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Remarks on null solutions of linear partial differential equations

By

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Introduction.

Null solution. Let $P(x; \partial_x)$ be a linear partial differential operator of order m defined in a neighborhood of the origin in $\boldsymbol{R}^d$. Let $\varphi(x)$ be a real-valued function such that $\varphi(0)=0$ and $\varphi_x(0)\neq 0$. Let S stand for the hypersurface defined by $\varphi(x)=0$. We assume S is characteristic to $P(x; \partial_x)$, i.e., $P_m(x; \varphi_x(x))=0$ on S. Here P_m denotes the principal part of P.

We call a solution u of $Pu=0$ a null solution if $\{0\}\in \operatorname{supp}[u]\subset\{x;\ \varphi(x)\geqq 0\}$. We are concerned, in the present paper, with the question firstly raised by Petrowski whether there exists a null solution of $Pu=0$. When all the coefficients of P and $\varphi(x)$ are analytic, the question is related to the inverse of Holmgren's uniqueness theorem. Since null solution is non-analytic at S, the existence of null solution implies also that the operator is not analytic hypo-elliptic.

Multiplicity. By the way, we defined the multiplicity of characteristic hypersurface, [5]. Let $x\in S$,

$$\Lambda_x=\{(\alpha, \beta);\ P^{(\alpha)}_{m(\beta)}(x, \varphi_x(x))\neq 0\},$$

$$k=\min\{|\alpha|+|\beta|\}, \qquad \text{for} \quad (\alpha, \beta)\in\Lambda_x$$

$$l=\min|\beta|, \qquad \text{for} \quad (\alpha, \beta)\in\Lambda_x\cap\{|\alpha|+|\beta|=k\}.$$

Here $P^{(\alpha)}_{m(\beta)}(x; \xi)=\partial^\beta_x\partial^\alpha_\xi P_m(x; \xi)$, and if Λ_x is empty, we put $k=l=\infty$. We call the pair $(k, l)_x$ the multiplicity of the characteristic hypersurface S at $x\in S$. Evidently, $k\geqq 1$, $0\leqq l\leqq k$, $k-l\leqq m$. This is an invariant notion with respect to the change of variables and also to the choice of $\varphi(x)$, see [5].

Already known facts. Let us assume that all the coefficients of $P(x; \partial_x)$ and $\varphi(x)$ are analytic and the multiplicity (k, l) of characteristic hypersurface S is constant on S itself. If $l<k$, whatever the lower order terms are, there exists a C^∞ null solution which is analytic for $x\notin S$. This fundamental theorem was proved by S. Ouchi [10] preceded by the works of S. Mizohata [8], L. Hörmander [2], J. Persson [12], H. Komatsu [7], and so on. However, when $l=k$, the question seems to take a different aspect. Among such operators there are Fuchs type ones

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defined by M. S. Baouendi and C. Goulaouic [1], see also [5]. We have constructed null solutions for them, [4]. They are distributions in general but analytic for $x \notin S$. We note that there are no C^∞ null solutions for Fuchs type equations, see [1]. Except Fuchs type equations, the author knows no general results yet.

If we don't assume the multiplicity to be constant on S, the problem becomes naturally much more delicate and difficult. We know only some typical examples, see, e.g. F. Trèves [14], S. Mizohata [9], and so on.

Aim. In the present note, restricting ourselves to the first order equations, we make further investigation into the question and reveal its new aspect. We will need there a certain class of distributions (pseudo-functions), which will be defined by means of improper integrals.

§1. Results.

1.1. Null solutions when the multiplicity is constant. Let $P(x, y; \partial_x, \partial_y)$ be a first order linear partial differential operator defined in a neighborhood of the origin $(x, y)=(0, 0)$ in $\boldsymbol{R}\times\boldsymbol{R}^d$. We assume the hyperplane $x=0$ to be characteristic to the operator P, namely $P_1(0, y; 1, 0)=0$. Then we may write

$$P=ax^m\partial_x+bx^n\partial_y+c, \quad (x, y)\in\boldsymbol{R}\times\boldsymbol{R}^d \tag{1.1}$$

where m, n, $d\in\boldsymbol{N}=\{0, 1, 2,\dots\}$, $m\geqq 1$, $\partial_x=\partial/\partial x$, $b=(b_1,\dots, b_d)$, $\partial_y=(\partial_1,\dots, \partial_d)$, $\partial_j=\partial/\partial y_j$, $b\partial_y=b_1\partial_1+\cdots+b_d\partial_d$, and the coefficients are defined in a neighborhood of the origin. We assume the coefficients to be analytic, or to be of C^∞ class when a and b_j are real-valued. For convenience, in this paper, the operator is said to be *analytic* in the former case and to be *hyperbolic* in the latter case.

