B_\gamma\notin(\mathfrak{g}_{-\mu})^\perp$. Let X be a non-zero element in $\mathfrak{g}_{-\mu}$ and $y=\exp Y$ with $Y\in\mathfrak{u}(F(\tilde{\gamma}))\cap\mathrm{Ad}\,(s)\mathfrak{u}_{F(\beta)}$. Then one has

$$Q(B_\gamma, \mathrm{Ad}\,(y)X)=Q(B_\gamma, X)+\sum_{1\leq j<+\infty}\frac{1}{j\,!}Q(B_\gamma, (\mathrm{ad}\,Y)^j X)\,.$$

The condition $Q(B_\gamma, \mathfrak{g}_{-\mu})\neq\{0\}$ implies that $Q(B_\gamma, \mathfrak{g}_{-\mu-\lambda})=\{0\}$ for all $\lambda\in\Delta^+$, because $\mathrm{Ad}\,(w_\gamma)^{-1}B_\gamma=B_\gamma^0\in\sum_{\lambda\in\Pi}\mathfrak{g}_\lambda$. Noting that $(\mathrm{ad}\,Y)^j X\in\sum_{\lambda\in\Delta^+}\mathfrak{g}_{-\mu-\lambda}$ for $j\geqq 1$, we have $Q(B_\gamma, \mathrm{Ad}\,(y)X)\neq 0$. This means that $\mathrm{Ad}\,(ys)^{-1}B_\gamma\notin\mathfrak{n}_{F(\beta)}$.

By the above, we showed that the former condition in the statement of proposition is equivalent to "$\mathrm{Ad}\,(s)B_\gamma^0\in\mathfrak{n}_{F(\beta)}$ for some $s\in W$". In turn, this condition is equivalent to the latter one in the statement of proposition. Q.E.D.

Remark 3.11. By [5, Lemma 3.2], the condition in Proposition 3.10 is equivalent to $\mathrm{Ad}\,(G)B_\gamma\cap\mathfrak{n}_{F(\beta)}\neq\emptyset$. This is the condition that the nilpotent $\mathrm{Ad}\,(G)$-orbit corresponding to γ is in the closure of that corresponding to the dual partition ${}^t\beta=(k_1, k_2, \cdots)$ of β.

Part II. Construction of Whittaker vectors

§ 4. Whittaker integrals and intertwining operators

Let G be a connected real semisimple Lie group with finite center. In this section, we introduce integral operators, so called Whittaker integrals, which give Whittaker vectors of the principal series representations.

4.1. In this subsection, we consider the principal series representations of G induced from the minimal parabolic subgroup P. For an $s\in W$, denote by $\langle\!\langle s\rangle\!\rangle$ the set of positive roots λ such that $2^{-1}\lambda\notin\Delta^+$ and $s\lambda\in-\Delta^+$. Put $\mathfrak{u}_s=\mathfrak{n}\cap\mathrm{Ad}(s)^{-1}\mathfrak{n}$ and $U_s=\exp\mathfrak{u}_s$. Then $\mathfrak{u}_s=\sum_{\lambda\in\langle\!\langle s\rangle\!\rangle}(\mathfrak{g}_{-\lambda}\oplus\mathfrak{g}_{-2\lambda})$. The set $\{U_s;\ s\in W\}$ includes the set $\{U_{F'};\ F'\subseteq\Pi\}$.

Let $s\in W$ and η be a unitary character of U_s. Let (σ_0, E_{σ_0}) be a finite dimensional irreducible representation of M, and $\nu_0\in\mathfrak{a}_C^*$. For $e^\vee\in E_{\sigma_0}^\vee$, we put

$$(4.1)\qquad (W^{e^\vee}(\sigma_0,\nu_0,\eta)f)(x)=\int_{U_s}\langle e^\vee, f(xu)\rangle\eta(u)^{-1}du\qquad (f\in H_{\sigma_0,\nu_0},\ x\in G),$$

where du denotes a Haar measure on U_s. This integral is called the *Whittaker integral* of $f\in H_{\sigma_0,\nu_0}$ of type (U_s,η). Define for $s\in W$ an open convex tubular domain D_s in $\mathfrak{a}_C^*$ by

$$D_s=\{\nu\in\mathfrak{a}_C^*;\ \langle\mathrm{Re}\,\nu,\lambda\rangle>0\qquad\text{for all }\lambda\in\langle\!\langle s\rangle\!\rangle\},$$

where $\langle\ ,\ \rangle$ denotes the inner product on $\mathfrak{a}^*$ defined through the Killing form. For $g\in G$, write $g=k(g)a(g)n(g)$ with $k(g)\in K$, $a(g)\in A$, $n(g)\in N$. The following lemma is well known.

Lemma 4.1 ([15, Theorem 8.10.16]). *For every $s\in W$, the integral $\int_{U_s}a(u)^{-\nu-\rho}du$ is absolutely convergent for $\nu\in D_s$.*

The following proposition is a slight generalization of Proposition 2.4 in [4]. One can prove it just in the same way as there with the aid of Lemma 4.1. So we omit the proof.

Proposition 4.2. *Let $\nu_0\in D_s$. The integral* (4.1) *is absolutely convergent for all $f\in H_{\sigma_0,\nu_0}$ and $e^\vee\in E_{\sigma_0}^\vee$. Moreover $W^{e^\vee}(\sigma_0,\nu_0,\eta)f(x)$ is a smooth function of $x\in G$ and holomorphic with respect to $\nu_0\in D_s$. The map $f\to W^{e^\vee}(\sigma_0,\nu_0,\eta)f$ gives a non-zero interwining operator from H_{σ_0,ν_0} to $C_\eta^\infty(G)$ for $e^\vee\neq 0$.*

Consider the case $U_s=U_{F'}$ for some non-empty subset F' of Π. Let $\nu_0\in D_s$. For $y\in U(F')$, the operator $R_y\cdot W^{e^\vee}(\sigma_0,\nu_0,\eta^{y^{-1}})$ with $\eta^{y^{-1}}(u)=\eta(yuy^{-1})$ also gives an intertwining operator from H_{σ_0,ν_0} to $C_\eta^\infty(G)$, where R_y denotes the right translation by y of functions on G. Let $T_y\in\mathcal{T}_\pi(G)$ be the E_{σ_0}-distribution corresponding to $R_y\cdot W^{e^\vee}(\sigma_0,\nu_0,\eta^{y^{-1}})$ by Lemmas 1.2 and 2.2. Then T_y is given by

$$(4.2)\qquad \langle T_y,\ \phi\rangle=\int_{U_{F'}\times M\times A\times N} a^{\nu+\rho}\langle e^{\vee},\ \sigma_0(m)\phi(uyman)\rangle\eta(u)^{-1}dudmdadn$$
$$(\phi\in\mathcal{D}(G,\ E_{\sigma_0}))\,,$$

where dm, da and dn denotes a Haar measure on M, A and N respectively. (4.2) implies that the restriction of T_y to UP is non-zero for $e^{\vee}\neq 0$, and $\operatorname{spt}(T_y|_{UP})\subseteq U_{F'}yP$. Hence $\{T_y\,;\ y\in U(F')\}$ is a linearly independent subset of $\mathcal{T}_\pi(G)$. This means that

$$\dim \operatorname{Hom}_G(\pi_{\sigma_0,\nu_0},\ \pi_\eta^{\vee})=+\infty \qquad \text{for } \nu_0\in D_s\,.$$

4.2. Now we consider the principal series representations induced from the parabolic subgroup P_F. Let s and η be as in 4.1. Let σ be an irreducible admissible smooth representation of M_F on a Fréchet space E_σ. A representation (σ, E_σ) of M_F is called *admissible* if every irreducible finite dimensional representation of $K(F)$ occurs in E_σ with finite multiplicity. We are very interested in constructing Whittaker vectors in $\mathrm{Wh}_\eta(H_{\sigma,\nu}^{\vee})$ mainly in cases that $\dim \mathrm{Wh}_\eta(H_{\sigma,\nu}^{\vee})<+\infty$, for instance in the case of Theorem 3.7 (2). By Caselman's subrepresentation theorem (e.g., [1]), some of Whittaker vectors will be obtained by the composition of Whittaker integrals in 4.1 and the embeddings, of $H_{\sigma,\nu}$ into the principal series representations induced from P.

