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Some functional equations which generate both crinkly broken lines and curves

By

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§ 0. Preliminary.

Since the discovery of Peano's plane filling curve, we have known many irregular curves as von Koch curve [5], Pólya curve [11] and Lévy curve [6]. In recent years, these curves have found to be related to Mandelbrot's Fractal theory and studied by many authors. They have shown that these curves can be defined as limits of sequences of broken lines generated by a kind of transformation. In fact, such an idea has already given in [9] by E. H. Moore.

A sequence of complex numbers $a=\{a_n\}_{n=0}^{+\infty} \in \boldsymbol{C}^{\boldsymbol{N}}$ generates a broken line $L(a)$ in the comprex plane whose turning points $\{z_n\}_{n=0}^{+\infty}$ are given by

$$z_0=0, \quad z_n=\sum_{k=0}^{n-1} a_k \quad n=1, 2, \ldots .$$

Definition. For $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) \in \boldsymbol{C}^n-\{0\}$, we define T_γ: $\boldsymbol{C}^{\boldsymbol{N}} \to \boldsymbol{C}^{\boldsymbol{N}}$ such that

$$T_\gamma(a_0, a_1, \ldots)=(\gamma_0 a_0, \gamma_1 a_0, \ldots, \gamma_{n-1} a_0, \gamma_0 a_1, \gamma_1 a_1, \ldots).$$

This transformation T_γ replaces a segment a_j by segments $\gamma_0 a_j$, $\gamma_1 a_j$, $\gamma_2 a_j, \ldots$, $\gamma_{n-1} a_j$. We will treat such a type of transformation in a new view point, which is different from previous studies, in the following sections. In §1, we will direct our attention to the broken lines which are invariant under T_γ. Roughly speaking, the invariant broken line is the eigen vector of the linear map T_γ and in the course of the discussion, the eigen vector turns out to be the solution of a functional equation on the formal power series $\boldsymbol{C}[[z]]$. In §2, it will be also shown that the solution of the functional equation has the natural boundary at the unit circle. In the rest of this section, we will review the previous results in our formulation.

The following theorem gives the process of generation of the irregular curves by T.

Theorem 1. *Let $a=\{a_j\} \in \boldsymbol{C}^{\boldsymbol{N}}$ and $\gamma=(\gamma_0, \ldots, \gamma_{n-1}) \in \boldsymbol{C}^n$. If $\sum_{j=0}^{n-1} \gamma_j=1$, $|\gamma_j|<1$ for all $j=0, 1, \ldots, n-1$ and $L(a)$ is a compact subset of the plane, then $L(T_\gamma^n(a))$ is compact and converges as $n \to +\infty$ in the sense of Hausdorff metric.*

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In the book [7] by Mandelbrot, such γ is called the generator of Fractal curves

Example 1.

(1) Let $\gamma=\left(\frac{1+i}{2}, \frac{1-i}{2}\right)$ and $a=(1, 0, 0,\dots)\in C^N$, then $L(T_\gamma^n(a))$ is convergent to Lévy curve as $n\to+\infty$.

(2) Let $\gamma=\left(\frac{1}{3}, \frac{1}{6}+\frac{\sqrt{3}}{6}i, \frac{1}{6}-\frac{\sqrt{3}}{6}i, \frac{1}{3}\right)$ and $a=\left(1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}, 0, 0,\dots\right)$, then $L(T_\gamma^n(a))$ is convergent to the snowflake curve (i.e. von Koch vurve).

(3) Let $=\left(\frac{1}{2}, \frac{1}{2}i, -\frac{1}{2}i, \frac{1}{2}\right)$ and $a=(1, 0, 0,\dots)$, then $L(T_\gamma^n(a))$ is convergent to Pólya's plane filling curve.

Similar theorem is shown by Dekking [2]. His theory of recurrent sets is able to apply some of the curves generated by T_γ. Hata [4] has studied different type of transformations viewing from the point of self similarity.

