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On certain d-sequence on Rees algebra

By

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Introduction.

In this paper we study the d-sequences on the associated graded ring of an ideal in a Noetherian local ring.

Let A be a Noetherian local ring with maximal ideal $\underline{m}$ and $\underline{q}=(a_1,\dots, a_r)$ an ideal of A. We define the Rees algebra of $\underline{q}$ as the subalgebra

$$A[a_1X,\dots, a_rX]$$

of the polynomial ring $A[X]$ in the indeterminate X over the local ring A, and denote it by

$$R=R(\underline{q}).$$

A sequence of elements $x_1,\dots, x_r$ of a commutative ring R is called a d-sequence if for all $i\geqq 0$ and $k\geqq i+1$, we have the equality

$$[(x_1,\dots, x_i): x_{i+1}x_k]=[(x_1,\dots, x_i): x_k].$$

Any regular sequence is obviously a d-sequence and every system of parameters in a Buchsbaum ring is a d-sequence ([7] Prop. 1.7.).

Let us state here some remarkable properties of the Rees algebra of an ideal $\underline{q}$ generated by a d-sequence $a_1,\dots, a_r$ of a local ring A.

Firstly, if $a_1,\dots, a_r$ form a regular sequence, then so do a_1, $a_2-a_1X,\dots, a_r-a_{r-1}X$, a_rX in the Rees algebra $R(\underline{q})$. Hence if A is a Cohen-Macaulay ring so is $R(\underline{q})$ ([1]). However, the converse of the above is not true in general. It has been quite an important problem to describe the condition of the Rees ring to be a Cohen-Macaulay ring. This has been partially settled in some papers [2], [5], [8], [10].

Secondary, if $a_1,\dots, a_r$ form a d-sequence, the Rees algebra is isomorphic to the symmetric algebra [7]. By virtue of this fact, J. Herzog, A. Simis and W. V. Vasconcelos have given a homological characterization of a d-sequence [4]. Recently, S. Goto and K. Yamagishi have shown results of a d-sequence in more detail than the above ([3]).

We treat in this paper the following question: If $a_1,\dots, a_r$ form a d-sequence, then do a_1, $a_2-a_1X,\dots, a_r-a_{r-1}X$, a_rX form a d-sequence in the Rees algebra? This is not true in general (see example (4.2)). We give in this paper a sufficient

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condition for a_1, $a_2-a_1X,\dots,a_r-a_{r-1}X$, a_rX to form a d-sequence in the Rees algebra.

Let M denote the ideal, introduced in Definition 1.1, of a generalized Cohen-Macaulay ring (abrev. a G.C.M. ring, see also Definition 1.1 for the definition). Then our first result is stated as follows:

Theorem 1. *Let A be a G.C.M. ring and $a_1,\dots,a_r$ a subsystem of parameters of A contained in M. Put $\underline{q}=(a_1,\dots,a_r)$. Then a_1, $a_2-a_1X,\dots,a_r-a_{r-1}X$, a_rX form a d-sequence in $R(\underline{q})$.*

As a consequence to the above theorem, we see that there exists a d-sequence a_1, $a_2-a_1X,\dots,a_r-a_{r-1}X$, a_rX such that $\dim(R(\underline{q})/(a_1, a_2-a_1X,\dots,a_r-a_{r-1}X, a_rX))=\dim A-r$.

On the other hand, C. Huneke has shown that a_1X, $a_2X,\dots,a_rX$ also form a d-sequence in $R(\underline{q})$ if $a_1,\dots,a_r$ form a d-sequence [6]. Notice that $\dim R(\underline{q})/(a_1X, a_2X,\dots,a_rX)=\dim A$.

The main result of this paper presents a d-sequence in the Rees algebra such that the ideal generated by the sequence can have the arbitrary dimension not greater than $\dim A$. It is stated as follows:

Theorem 2. *Let A be a G.C.M. ring and $a_1,\dots,a_r$ a subsystem of parameters of A contained in M. Let n be an integer with $0\leqq n<r$ and define a sequence of elements in $R(\underline{q})$ as:*

$$f_i=\begin{cases} a_i-a_{i-1}X & (0\leqq i\leqq n)\\ a_iX & (n+1\leqq i\leqq r),\end{cases}$$

where we set $a_0=a_{-1}=0$.

Then the sequence

$$\underline{f}=f_1,\dots,f_n,f_{n+1},\dots,f_r$$

form a d-sequence in $R(\underline{q})$ and

$$\dim(R(\underline{q})/(\underline{f}))=\dim A-n.$$

We will prove the above theorems in Section 4. In Section 2, some fundamental lemmas on d-sequences will be prepared.

If $a_1,\dots,a_r$ form a regular sequence, one can find the generators of ideals $[(a_1, a_2-a_1X,\dots,a_j-a_{j-1}X)\colon a_2]$ and $[(a_1, a_2-a_1X,\dots,a_j-a_{j-1}X, a_2)\colon(a_{j+1}-a_jX)]$ ([12]). We also find the generators of the above ideals in the case of d-sequence. This is given in Section 3.