We explain this in more details. Put $(E_\sigma^{\vee})^{N(F)}=\{e^{\vee}\in E_\sigma^{\vee}\,;\ \sigma^{\vee}(n)e^{\vee}=e^{\vee}$ for all $n\in N(F)\}$. Then $(E_\sigma^{\vee})^{N(F)}$ is a finite dimensional $MA(F)N(F)$-module in the natural action. Indeed, it is clear that $(E_\sigma^{\vee})^{N(F)}$ is $MA(F)N(F)$-stable. By restriction of elements of $(E_\sigma^{\vee})^{N(F)}$ to the $((\mathfrak{m}_F)_C,\ K(F))$-module $(E_\sigma)_{K(F)}$ of all $K(F)$-finite vectors in E_σ, one has an $(\mathfrak{m}\oplus\mathfrak{a}(F)\oplus\mathfrak{n}(F))$-module embedding

$$(4.3)\qquad (E_\sigma^{\vee})^{N(F)}\hookrightarrow ((E_\sigma)_{K(F)}/\sigma(\mathfrak{n}(F)_C)(E_\sigma)_{K(F)})^{\vee}\,.$$

Since $(E_\sigma)_{K(F)}$ is finitely generated as $U(\mathfrak{n}(F)_C)$-module, $\sigma(\mathfrak{n}(F)_C)(E_\sigma)_{K(F)}$ is codimension finite in $(E_\sigma)_{K(F)}$. Hence $(E_\sigma^{\vee})^{N(F)}$ is finite dimensional and (4.3) gives an $MA(F)N(F)$-module embedding. By Casselman, $(E_\sigma)_{K(F)}\neq\sigma(\mathfrak{n}(F)_C)(E_\sigma)_{K(F)}$. This means that there exists a non-zero vector e' in the algebraic dual of $(E_\sigma)_{K(F)}$ such that $\langle e',\ \sigma(\mathfrak{n}(F)_C)(E_\sigma)_{K(F)}\rangle=\{0\}$. Suggested by this fact, we assume here that $(E_\sigma^{\vee})^{N(F)}\neq\{0\}$.

Let V be a finite dimensional irreducible $MA(F)N(F)$-submodule of $(E_\sigma^{\vee})^{N(F)}$. As an $MA(F)N(F)$-module, $V\simeq\sigma_0\otimes\exp\{\nu_0+(\rho|_{\mathfrak{a}(F)})\}\otimes 1$ for some irreducible finite dimensional representation σ_0 of M and $\nu_0\in\mathfrak{a}(F)_C{}^*$. Let $(\pi^F_{\sigma_0,\nu_0},\ H^F_{\sigma_0,\nu_0})$ be the representation of M_F induced smoothly from the representation $\sigma_0\otimes e^{\nu_0}\otimes 1$ of $MA(F)N(F)$. Then the map $\iota: E_\sigma\to C^\infty(M_F,\ V)$ defined by

$$\langle(\iota\xi)(x),\ v^{\vee}\rangle=\langle v^{\vee},\ \sigma(x^{-1})\xi\rangle \qquad (\xi\in E_\sigma,\ x\in G,\ v^{\vee}\in V^{\vee})$$

gives a continuous embedding of M_F-module E_σ into $H^F_{\sigma_0,\nu_0}$. Here we identify V with $(V^{\vee})^{\vee}$ in the canonical way.

Remark 4.3. If σ is finite dimensional, $(E_\sigma^{\vee})^{N(F)}$ is an irreducible

$MA(F)N(F)$-module, and naturally non-trivial by the highest weight theory of irreducible finite dimensional representations of $(\mathfrak{m}_F)_C$ ([15, Lemma 8.5.3]).

Let $\nu\in(\mathfrak{a}_F)_C{}^*$. The above ι induces a continuous embedding δ of G-module $H_{\sigma,\nu}$ into $H_{\sigma_0,\nu+\nu_0}$ by $(\delta F)(x)=\iota(F(x))(e)$ $(x\in G)$ for $F\in H_{\sigma,\nu}$. Here we regard $(\mathfrak{a}_F)_C{}^*\subseteqq\mathfrak{a}_C{}^*$ (resp. $\mathfrak{a}(F)_C{}^*\subseteqq\mathfrak{a}_C{}^*$) by $\langle(\mathfrak{a}_F)_C{}^*, \mathfrak{a}(F)\rangle=\{0\}$ (resp. $\langle\mathfrak{a}(F)_C{}^*, \mathfrak{a}_F\rangle=\{0\}$).

For $v^{\vee}\in V^{\vee}$, we define Whittaker integral just as in (4.1)

$$(4.4)\qquad (W^{v^{\vee}}(\sigma, \nu, \eta)F)(x)=\int_{U_s}\langle v^{\vee}, F(xu)\rangle\eta(n)^{-1}du \qquad (F\in H_{\sigma,\nu},\ x\in G).$$

Then we have the following proposition.

Proposition 4.4. *Let the notations and assumptions be as above. Assume that* $U_F\supseteqq U_s$. *Then one has*

(1) *the set* $(D_s-\nu_0)\cap(\mathfrak{a}_F)_C{}^*=\{\nu\in(\mathfrak{a}_F)_C{}^*\,;\,\nu+\nu_0\in D_s\}$ *is a non-empty open convex domain in* $(\mathfrak{a}_F)_C{}^*$.

(2) *The integral* (4.4) *is absolutely convergent for* $\nu\in(D_s-\nu_0)\cap(\mathfrak{a}_F)_C{}^*$. *Moreover* $W^{v^{\vee}}(\sigma, \nu, \eta)$ *gives a non-zero intertwing operator from* $H_{\sigma,\nu}$ *to* $C_\eta^\infty(G)$ *if* $v^{\vee}\neq 0$.

(3) *For* $F\in H_{\sigma,\nu}$ *and* $x\in G$, *the function* $\nu\rightarrow W^{v^{\vee}}(\sigma, \nu, \eta)F(x)$ *is a holomorphic function of* ν *in* $(D_s-\nu_0)\cap(\mathfrak{a}_F)_C{}^*$.

Proof. We easily have that $W^{v^{\vee}}(\sigma, \nu, \eta)=W^{v^{\vee}}(\sigma_0, \nu+\nu_0, \eta)\circ\delta$. Then this proposition follows from Proposition 4.2. Q. E. D.