Proof of Theorem 1. We will show that

$$d_H(L(T_\gamma a), L(T_\gamma^2 a))\leqq r d_H(L(a), L(T_\gamma a)), \tag{0.1}$$

where $r=\sup\{|\gamma_j|: j=0, 1,\dots, n-1\}$ and $d_H(\cdot,\cdot)$ denotes Hausdorff metric. For the simplicity of the discussion, we assume $a=(1, 0, 0,\dots)$. In this case, $T_\gamma a=(\gamma_0, \gamma_1,\dots, \gamma_{n-1}, 0, 0,\dots)$. Let $a^j=(0, 0,\dots, 0, \gamma_j, 0,\dots)$, then

$$d_H(L(T_\gamma a), L(T_\gamma^2 a))\leqq \sup_{j=0,\dots,n-1} d_H(L(a^j), L(T_\gamma a^j)).$$

Here $d_H(L(a^j), L(T_\gamma a^j))=|\gamma_j| d_H(L(a), L(T_\gamma a))$. Therefore we obtain (0.1). Now, using (0.1) inductively, we have

$$d_H(L(T_\gamma^n a), L(T_\gamma^{n-1} a))\leqq r^n d_H(L(a), L(T_\gamma a)).$$

Hence, for $n<m$, we have

$$\begin{aligned} d_H(L(T_\gamma^n a), L(T_\gamma^m a)) &\leqq \textstyle\sum_{k=n}^{m-1} d_H(L(T_\gamma^k a), L(T_\gamma^{k+1} a)) \\ &\leqq (\textstyle\sum_{k=n}^{m-1} r^k) d_H(L(a), L(T_\gamma a)) \\ &\leqq \frac{r^n}{1-r} d_H(L(a), L(T_\gamma a)). \end{aligned}$$

Thus, $L(T_\gamma^n a)$ is convergent in Hausdorff metric as $n\to+\infty$.

§ 1. Invariant broken lines.

We will establish a new viewpoint on the transformation T_γ from the fact that T_γ is a linear map from C^N to itself. At first, let us consider a broken line which is invariant in shape under T_γ. Here, the shape of a broken line is determined by a equivalence in C^N: $a, b\in C^N$ are equivalent if and only if there exists $\alpha\in C-\{0\}$ and

$a_n = \alpha b_n$ for all $n \in \boldsymbol{N}$. Now, a broken line a is invariant in shape under T_γ when $T_\gamma a$ and a are equivalent in above sense, in other words, a is an eigen vector of T_γ for a nonzero eigen value.

Proposition 1. *T_γ has a nontrivial eigen vector if and only if $\gamma_0 \neq 0$. And in this case, T_γ has only one eigen value γ_0 and the eigen vector corresponds to this eigen value.*

Proof. First, we rewrite explicitely the definition of T_γ. Let $T_\gamma a = (b_0, b_1, \ldots)$, then
$$\gamma_k a_p = b_{nq+k} \tag{1.1}$$
for $k = 0, 1, \ldots, n-1$ and $p = 0, 1, 2, \ldots$.
Let $a = (a_0, a_1, \ldots) \neq 0$ be an eigen vector of T_γ with an eigen value λ. By (1.1), we have
$$\gamma_k a_p = \lambda a_{np+k}. \tag{1.2}$$
For some k and p, $\gamma_k a_p \neq 0$ and this implies $\lambda \neq 0$ by (1.2). Now using the formula $a_{np+k} = \lambda^{-1}\gamma_k a_p$ inductively, it turns out that $a_0 = 0$ implies $a_m = 0$ for all m. Therefore, $a_0 \neq 0$, and hence by $a_0 = \gamma_0 a_0$, we have $\lambda = \gamma_0$. Thus, we obtain $\gamma_0 = \lambda \neq 0$, and the eigen vector is determined inductively from a_0 by the formula $a_{np+k} = \lambda^{-1}\gamma_k a_p$.

By Proposition 1, it turns out that $\gamma_0 \neq 0$ must be assumed to study the broken lines invariant in shape under T_γ. And then, the eigen vector of T_γ is the fixed point of $(\gamma_0)^{-1} T_\gamma = T_{\gamma'}$ where $\gamma' = (1, \gamma_1/\gamma_0, \gamma_2/\gamma_0, \ldots, \gamma_{n-1}/\gamma_0)$. So, let $\omega = (1, \omega_1, \omega_2, \ldots, \omega_{n-1})$ and study the fixed points of T_ω.

