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On Hadamard gap series and errata to Plessner, Julia, and ρ^*-points

Dedicated to Professor Yukio Kusunoki on his 60th birthday

By

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1. Introduction.

This note contains three purposes: First, we present an alternative solution to our previous problem on gap series raised in this journal (Kyoto University 1978). Second, we give a short proof of Fuchs Theorem on gap series. Finally, we correct three of our theorems which were pointed out by Gavrilov (from Moscow State University) to whom the author is indebted.

Let $D=\{z: |z|<1\}$ be the unit disk, $C=\{z: |z|=1\}$ the unit circle, and $f(z)$ a function meromorphic in D. As in K. Noshiro [19, p. 87], we say that $f(z)$ is normal in D if and only if

$$(1-|z|^2)|f'(z)|/(1+|f(z)|^2)\leqq M, \qquad \text{for all} \quad z\in D,$$

where M is a constant independent of points z in D.

In this journal [11, Theorem 7], we proved the following necessary and sufficient conditions of gap series to be normal in D.

Theorem 1. *Let $f_m(z)=\sum_{k=0}^{\infty} n_k^m z^{n_k}$, where $n_{k+1}/n_k\to\infty$, as $k\to\infty$, then f_m is normal if $m\leqq 0$ and f_m is not normal if $m\geqq 1$.*

We conjectured [11, p. 188] that f_m is normal if and only if $m\leqq 0$, where $n_{k+1}/n_k\to q>1$. We posed this problem for $q=\infty$ in Detroit Meeting and it was solved by L. R. Sons [23]. The general case $q>1$ was finally solved by the following theorem of Sons and Campbell [24].

Theorem 2. *Let $f(z)$ be a Hadamard gap series defined by*

$$f(z)=\sum_{k=0}^{\infty} c_k z^{n_k}, \quad n_{k+1}/n_k\geqq q>1, \tag{1}$$

where the series is convergent in D, then f is normal if and only if $\limsup|c_k|<\infty$.

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* I am indebted to Pommerenke for many valuable comments in this work.

With regard to gap series, the class of normal functions is disjoint from that of annular functions. Recall that a function $f(z)$ is annular if and only if there is a sequence of Jordan curves J_n in D such that each J_n lies in the interior of J_{n+1}, $\min\{|z|: z\in J_n\}\to 1$, and $\min\{|f(z)|: z\in J_n\}\to\infty$, as $n\to\infty$. In [24, p. 117], the authors ask the question whether $\sup|c_k|=\infty$ implies that f is annular, where f is a gap series defined in Theorem 2. This problem was recently solved by the author and Campbell in [13, Theorem 1] as follows.

Theorem 3. *Let f be a gap series defined in Theorem 2, then f is annular if and only if f is not normal.*

The proof in [13, Theorem 1] relies on the following theorems: K. G. Binmore [2, Theorem 1], J. Warren [25, Theorem 1], and D. D. Bonar [3, Theorem 4]. We shall present an elementary proof of Theorem 3 which is different from the one in [13, Theorem 1].

Our original intention is to consider a type of multiple gap series introduced by Ch. Pommerenke [20], that is

$$f(z)=\sum_{k=0}^{\infty}(c_k z^{n_k}+\cdots+c_{k+p}z^{n_k+p}), \qquad n_{k+1}/n_k\geqq q>1,$$

where p is a non-negative integer.

The key point here depends mainly on whether the following theorem of Binmore [2] can be extended to the multiple case? Based on his recent work [21], Pommerenke conjectured (to the author) the answer to be negative.

Theorem B. *Let $f(z)$ be a function defined by* (1), *then there is a constant $M(q)$ such that*

$$(2) \qquad |c_k|\leqq M(q)\sup_{z\in\Gamma}|f(z)| \qquad (k=0, 1, \cdots),$$

where Γ is an arbitrary path lying in D and tending to C.

In fact, Pommerenke's comment yields the following.

Theorem 4. *Theorem B is false for the multiple gap series.*

2. Proof of Theorem 3.

Necessity. If f is annular, then f is not normal since it has Koebe arcs. By our previous theorem [11, Theorem 4] the non-normality of f implies $\limsup|c_k|=\infty$.