§ 5. Analytic continuation of Whittaker integrals

In this section, let G be a connected complex semisimple Lie group. We deal with the analytic continuation of Whittaker integrals for generalized Gelfand-Graev representations.

5.1. A method of analytic continuation. First we present a method that gives a domain of $\mathfrak{a}_C{}^*$ to which Whittaker integral (4.1) can be extended meromorphically.

We regard G as a real group as in 3.1. For an $s\in W$, let η be a unitary character of U_s. Consider the Whittaker integral

$$W(\sigma, \nu, \eta)f(g)=\int_{U_s}f(gu)\eta(u)^{-1}du \qquad (\sigma\in\hat{M},\ \nu\in\mathfrak{a}_C{}^*,\ f\in H_{\sigma,\nu},\ g\in G),$$

where $\hat{M}$ denotes the set of equivalence classes of irreducible unitary representations of M. Note that M is a torus. So every irreducible unitary representation of M is a character. We put $A_s(\sigma, \nu)f(g)=W(\sigma, \nu, 1_{U_s})f(gs)$ for $s\in W$, taking a representative (denoted again by s) in $N_K(\mathfrak{a})$. Then $A_s(\sigma, \nu)$ gives an intertwining operator from $H_{\sigma,\nu}$ to $H_{s\sigma,s\nu}$ by meromorphic continuation.

Let $\Pi=\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Denote by s_i $(1\leqq i\leqq n)$ the simple reflection corresponding to λ_i. For an $s\in W$, let $l(s)$ be the length of s with respect to $\{s_i\}$. As is well-known, for $s, t, w\in W$ such that $s=tw$ and $l(s)=l(t)+l(w)$, one has

$$\langle\!\langle s\rangle\!\rangle=\langle\!\langle w\rangle\!\rangle\cup w^{-1}\langle\!\langle t\rangle\!\rangle \qquad \text{(disjoint union)},$$

$$D_s=D_w\cap w^{-1}D_t, \qquad \mathfrak{u}_s=\mathfrak{u}_w\oplus \mathrm{Ad}\,(w)^{-1}\mathfrak{u}_t.$$

Lemma 5.1. *Let $s, t, w\in W$ be as above. Let η be a unitary character of U_s which is trivial on U_w. Then one has*

$$W(\sigma, \nu, \eta)f(g)=\{W(w\sigma, w\nu, \eta_t^w)A_w(\sigma, \nu)f\}(gw^{-1})$$

for $f\in H_{\sigma,\nu}$, $g\in G$ and $\nu\in D_s$, where η_t^w is a unitary character of U_t defined as $\eta_t^w(u)=\eta(w^{-1}uw)$ $(u\in U_t)$.

Proof. Since $\mathfrak{u}_s=\mathfrak{u}_w\oplus\mathrm{Ad}\,(w^{-1})\mathfrak{u}_t$, one has

$$\begin{aligned} W(\sigma, \nu, \eta)f(g) &=\int_{U_t\times U_w} f(gw^{-1}u_twu_w)\eta(w^{-1}u_tw)^{-1}du_tdu_w \\ &=\int_{U_t} A_w(\sigma, \nu)f(gw^{-1}u_t)\eta_t^w(u_t)^{-1}du_t \\ &=\{W(w\sigma, w\nu, \eta_t^w)A_w(\sigma, \nu)f\}(gw^{-1}), \end{aligned}$$

where du_t (resp. du_w) denotes a Haar measure on U_t (resp. U_w) normalized so that the first equality holds. The above calculation is valid for $\nu\in D_s$. Q.E.D.

For a unitary character η of U_s, let $W'_{s,\eta}$ be the subgroup of W generated by the reflections corresponding to the elements $\lambda\in\Pi$ such that (i) $\mathfrak{g}_{-\lambda}\subseteqq\mathfrak{u}_s$ and (ii) the restriction of η to $\exp\mathfrak{g}_{-\lambda}$ is non-trivial.

The following theorem is due to H. Jacquet.

Theorem 5.2 ([6, Theorem 3.4]). *Let the notations be as above. For $f\in (H_{\sigma,\nu})_K$, the function $D_s\ni\nu\to W(\sigma, \nu, \eta)f(g)$ extends to a holomorphic function on $[W'_{s,\eta}D_s]$, where $[\omega]$ denotes the convex hull of a subset ω of $\mathfrak{a}_C{}^*$. Here $(H_{\sigma,\nu})_K$ denotes the space of K-finite vectors in $H_{\sigma,\nu}$.*

By Lemma 5.1 and Theorem 5.2, we can give a domain to which Whittaker integral can be extended meromorphically in the following way. Let $f\in(H_{\sigma,\nu})_K$. First apply Lemma 5.1. Then the analytic continuation of $W(\sigma, \nu, \eta)f(g)$ is reduced to that of $W(\sigma, \nu, \eta_t^w)f(g)$ in the notations in Lemma 5.1. Secondly, apply Theorem 5.2 to the latter.

Before we apply this method in 5.3∼5.7, we give another proof of the key theorem (Theorem 5.2) in the following subsection 5.2.

5.2. On another proof of Theorem 5.2. Using the results of G. Schiffmann, M. Hashizume proved in [4] the result corresponding to Theorem 5.2 for the spherical principal series representations of reductive algebraic groups over $\boldsymbol{R}$.

The proof is quite general, which works for non-spherical principal series representations if one overcomes some check points for real rank one groups. These point would be overcome if one succeeds in calculating explicitly the intertwining operators between principal series representations and the Fourier transform of its kernel.

Here we make these calculations for $G=SL(2, \boldsymbol{C})$. As its consequence, one finds out that Hashizume's method works for non-spherical principal series representations of connected complex semisimple Lie groups. So one can give another proof of Theorem 5.2.

Let $G=SL(2, \boldsymbol{C})$. Keep to the notations in 3.4. For a non-negative integer l, let (γ_l, P^l) be an $(l+1)$-dimensional representation of $K=SU(2)$ defined as follows. Let P^l be the space of all complex polynomials in ${}^t(w_1, w_2)\in \boldsymbol{C}^2$ homogeneous of degree l. The natural action of K on $\boldsymbol{C}^2$ induces a representation γ_l of K on P^l: $(\gamma_l(k)f)(w)=f(k^{-1}w)$ for $f\in P^l$, $k\in K$ and $w={}^t(w_1, w_2)\in \boldsymbol{C}^2$. Then the representations γ_l are irreducible and $\{\gamma_l\,;\, l\geqq 0\}$ is a complete system of representatives in $\hat{K}$, the set of all equivalence classes of irreducible unitary representations of K. The group $M=\{m(\theta)=\mathrm{diag}\,(e^{-\sqrt{-1}\theta}, e^{\sqrt{-1}\theta})\,;\, \theta\in \boldsymbol{R}\}$ is a maximal torus of K. For an integer p, define a weight σ_p of M by $\sigma_p(m(\theta))=e^{\sqrt{-1}p\theta}$ $(\theta\in \boldsymbol{R})$. Then $\hat{M}=\{\sigma_p\,;\, p\in \boldsymbol{Z}\}$. For $0\leqq j\leqq l$, put $f_j^{(l)}=w_1^j w_2^{l-j}\in P^l$. Then one has $P^l=\bigoplus_{0\leq j\leq l}\boldsymbol{C}f_j^{(l)}$ and $\boldsymbol{C}f_j^{(l)}$ is the M-weight space corresponding to σ_{2j-l}. Let λ be the positive root of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We identify $\mathfrak{a}_C{}^*$ with $\boldsymbol{C}$ so that $\lambda=2$. For $\eta\in \boldsymbol{C}$, $u_z=\begin{bmatrix}1 & z\\ 0 & 1\end{bmatrix}\to \exp\{\sqrt{-1}\,\mathrm{Re}\,(\eta z)\}$ $(z\in \boldsymbol{C})$ defines a unitary character of $U=\{u_z\,;\, z\in \boldsymbol{C}\}$. Denote this character by the same letter η. Put $s=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$. Then s is a representative of the non-trivial element of W. By Bruhat's decomposition, $s^{-1}u_z$ is in $UMAN$ for all $z\in \boldsymbol{C}\backslash\{0\}$. Write $s^{-1}u_z=u_z{}'m_z h_z n_z$ with $u_z{}'\in U$, $m_z\in M$, $h_z\in A$ and $n_z\in N$. Then we have