It is convenient to identify $\boldsymbol{C}^{\boldsymbol{N}}$ with the formal power series $\boldsymbol{C}[[z]]$ as $(a_0, a_1, a_2, \ldots) \leftrightarrow a_0 + a_1 z + a_2 z^2 + \ldots$.
In this expression, for $f \in \boldsymbol{C}[[z]]$
$$T_\omega f(z) = \psi_\omega(z) f(z^n),$$
where $\psi_\omega(z) = 1 + \omega_1 z + \omega_2 z^2 + \ldots + \omega_{n-1} z^{n-1}$. Therefore, the fixed points of T_ω are the solutions of the functional equation in $\boldsymbol{C}[[z]]$
$$\psi_\omega(z) f(z^n) = f(z). \tag{1.3}$$

Remark. Such a functional equation as (1.3) is seen on several occasions. In the kneading theory of iteration of the interval maps, the functional equation $(1-t)D(f, t^2) = D(f, t)$ is used to characterize the kneading determinant D of the unimodal map f lying on a critical state. See §9 of [8] for the details. In [10], Odlyzko has studied functional equations
$$f(z) = P(z) + f(Q(Z)) \tag{1.4}$$
from the interest in the enumeration of 2, 3-trees. Naturally, this type of functional equations are closely related with the famous works of Fatou and Julia (See [1]). We will treat the details in §3.

Using (1.3) for $f(z^n)$, the solution f of (1.3) turns out to satisfy $\psi_\omega(z)\psi_\omega(z^n) f(z^{n^2}) = f(z)$. So, repeating this process infinitely, we can find the solution of (1.3) as follows.

Theorem 2. *Let $f_\omega(z) = \prod_{k=0}^{+\infty} \psi_\omega(z^{n^k})$, then f_ω is well defined as an element of*

$C[[z]]$, and the solution of (1.3) is given by $\alpha f_\omega(z)$ for $\alpha \in C$.

Proof of Theorem 2. Let $f^{(k)}(z)=\prod_{j=0}^{k-1}\psi_\omega(z^{n^j})$, then $\psi_\omega(z^{n^k})f^{(k)}(z)=f^{(k+1)}(z)$. Here $\psi_\omega(z^{n^k})=1+\omega_1 z^{n^k}+\dots+\omega_{n-1}z^{(n-1)n^k}$, therefore we have

$$f^{(k)}(z)-f^{(k+1)}(z)\in z^{n^k}C[[z]]. \tag{1.5}$$

Hence $f^{(k)}(z)$ is convergent in the formal power series topology and the limit $f_\omega(z)$ has the property

$$f_\omega(z)-f^{(k)}(z)\in z^{n^k}C[[z]]. \tag{1.6}$$

By (1.5) and (1.6), we have $\psi_\omega(z)f_\omega(z^n)-f_\omega(z)\in z^{n^k}C[[z]]$ for all k. Hence $f_\omega(z)$ is the solution of (1.3). By (1.2), for a solution $f(z)=\sum_{j=0}^{+\infty}a_j z^j$ of (1.3), a_n is uniquely determined by a_0. And so, we obtain $f(z)=a_0 f_\omega(z)$.

Definition. For $m\in N$, $n\in N-\{0\}$ and $j=0, 1,\dots, n-1$, we define $S(m: n, j)$ as follows. If $m=\sum_{k=1}^{p} j_k\cdot n^k$ such that $j_k\in\{0, 1,\dots, n-1\}$, then

$$S(m: n, j)=|\{k: j_k=j\}|,$$

where $|\cdot|$ represents the number of elements of the set.
Especially, $S(m)=S(m: 2, 1)$.

The following lemma is immediately verified by above definition.

Lemma. *$S(nm+k: n, j)=S(m: n, j)+\delta_{jk}$ for $k=0, 1, 2,\dots, n-1$.*

Making use of $S(m: n, j)$, $f_\omega(z)$ can be expressed by its coefficients.

Theorem 3. *$f_\omega(z)=\sum_{m=0}^{+\infty}(\prod_{j=1}^{n-1}\omega_j^{S(m:n,j)})z^m$.*

Remark. In this paper, we define $0^0=1$.