1. Definition and notation.

Definition 1.1. ([11]). Let A be a Noetherian local ring with maximal ideal $\underline{m}$. Then A is called a Generalized Cohen-Macaulay ring (abrev. a G.C.M. ring) if $H^i_{\underline{m}}(A)$

has a finite length for all $i<\dim A=d$. Here $H^i_{\underline{m}}(A)$ denotes the i-th local cohomology module.

This is equivalent to the condition that there exists an ideal M with $A\supseteqq\sqrt{M}\supseteqq\underline{m}$ such that the equality

$$(a_1,\dots,a_j)\colon a_{j+1}=(a_1,\dots,a_j)\colon M$$

holds for every $0\leqq j\leqq d-1$ and for every system of parameters $a_1,\dots,a_d$ contained in M.

Definition 1.2. $U(a_1,\dots,a_{j-1})=(a_1,\dots,a_{j-1})\underset{A}{:}a_j$ for $1\leqq j\leqq r$ and for a sequence of elements $a_1,\dots,a_r$. If A is a G.C.M. ring, then we have

$$U(a_1,\dots,a_j)=(a_1,\dots,a_j)\colon a=(a_1,\dots,a_j)\colon b$$

for any two subsystems of parameters $\{a_1,\dots,a_j,a\}$ and $\{a_1,\dots,a_j,b\}$ contained in M.

Notation 1.3. If f is an element of Rees algebra $R(\underline{q})$, we denote by $f^{(n)}$ the coefficient of the term X^n in f.

Notation 1.4. Let R be a Noetherian ring, $x_1,\dots,x_r$ a sequence of elements, and I an ideal of R. We always denote by $\bar{R}$ the factor ring R/I and $\bar{a}$ the image of an element a of R in the ring $\bar{R}$. Moreover, we denote by $\bar{U}(\bar{x}_1,\dots,\bar{x}_{j-1})$ the ideal $[U(x_1,\dots,x_{j-1})+I]/I$ for every $1\leqq j\leqq r$.

Notation 1.5. Let $a_1,\dots,a_r$ be a sequence of elements of A and put $\underline{q}=(a_1,\dots,a_r)$. For an element f of Rees algebra $R(\underline{q})$ we denote by $\bar{f}$ the image of f in the ring $\bar{R}=R(\underline{q}+U(a_1,\dots,a_j)/U(a_1,\dots,a_j))$ for $1\leqq j\leqq r-1$.

2. Preliminary.

Throughout this paper, let A be a G.C.M. ring of $\dim A=d$ and M the ideal in Definition 1.1. Let $a_1,\dots,a_r$ be a subsystem of parameters for A contained in M and put $\underline{q}=(a_1,\dots,a_r)$. Define a sequence of elements in the Rees algebra $R(\underline{q})$ as:

$$g_j=a_j-a_{j-1}X \quad (1\leqq j\leqq r+1)$$

where we set $a_0=a_{r+1}=0$. We always denote by Q_j the ideal of $R(\underline{q})$ generated by $g_1,\dots,g_j$ for every $1\leqq j\leqq r+1$.

Our first lemma, which we will use frequently, is as follows:

Lemma 2.1. *Let $b_1,\dots,b_t$ be a subsystem of parameters for A contained in M. Then both ring $A/(b_1,\dots,b_t)$ and $A/U(b_1,\dots,b_t)$ are again G.C.M. rings.*

Proof. Let $b_{t+1},\dots,b_d$ be a sequence of elements of A such that $b_1,\dots,b_t$, $b_{t+1},\dots,b_d$ form a system of parameters for A contained in M. Then $b_{t+1},\dots,b_d$ form a system of parameters for $A/(b_1,\dots,b_t)$ and $A/U(b_1,\dots,b_t)$. Since A is a

G.C.M. ring and since $b_1,\dots, b_t, b_{t+1},\dots, b_j, b_{j+1}^2$ form a subsystem of parameters for A, we have

$$\begin{aligned}(b_1,\dots, b_t, b_{t+1},\dots, b_j)&: b_{j+1}\\ &\subset [U(b_1,\dots, b_t), b_{t+1},\dots, b_j]: b_{j+1}\\ &\subset (b_1,\dots, b_t, b_{t+1},\dots, b_j): b_{j+1}^2\\ &= (b_1,\dots, b_t, b_{t+1},\dots, b_j): M\\ &\subset [U(b_1,\dots, b_t), b_{t+1},\dots, b_j]: M\end{aligned}$$

for every $t+1 \leqq j \leqq d$. Thus we have the desired result.

Lemma 2.2. *$U(a_1,\dots, a_{i-1}) \cap \underline{q}^k = (a_1,\dots, a_{i-1})\underline{q}^{k-1}$ for every $1 \leqq i \leqq r$ and all $k>0$.*

Proof. See [2] Lemma 4.2 and [10] Lemma 2.2.

Lemma 2.3. *$U(a_{i+1}) \cap U(a_1,\dots, a_i) \subseteqq (a_{i+1}) + U(0)$ for every $1 \leqq i \leqq r-1$.*

Proof. Suppose that $i=1$. Let x be an element of A and assume that

$$xa_2 = sa_1 \quad \text{and} \quad xa_1 = ta_2$$

for some $s, t \in A$. Then we have $s \in U(a_2^2)$ since $ta_2^2 = xa_1a_2 = sa_1^2$ and a_2^2, a_1 form a d-sequence. Let $xa_2 = s'a_2^2$ for some $s' \in A$, and we have $x - s'a_2 \in U(0)$, i.e., $x \in (a_2) + U(0)$.