$$m_z=\begin{pmatrix}\dfrac{\bar{z}}{|z|} & 0\\ 0 & \dfrac{z}{|z|}\end{pmatrix},\qquad h_z=\begin{pmatrix}\dfrac{1}{|z|} & 0\\ 0 & |z|\end{pmatrix}.$$

For $\sigma_p\in\hat{M}$ and $\nu\in\mathfrak{a}_C{}^*$, put $\Phi_{s,\sigma_p,\nu}(z)=\sigma_p(m_z)h_z{}^{\nu-2}$. If $\mathrm{Re}\,\nu\geqq 0$, the function $\Phi_{s,\sigma_p,\nu}$ on $\boldsymbol{C}\simeq \boldsymbol{R}^2$ is locally integrable, hence defines a distribution. Moreover, if $0\leqq\mathrm{Re}\,\nu\leqq 1$, this distribution is tempered, and its Fourier transform

$$\hat{\Phi}_{s,\sigma_p,\nu}(\eta)=\int_{\boldsymbol{R}^2}\Phi_{s,\sigma_p,\nu}(z)\eta(u_z)^{-1}dx\,dy\qquad (z=x+\sqrt{-1}\,y)$$

is a function. The following proposition is a special case of the result of G. Schiffmann [13].

Proposition 5.3. *Let $\eta\in\boldsymbol{C}\backslash\{0\}$. Then one has*

(1) *$\hat{\Phi}_{s,\sigma_p,\nu}(\eta)$ extends to a meromorphic function of ν on the whole $\mathfrak{a}_C{}^*$.*

(2) *For $f \in (H_{\sigma_p,\nu})_K$, the Whittaker integral $W(\sigma_p, \nu, \eta)f(g)$ $(g \in G)$ extends to an entire function of ν on $\mathfrak{a}_C{}^*$. Moreover, ond has the equality, by analytic continuation,*

$$(5.1) \qquad W(\sigma_{-p}, -\nu, \eta)A_s(\sigma_p, \nu)f = \hat{\Phi}_{s,\sigma_p,\nu}(\eta)W(\sigma_p, \nu, \eta)f\,.$$

Until the end of 5.2, we assume that $\eta \neq 0$, and calculate the intertwining operator $A_s(\sigma_p, \nu)$ and the Fourier transform $\hat{\Phi}_{s,\sigma_p,\nu}(\eta)$ of its kernel. These calculations show that $\hat{\Phi}_{s,\sigma_p,\nu}(\eta)^{-1}A_s(\sigma_p, \nu)$ is holomorphic in $\mathrm{Re}\,\nu<0$. This is the most important fact when we apply the argument in [4].

Lemma 5.4. *The Fourier transform is expressed as*

$$\hat{\Phi}_{s,\sigma_p,\nu}(\eta) = \pi(\sqrt{-1})^{|p|}\Big(\frac{2}{|\eta|}\Big)^{\nu}\Big(\frac{-\eta}{|\eta|}\Big)^{-p}\frac{\Gamma\big(\frac{\nu}{2}+\frac{|p|}{2}\big)}{\Gamma\big(1-\frac{\nu}{2}+\frac{|p|}{2}\big)}\,.$$

In particular, $\hat{\Phi}_{s,\sigma_p,\nu}(\eta) \neq 0$ if $\mathrm{Re}\,\nu<0$, and $\hat{\Phi}_{s,\sigma_p,\nu}(\eta)$ has a simple pole at $\nu = -(|p|+2r)$, for every non-negative integer r.

Proof. For $0<\mathrm{Re}\,\nu<1$, one has

$$\hat{\Phi}_{s,\sigma_p,\nu}(\eta) = \lim_{\varepsilon\to+0}\int e^{-\varepsilon|z|}\Big(\frac{z}{|z|}\Big)^p |z|^{\nu-2}e^{-\sqrt{-1}\mathrm{Re}(\eta z)}dxdy \qquad (z=x+\sqrt{-1}\,y)$$

$$= \lim_{\varepsilon\to+0}\int_{0\le r<+\infty} r^{\nu-1}e^{-\varepsilon r}dr\Big\{\int_{0\le\theta\le 2\pi} e^{\sqrt{-1}p\theta}e^{-\sqrt{-1}r\,\mathrm{Re}(\eta e^{\sqrt{-1}\theta})}d\theta\Big\}\,.$$

On the other hand,

$$\int_{0\le\theta\le 2\pi} e^{\sqrt{-1}p\theta}e^{-\sqrt{-1}r\,\mathrm{Re}(\eta e^{\sqrt{-1}\theta})}d\theta = 2\Big(\frac{-\eta}{|\eta|}\Big)^{-p}\int_{0\le\theta\le\pi}\cos(|p|\theta)e^{\sqrt{-1}|\eta|r\cos\theta}d\theta$$

$$= 2\pi(\sqrt{-1})^{|p|}\Big(\frac{-\eta}{|\eta|}\Big)^{-p}J_{|p|}(r|\eta|)\,,$$

where J_μ denotes the Bessel function of order μ. Hence one has

$$\hat{\Phi}_{s,\sigma_p,\nu}(\eta) = 2\pi(\sqrt{-1})^{|p|}\Big(\frac{-\eta}{|\eta|}\Big)^{-p}|\eta|^{-\nu}\lim_{\varepsilon\to+0}\int_{0\le r<+\infty}e^{-\varepsilon r}r^{\nu-1}J_{|p|}(r)dr\,.$$

By the well-known formula

$$\lim_{\varepsilon\to+0}\int_{0\le r<+\infty}e^{-\varepsilon r}J_\mu(r)r^{\nu-1}dr = \frac{2^{\nu-1}\Gamma\big(\frac{\nu+\mu}{2}\big)}{\Gamma\big(\frac{-\nu+\mu}{2}+1\big)}\,,$$

we complete the proof of the lemma. Q. E. D.