Proof of Theorem 3. Let $a_n=\prod_{j=1}^{n-1}\omega_j^{S(m:n,j)}$, then by the lemma, we have

$$\begin{aligned}\omega_k a_p &= \omega_k^{S(p:n,k)+1}\prod_{j=1, j\neq k}^{n-1}\omega_j^{S(p:n,j)}\\ &=\prod_{j=1}^{n-1}\omega_k^{S(np+k:n,j)}=a_{np+k},\end{aligned} \tag{1.7}$$

where $\omega_0=1$. Using (1.2) in the proof of Proposition 1, (1.7) implies that $\sum_{m=0}^{+\infty}a_m z^m$ is a solution of (1.3). By Theorem 2 and the fact that $a_0=1$, we obtain

$$\sum_{m=0}^{+\infty}a_m z^m=f_\omega(z).$$

Example 2. Let $\omega=(1, e^{2\pi i\alpha})$ where $\alpha\in R$, then the solution of $(1+e^{2\pi i\alpha}z)f(z^2)=f(z)$ is $f_\omega(z)=\sum_{m=0}^{+\infty}e^{2\pi i\alpha S(m)}z^m$. If $\alpha=\frac{3}{4}$, then $\omega=\left(1,\frac{1-i}{1+i}\right)$ and $f_\omega(z)$ represents the broken line which is invariant under T_γ of Example 1-(1). For arbitral α, Dekking and Mendès France have studied the broken lines represented by $f_\omega(z)$ in Example 4.3 of [3].

§2. Radious of convergence and natural boundary of $f_\omega(z)$.

We have considered the functional equation (1.3) on the formal power series $C[[z]]$. The solution of (1.3) is given by theorems in §1 as a formal power series. Now, we will discuss some properties of the solution $f_\omega(z)$ as an analytic function in this section. By the way, if $\omega=(1, 0, 0,\dots, 0)$, then $f_\omega(z)\equiv 1$ and this case is often a trivial exception in the below discussion. Therefore we will eliminate this case from our consideration.

Proposition 2. *The radious of convergence of f_ω is* 1.

Proof of Proposition 2. Let a_m be the same as in the proof of Theorem 3 and $w=\sup_{j=0,1,\dots,n-1}|\omega_j|$ where $\omega_0=1$. By the definition of $S(m\colon n, j)$ we have $\sum_{j=1}^{n-1}\cdot S(m\colon n, j)\leqq\log_n m+1$. Therefore, $|a_m|=\prod_{j=1}^{n-1}|\omega_j|^{S(m:n,j)}\leqq w^{\log_n m+1}$. Hence we have $\limsup_{m\to+\infty}|a_m|^{1/m}\leqq\lim_{m\to+\infty}w^{(\log_n m+1)/m}=1$.
On the other hand, if $\omega_k\neq 0$ for some k, then $a_{m(p)}=\omega_k$ where $m(p)=kn^p$. Hence we have $\limsup_{m\to+\infty}|a_m|^{1/m}\leqq\lim_{p\to+\infty}|a_{m(p)}|^{1/m(p)}=1$. Thus we obtain $\limsup_{m\to+\infty}|a_m|^{1/m}=1$.

It is also obtained that $f^{(k)}(z)$ converges uniformly on the unit disk $D=\{z\colon |z|<1\}$ and for $|z|>1$, $f^{(k)}(z)\to+\infty$ as $k\to+\infty$. And then, the next question is naturally on the possibility of analytic continuation of f_ω.

Theorem 4. *f_ω has the natural boundary at the unit circle $C=\{z\colon |z|=1\}$ with the exception of the following case.*

Exception. Let $\zeta_{n-1}=e^{2\pi i/n-1}$ and $\omega=(1, \zeta_{n-1}^m, \zeta_{n-1}^{2m},\dots, \zeta_{n-1}^{(n-2)m}, 1)$ where $m=1, 2,\dots, n-1$, then $f_\omega(z)=1/(1-\zeta_{n-1}^m z)$.

The natural boundary of f_ω can be thought to represent the complexity of the corresponding broken line. For example, $f_\omega(z)$ in Exception corresponds to a regular $(n-1)$ polygon. The rest of this section is filled with the proof of Theorem 4.