Now suppose that $i>1$ and the assertion holds for $i-1$. Let x be an element of $U(a_{i+1}) \cap U(a_1,\dots, a_i)$ and put $\bar{A} = A/(a_1,\dots, a_{i-1})$. Then $x \in \bar{U}(\bar{a}_{i+1}) \cap \bar{U}(\bar{a}_i)$. By virtue of the above result, we have $\bar{x} \in U(\bar{0}) + (\bar{a}_{i+1})$, which implies

$$x \in (a_{i+1}) + U(a_1,\dots, a_{i-1}).$$

Hence $x \in (a_{i+1}) + [U(a_{i+1}) \cap U(a_1,\dots, a_{i-1})]$ and by induction we have $x \in (a_{i+1}) + U(0)$.

Corollary 2.4. *$[\underline{q} + U(a_i) + \cdots + U(a_{i+k})] \cap U(a_1,\dots, a_{i-1}) = (a_1,\dots, a_{i-1}) + U(0)$ for every $1 \leqq i < r$ and $0 \leqq k \leqq r-i$.*

Proof. If $r=1$, then there is nothing to prove. So suppose that $r>1$. If $i=1$, the assertion is obvious. Thus we may assume that $i>1$. Since $\bar{a}_1,\dots, \bar{a}_{i-1}, \bar{a}_{i+1},\cdots,\bar{a}_r$ form a subsystem of parameters in the ring $\bar{A} = A/U(a_i)$ and since $U(\bar{0}) = (\bar{0})$, by induction on r we have

$$[\bar{\underline{q}} + \bar{U}(\bar{a}_{i+1}) + \cdots + \bar{U}(\bar{a}_{i+k})] \cap \bar{U}(\bar{a}_1,\dots, \bar{a}_{i-1}) = (\bar{a}_1,\dots, \bar{a}_{i-1}),$$

which implies

$$\begin{aligned}&[\underline{q} + U(a_i, a_{i+1}) + \cdots + U(a_i, a_{i+k})] \cap U(a_1,\dots, a_i)\\ &\quad = (a_1,\dots, a_{i-1}) + U(a_i).\end{aligned}$$

Thus we have

$$
\begin{aligned}
&[\underline{q}+U(a_i)+\cdots+U(a_{i+k})]\cap U(a_1,\dots,a_{i-1})\\
&\quad\subset[\underline{q}+U(a_i,a_{i+1})+\cdots+U(a_i,a_{i+k})]\cap U(a_1,\dots,a_{i-1})\\
&\quad=(a_1,\dots,a_{i-1})+[U(a_i)\cap U(a_1,\dots,a_{i-1})]\\
&\quad=(a_1,\dots,a_{i-1})+(a_i)+U(0)\\
&\quad=(a_1,\dots,a_i)+U(0).
\end{aligned}
$$

Now, we will give some results about Rees algebra when $a_1,\dots,a_r$ form a d-sequence.

Lemma 2.5. $R(\underline{q}+U(a_1,\dots,a_n)/U(a_1,\dots,a_n))\cong R(\underline{q})/[U(a_1,\dots,a_n),a_1X,\dots,a_nX]$ *for every* $0\leqq n\leqq r-1$.

Proof. See Proposition 4.4 in [2].

Lemma 2.6. $[\underline{q},U(a_1,\dots,a_{i-1}),a_1X,\dots,a_{i-1}X]:a_iX=[\underline{q},U(a_1,\dots,a_{i-1}),a_1X,\dots,a_{i-1}X]$ *for every* $1\leqq i\leqq r$.

Proof. The isomorphism $R/[U(a_1,\dots,a_{i-1}),a_1X,\dots,a_{i-1}X]\cong R(\underline{q}+U(a_1,\dots,a_{i-1})/U(a_1,\dots,a_{i-1}))$ allows us to assume that $i=1$ and that a_1 is a nonzero divisor element. So it suffices to prove that $\underline{q}:a_1X=\underline{q}R$.

Now, let cX^t be an element of $\underline{q}:a_1X$. Then $ca_1\in\underline{q}^{t+2}\cap U(a_1)$ and so by Lemma 2.2, we have $ca_1=c'a_1$ for some $c'\in\underline{q}^{t+1}$. Hence $cX^t\in\underline{q}R$ as a_1 is a nonzero divisor.