Now we calculate the intertwining operator $A_s(\sigma_p, \nu)$. For $(\gamma, V_\gamma) \in \hat{K}$ and $\sigma \in \hat{M}$, the multiplicity of M-weight σ in γ is at most one: $\dim \mathrm{Hom}_M(\gamma|_M, \sigma) \le 1$. Suppose $\mathrm{Hom}_M(\gamma|_M, \sigma) \neq \{0\}$. Let v_σ be an element in V_γ such that $\|v_\sigma\|$

$=1$ and $\gamma(m)v_\sigma=\sigma(m)v_\sigma$ for $m\in M$. Here $\|v\|=\sqrt{(v|v)}$ for $v\in V_\gamma$, and $(\cdot|\cdot)$ denotes a K-invariant inner product in V_γ. Put $T_\sigma v=(v|v_\sigma)$ for $v\in V_\gamma$. Then $T_\sigma\in\mathrm{Hom}_M(\gamma|_M, \sigma)$. The isotypic K-submodule $(H_{\sigma,\nu})_\gamma$ of type γ in $H_{\sigma,\nu}$ is identified with V_γ by

$$V_\gamma\ni v \longrightarrow [g\to a(g)^{-\nu-\rho}T_\sigma(\gamma(k(g)^{-1})v)]\in(H_{\sigma,\nu})_\gamma .$$

Similarly we identify $(H_{s\sigma,s\nu})_\gamma$ with V_γ.

Now we put

$$T'(\gamma, \nu)=\int_U a(u)^{-\nu-\rho}\gamma(k(u))du .$$

It is easy to see that $T'(\gamma, \nu)\in\mathrm{Hom}_M(\gamma|_M, \gamma|_M)$. Since $\dim\mathrm{Hom}_M(\gamma|_M, \sigma)=1$, there exists a constant $C_\sigma(\gamma, \nu)$ such that $T'(\gamma, \nu)v_\sigma=C_\sigma(\gamma, \nu)v_\sigma$. Then the restriction of $A_s(\sigma, \nu)$ to $(H_{\sigma,\nu})_\gamma\simeq V_\gamma$ is given by

$$A_s(\sigma, \nu)|_{V_\gamma}=\overline{C_\sigma(\gamma, \bar{\nu})}Id .$$

As is well-known, $C_\sigma(\gamma, \nu)$ is a meromorphic function of $\nu\in\mathfrak{a}_C{}^*$.

Lemma 5.5. *For integers l, j such that $0\leqq j\leqq l$, one has*

$$C_{\sigma_{2j-l}}(\gamma_l, \nu)=\sum_{0\leqq q\leqq\min(j,l-j)}\pi q!(-2)^q\binom{j}{j-q}\binom{l-j}{q}\prod_{0\leqq r\leqq q}\frac{1}{\nu+l-2r}.$$

In particular, $C_{\sigma_{2j-l}}(\gamma_l, \nu)$ is holomorphic in ν except $\nu=-(l-2r)$ for some $0\leqq r\leqq\min(j, l-j)$.

Proof. We may assume that $f_j{}^{(l)}=v_{\sigma_{2j-l}}\in V_{\gamma_l}=P^l$. For $z\in\boldsymbol{C}$, the Iwasawa decomposition of u_z gives

$$k(u_z)=\frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix}1 & z\\ -\bar{z} & 1\end{pmatrix}, \qquad a(u_z)=\begin{pmatrix}\dfrac{1}{\sqrt{1+|z|^2}} & 0\\ 0 & \sqrt{1+|z|^2}\end{pmatrix}.$$

By direct calculations, one has

$$(a(u_z)^{-\nu-\rho}\gamma_l(h(u_z))f_j{}^{(l)}|f_j{}^{(l)})=\sum_{0\leqq q\leqq\min(j,l-j)}(-1)^q\binom{j}{j-q}\binom{l-j}{q}(1+|z|^2)^{-(\nu+l)/2-1}|z|^{2q}.$$

Therefore,

$$C_{\sigma_{2j-l}}(\gamma_l, \nu)=\sum_{0\leqq q\leqq\min(j,l-j)}(-1)^q\binom{j}{j-q}\binom{l-j}{q}2\pi\int_{0\leqq r<+\infty}(1+r^2)^{-(\nu+l)/2-1}r^{2q+1}dr .$$

By [15, Lemma 8.10.15],

$$\int_{0\leqq r<+\infty}(1+r^2)^{-(\nu+l)/2-1}r^{2q+1}dr=\frac{\Gamma(q+1)\Gamma\left(\frac{\nu+l}{2}-q\right)}{2\Gamma\left(\frac{\nu+l}{2}+1\right)}$$

$$=2^{q-1}q!\prod_{0\leqq r\leqq q}\frac{1}{\nu+l-2r}.$$

This completes the proof. Q. E. D.

Proposition 5.6. *Let* $\sigma\in\hat{M}$ *and* $\nu\in\mathfrak{a}_C^*$. *For* $f\in(H_{\sigma,\nu})_K$, *the function* $\nu\to\hat{\Phi}_{s,\sigma,\nu}(\eta)^{-1}(A_s(\sigma,\nu)f)(g)$ *is holomorphic in* $\mathrm{Re}\,\nu<0$ *for every* $g\in G$.

Proof. Let l, j be as in Lemma 5.5. By Lemmas 5.4 and 5.5, the function $\hat{\Phi}_{s,\sigma_{2j-l},\nu}(\eta)^{-1}C_{\sigma_{2j-l}}(\gamma_l,\nu)$ of $\nu\in\mathfrak{a}_C^*$ is holomorphic in $\mathrm{Re}\,\nu<0$. This proves the proposition. Q. E. D.

Thanks to Proposition 5.6, we can prove Theorem 5.2 just as in the same way as in [4, §§3~4]. However the proof is so long, so we omit it here.

5.3. Analytic continuation of intertwining operators $W(\sigma,\nu,\eta_A)$. Let η be a unitary character of U_s, where s is a fixed element in W. The following lemma gives a sufficient condition for $[W'_{s,\eta}D_s]=\mathfrak{a}_C^*$, which implies that the function $W(\sigma,\nu,\eta)f(g)$ extends to an entire function on the whole $\mathfrak{a}_C^*$.

Lemma 5.7. *If there exists* $w'\in W'_{s,\eta}$ *such that* $\langle\!\langle sw'^{-1}\rangle\!\rangle\cap\langle\!\langle s\rangle\!\rangle=\emptyset$, *then* $[W'_{s,\eta}D_s]=\mathfrak{a}_C^*$ *holds.*

Proof. Put $J_s=\{w\in W;\ w^{-1}D\subseteqq D_s\}$, where $D=D_{s_0}$ with the longest element $s_0\in W$. Then we note that $J_s=\{w\in W;\ \langle\!\langle w\rangle\!\rangle\cap\langle\!\langle s\rangle\!\rangle=\emptyset\}$. In fact, for $w\in W$, $D_s\supseteqq w^{-1}D$ holds if and only if $\langle\mathrm{Re}\,x, w\lambda\rangle>0$ for all $x\in D$ and $\lambda\in\langle\!\langle s\rangle\!\rangle$. This means that $w\lambda\in\Lambda^+$ for all $\lambda\in\langle\!\langle s\rangle\!\rangle$, or $\langle\!\langle w\rangle\!\rangle\cap\langle\!\langle s\rangle\!\rangle=\emptyset$.

Put $w_1=s_0s$ and $w_2=sw'^{-1}$, then $w_1, w_2\in J_s$ by the above. Hence $W'_{s,\eta}D_s$ contains $w_2^{-1}D$ and $w'w_1^{-1}D=-w_2^{-1}D$. Therefore, $[W'_{s,\eta}D_s]=\mathfrak{a}_C^*$. Q. E. D.