We consider the homogenuous equation

$$Pu=0. \tag{1.2}$$

We call a solution u of (1.2) a null solution if

$$(0, 0)\in\operatorname{supp}[u]\subset\{x\geqq 0\}. \tag{1.3}$$

If we assume the multiplicity of the characteristic hyperplane $x=0$ to be constant on itself and moreover to be finite (when all the coefficients are analytic, it is always finite), then the following three cases occur.

Case A: $1=m\leqq n$, $a(0, 0)\neq 0$.

Case B: $2\leqq m\leqq n$, $a(0, 0)\neq 0$.

Case C: $0\leqq n<m$, $b_j(0, 0)\neq 0$ for some j.

The multiplicity of the characteristic hyperplane $x=0$ is $(1, 1)$ in the case A, (m, m) in the case B, and $(n+1, n)$ in the case C.

Theorem A. *In the case A, there exists a $\mathscr{D}'$ (distribution) null solution which is analytic (C^∞) for $x\neq 0$ when the equation is analytic (hyperbolic respectively).*

There are no C^∞ null solutions.

To state the result for the case B, we define $\{\lambda_k(y)\}_{k=m,\dots,2}$ and $\{c_k(x, y)\}_{k=m,\dots,1}$ by the recurrence relations

$$(1.4)\qquad c_k(0, y)+(-k+1)a(0, y)\lambda_k(y)=0,$$
$$c_{k-1}=\{c_k+(-k+1)a\lambda_k+bx^{n-m+1}\partial_y\lambda_k(y)\}x^{-1},$$

where $c_m=c$. Besides, we put

$$(1.5)\qquad Q(x, y)=\lambda_m(y)x^{-m+1}+\cdots+\lambda_2(y)x^{-1}.$$

Theorem B. *In the case B, the following* 1), 2) *and* 3) *hold, provided that the condition* $c(0, 0)\neq 0$ *is assumed in* 2) *and* 3).

1) *If for any integer* $v>0$, *there are two constants* $0<\delta$, $\delta_v<1$ (δ *does not depend on* v) *such that*

$$(1.6)\qquad \operatorname{Re} Q(x, y)\leqq v\log x, \qquad \text{for}\quad 0<x<\delta_v,\quad |y|<\delta,$$

then there exists a C^∞ null solution which is analytic for $x\neq 0$ when the equation is analytic.

2) *If there are two constants* $C>0$ *and* $0<\delta<1$ *such that*

$$(1.7)\qquad \operatorname{Re} Q(x, y)\leqq C\log(1/x), \qquad \text{for}\quad 0<x<\delta,\quad |y|<\delta,$$

then there exists a $\mathscr{D}'$ null solution which is analytic (C^∞) for $x\neq 0$ when the equation is analytic (hyperbolic respectively).

3) *If for any integer* $v>0$, *there are two constants* $0<\delta$, $\delta_v<1$ (δ *does not depend on* v) *such that*

$$(1.8)\qquad \operatorname{Re} Q(x, y)\geqq v\log(1/x), \qquad \text{for}\quad 0<x<\delta_v,\quad |y|<\delta,$$

then even a distribution null solution does not exist.

Theorem C. *In the case C, there exists a C^∞ null solution which is analytic for $x\neq 0$ when the equation is analytic.*

Remark 1. In the case A, the equation is said to be of Fuchs type. When the equation is analytic, the former part of Theorem A is a special case of *K.* Igari [4], the latter part is of M. S. Baouendi and C. Goulaouic [1], and Theorem C is of S. Ouchi [10].