Proof of Theorem 4. We will show that if $f_\omega(z)$ can be continuated on an open set $V\supsetneqq D$, then $f_\omega(z)$ is one of the exceptional cases.

Lemma 1. *If $f_\omega(z)$ can be continuated to an analytic function on an open set $V\supsetneqq D$, then $f_\omega(z)$ can be continuated to a rational function on an open set $U\supset\bar{D}$.*

Proof. Let $P(z)=z^n$, then for all $\zeta\in P(U)\cap C$, there exists $z_0\in U\cap C$ such that $P(z_0)=\zeta$. Taking a branch of P^{-1} as $Q(z)$ such that $Q(\zeta)=z_0$, we can continuate $f_\omega(z)$ to a rational function on a neighborhood of ζ by $f_\omega(z)=f_\omega(Q(z))/\psi_\omega(Q(z))$. And so, it is easily verified that $\bigcup_{m\geqq 1}P^n(U)=C$, therefore for all $\zeta\in C$, we can continuate $f_\omega(z)$ to a rational function on a neibhforhood of ζ by repeating above process. And then, all of the function element on a neighborhood of ζ are compatible as the direct analytic continuation from D. Thus we can continuate $f_\omega(z)$ to a rational function $F(z)$ on an open set $U\supset\bar{D}$.

We can choose an open set W such that $U\supset W\supset \bar{D}$ and $P(W)\subset U$, then for all $z\in W$,

$$\psi_{\omega}(z)F(P(z))=F(z). \tag{2.1}$$

Now, the radious of convergence of $f(z)$ is one and there is a pole η of $F(z)$ such that $|\eta|=1$.

Lemma 2. *η is a periodic point of P.*

Proof. By (2.1), we have $\psi_{\omega}(\eta)F(p(\eta))=F(\eta)$. Hence, if $F(\eta)=\infty$, then $F(P(\eta))=\infty$ and $P(\eta)$ is a pole of $F(z)$. The same discussion shows that $P^n(\eta)$ is also a pole of $F(z)$ for $n=0, 1,\dots$. Here, if η is not a periodic point of P, then $\{P^n(\eta)\}_{n=0}^{+\infty}$ has some accumulated points on C. This contradicts the fact that F is a rational function on U.

Let η be a q-periodic point of P and $\eta_0=P^{q-1}(\eta)$. Then $P^{-1}(\eta)$ has exactly n-different points $\eta_0, z_1, z_2,\dots, z_{n-1}$.

Lemma 3. *$\psi_{\omega}(z_j)=0$ for $j=, 1,\dots, n-1$.*

Proof. If $\psi_{\omega}(z_j)\neq 0$, then by (2.1), $F(z_j)=\psi_{\omega}(z_j)F(\eta)=\infty$. Therefore, z_j is a pole of F. Then by the same discussion as Lemma 1, we have z_j is a periodic point of P. This contradicts the fact that η is a q-periodic point and $P^{q-1}(\eta)=\eta_0\neq z_j$.

By the above lemma, $\psi_{\omega}(z)$ must have at least $n-1$ different zeros. On the other hand, $\psi_{\omega}(z)$ is a polynominal of degree $n-1$ at most. Hence we obtain $\psi_{\omega}(z)=\prod_{j=1}^{n-1}(1-z/z_j)$. And now, making the same discussion as above, we also obtain $P^{-1}(\eta_0)=\{P^{q-1}(\eta_0), z_1, z_2,\dots, z_{n-1}\}$. Therefore $\eta_0=\eta$ and η is a fixed point of P. Hence we obtain $\psi_{\omega}(z)=\prod_{j=1}^{n-1}(1-z/z_j)$ where $P^{-1}(P(z_j))=\{P(z_j), z_1, z_2,\dots, z_{n-1}\}$. By the elementary caluclation, the above condition implies that $\psi_{\omega}(z)$ is one of the exceptional cases.

§ 3. The natural boundary and Julia set.