We shall consider the analytic continuation of Whittaker integrals for generalized Gelfand-Graev representations in accordance with the method given in 5.1.

Let $\{A, H, B\}\subseteqq\mathfrak{g}$ be an $\mathfrak{sl}_2$-triplet:

$$[H, A]=2A\,,\qquad [H, B]=-2B\,,\qquad [A, B]=H \tag{5.2}$$

such that H is in the closure of negative Weyl chamber of $\mathfrak{a}$. Then we have the following proposition.

Proposition 5.8. *Assume that* $U(1.5)_A=U_s$ *for some* $s\in W$. *Then the Whittaker integral* $W(\sigma,\nu,\eta_A)f(g)$ *extends to a meromorphic function of* ν *on the whole* $\mathfrak{a}_C^*$ *for every* $\sigma\in\hat{M}$, $f\in(H_{\sigma,\nu})_K$ *and* $g\in G$, *if there exists* $w\in W$ *satisfying the following conditions* (1)~(3).

(1) $\langle\!\langle w\rangle\!\rangle\subseteqq\langle\!\langle s\rangle\!\rangle$,

(2) $\mathrm{Ad}(w)B\in\mathfrak{n}$.

(3) *Put* $t=sw^{-1}$. *Then* $[W'_{t,A}D_t]=\mathfrak{a}_C^*$, *where* $W'_{t,A}$ *is the subgroup of* W *generated by simple reflections* s_λ ($\lambda\in\Pi$) *such that* $Q(\mathrm{Ad}(w)B, \mathfrak{g}_{-\lambda})\neq\{0\}$ *and* $\mathfrak{g}_{-\lambda}\subseteqq\mathfrak{u}_t$.

Proof. One has $s=tw$, $l(s)=l(t)+l(w)$ by (1). And η_A is trivial on U_w by (2). By Lemma 5.1, meromorphic continuation of the Whittaker integral in question is reduced to that of $W(\sigma, \nu, (\eta_A)_t^w)f(g)$. On the other hand, $W'_{t,A}=W'_{t,\eta'}$ with $\eta'=(\eta_A)_t^w$. By Theorem 5.2, $W(\sigma, \nu, \eta')f(g)$ extends to an entire function of ν on $\mathfrak{a}_C^*$. Q. E. D.

Let A_0 be a non-zero nilpotent element of $\mathfrak{g}$. Assume that A_0 is in $\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$. Then there exists a subset F of Π such that A_0 is a regular nilpotent element of $\mathfrak{m}_F$. Let $\{A_0, H_0, B_0\}\subseteq\mathfrak{m}_F$ be the $\mathfrak{sl}_2$-triplet: $[H_0, A_0]=2A_0$, $[H_0, B_0]=-2B_0$, $[A_0, B_0]=H_0$, such that $H_0\in\mathfrak{a}(F)$.

We note the following well-known fact.

Lemma 5.9 ([15, Lemma 8.9.11]). *For a subset Ψ of Λ^+, $\Psi=\langle\!\langle s\rangle\!\rangle$ for some $s\in W$ if and only if Ψ satisfies following conditions* (1) *and* (2).

(1) *If $\lambda\in\Psi$ and $\lambda=\mu_1+\mu_2$, $\mu_1, \mu_2\in\Lambda^+$, then μ_1 or μ_2 is in Ψ.*

(2) *If $\lambda, \mu\in\Psi$ and $\lambda+\mu\in\Lambda^+$, then $\lambda+\mu\in\Psi$.*

The set $\{\lambda\in\Lambda^+; \langle\lambda, H_0\rangle>0\}$ satisfies the above (1) and (2). So, let w be the element in W such that $\langle\!\langle w^{-1}\rangle\!\rangle=\{\lambda\in\Lambda^+; \langle\lambda, H_0\rangle>0\}$. Put $A=\mathrm{Ad}(w)^{-1}A_0$, $H=\mathrm{Ad}(w)^{-1}H_0$ and $B=\mathrm{Ad}(w)^{-1}B_0$. Then one has

Lemma 5.10. (1) *$\{A, H, B\}\subseteq\mathfrak{g}$ is an $\mathfrak{sl}_2$-triplet such that H is in the closure of the negative Weyl chamber of $\mathfrak{a}$.*

(2) $\mathfrak{u}_w\subseteq\mathfrak{u}(1)_A$.

(3) *The subalgebra $(\mathfrak{u}_w\cap\mathfrak{g}(1)_A)\oplus\mathfrak{u}(2)_A$ of $\mathfrak{u}(1)_A$ is subordinate to the linear form $B^*: X\to Q(X, B)$ on $\mathfrak{u}(1)_A$.*

Proof. (1) One has $w\Lambda^+=(w\Lambda^+\cap\Lambda^-)\cup(w\Lambda^+\cap\Lambda^+)=(-\langle\!\langle w^{-1}\rangle\!\rangle)\cup(\Lambda^+\backslash\langle\!\langle w^{-1}\rangle\!\rangle)$. By the definition of w, $\langle\lambda, H\rangle=\langle w\lambda, H_0\rangle\leqq 0$ for all $\lambda\in\Lambda^+$. Hence H is in the closure of the negative Weyl chamber of $\mathfrak{a}$.

(2) Note that $-\langle\!\langle w\rangle\!\rangle=w^{-1}\langle\!\langle w^{-1}\rangle\!\rangle$. For $\lambda=w^{-1}\lambda'\in-\langle\!\langle w\rangle\!\rangle$ with $\lambda'\in\langle\!\langle w^{-1}\rangle\!\rangle$, $\langle\lambda, H\rangle=\langle\lambda', wH\rangle=\langle\lambda', H_0\rangle>0$. Since $\mathfrak{u}_w=\sum_{\lambda\in-\langle w\rangle}\mathfrak{g}_\lambda$, $\mathfrak{u}_w\subseteq\mathfrak{u}(1)_A$.

(3) $Q(\mathfrak{u}_w, B)=Q(\mathrm{Ad}(w)\mathfrak{u}_w, B_0)\subseteq Q(\mathfrak{n}, B_0)=\{0\}$. This proves the assertion. Q. E. D.

Let us consider the next condition (*).

(*) There exists $s\in W$ such that $\mathfrak{u}_s\supseteq(\mathfrak{u}_w\cap\mathfrak{g}(1)_A)\oplus\mathfrak{u}(2)_A$ and $\mathfrak{u}(1.5)_A$ can be taken as $\mathfrak{u}_s$.

In the following proposition, we give a sufficient condition for the meromorphic continuation of Whittaker integrals $W(\sigma, \nu, \eta_A)f(g)$ to the whole $\mathfrak{a}_C^*$.