Remark 2. Though $Q(x, y)$ is a polynomial in x^{-1}, if we admit for $\lambda_j(y)$ C^∞ functions, it can actually occur that

$$\sup_{|y|<\delta} \operatorname{Re} Q(x, y)=C\log x^{-1}(1+o(1)), \quad \text{as}\quad x\longrightarrow +0,$$

where C and δ are some positive constants. We show it by an example. Let

$$f(y)=\int_0^y y^{-3}e^{-2/y}dy, \qquad \text{for}\quad y>0, \quad =0 \qquad \text{for}\quad y\leqq 0,$$

$$g(y)=\int_0^y y^{-3}e^{-1/y}dy, \qquad \text{for} \quad y>0, \quad =0 \qquad \text{for} \quad y\leqq 0,$$

$$\operatorname{Re} Q(x, y)=-f(y)x^{-2}+g(y)x^{-1}.$$

Then, by an elementary calculation, we see that

$$\sup_{|y|<\delta} \operatorname{Re} Q(x, y)=\frac{1}{2}\log x^{-1}+\frac{3}{4}, \quad \text{as} \quad x \longrightarrow +0,$$

where δ is a small positive constant.

Contrarily, when all $\lambda_j(y)$ are polynomials in y, such a logarithmic behavior doesn't happen, instead does an algebraic one, see e.g. L. Hörmander [2], Appendix.

We note that a part of the results stated above were announced in [3].

1.2. Introducing a kind of pseudo-functions. There are apparently differences among the three cases A, B and C. In the former two cases, one can find some similarities to the ordinary differential equations. The main part of this article is Theorem B. We explain the idea to construct null solutions in the case B by considering as an example the ordinary differential equation

$$x^2\frac{du}{dx}+(a-bx)u=0, \quad x\in \boldsymbol{R}, \tag{1.9}$$

where $a\neq 0$ and b are complex constants.

The function $e^{a/x}x^b$ is its solution. The conditions (1.6) and (1.7) in Theorem B correspond to $\operatorname{Re} a<0$ and $\operatorname{Re} a=0$ respectively. In the former case, there are no problems, because $\{e^{a/x}x^b\}_{x>0}$ is evidently a C^∞ null solution of (1.9), and so we consider the second case, writing $a=i\alpha$, $i=\sqrt{-1}$, $\alpha\in \boldsymbol{R}$.

We are to define a distribution (pseudo-function) $\mathrm{Pf.}\,(e^{i\alpha/x}x^b)_{x>0}$ as an analoguous one to the pseudomonomial $\mathrm{Pf.}\,(x^m)_{x>0}$, $m\in \boldsymbol{C}$, which is a distribution defined through the notion of finite part (partie finie) due to J. Hadamard, cf. [13]. We want to define it in the form

$$\mathrm{Pf.}\,(e^{i\alpha/x}x^b)_{x>0}=\partial_x^\mu\{\partial_x^{(-\mu)}(e^{i\alpha/x}x^b)\}_{x>0}, \tag{1.10}$$

where μ is an appropriate non-negative integer and the differentiation is in the distribution sense.

To complete the definition, we must define before-hand $\partial_x^{(-\mu)}(e^{i\alpha/x}x^b)$, a kind of improper integral. If $\operatorname{Re} b>-1$, we define

$$\partial_x^{(-1)}(e^{i\alpha/x}x^b)=\int_0^x e^{i\alpha/x}x^b dx.$$

Integrating by parts, we have

$$\partial_x^{(-1)}(e^{i\alpha/x}x^b)=\frac{-1}{i\alpha}e^{i\alpha/x}x^{b+2}+\frac{b+2}{i\alpha}\partial_x^{(-1)}(e^{i\alpha/x}x^{b+1}) \tag{1.11}$$

for $\operatorname{Re} b>-1$. Let us remark that the right hand side has a definite meaning for