Sufficiency. Let $\limsup|c_k|=\infty$. We define $C_n=\{z: |z|=1-1/(n+1)\}$ and $G_n=\{z: z\in D$ and $|f(z)|<n\}$. Then clearly we have

$$(3) \qquad |f(z)|\geqq n, \qquad \text{for} \quad z\in D-G_n.$$

Let G_n be decomposed into components

$$G_n = \cup G_{ni}, \quad \text{where} \quad G_{ni} \cap G_{nj} = \emptyset, \quad i \neq j.$$

By the maximum principle each component G_{ni} is a simply connected domain. Binmore's bound (2) and the assumption that $\limsup |c_k| = \infty$ prevent a component from going to C. Thus each component lies strictly inside D. By the minimum principle each G_{ni} contains at least one zero of f.

We now begin with $n=1$. We prove that the first circle C_1 meets at most finitely many components G_{1i}. Suppose on the contrary that C_1 meets infinitely many components, say G_{1i}, $i=1, 2, \cdots$. Then by Binmore's theorem we find that all the G_{1i} are forced to lie strictly in the interior of some circle $C_{N(1)}$, where $N(1)>1$. Since each G_{1i} contains at least one zero of f, it follows that the interior of $C_{N(1)}$ contains infinitely many zeros of f, in violation to the analyticity of f.

Next, we want to prove that there is a Jordan curve $J_1 \subset D$ whose interior contains C_1 and further satisfies

$$(4) \qquad |f(z)| \geqq 1, \quad \text{for each} \quad z \in J_1.$$

For this, we let R_1 be the ring between C_1 and $C_{N(1)}$. Since each component meeting C_1 must lie strictly in the interior of $C_{N(1)}$, it then follows that no components G_{1i} can meet both C_1 and $C_{N(1)}$. Since R_1 contains only finitely many components, we thus conclude that the complement

$$(5) \qquad S_1 = R_1 - \bigcup_i G_{1i} = R_1 - G_1,$$

contains a Jordan curve J_1 such that C_1 lies within the interior of J_1. Furthermore, from (5) we can see that if $z \in J_1$, then $z \notin G_1$, so by (3) we have $|f(z)| \geqq 1$. This proves (4).

We now consider $C_{N(1)}$ in place of C_1. By the same argument as before, there is a circle $C_{N(2)}$, where $N(2)>N(1)$, such that all components G_{2i} meeting $C_{N(1)}$ must lie strictly in the interior of $C_{N(2)}$. Thus, no components G_{2i} can meet both $C_{N(1)}$ and $C_{N(2)}$, so that the complement

$$S_2 = R_2 - \bigcup_i G_{2i} = R_2 - G_2,$$

where the domain R_2 is the ring between $C_{N(1)}$ and $C_{N(2)}$, contains a Jordan curve J_2 whose interior contains $C_{N(1)}$ and

$$|f(z)| \geqq 2, \quad \text{for each} \quad z \in J_2.$$

By continuing this process, we finally obtain a sequence of disjoint Jordan curves J_n such that

$$J_n \to C \quad \text{and} \quad f(z) \to \infty \quad \text{on} \quad J_n, \quad \text{as} \quad n \to \infty.$$

This shows that the function f is annular and the proof is complete.

3. Proof of Theorems 1-4.

We first prove Theorem 2. Let the coefficients c_k be bounded, then by [11, Theorem 4] the function f is Bloch and therefore is normal.

Conversely, if the coefficients are unbounded, then by Theorem 3 the function f is annular and hence is not normal due to a theorem of Bagemihl and Seidel [1, Theorem 1].

Next, we prove Theorem 1. We need only observe that the coefficients n_k^m is bounded if and only if $m \leqq 0$ and hence the function f_m is normal if and only if $m \leqq 0$.

Finally, we shall prove Theorem 4. It suffices to construct a sum f of two Hadamard gap series such that the coefficients of f is unbounded, but the range $f(z)$ is bounded on $[0, 1)$. This construction is due to W. Rudin [22]. In fact, we may choose

$$f(z)=\sum_{k=1}^{\infty} k(z^{n_k}-(n_k/(n_k+1))z^{n_k+1}), \quad n_{k+1}/n_k \geqq q>1, \quad n_k \geqq k^3.$$

4. Fuchs Theorem.

In [7, Theorem 4.1], W. H. J. Fuchs proved that if f is a gap series defined in Theorem 2 and if $\limsup|c_k|>0$, then the function $f(z)$ assumes every complex value infinitely often in every sector $\Delta(\alpha, \beta)=\{z\in D \text{ and } \alpha<\arg z<\beta\}$. The original proof of Fuchs is very complicated which relies mainly on the Petrenko formula [7, Lemma 4.3] in Nevanlinna theory. Later, I. L. Chang [4] gave a different proof, but still complicated. In [10], we gave a simple proof of Fuchs Theorem provided the coefficients are bounded. We now want to complete the proof of Fuchs Theorem by relaxing the boundedness of coefficients.