Lemma 2.7. *Let* $3\leqq k\leqq j\leqq r-1$ *be integers and* c *an element of* $U(a_1,\dots,a_{j+2-k})$. *Then*

$$a_kc(a_{j+1}-a_jX)\equiv a_2\left[\sum_{t=1}^{j+2-k}(c'_ta_{t+k-3}X-c_ta_{t+k-2}X)\right]\mod Q_j.$$

Here, $c_1,\dots,c_{j+2-k}$ *(resp.* $c'_1,\dots,c'_{j+2-k}$*) are elements of* A *satisfying the following equalities*

$$ca_j=\sum_{t=1}^{j+2-k}c_ta_t\quad(\textit{resp. } ca_{j+1}=\sum_{t=1}^{j+2-k}c'_ta_t).$$

Proof.

$$
\begin{aligned}
a_kc(a_{j+1}-a_jX)&=a_k\sum_{t=1}^{j+2-k}(c'_ta_t-c_ta_tX)\\
&\equiv\sum_{t=1}^{j+2-k}(a_{k-1}Xc'_ta_t-a_kc_ta_{t+1})\\
&\equiv\sum_{t=1}^{j+2-k}(a_{k-1}c'_ta_{t+1}-a_{k-1}Xc_ta_{t+1})\\
&\equiv\cdots\\
&\equiv\sum_{t=1}^{j+2-k}a_2(c'_ta_{t+k-3}X-c_ta_{t+k-2}X)
\end{aligned}
$$

mod Q_j, which follows from the equation $a_i\equiv a_{i-1}X$ mod Q_j for every $1\leqq i\leqq j$.

3. Generators of the two ideals.

The purpose of this section is to find the generators of the following two ideals:

$$(a_1, a_2-a_1X,\dots, a_j-a_{j-1}X): a_2$$

and

$$(a_1, a_2-a_1X,\dots, a_j-a_{j-1}X, a_2): (a_{j+1}-a_jX)$$

for every $1\leqq j\leqq r$ (Henceforth we denote the above two ideals by $(Q_j): a_2$ and $(Q_j, a_2): g_{j+1}$). However, we will find that the system of generators of the ideal $(Q_j, a_2): g_{j+1}$ is somewhat complicated in the case $j\geqq 3$. Thus, firstly, we will treat the cases $j=1, 2$.

Lemma 3.1. (α_1) $(a_1, a_2): g_2=[\underline{q}, U(0)]$.

(β_1) $(a_1): a_2=[U(a_1), a_1X]$.

(α_2) $(Q_2, a_2): g_3=[Q_2, a_2, U(a_1)]$.

(β_2) $(Q_2): a_2=[\underline{q}, U(a_1), a_1X]$.

Proof. (α_1): First, notice that $(a_1, a_2): g_2=(a_1, a_2): a_1X\subset[\underline{q}, U(0)]: a_1X$. Then we have $(a_1, a_2): g_2\subset[\underline{q}, U(0)]$ from Lemma 2.6.

(β_1): Let cX^n be an element of $(a_1): a_2$. Then we have

$$c\in U(a_1)\cap \underline{q}^n=\begin{cases} U(a_1) & (n=0)\\ (a_1)\underline{q}^{n-1} & (n>0)\end{cases}$$

by Lemma 2.2. Thus, in any case, we find $cX^n\in[U(a_1), a_1X]$.

(α_2): Let f be an element of $(Q_2, a_2): g_3$ and put $\bar{R}=R(\underline{q}+U(a_1)/U(a_1))=R(q)/[U(a_1), a_1X]$. Then, we have

$$\bar{f}\in(\bar{Q}_2, \bar{a}_2): \bar{g}_3=(\bar{a}_2): \bar{g}_3\subseteqq(\bar{a}_2, \bar{a}_3): \bar{g}_3=\bar{\underline{q}}\bar{R}$$

by the claim (α_1) in the ring $\bar{R}$. Hence we have $f\in[\underline{q}, U(a_1), a_1X]$ and so f may be expressed as

$$f=h+h'+h''a_1X$$

with $h\in\underline{q}R$, $h'\in U(a_1)R$ and $h''\in R$. Recalling $h(a_3-a_2X)\in(Q_2, a_2)$, we have $ha_3\in(a_1X, a_2X, a_1, a_2)$.

Now, put $h=h^{(0)}+h^{(1)}X+\cdots+h^{(k)}X^k$. Then $h^{(i)}\in\underline{q}^{i+1}\cap U(a_1, a_2)$ for every $0\leqq i\leqq k$, so, by Lemma 2.2, we have $h^{(i)}\in(a_1, a_2)\underline{q}^i$. Thus we see $h\in(a_1, a_2)R\subset(Q_2, a_2)$, which implies $f\in[Q_2, a_2, U(a_1)]$.

(β_2): Let f be an element of R and assume that $fa_2=h_1a_1+h_2g_2$ with some h_1, $h_2\in R$. Then $h_2\in(a_1, a_2): g_2=\underline{q}R+U(0)$ by the claim (α_1). Therefore, we may express

$$fa_2=h_1a_1+h_2a_2-a_1h_2X,$$

which implies $f-h_2 \in (a_1): a_2$. By the claim (β_1), we get $f \in [\underline{q}, U(a_1), a_1X]$.

Proposition 3.2.

(α_j) (Q_j, a_2): $g_{j+1}=[Q_j, U(a_1), a_2, \sum_{k=3}^{j-1} a_kU(a_1,\dots, a_{j+2-k})]$ *for every* $3 \leqq j \leqq r$.

(β_j) (Q_j): $a_2=[\underline{q}, U(a_1),\dots, U(a_{j-1}), a_1X,\dots, a_{j-1}X]$ *for every* $3 \leqq j \leqq r+1$.