$\operatorname{Re} b > -2$. We define the left hand side for $\operatorname{Re} b > -2$ by this relation. The right hand side has, in turn, a definite meaning for $\operatorname{Re} b > -3$. Using the relation (1.11) again, we define the left hand side for $\operatorname{Re} b > -3$. Repeating this argument, we define $\partial_x^{(-1)}(e^{i\alpha/x}x^b)$ for all $b \in \boldsymbol{C}$. We define furthere $\partial_x^{(-\mu)}(e^{i\alpha/x}x^b)$ for every $\mu \in \boldsymbol{N}$ and $b \in \boldsymbol{C}$ in the same way.
We see then that

$$\partial_x^{(-\mu)}(e^{i\alpha/x}x^b) = O(x^{\operatorname{Re} b + 2\mu}) \quad \text{as} \quad x \longrightarrow +0.$$

Taking a non-negative integer μ satisfying $\operatorname{Re} b + 2\mu > -1$, we define Pf. $(e^{i\alpha/x}x^b)_{x>0}$ by the relation (1.10). We are able to show that this definition doesn't depend on the choice of μ, and that the pseudo-function Pf. $(e^{i\alpha/x}x^b)_{x>0}$ coincides with $e^{i\alpha/x}x^b$ for $x>0$ and satisfies the equation (1.9) in the distribution sense.

By the way, the equation (1.2) has a solution of the form

$$e^{Q(x,y)}x^{\lambda_1(y)}f(x, x\log x, y)$$

where $\lambda_1(y) = -c_1(0, y)/a(0, y)$, $f(x, \xi, y)$ is a function (analytic or C^∞) defined in a neighborhood of the origin $(x, \xi, y) = (0, 0, 0)$ in $\boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}^d$, and we can take $f(0, 0, y)$ arbitrarily. We want to define a pseudo-function corresponding to the above solution under the condition (1.7) and to show that it satisfies the equation in the distribution sense. For this purpose, in §2, we will introduce in a systematic way a class of pseudo-functions by means of improper integrals and state some fundamental properties of them. We note that such an idea as this has already appeared in Y. Kannai [6].

1.3. Variable multiplicity case, An example. Concerning the case of variable multiplicity, we only add a simple example. Let us consider the equation

$$ax^m\partial_x u + by^n\partial_y u = 0, \quad (x, y) \in \boldsymbol{R}^2, \tag{1.12}$$

where $a, b \in \boldsymbol{C}$, $b \neq 0$; $m, n \in \boldsymbol{N}$, $m \geqq 1$, $n \geqq 1$. The line $x=0$ is characteristic to the equation; its multiplicity is $(1, 0)$ for $y \neq 0$, but at the origin it is equal to (m, m) when $m \leqq n$ and to $(n+1, n)$ when $m > n$.

Proposition D. *About the equation* (1.12), *the following* 1) *and* 2) *hold.*

1) *When n is even, there is a C^∞ null solution.*

2) *When n is odd, if a/b is not real-positive, there is a C^∞ null solution; on the contrary, if a/b is real positive, even a continuous null solution does not exist. Here null solution is a solution* (*local*) *such that* $(0, 0) \in \operatorname{supp}[u] \subset \{x \geqq 0\}$.

§2. A kind of pseudo-functions.

As stated in the paragraph 1.2, we introduce a kind of pseudo-functions, which will be used in the following section to prove Theorem B, 2). The author believes they will be useful to some other problems.

Let $\Omega_+ = \{0 < x < \delta, |y| < \delta\}$ and $\Omega = \{|x| < \delta, |y| < \delta\}$. Here $x \in \boldsymbol{R}$, $y = (y_1, \ldots, y_d) \in \boldsymbol{R}^d$, $d \in \boldsymbol{N}$, $0 < \delta < 1$ is a constant.

Definition 2.1. Let $\sigma \in \boldsymbol{R}$. We say that a function $a(x, y)$ defined in Ω_+ belongs to A^σ, i.e. $a \in A^\sigma$, if $a(x, y)$ and all of its derivatives with respect to x are continuous in Ω_+ and satisfy the inequalities

$$(2.1) \qquad |\partial_x^j a(x, y)| \leqq C_j x^{\sigma-j}, \quad (x, y) \in \Omega_+$$

for every $j \in \boldsymbol{N}$. Here C_j are some constants which may depend on j. We denote $A = \bigcup_{\sigma \in R} A^\sigma$.