Theorem 5. *Let f be a Hadamard gap series defined by* (1). *If* $\limsup|c_k|>0$, *then the function value $f(z)$ assumes every complex value infinitely often in every sector $\Delta(\alpha, \beta)$.*

To prove Theorem 5, we shall need the following theorem of Murai [17] which answers affirmatively a problem in the MacLane class [15].

Theorem M. *Let f be a Hadamard gap series defined by* (1) *with* $\limsup|c_k|=\infty$. *Then f has an asymptotic value ∞ at every point of C.*

Note that the above Theorem M cannot be extended to the case of multiple gap series due to a recent result of Pommerenke [21, Theorem 2]. However, we believe that in the multiple case the function f should have the asymptotic value ∞ on a dense subset of C.

5. Proof of Theorem 5.

Let f be a Hadamard gap series defined by (1). If the coefficients c_k are bounded then the assertion was already proved in [10]. We may therefore assume that the coefficients c_k are unbounded, so that Theorem M is applicable. To prove the assertion, we suppose on the contrary that $f(z)$ assumes a value v finitely often in a sector $\Delta(\alpha, \beta)$. Then by Theorem M and the standard method (see [12]) there can be constructed a path Γ lying in $\Delta(\alpha, \beta)$ and tending to C such that

$$f(z) \to v, \quad \text{as} \quad |z| \to 1 \quad \text{and} \quad z \in \Gamma.$$

It follows from Theorem B that the coefficients c_k are bounded, a contradiction. This concludes that the function $f(z)$ assumes every complex value infinitely often in every sector.

6. Pommerenke's Theorem.

As an application of Theorem 5, we have the following theorem of Pommerenke [20].

Theorem 6. *Let f be a Hadamard gap series defined by* (1). *If the function value $f(z)$ is defferent from* 0 *throughout D, then the coefficients $c_k \to 0$.*

Proof. Suppose on the contrary that $\limsup |c_k| > 0$. Then by Theorem 5 the function $f(z)$ assumes the value 0 infinitely often in D. This contradicts the hypothesis and the theorem is proved.

We remark that the original theorem of Pommerenke [20] is true for multiple gap series. This would be a consequence of the conjecture we made at the end of Section 4.

7. Paley's Conjecture.

In [16], Murai solved a conjecture of Paley by showing the following.

Theorem M′. *Let f be a gap series defined in Theorem* 2. *If $\sum |c_k| = \infty$, then the function $f(z)$ assumes every complex value infinitely often in D.*

Murai [16, p. 155] remarked that the same line proof as in Theorem M′ gives the following extension.

Theorem 7. *Under the hypotheses of Theorem M′, the function value $f(z)$ assumes every complex value infinitely often in every sector $\Delta(\alpha, \beta)$.*

Proof. Instead of the same line proof (20 pages), we shall give a simple one based upon Theorem M′ only. By a rotation, we may assume that $\alpha = 0$.

Choose a positive integer p such that $2\pi/p<\beta$. Let $z=w^{1/p}$ and $g(w)=f(w^{1/p})$, then we have

$$g(w)=\sum_{k=0}^{\infty} c_k w^{n_k/p}, \qquad \text{where} \quad (n_{k+1}/p)/(n_k/p)\geqq q>1\,,$$

which is holomorphic in $D-\{0\}$. A careful checking in [16], we see that Theorem M′ is still true no matter the positive numbers n_k are integer or not. It follows that the function $g(w)$ assumes every complex value infinitely often in $D-\{0\}$, so does the function $f(z)$ in $\Delta(0, 2\pi/p)$. This proves the theorem.

As a consequence of the above theorem, we obtain immediately Theorem 6. Also note that Theorem 7 should be extended to a multiple gap series instead of a single one.

8. Errata and corrections.

We recall the definitions of Julia points and ρ^*-points. Following Collingwood and Piranian [6], we say that $e^{i\theta}$ is a Julia point of a function $f(z)$ if in each Stolz angle having one vertex at $e^{i\theta}$ the function $f(z)$ assumes all values on the Riemann sphere Ω except possibly two.