Proposition 5.11. *Let the notations be as above. Assume that the condition* (*) *holds. Then the function $D_s\ni\nu\to W(\sigma, \nu, \eta_A)f(g)$ extends to a meromorphic function of ν on $\mathfrak{a}_C^*$, for $\sigma\in\hat{M}$, $f\in(H_{\sigma,\nu})_K$ and $g\in G$.*

Proof. Let w be as above. For this w, we check the conditions (1)~(3) of Proposition 5.8. The conditions (1) and (2) are satisfied by the definition of s and w. We check (3). Put $t=sw^{-1}$, then $W'_{t,A}=W_F$. Let $w'_0\in W_F$ be the element such that $w'_0F=-F$. Then $w'_0H_0=-H_0$, because H_0 is the unique element in $\mathfrak{a}(F)$ such that $\langle\lambda, H_0\rangle=-2$ for all $\lambda\in F$. Let $\lambda\in\langle\langle tw_0'^{-1}\rangle\rangle$. Since $\langle\langle tw_0'^{-1}\rangle\rangle=w'_0\langle\langle t\rangle\rangle\cap\Delta^+$, one has $w_0'^{-1}\lambda\in\langle\langle t\rangle\rangle$, whence $\langle w_0'^{-1}\lambda, H_0\rangle<0$, or $\langle\lambda, H_0\rangle>0$. This shows that $\lambda\notin\langle\langle t\rangle\rangle$. Therefore, $\langle\langle t\rangle\rangle\cap\langle\langle tw_0'^{-1}\rangle\rangle=\emptyset$. By Lemma 5.7, we have $[W'_{t,A}D_t]=\mathfrak{a}_C^*$. Q. E. D.

A nilpotent element $A\in\mathfrak{g}$ is called *even*, if $\mathfrak{g}(1)_A=\{0\}$. In case that A is even, the condition (*) is clearly satisfied. Therefore we have

Corollary 5.12. *Let $A_0\in\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$ be a non-zero even nilpotent element. Under the above notations, the function $D_s\ni\nu\to W(\sigma, \nu, \eta_A)f(g)$ extends to a meromorphic function on the whole $\mathfrak{a}_C^*$, for every $\sigma\in\hat{M}$, $f\in(H_{\sigma,\nu})_K$ and $g\in G$.*

5.4. Now we apply Proposition 5.11 to complex simple Lie groups. For this purpose, we examine if the condition (*) in 5.3 is satisfied.

Case of type (A_l). Let G be a connected complex simple Lie group of type (A_{n-1}). Then $\mathfrak{g}\simeq\mathfrak{sl}(n, \boldsymbol{C})$, so we identify $\mathfrak{g}$ with $\mathfrak{sl}(n, \boldsymbol{C})$ and keep to the notations in 3.4. For $\gamma\in P_n$, put $A_0=A_\gamma^0$, $H_0=H_\gamma^0$, $B_0=B_\gamma^0$. Then $w=w_\gamma^{-1}$ and $A=A_\gamma$, $H=H_\gamma$, $B=B_\gamma$. Moreover we see easily

$$(\mathfrak{u}_w\cap\mathfrak{g}(1)_A)\oplus\mathfrak{u}(2)_A=\mathfrak{u}_\gamma\,,$$

where $\mathfrak{u}_\gamma$ is the Lie algebra of U_γ. Therefore, the condition (*) is satisfied by putting $\mathfrak{u}_\gamma=\mathfrak{u}_s$. Consequently, we have the following theorem.

Theorem 5.13. *Let G be a connected complex simple Lie group of type (A_{n-1}). Let $\gamma\in P_n$. Then the function $W(\sigma, \nu, \eta_{A_\gamma})f(g)$ extends to a meromorphic function on the whole $\mathfrak{a}_C^*$, for every $\sigma\in\hat{M}$, $f\in(H_{\sigma,\nu})_K$ and $g\in G$.*

By the above theorem, Whittaker integrals $W(\sigma, \nu, \eta_A)f(g)$ can be extended meromorphically to $\mathfrak{a}_C^*$ for all nolpotent elements $A\in\mathfrak{g}$ when $\mathfrak{g}$ is of type (A_l). This is a complete result in case of type (A_l).

We proceed to the cases of simple groups of other types. We should treat the cases of rank 2 in the first place.

5.5. Case of type $(B_2)=(C_2)$. Suppose that $\mathfrak{g}$ is of type (C_2). Then every nilpotent $\mathrm{Ad}(G)$-orbit of $\mathfrak{g}$ intersects $\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$. Moreover, the condition (*) is essentially satisfied for all nilpotent elements. Therefore, the corresponding Whittaker integrals extend meromorphically to the whole $\mathfrak{a}_C^*$ (see Table 5.14).

5.6. Case of type (G_2). We consider the case of type (G_2). The Whittaker integrals $W(\sigma, \nu, \eta_A)f(g)$ extend meromorphically to the whole $\mathfrak{a}_C^*$ except only a unique case. In this exceptional case, the best we can obtain within our present method is to extend the Whittaker integral meromorphically to a half space. We do not know whether it extends meromorphically to a larger domain or not. In the following, we explain these in more details.

Beforehand, we need to make some comments on the classification of nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$ ([14]). Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. For a non-zero nilpotent element $A\in\mathfrak{g}$, consider an $\mathfrak{sl}_2$-triplet $\{A, H, B\}\subseteq\mathfrak{g}$ as in (5.2). By taking a suitable $\mathrm{Ad}(G)$-conjugate of A instead of A, we may assume that H is in the closure of the negative Weyl chamber of $\mathfrak{a}$. Put $d(A)=(-\lambda_1(H), \cdots, -\lambda_n(H))$. Note that $-\lambda_i(H)=0$ or 1 or 2 for $1\leqq i\leqq n$. Then the nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$, except for $\{0\}$, are parametrized by the set $H(\mathfrak{g})=\{d(A); A\}$ of "weighted Dynkin diagrams".

Suppose that $\mathfrak{g}$ is of type (G_2). Then Δ^+ is given as

$$\Delta^+=\{\lambda_1, \lambda_2, \lambda_1+\lambda_2, 2\lambda_1+\lambda_2, 3\lambda_1+\lambda_2, 3\lambda_1+2\lambda_2\}$$

with $\Pi=\{\lambda_1, \lambda_2\}$ and the Dynkin diagram $\lambda_1 \circ\!\Lleftarrow\!\circ\, \lambda_2$. By [2, Table 16], one has $H(\mathfrak{g})=\{(1, 0), (0, 1), (0, 2), (2, 2)\}$.

Let A_0 be a non-zero element in $\mathfrak{g}_{-\lambda_1}\oplus\mathfrak{g}_{-\lambda_2}$. Let w and $\{A, H, B\}$ be as in Lemma 5.10.

Case 1. $A_0=X_{-\lambda_1}+X_{-\lambda_2}$ with $X_{-\lambda_i}\in\mathfrak{g}_{-\lambda_i}\backslash\{0\}$ $(i=1, 2)$. In this case, A_0 is a regular nilpotent element of $\mathfrak{g}$ with $d(A_0)=(2, 2)$, so the condition $(*)$ is satisfied. Corollary 5.12 assures that the corresponding Whittaker integral extends to a meromorphic function on $\mathfrak{a}_C{}^*$. Moreover, it is an entire function on $\mathfrak{a}_C{}^*$ by Theorem 5.2.