Proposition 2.1. 1) *If* $a \in A^\sigma$ *and* $m \in \boldsymbol{R}$, *then* $x^m a \in A^{\sigma+m}$.

2) *If* $a \in A^\sigma$ *and* $b \in A^{\sigma'}$, *then* $ab \in A^{\sigma+\sigma'}$ *and* $a+b \in A^{\sigma''}$, $\sigma'' = \min\{\sigma, \sigma'\}$.

3) *If* $f \in \mathscr{B}^\infty(\Omega_+)$, *then* $f \in A^0$.

4) *If* $a \in A^\sigma$, *then* $\partial_x a \in A^{\sigma-1}$.

The proof is evident.

Let $Q(x, y)$ be a continuous function defined in Ω_+. We assume it to satisfy the following condition:

Condition ($\boldsymbol{\Phi}$). There exists a real constant K such that

$$(2.2) \qquad \operatorname{Re} Q(x, y) \leqq K \log(1/x), \quad (x, y) \in \Omega_+ ;$$

and 2) $Q_x = (\partial/\partial x)Q$ does not vanish in Ω_+ and there is a constant $m > 1$ such that $Q_x \in A^{-m}$ and $Q_x^{-1} = 1/Q_x \in A^m$.

We define a kind of improper integral.

Definition 2.2. $-\partial_x^{(-\mu)}(e^Q a)$, $a \in A$, $\mu \in \boldsymbol{N}-$
We define

$$(2.3) \qquad \partial_x^{(0)}(e^Q a) = e^Q a, \qquad \text{for every} \quad a \in A$$

$$\partial_x^{(-\mu)}(e^Q a) = \int_0^x \cdots \int_0^x e^{Q(x,y)} a(x, y)(dx)^\mu, \qquad \text{for} \quad \mu \in \boldsymbol{N}$$

and $a \in \bigcup_{\sigma > K-1} A^\sigma$. By integration by parts we have

$$(2.4) \qquad \partial_x^{(-\mu)}(e^Q a) = \partial_x^{(-\mu+1)}(e^Q Q_x^{-1} a) - \partial_x^{(-\mu)}(e^Q \partial_x(Q_x^{-1} a))$$

for $a \in \bigcup_{\sigma > K-1} A^\sigma$. To define $\partial_x^{(-\mu)}(e^Q a)$ for all $a \in A$, we make use of this relation, and concering μ, the mathematical induction. Assume that $\partial_x^{(-\mu+1)}(e^Q a)$ has been defined for every $a \in A$. Given a real number σ' arbitrarily, we assume also that $\partial_x^{(-\mu)}(e^Q a)$ has been defined for every $a \in \bigcup_{\sigma > \sigma'} A^\sigma$. Note that if $a \in A^\sigma$, then $Q_x^{-1} a \in A^{\sigma+m}$, $\partial_x(Q_x^{-1} a) \in A^{\sigma+m-1}$, and that $m > 1$. We see then that for $a \in A^\sigma$ with $\sigma + m - 1 > \sigma'$ the right hand side of the relation (2.4) has a definite meaning. We define $\partial_x^{(-\mu)}(e^Q a)$ for $a \in A^\sigma$ with $\sigma > \sigma' - m + 1$ by the relation (2.4). Since σ' is an arbitrary number and $m > 1$, $\partial_x^{(-\mu)}(e^Q a)$ is thus defined for all $a \in A$.

Proposition 2.2. *For every $\mu \in N$,*

1) $\partial_x^\mu \partial_x^{(-\mu)}(e^Q a) = e^Q a$, *for every $a \in A$,*

2) $\partial_x^{(-\mu-1)}\partial_x(e^Q a) = \partial_x^{(-\mu)}(e^Q a)$, *for every $a \in A$.*

Proof. 1) For $\mu = 0$ it is evident. Assume now that the identity is true for $\mu - 1$. For μ, if $a \in A^\sigma$ with $\sigma > K-1$, it is also evident. Given a real number σ' arbitrarily, assume it is true for every $a \in A^\sigma$ with $\sigma > \sigma'$. By the relation (2.4) we have

$$\partial_x^\mu \partial_x^{(-\mu)}(e^Q a) = \partial_x^\mu \partial_x^{(-\mu+1)}(e^Q Q_x^{-1} a) - \partial_x^\mu \partial_x^{(-\mu)}(e^Q \partial_x(Q_x^{-1} a)).$$