In the preceding section, we show that the solution of (1.3) has the natural boundary at $|z|=1$ except for a few cases. It is also known that Julia set of $z\to z^n$ is $|z|=1$. This correspondence will lead us to a new problem. Let us consider a functional equation

$$f(z)=\psi(z)f(P(z)), \tag{3.1}$$

where ψ is a given entire function and P is a given polynominal which satisfy $\psi(0)=1$, $P(0)=0$ and $|P'(0)|<1$. A continuous function f on an open set U is said to be the solution of (3.1) if $P(U)\subset U$ and $f(z)=\psi(z)f(P(z))$ for all $z\in U$. The problem is the relation between the natural boundary of the solution of (3.1) and the Julia set of P.

Definition. The stable set of 0 is the set

$$W_0^s=\{z: P^n(z)\to 0 \text{ as } n\to +\infty\}.$$

The immediate stable set A_0 of 0 is the component of W_0^s containing 0. $J(P)$ denotes the Julia set of P.

The following proposition is the known fundamental facts about W_0^s, A_0 and $J(P)$. See [1] for the proofs and the details.

Proposition 3. (1) *W_0^s and A_0 are open sets. The frontier of W_0^s is contained in $J(P)$ and the frontier of A_0 is also contained in $J(P)$.* (2) *W_0^s is completely invariant by P and A_0 is foreward invariant by P.*

We can find the solution of (3.1) on W_0^s by the same method used in Theorem 2.

Theorem 5.

(1) *$\prod_{k=0}^{n} \psi(P^k(z))$ is uniformly convergent on W_0^s as $n \to +\infty$.*

(2) *The limit $F(z) = \prod_{k=0}^{+\infty} \psi(P^k(z))$ is the solution of (3.1) on W_0^s.*

(3) *Every solution of (3.1) on a neighborhood of 0 coincides with $\alpha F(z)$ for some $\alpha \in C$ on some neighborhood of 0.*

Proof. (1) We can choose λ and a bounded open set $V \ni 0$ so as to satisfy $|P'(0)| < \lambda < 1$, $P(V) \subset V$ and $|P(z)| < \lambda |z|$ for all $z \in V$. Then let $R = \sup_{z \in V} |z|$, we have

$$|P^n(z)| < \lambda^n R \qquad \text{for all} \quad n = 1, 2, \dots \quad \text{and} \quad z \in V.$$

Hence, $\log |\psi(P^k(z))| < c\lambda^k$ for some $c > 0$. Therefore, $e^{c\lambda^k}$ is the majorant of $\psi(P^k(z))$. $\prod_{k=0}^{+\infty} e^{c\lambda^k}$ is obviously convergent to $e^{c\lambda/\lambda-1}$ and this implies that $\prod_{k=0}^{n} \psi(P^k(z))$ is uniformly convergent on V. Now, for all compact set $K \subset W_0^s$, we have $P^k(K) \subset V$ for sufficiently large k. Hence we can also obtain that $\prod_{k=0}^{n} \psi(P^k(z))$ is uniformly convergent on K. Thus, $\prod_{k=0}^{n} \psi(P^k(z))$ is uniformly convergent on W_0^s.

(2) This is obvious by the definition of $F(z)$ and (1).

(3) Let $f(z)$ be a solution of (3.1) on an open set $U \ni 0$. We can choose an open set $W \ni 0$ such that $U \cap A_0 \supset W$ and $P(W) \subset W$. Then for all $z \in W$, we have $f(z) = \psi(z)\psi(P(z)) \dots \psi(P^{n-1}(z)) f(P^n(z))$. Here, $\prod_{j=0}^{n-1} \psi(P^j(z))$ and $f(P^n(z))$ are uniformly convergent to $F(z)$ and $f(0)$ on W respectively. Thus, we obtain

$$f(z) = f(0)F(z) \quad \text{on} \quad W.$$

Now we can state our problem precisely.

Problem. When does the natural boundary of $F(z)|_{A_0}$ coincide with the frontier of A_0?

The following theorem is one of the sufficient conditions to our problem, which is very simple and meaningful.