Case 2. $A_0=X_{-\lambda_1}$. Then $w=s_2s_1$ and $d(A)=(1, 0)$. Hence one has

$$\mathfrak{g}(1)_A\cap\mathfrak{u}_w=\mathfrak{g}_{-\lambda_1}, \qquad \mathfrak{u}(2)_A=\mathfrak{g}_{-2\lambda_1}\oplus\mathfrak{g}_{-3\lambda_1-\lambda_2}\oplus\mathfrak{g}_{-3\lambda_1-2\lambda_2}.$$

The subalgebra $\mathfrak{u}(1.5)_A$ can be taken as

$$\mathfrak{u}(1.5)_A=(\mathfrak{g}(1)_A\cap\mathfrak{u}_w)\oplus\mathfrak{u}(2)_A=\mathfrak{u}_s \qquad \text{with } s=s_2s_1s_2s_1.$$

Therefore, the condition $(*)$ is satisfied.

Case 3. $A_0=X_{-\lambda_2}$. Then $w=s_1s_2$ and $d(A)=(0, 1)$. Hence $\mathfrak{g}(1)_A\cap\mathfrak{u}_w=\mathfrak{g}_{-\lambda_2}\oplus\mathfrak{g}_{-\lambda_1-\lambda_2}$, $\mathfrak{u}(2)_A=\mathfrak{g}_{-3\lambda_1-2\lambda_2}$. We can take $\mathfrak{u}(1.5)_A=(\mathfrak{u}_w\cap\mathfrak{g}(1)_A)\oplus\mathfrak{u}(2)_A=\mathfrak{u}_s$ with $s=s_2s_1s_2$. Therefore the condition $(*)$ is satisfied.

Case 4. The nilpotent $\mathrm{Ad}(G)$-orbit corresponding to $(0, 2)$ does not intersect $\mathfrak{g}_{-\lambda_1}\oplus\mathfrak{g}_{-\lambda_2}$. This case is beyond application of Proposition 5.11. So we return to our original method in 5.1. Let A be an element of this orbit. Consider an $\mathfrak{sl}_2$-triplet $\{A, H, B\}$ with H in the closure of the negative Weyl chamber of $\mathfrak{a}$. Within the limit of our present method, we know only that the function $W(\sigma, \nu, \eta_A)f(g)$ extends to a meromorphic function on the half space $\{\nu\in\mathfrak{a}_C{}^*; \langle\mathrm{Re}\,\nu, 3\lambda_1+2\lambda_2\rangle>0\}$. It is left open whether or not it extends meromorphically to a larger domain.

5.7. In case that $\mathfrak{g}$ is of type (B_n), (C_n), (D_n) $(n \geqq 3)$, (E_n) or (F_4), we have only one result, Corollary 5.12. The phenomenon of the same kind as Case 4 of type (G_2) occurs already when $\mathfrak{g}$ is of type (C_3).

At last, we list up, for complex simple Lie groups of rank 2, our results of analytic continuation of Whittaker integrals $W(\sigma, \nu, \eta_A)f(g)$.

Table 5.14.

type of $\mathfrak{g}$	Dynkin diagram	weighted Dynkin diagram $d(A)$	a representative A	$U(1.5)_A$	analytic continuation of $W(\sigma, \nu, \eta_A)f(g)$
A_2	λ_1 λ_2	(2, 2)	$X_{-\lambda_1}+X_{-\lambda_2}$	U	entire on $\mathfrak{a}_C{}^*$
		(1, 1)	$X_{-\lambda_1-\lambda_2}$	$U_{s_2s_1}$	meromorphic on $\mathfrak{a}_C{}^*$
$B_2=C_2$	λ_1 λ_2	(2, 2)	$X_{-\lambda_1}+X_{-\lambda_2}$	U	entire on $\mathfrak{a}_C{}^*$
		(0, 2)	$X_{-\lambda_1-\lambda_2}$	$U_{s_2s_1s_2}$	meromorphic on $\mathfrak{a}_C{}^*$
		(1, 0)	$X_{-2\lambda_1-\lambda_2}$	$U_{s_2s_1}$	meromorphic on $\mathfrak{a}_C{}^*$
G_2	λ_1 λ_2	(2, 2)	$X_{-\lambda_1}+X_{-\lambda_2}$	U	entire on $\mathfrak{a}_C{}^*$
		(0, 1)	$X_{-3\lambda_1-2\lambda_2}$	$U_{s_2s_1s_2}$	meromorphic on $\mathfrak{a}_C{}^*$
		(1, 0)	$X_{-2\lambda_1-\lambda_2}$	$U_{s_2s_1s_2s_1}$	meromorphic on $\mathfrak{a}_C{}^*$
		(0, 2)	$X_{-\lambda_1-\lambda_2}+X_{-3\lambda_1-\lambda_2}$	$U_{s_1s_0}$	meromorphic on a half space $H(3\lambda_1+2\lambda_2)$

X_λ: a non-zero element of the root space $\mathfrak{g}_\lambda$ of a root λ.
s_i $(i=1, 2)$: the simple reflection corresponding to λ_i.
s_0: the longest element in W.
$H(3\lambda_1+2\lambda_2)=\{\nu \in \mathfrak{a}_C{}^*;\ \langle \operatorname{Re}\nu, 3\lambda_1+2\lambda_2\rangle > 0\}$.

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References

[1] W. Casselman and D. Milicić, Asymptotic behavior of matrix coefficients of admissible representations, Duke Math. J., **48**-4 (1982), 869-930.

[2] E.B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, Amer. Math. Transl., Ser. 2, **6** (1957), 111-245.

[3] I.M. Gelfand and M.I. Graev, Construction of irreducible representations of simple algebraic groups over a finite field, Soviet Math. Dokl., **3** (1962), 1646-1649.

[4] M. Hashizume, Whittaker models for real reductive groups, Japan. J. Math., **5** (1979), 349-401.
[5] T. Hirai, Structure of unipotent orbits and Fourier transform of unipotent orbital integrals for semisimple Lie groups, Lec. in Math., Kyoto univ., No. **14** (1982), 75-138.
[6] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France, **95** (1967), 243-309.
[7] M.L. Karel, Functional equations of Whittaker functions on p-adic groups, Amer. J. Math., **101**-6 (1979), 1303-1325.
[8] N. Kawanaka, Generalized Gelfand-Graev representations and Ennola duality, Advanced Studies in Pure Math., **6** (1985), 175-206.
[9] B. Kostant, The principal three dimensional subgroup and Betti numbers of a complex simple Lie group, Amer. J. Math., **81** (1959), 973-1032.
[10] B. Kostant, On Whittaker vectors and representation theory, Invent. Math., **48** (1978), 101-184.
[11] H. Ozeki and M. Wakimoto, On polarizations of certain homogeneous spaces, Hiroshima Math. J., **2** (1972), 445-482.
[12] J.A. Shalika, The multiplicity one theorem for $GL(n)$, Ann. of Math., **100** (1974), 171-193.
[13] G. Shiffman. Intégrales d'entrelacement et fonctions de Whittaker, Bull. Soc. Math. France, **99** (1971), 3-72.
[14] T.A. Springer and R. Steinberg, Conjugacy classes, Lec. Notes in Math. Springer-Verlag, vol. 131 (1970), 167-266.
[15] N.R. Wallach, Harmonic analysis on homogeneous spaces, Dekker, New-York, 1973.
[16] G. Warner, Harmonic analysis on semisimple Lie groups, vol. 1, Springer-Verlag, Berlin, 1973.

Added in proof: Main results of this article have been reported in the following note.

H. Yamashita, On Whittaker vectors for generalized Gelfand-Graev representations of semisimple Lie groups, Proc. Japan Acad., **61**, Ser. A (1985), 213-216.