If $\sigma + m - 1 > \sigma'$, we see by the above assumption that the right hand side is equal to

$$\partial_x(e^Q Q_x^{-1} a) - e^Q \partial_x(Q_x^{-1} a) = e^Q a.$$

Thus the identity is true for every $a \in A^\sigma$ with $\sigma > \sigma' - m + 1$. Since σ' is an arbitrary number, we get the claim.

2) $$\partial_x^{(-\mu-1)}\partial_x(e^Q a) = \partial_x^{(-\mu-1)}(e^Q(Q_x a + a_x))$$
$$= \partial_x^{(-\mu)}(e^Q a) - \partial_x^{(-\mu-1)}(e^Q \partial_x(Q_x^{-1} Q_x a)) + \partial_x^{(-\mu-1)}(e^Q a_x)$$
$$= \partial_x^{(-\mu)}(e^Q a).$$

Thus we have the second claim. Q. E. D.

Let $a \in A^\sigma$. Using the relation (2.4), with a non-negative integer ν such that $\sigma + \nu(m-1) > K-1$, we have

$$\partial_x^{(-\mu)}(e^Q a) = \partial_x^{(-\mu+1)}(e^Q a_1) + \int_0^x \cdots \int_0^x e^Q r_1 (dx)^\mu,$$

where $a_1 = Q_x^{-1} \sum_{k=0}^{\nu-1} (-\partial_x Q_x^{-1})^k a$, $r_1 = (-\partial_x Q_x^{-1})^\nu a$.
Clearly $a_1 \in A^{\sigma+m}$ and $r_1 \in \bigcup_{\sigma > K-1} A^\sigma$, which depend on ν but not on μ. Moreover, for every $\mu' \in N$, there are $a_{\mu'} \in A^{\sigma + m\mu'}$ and $r_k \in \bigcup_{\sigma > K-1} A^\sigma$ such that for every $\mu \geqq \mu'$

$$\partial_x^{(-\mu)}(e^Q a) = \partial^{(-\mu+\mu')}(e^Q a_{\mu'}) + \sum_{k=1}^{\mu'} \int_0^x \cdots \int_0^x e^Q r_k (dx)^{\mu-k+1}.$$

If $\sigma + m\mu' > K-1$,

$$\partial_x^{(-\mu)}(e^Q a) = \int_0^x \cdots \int_0^x \partial_x^{(-\mu')}(e^Q a)\,(dx)^{\mu-\mu'} \tag{2.5}$$

and further, $\partial_x^{(-\mu')}(e^Q a)$ satisfies the inequality

$$|\partial_x^{(-\mu')}(e^Q a)| \leqq \text{const.}\, x^{-1+\alpha}, \quad (x, y) \in \Omega_+ \tag{2.6}$$

with some constant $\alpha > 0$.

Proposition 2.3. *Let $a \in A^\sigma$, and μ, μ' be nonnegative integers satisfying $\mu \geqq \mu'$ and $\sigma + m\mu' > K-1$. Then as a distribution in Ω,*

$$\partial_x^{\mu}\{\partial_x^{(-\mu)}(e^Q a)\}_{x>0}=\partial_x^{\mu'}\{\partial_x^{(-\mu')}(e^Q a)\}_{x>0}.$$

Here $\{f\}_{x>0}$ *stands for the function which is equal to* f *for* $x>0$ *but identically vanishes for* $x\leqq 0$.

By the following lemma, this proposition follows immediately from (2.5) and (2.6).

Lemma 2.4. *Let* $f(x, y)$ *be a measurable function defined on* Ω_+ *and satisfy*

$$|f(x, y)|\leqq \text{const. } x^{-1+\alpha}, \quad (x, y)\in\Omega_+$$

with some constant $\alpha>0$. *Then*

$$\{f\}_{x>0}=\partial_x\left\{\int_0^x f(x, y)dx\right\}_{x>0}, \quad \textit{in } \mathscr{D}'(\Omega) \textit{ sense.}$$

The proof is very elementary, and so we omit it.