Theorem 6. *If there is $\eta \in A_0$ such that $\psi(\eta) = 0$, then the natural boundary of $F(z)|_{A_0}$ is the frontier of A_0.*

Proof. **Lemma.** *Let $\zeta \in A_0$ and $P^m(\zeta) = \eta$, then $F(\zeta) = 0$.*

Proof. This is obvious by the equation

$$F(\zeta)=\psi(\zeta)\ldots\psi(P^m(\zeta))F(P^{m+1}(\zeta)).$$

It is easily verified that $P: A_0 \rightarrow A_0$ is surjective. Therefore, we have $(P|_{A_0})^{-m}(\eta) \neq \phi$ for all $m \geqq 1$. Then, by the above lemma, the accumulated points of $\bigcup_{m\geqq 1}(P|_{A_0})^{-m}(\eta)$ is the accumulated points of the zero's of $F|_{A_0}$. By the theorem of identity, the accumulated points of the zero's of $F|_{A_0}$ are contained in ∂A_0. Therefore we obtain the following lemma.

Lemma. *There exists $\xi \in \partial A_0$ such that $F|_{A_0}$ can not be continuated to any rational map on a neighborhood of ξ.*

Now, if $F|_{A_0}$ can be continuated to some analytic function on an open set $U \supsetneqq A_0$, then making use of the same arguement as is used in the proof of Theorem 4, $F|_{A_0}$ can be continuated to a rational map on an neighborhood of ξ. This contradicts to the preceding lemma.

Remark. In [10], Odlyzko has studied the functional equation (1.4) and encountered a problem similar to ours. To avoid the confusion with our notation, we rewrite (1.4) as

$$g(z)=\phi(z)+g(P(z)). \tag{3.2}$$

Odlyzko assume that $P(z)$ and $\phi(z)$ are nonzero polynominal with real nonnegative coefficients, which satisfy $\phi(0)=P(0)=P'(0)=0$. If we consider the exponential form of (3.2), $e^{g(z)}=e^{\phi(z)}e^{g(P(z))}$ and let $e^{g(z)}=f(z)$ and $e^{\phi(z)}=\psi(z)$, then it comes to our functional equation (3.1). Odlyzko has proved that the natural boundary of the solution of (3.1) is the Julia set of P under some extensive assumptions.

Remark. If $\psi(z)$ and $P(z)$ are formal power series, we can define a linear map $T_\psi^P: C[[z]] \rightarrow C[[z]]$ by $T_\psi^P(f)(z)=\psi(z)f(P(z))$. The eigen vector of T_ψ^P is given by the formal power series $\prod_{k=0}^{+\infty}\left(\frac{1}{\psi(0)}\psi(P^k(z))\right)$ when $\psi(0)\neq 0$, $P(0)=0$ and $P'(0)=0$. This formal power series represents an invariant broken line under T_ψ^P. So, we can generate a broad class of crinkly broken lines by the transfomations T_ψ^P.

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References

[1] P. Blanchard, Complex analytic dynamics on the Riemann sphere, Bull. Amer. Math. Soc. (NS), **11** (1984), 85–141.
[2] F. M. Dekking, Recurrent sets, Adv. in Math., **44** (1982), 78–104.
[3] F. M. Dekking and M. Mendès France, Uniform distribution modulo one: a geometrical view point, J. Reine Angew. Math., **329** (1981), 143–153.
[4] M. Hata, On the structure of self similar sets, to appear in Japan J. Appl. Math., 2.

[5] H. von Koch, Sur une courbe continue sans tangente obtenue par une construction géométrique élémentarire, Arkiv för Mat. Astronomi och Fysik, **1** (1904), 681–702.
[6] P. Lévy, Les courbes planes ou gauches et les surface composées de parties semblables au tout, J. l'Ecole Poly., 1939, 227–292.
[7] B. B. Mandelbrot, The Fractal Geometry of Nature, Freeman, 1982.
[8] J. Milnor and W. Thurston, On iterated maps of the interval II, preprint, Princeton, 1977.
[9] E. H .Moore, On certain crinkly curves, Trans. Amer. Math. Soc., **1** (1900), 72–90.
[10] A. M. Odlyzko, Periodic oscillation of coefficients of power series that satisfy functional equations, Adv. in Math., **44** (1982), 180–205.
[11] G. Pólya, Uber eine Peanoche Kurve, Bull. Acad Sci. Cracovie, A, 1913, 305–313.