Proof. Since (α_2) and (β_2) follows from Lemma 3.1, it suffices to prove

(i) If $s=2,\dots, r$ and both (α_s), (β_s) hold, then (β_{s+1}) holds.

(ii) If $t=2,\dots, r-1$ and both (α_t), (β_t) hold, then (α_{t+1}) holds.

(i): First we will show that $[\underline{q}, U(a_1),\dots, U(a_s), a_1X,\dots, a_sX] \subset (Q_{s+1}): a_2$. Let y be an element of $\underline{q}$. Then $ya_2=(a_2-a_1X)y+a_1yX \in Q_{s+1}$. Thus $\underline{q}R \subset (Q_{s+1}): a_2$.

Next, let $1 \leqq k \leqq s$ be an integer. Let z be an element of $U(a_k)R$ and express $z=cX^n$ $(c \in \underline{q}^n)$. Then by Lemma 2.2 we have $ca_1=da_k$ for some $d \in \underline{q}^n$. Moreover, $da_{k+1}=d'a_1$ for some $d' \in \underline{q}^n$, since $d \in U(a_1)$. On the other hand we observe that

$$cX^na_2 \equiv ca_1X^{n+1} \equiv da_kX^{n+1} \equiv da_{k+1}X^n \equiv d'a_1X^n \equiv a_1d'X^n$$

mod Q_{s+1}, since $a_i \equiv a_{i-1}X$ mod Q_{s+1} for every $2 \leqq i \leqq s+1$. Thus $z=cX^n \in (Q_{s+1}): a_2$.

Finally, observing that

$$\begin{aligned} a_2(a_kX) &= -(a_{k+1}-a_kX)a_2+a_2a_{k+1} \\ &= -g_{k+1}a_2+g_2a_{k+1}-a_1a_{k+1}X \\ &\equiv 0 \end{aligned}$$

mod Q_{s+1} for every $1 \leqq k \leqq s$, we have $a_kX \in (Q_{s+1}): a_2$.

Now, we will consider the opposite inclusion. Let f be an element of R and assume that

$$fa_2=g+ug_{s+1}$$

where $g \in Q_s$ and $u \in R$. Then $ug_{s+1} \in (Q_s, a_2)$, and so $u \in [Q_s, U(a_1), a_2, \sum_{k=3}^{s-1} a_k \cdot U(a_1,\dots, a_{s+2-k})]$ by the claim (α_s). Thus u may be expressed as

$$u=v+y+za_2+\textstyle\sum_{k=3}^{s-1} c_ka_k \tag{3.2.a}$$

where $v \in Q_s$, $y \in U(a_1)R$, $z \in R$ and $c_k \in U(a_1,\dots, a_{s+2-k})R$.

First, since $y \in U(a_1)R$, notice that

$$yg_{s+1}=y_1a_1-y_2a_1X=y_1a_1+y_2g_2-y_2a_2 \tag{3.2.b}$$

for some y_1, $y_2 \in R$.

On the other hand, we have that

$$a_kc_kg_{s+1} \equiv a_2 \textstyle\sum_{p=1}^{s+2-k} (c'_{k,p}a_{p+k-3}X-c_{k,p}a_{p+k-2}X) \tag{3.2.c}$$

mod Q_s for every $3 \leqq k \leqq s-1$, where $c_ka_s=\sum_{p=1}^{s+2-k} c_{k,p}a_p$ and $c_ka_{s+1}=\sum_{p=1}^{s+2-k} c'_{k,p}a_p$.

Put $c=\sum_{p=1}^{s+2-k}(c'_{k,p}a_{p+k-3}-c_{k,p}a_{p+k-2})$. Then clearly $cX\in(a_1X,\dots,a_sX)$ as $p+k-3\leqq(s+2-k)+k-3=s-1$ and $p+k-2\leqq(s+2-k)+k-2=s$.

Combining (3.2.a), (3.2.b) and (3.2.c), we get

$$fa_2\equiv ug_{s+1}\equiv(y+za_2+\textstyle\sum_{k=3}^{s-1}c_ka_k)g_{s+1}$$
$$\equiv -y_2a_2+za_2g_{s+1}+a_2cX$$

mod Q_s. Hence $f+y_2-zg_{s+1}-cX\in(Q_s)\colon a_2$. As (β_s) holds, we have $f+y_2-zg_{s+1}-cX\in[\underline{q},U(a_1),\dots,U(a_{s-1}),a_1X,\dots,a_{s-1}X]$, which implies $f\in[\underline{q},U(a_1),\dots,U(a_s),a_1X,\dots,a_sX]$.

(ii): First, we will show that

$$(Q_{t+1},a_2)\colon g_{t+2}\supset[Q_{t+1},U(a_1),a_2,\textstyle\sum_{k=3}^{t}a_kU(a_1,\dots,a_{t+3-k})].$$

Notice that $g_{t+2}U(a_1)\subset(a_1,a_1X)$, and we have $U(a_1)R\subset(Q_{t+1},a_2)\colon g_{t+2}$. Now, let cX^n be an element of $U(a_1,\dots,a_{t+3-k})R$. Then we have

$$a_kcX^ng_{t+2}\equiv a_2\textstyle\sum_{p=1}^{t+3-k}(c'_pa_{p+k-3}X-c_pa_{p+k-2}X)X^n$$

mod Q_{t+1} by Lemma 2.7. This implies $a_kU(a_1,\dots,a_{t+3-k})R\subseteqq(Q_{t+1},a_2)\colon g_{t+2}$.