Definition 2.3. Let $a\in A^{\sigma}$ and $\mu\in N$ such that $\sigma+m\mu>K-1$. We define the pseudo-function Pf. $(e^Q a)_{x>0}$ by

$$\text{(2.7)} \qquad \text{Pf.}\,(e^Q a)_{x>0}=\partial_x^{\mu}\{\partial_x^{(-\mu)}(e^Q a)\}_{x>0},$$

which is a distribution in Ω.

Surely the right hand side of (2.7) is a distribution in Ω and does not depend on μ, so this is a well-defined notion. We have from (2.4) an important relation

$$\text{(2.8)} \qquad \text{Pf.}\,(e^Q a)_{x>0}=\partial_x\,\text{Pf.}\,(e^Q Q_x^{-1}a)_{x>0}-\text{Pf.}\,(e^Q\partial_x(Q_x^{-1}a))_{x>0}.$$

Theorem E. *The following* 1), 2) *and* 3) *hold.*

1) $\partial_x\,\text{Pf.}\,(e^Q a)_{x>0}=\text{Pf.}\,(\partial_x(e^Q a))_{x>0}$.

2) *Let* $k\in N$ *and* $A^{\sigma;k}=\{a\in A;\ \partial_y^{\beta}a\in A^{\sigma-|\beta|}$ *for* $|\beta|\leqq k\}$. *Assume* $Q\in A^{-m+1;k}$. *Then for every* $a\in\bigcup_{\sigma\in R}A^{\sigma;k}$,

$$\partial_y^{\beta}\,\text{Pf.}\,(e^Q a)_{x>0}=\text{Pf.}\,(\partial_y^{\beta}(e^Q a))_{x>0} \qquad \text{for} \quad |\beta|\leqq k.$$

3) *If* $f\in\mathscr{B}^{\infty}(\Omega)$, *then* $f\,\text{Pf.}\,(e^Q a)_{x>0}=\text{Pf.}\,(e^Q fa)_{x>0}$.

Proof. 1) $\partial_x\partial_x^{\mu}(\partial_x^{(-\mu)}(e^Q a))_{x>0}$

$$=\partial_x^{\mu+1}(\partial^{(-\mu-1)}\partial_x(e^Q a))_{x>0}=\text{Pf.}\,(\partial_x(e^Q a))_{x>0}.$$

Here we used Proposition 2.2, 2).

2) For simplicity we prove the claim only for $|\beta|=1$. By the assumption, $\partial_y^{\beta}a+a\partial_y^{\beta}Q\in A^{\sigma-m}$, if $a\in A^{\sigma;k}$. Hence if $\sigma-m>K-1$, what we want to prove is evident. Now given an arbitrary number σ', we suppose the claim is true for $\sigma>\sigma'$. By means of (2.8) we see that if $\sigma+m-1>\sigma'$, then

$$\partial_y^{\beta}\,\text{Pf.}\,(e^Q a)_{x>0}$$

$$=\partial_x\,\text{Pf.}\,(\partial_y^{\beta}(e^Q Q_x^{-1}a))_{x>0}-\text{Pf.}\,(\partial_y^{\beta}(e^Q\partial_x(Q_x^{-1}a)))_{x>0}$$

$$= \mathrm{Pf.}\,(\partial_y^\beta(e^Q a))_{x>0}.$$

Thus the claim is true for $\sigma > \sigma' - m + 1$, too. Here we used 1) proved above. Since σ' is an arbitrary number, we get the claim for every σ.

3) If $a \in A^\sigma$ with $\sigma > K-1$, the claim is evident. Now let σ' be an arbitrary number, and assume the claim to be true for $\sigma > \sigma'$. If $\sigma + m - 1 > \sigma'$, then

$$\begin{aligned} f\,\mathrm{Pf.}\,(e^Q a)_{x>0} &= \mathrm{Pf.}\,(\partial_x(e^Q f Q_x^{-1} a))_{x>0} \\ &\quad - \mathrm{Pf.}\,(e^Q f_x Q_x^{-1} a))_{x>0} - \mathrm{Pf.}\,(e^Q f \partial_x(Q_x^{-1} a))_{x>0} \\ &= \mathrm{Pf.}\,(e^Q f a)_{x>0}. \end{aligned}$$

Here we used the relation (2.8) and 1) proved above. Since σ' is an arbitrary number, we have the claim for every σ. Q. E. D.