Also as in Gavrilov [9], we call a sequence of points $\{z_n\}$ in D, $|z_n|\to 1$, a ρ-sequence (or P-sequence) of a function $f(z)$ if any $\varepsilon>0$ and any infinite subsequence $\{z_{n_k}\}$, the function $f(z)$ assumes in the set $\bigcup_k\{z\in D,\ \rho(z, z_{n_k})<\varepsilon\}$ each value in Ω infinitely often, with at most two exceptions, where $\rho(a, b)=(1/2)\ln[(1+u)/(1-u)]$, $u=|(a-b)/(1-\bar{a}b)|$. We then call $e^{i\theta}$ a ρ^*-point of $f(z)$ if each Stolz angle with one vertex at $e^{i\theta}$ possesses a ρ-sequence.

For each function $f(z)$, we let $J(f)$ and $\rho^*(f)$ be the set of Julia and ρ^*-points of $f(z)$ respectively. In [11, Theorem 3], we proved the following

Theorem. *There is a function $f(z)$ holomorphic in D such that $\rho^*(f)=\emptyset$ and* mea. $J(f)=2\pi$.

It was pointed out by Gavrilov few years ago that we made a mistake by using the Gross-Iversen Theorem to conclude the asymptotic value at a Plessner point which is not a Julia point. Unfortunately, we have not been able to prove the above Theorem. What we can do here is to apply some theorems of Gavrilov to give a desired meromorphic function instead of a holomorphic one.

Theorem 8. *There is a function $f(z)$ meromorphlc in D such that $\rho^*(f)=\emptyset$ and* mea. $J(f)=2\pi$.

Proof. Let $f(z)$ be a normal meromorphic function of genus one in the sense of Noshiro [18]. Then by a theorem of Gavrilov [8], we have that $\rho^*(f)=\emptyset$. Moreover, the last assertion mea. $J(f)=2\pi$ follows from [9, Theorem 3].

Finally, we shall correct two theorems in the same paper [11, Theorems 12 and 13]. This requires only a change of definition. Recall that a point $e^{i\theta}$ is called a Meier point of a function f if (i) the cluster set $C(f, e^{i\theta})$ at $e^{i\theta}$ is subtotal (i.e. a proper subset of Ω) and (ii) the chordal cluster set $C_{\rho(\phi)}(f, e^{i\theta})$ $=C(f, e^{i\theta})$ for all values of ϕ in $[-\pi/2, \pi/2]$, (see [5, p. 153]).

We now omit the first condition and we call $e^{i\theta}$ a weak Meier point if the above condition (ii) holds. Denote by $M^*(f)$ the set of all weak Meier points of the function f. With this definition, what we have proved in [11, Theorems 12 and 13] are the following.

Theorem 9. *If $f(z)$ is meromorphic in D, then all points of C except a set of first category belong to $M^*(f) \cup \rho^*(f)$ and therefore to $M^*(f) \cup J(f)$.*

Theorem 10. *If $f(z)$ is normal in D, then all points of C except a set of first category belong to $M^*(f)$.*

Note that the set $M^*(f)$ cannot be replaced by $M(f)$, the set of all Meier points, in the above theorems. For instance, the elliptic modular function f is normal for which the cluster set $C(f, e^{i\theta})=\Omega$ for all $e^{i\theta}$ except at most a countable subset of C, so that the set $M(f)$ is at most countable. Hence both Theorems 9 and 10 are no longer true if $M^*(f)$ is replaced by $M(f)$.

9. Open problems.

In closing this note, let us pose the following three problems regard to multiple gap series.

Problem 1. *If $f(z)$ is a multiple gap series whose coefficients are both bounded from above and below, is it true that* mea. $J(f)=2\pi$.

Problem 2. *If $f(z)$ is a multiple gap series whose coefficients are bounded from below, is it true that f has radial limit ∞ on a dense subset of C.*

Note that if the coefficients of f are bounded above then the answer is affirmative due to [1, Theorem 3], so that Problem 2 needs to be answered only for the unbounded case.

Finally, we shall mention a problem on automorphic functions and gap series. In [14, Theorem 4], we present an answer to a problem of Rubel as follows:

If G is a Fuchsian group with $\Sigma \supset D$ as its domain of discontinuity, then there does not exist a non-constant Hadamard gap series $f(z)$ which is automorphic on G and meromorphic in Σ.

Recently, in a private communication, Pommerenke asked the following

Problem 3. *Is the above theorem still true if the function $f(z)$ is merely holomorphic in D (not over Σ).*

In fact, Pommerenke attributed this problem to the original question of Rubel.

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