Now, let us consider the opposite inclusion. Suppose that the assertion does not hold. Then there exists an element f of $R(\underline{q})$ which has minimal degree among the elements contained in $(Q_{t+1},a_2)\colon g_{t+2}$ but not contained in $[Q_{t+1},U(a_1),a_2,\sum_{k=3}^{t}a_kU(a_1,\dots,a_{t+3-k})]$ (We denote by Q the latter ideal for convenience). Put $\bar{R}=R/[U(a_1),a_1X]\cong R(\underline{q}+U(a_1)/U(a_1))$. Then we get

$$\bar{f}\in[\bar{Q}_{t+1},\bar{a}_3,U(\bar{a}_2),\textstyle\sum_{k=4}^{t}\bar{a}_kU(\bar{a}_2,\dots,\bar{a}_{t+3-k})]$$

from the claim (α_t), which implies

$$f\in[Q_{t+1},a_2,a_3,U(a_1,a_2),\textstyle\sum_{k=4}^{t}a_kU(a_1,a_2,\dots,a_{t+3-k})].$$

Thus, we may express $f=h_1+h_2a_3$ for some $h_1\in U(a_1,a_2)R$ and $h_2\in R$.

Now, put $h_1=h_1^{(p)}X^p+\cdots+h_1^{(0)}$. Then for each $i>0$, we have $h_1^{(i)}\in(a_1,a_2)\underline{q}^{i-1}$ by Lemma 2.2 and we are able to express $h_1^{(i)}=a_1y_1+a_2y_2$ for some $y_1,y_2\in\underline{q}^{i-1}$. Thus we see that

$$h_1^{(i)}X^i=a_1y_1X^i+a_2y_2X^i\equiv a_2y_1X^{i-1}+a_3y_2X^{i-1}$$

mod Q. This allows us to express $f=ha_3+c$ for some $c\in U(a_1,a_2)$. We will show that $c\in Q$. Put $ca_{t+1}=d_1a_1+d_2a_2$, then

$$(c+ha_3)g_{t+2}\equiv a_3hg_{t+2}-d_2a_3$$

mod (Q_{t+1},a_2), which implies

$$hg_{t+2}-d_2\in(Q_{t+1},a_2)\colon a_3.$$

On the other hand, using the claim (β_t) in the ring $\bar{R}=R/[U(a_1),a_1X]$, we have

(3.2.d) $$hg_{t+2}-d_2\in[\underline{q},U(a_1,a_2),\dots,U(a_1,a_t),a_1X,\dots,a_tX].$$

As $U(a_1, a_i) \subset U(a_1,\dots, a_t)$ for every $2 \leqq i \leqq t$, we see $d_2 \in [\underline{q} + U(a_1,\dots, a_t)] \cap U(a_1, a_{t+1}) = (a_{t+1}) + U(a_1)$ from Corollary 2.4. Therefore, we can express $ca_{t+1} = d'_1 a_1 + d'_2 a_2 a_{t+1}$ for some $d'_1, d'_2 \in A$, and hence $c \in (a_2) + U(a_1) \subset Q$.

Now, in order to establish the assertion, we will show that $f \in Q$, which implies the desired contradiction. By (3.2.d) we have $h \in [\underline{q}, U(a_1,\dots, a_t), a_1X,\dots, a_tX] : g_{t+2}$, and hence $h \in [\underline{q}, U(a_1,\dots, a_t), a_1X,\dots, a_tX]$ from Lemma 2.6. Recall that $a_3 U(a_1,\dots, a_t) \subset Q$, $a_3\underline{q} \subset (a_3 - a_2X)\underline{q} + a_2\underline{q}X \subset Q$, and that $a_3 a_j X = -a_3(a_{j+1} - a_jX) + (a_3 - a_2X)a_{j+1} + a_2 a_{j+1} X \in Q$ for every $1 \leqq j \leqq t$, we have $f = ha_3 + c \in Q$.

4. The proof of Theorems 1 and 2.

Proof of Theorem 1: For $1 \leqq j \leqq k \leqq r+1$ we will prove

$$(g_1,\dots, g_{j-1}) : g_j g_k = (g_1,\dots, g_{j-1}) : g_k .$$

If $j=1$, the assertion is obvious. Suppose that $j>1$. Put $\bar{A} = A/a_1A$. Then we have

$$(\bar{g}_2,\dots, \bar{g}_{j-1}) : \bar{g}_j \bar{g}_k = (\bar{g}_2,\dots, \bar{g}_{j-1}) : \bar{g}_k$$

in the ring $\bar{R} = R(\underline{q}/a_1A) = R(\underline{q})/(a_1, a_1X)$ since $\bar{A}$ is a G.C.M. ring and by induction on r. This implies

$$(g_1,\dots, g_{j-1}, a_1X) : g_j g_k = (g_1,\dots, g_{j-1}, a_1X) : g_k .$$

Now, let f be an element of $(g_1,\dots, g_{j-1}) : g_j g_k$. Then $f \in (g_1,\dots, g_{j-1}, a_2) : g_k$ by the above result. Express $fg_k = h + h'a_2$ for some $h \in Q_{j-1}$, $h' \in R$. Then we have

$$h' g_j \in [\underline{q}, U(a_1),\dots, U(a_{j-2}), a_1X,\dots, a_{j-2}X]$$

by the claim (β_{j-1}), since $h'g_j \in (Q_{j-1}) : a_2$. Hence $h' \in [\underline{q}, U(a_1,\dots, a_{j-2}), a_1X,\dots, a_{j-2}X]$ from Lemma 2.6.

On the other hand, put $h' = h'^{(p)}X^p + \cdots + h'^{(0)}$. Then we have

$$h'^{(0)} \in [\underline{q}, U(a_1),\dots, U(a_{k-1}), a_1X,\dots, a_{k-1}X]$$

by the claim (β_k), since $h' \in (Q_k) : a_2$, which is a homogeneous ideal. Thus we have

$$\begin{aligned} h'^{(0)} &\in [\underline{q} + U(a_1) + \cdots + U(a_{k-1})] \cap [\underline{q} + U(a_1,\dots, a_{j-2})] \\ &= \underline{q} + U(a_1) + \cdots + U(a_{j-2}) + [\underline{q} + U(a_{j-1}) + \cdots + U(a_{k-1})] \\ &\quad \cap U(a_1,\dots, a_{j-2}) \\ &= \underline{q} + U(a_1) + \cdots + U(a_{j-2}) \end{aligned}$$

from Corollary 2.4. Moreover, $h' - h'^{(0)} \in [\underline{q}, U(a_1,\dots, a_{j-2}), a_1X,\dots, a_{j-2}X]$, and hence $h' - h'^{(0)} \in [\underline{q}, a_1X,\dots, a_{j-2}X]$, since $h' - h'^{(0)} \in (a_1X,\dots, a_rX)R$. This implies

$$h' \in [\underline{q}, U(a_1),\dots, U(a_1),\dots, U(a_{j-2}), a_1X,\dots, a_{j-2}X] .$$

Recall that $(Q_{j-1}) : a_2 = [\underline{q}, U(a_1),\dots, U(a_{j-2}), a_1X,\dots, a_{j-2}X]$. Then we have

$fg_k \in Q_{j-1}=(g_1,\dots, g_{j-1})$. This completes the proof of Theorem 1.

Proof of Theorem 2:

We have only to prove

$$(f_1,\dots,f_j): f_{j+1}f_k=(f_1,\dots,f_j): f_k$$

for every $0 \leqq j<k \leqq r$. First, suppose that $j \leqq n-1$. Using the sequence

$$a_1,\ a_2-a_1X,\dots,\ a_n-a_{n-1}X$$

for $k \leqq n$ or

$$a_1,\ a_2-a_1X,\dots,\ a_n-a_{n-1}X,\ a_{n+1}-a_nX,\dots,\ a_{k-1}-a_{k-2}X,$$

$$a_{k+1}-a_{k-1}X,\ a_{k+2}-a_{k+1}X,\dots,\ a_r-a_{r-1}X,\ a_k-a_rX,\ a_kX$$

for $k \geqq n+1$, we have the desired result from Theorem 1.

Suppose that $j \geqq n$ and put $P_n=(f_1,\dots, f_n)$. Then we have to prove

(4.1.a) $\qquad (P_n,\ a_{n+1}X,\dots,\ a_jX):\ a_{j+1}Xa_kX=(P_n,\ a_{n+1}X,\dots,\ a_jX):\ a_kX$

for every $n \leqq j<k \leqq r$.

As $U(a_{n+1},\dots, a_j)R \subset (P_n, a_{n+1}X,\dots, a_jX): a_kX \subset (P_n, a_{n+1}X,\dots, a_jX): a_{j+1}Xa_kX$ and $(P_n,\ U(a_{n+1},\dots,\ a_j),\ a_{n+1}X,\dots,\ a_jX):\ a_kX=(P_n,\ a_{n+1}X,\dots,\ a_jX):\ a_kX$, we may consider the above (4.1.a) in the ring $\bar{R}=R/[U(a_{n+1},\dots,\ a_j),\ a_{n+1}X,\dots,\ a_jX] \cong R(\underline{q}+U/U)$, where$U=U(a_{n+1},\dots,\ a_j)$ and the isomorphism follows from Lemma 2.5. Thus we have only to prove

Claim $\quad P_n:\ a_{j+1}Xa_kX=P_n:\ a_kX$ for every $n \leqq j<k \leqq r$.

Proof. If $n=0$ or 1, the claim follows from Corollary 1.2 in [6]. Thus we may assume that $n \geqq 2$. Suppose that $g \in P_n:\ a_{j+1}Xa_kX$. Then we have $g \in (P_n,\ a_2):\ a_{j+1}Xa_kX$. Applying the induction hypothesis on r to the ring $\bar{R}= R/[U(a_1),\ a_1X] \cong R(\underline{q}+U(a_1)/U(a_1))$, we obtain that

$$g \in (P_n,\ U(a_1),\ a_2):\ a_kX=(P_n,\ a_2):\ a_kX.$$

Thus, we can express

(4.1.b) $\qquad ga_kX=h+h'a_2$

for some $h \in P_n$ and $h' \in R$. Notice that $P_n=Q_n$ by the definitions of f_i and g_i for every $1 \leqq i \leqq n$. Then we have

$$h' \in [\underline{q},\ U(a_1,\dots,\ a_{n-1}),\ a_1X,\dots,\ a_{n-1}X]:\ a_{j+1}X$$

$$=[\underline{q},\ U(a_1,\dots,\ a_{n-1}),\ a_1X,\dots,\ a_{n-1}X]$$

from the claim (β_n) and Lemma 2.6, since $h'a_{j+1}X \in P_n:\ a_2$. Now, put $h'=h''+c$ (c is the constant term of h'), and we have $h'' \in (\underline{q},\ a_1X,\dots,\ a_{n-1}X)$ since $h' \in [\underline{q},\ U(a_1,\dots,\ a_{n-1}),\ a_1X,\dots,\ a_{n-1}X]$.

On the other hand, we have

$$c\in[\underline{q},\ U(a_1),\ldots,\ U(a_r),\ a_1X,\ldots,\ a_rX]$$

from (β_r) and Lemma 2.6, since $h'a_2\in(a_1,\ a_2-a_1X,\ldots,\ a_{k-1}-a_{k-2}X,\ a_{k+1}-a_{k-1}X,\ a_{k+2}-a_{k+1}X,\ldots,\ a_r-a_{r-1}X,\ a_k-a_rX,\ a_kX)$. Therefore, we obtain that

$$\begin{aligned}c&\in[\underline{q}+U(a_1,\ldots,\ a_{n-1})]\cap[\underline{q}+U(a_1)+\cdots+U(a_r)]\\&=\underline{q}+U(a_1)+\cdots+U(a_{n-1})+(a_1,\ldots,\ a_{n-1})\\&=\underline{q}+U(a_1)+\cdots+U(a_{n-1})\end{aligned}$$

from Corollary 2.4, which implies $h'a_2\in P_n$ by (β_n).

Now, we will prove the second statement. Put $J=(f_1,\ldots,f_n,\ldots,f_r)$, $P=(a_1,\ldots,\ a_n,\ a_1X,\ldots,\ a_nX,\ldots,\ a_rX)$ and $Q=(f_1,\ldots,f_n,\ f_r)$. Then we have $\dim R/P\leqq\dim R/J\leqq\dim R/Q$. On the other hand, we have $\dim R/P=\dim A-n$, since $R/P\cong A/(a_1,\ldots,\ a_n)$.

Now, put $\bar{R}=R/Q$ and consider the set of elements $a_1,\ a_2-a_1X,\ldots,\ a_n-a_{n-1}X,\ a_{n+1}-a_nX,\ldots,\ a_r-a_{r-1}X,\ a_rX$. Put $Q'=(\bar{a}_{n+1}-\bar{a}_nX,\ldots,\ \bar{a}_r-\bar{a}_{r-1}X)$. Then we have $\dim\bar{R}/Q'=\dim A/(a_1,\ldots,\ a_r)=\dim A-r$. Thus, by Theorem 154 in [9], we have $\dim\bar{R}\leqq\dim A-r+(r-n)=\dim A-n$. Hence $\dim A-n\leqq\dim R/J\leqq\dim A-n$, which implies that $\dim R/J=\dim A-n$. This completes the proof of Theorem 2.

Example 4.2. Even if $a_1,\ldots,\ a_r$ form a d-sequence, a sequence of elements $a_1,\ a_2-a_1X,\ldots,\ a_{j+1}-a_jX,\ldots,\ a_r-a_{r-1}X,\ a_rX$ does not allways form a d-sequence. Indeed, let S, T, U, V and W be indeterminates over a field k. Put $A=k[[S,\ T,\ U,\ V,\ W]]/(TV-UW)=k[[s,\ t,\ u,\ v,\ w]]$. Then, s, t, u form a d-sequence, but s, $t-sX$, $u-tX$, uX don't form a d-sequence, since $w^3\in(s,\ t-sX,\ u-tX)\colon(uX)^2$ and $w^3\notin(s,\ t-sX,\ u-tX)\colon uX$.

Remark 4.3. A sequence of elements $a_1,\ a_2-a_1X,\ldots,\ a_n-a_{n-1}X,\ a_nX,\ldots,\ a_rX$ does not necessarily form a d-sequence.

For example, put $A=k[[Y,\ Z,\ W]]$, $a_1=Y$, $a_2=Z$ and $a_3=W$. Then $a_1,\ a_2-a_1X,\ a_2X,\ a_3X$ don't form a d-sequence. In fact, we have $a_3^2\in(a_1,\ a_2-a_1X)\colon a_2Xa_3X$, but a_3^2 is not contained in $(a_1,\ a_2-a_1X)\colon a_3X$.

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