§ 3. Proofs.

3.1. Proof of Theorem A. We may suppose that $a \equiv 1$ and $n = 1$, so the equation we consider is

$$Pu = \{x\partial_x + xb\partial_y + c\}u = 0. \tag{3.1}$$

This is an equation of Fuchs type. When the equation is analytic, the result is a particular case of K. Igari [4]. and so we consider only the hyperbolic case.

Let $\eta(x, y)$ be the solution of $\eta_x + b\eta_y = 0$ with $\eta(0, y) = y$. We denote by Ψ the change of variables: $\xi = x$, $\eta = \eta(x, y)$, and by Ψ^{-1} its inverse: $x = \xi$, $y = y(\xi, \eta)$. The equation (3.1) is transformed into

$$\tilde{P}v = \{\xi\partial_\xi + \tilde{c}\}v = 0, \tag{3.2}$$

where $\tilde{c} = c \circ \Psi^{-1} = c(\xi, \mathrm{y}(\xi, \eta))$.

Now let $\Gamma(z)$ be the gamma function. We know that $1/\Gamma(z)$ is an entire function of $z \in \boldsymbol{C}$. Let $\sigma(\eta)$ be a C^∞ function. We define the distribution Y_σ on a neighborhood of the origin $(\xi, \eta) = (0, 0)$ in $\boldsymbol{R} \times \boldsymbol{R}^d$ by

$$Y_\sigma = \partial_\xi^\mu\{(\xi^{\sigma(\eta)+\mu}/\Gamma(\sigma(\eta)+\mu+1)\}_{\xi>0} \tag{3.3}$$

where μ is a non-negative integer such that $\mathrm{Re}\,\sigma(0) + \mu > -1$. Note that $\sigma(\eta) + \mu + 1$ differs from the poles of $\Gamma(z)$ in a neighborhood of $\eta = 0$. It is easy to show the relations

$$\partial_\xi Y_\sigma = Y_{\sigma-1}, \quad \xi Y_\sigma = (\sigma+1)Y_{\sigma+1}. \tag{3.4}$$

By the mean value theorem, we write $-\tilde{c}(\xi, \eta) = \sigma(\eta) + \xi\rho(\xi, \eta)$. Let $\theta(\xi, \eta)$ be a C^∞ solution of $\theta_\xi - \rho\theta = 0$ with $\theta(0, 0) \neq 0$. If we put $V = \theta(\xi, \eta)Y_\sigma$, then V satisfies the equation (3.2), and consequantly $U = V \circ \Psi$ is a distribution null solution of (3.1). Q. E. D.

Remark 3.1. When $d = 0$, namely in the case of one independent variable, σ in the definition (3.3) is a complex constant and we can see easily that

$$Y_\sigma = \begin{cases} \text{Pf.}\,(x^\sigma)_{x>0}/\Gamma(\sigma+1), & \text{if} \quad \sigma \neq -1, -2, \dots, \\ \delta^{(-\sigma-1)}, & \text{if} \quad \sigma = -1, 2, \dots, \end{cases}$$

which is the distribution appeared in L. Schwartz [13]. We note further that $\{x(d/dx)-\sigma\}Y_\sigma=0$ in the distribution sense for every $\sigma \in \boldsymbol{C}$, but $\{x(d/dx)-\sigma\}\ \text{Pf.}\,(x^\sigma)_{x>0} \neq 0$ when $\sigma=-1, -2, \dots$.

3.2. Proof of Theorem B. We may suppose $a \equiv 1$ and $m=n$. Put

$$u = u_1 \exp\{Q(x, y)\}, \quad P_1 = x\partial_x + xb\partial_y + c_1, \tag{3.5}$$

then the equation becomes

$$Pu = \{x^m\partial_x + x^m b\partial_y + c\}u = x^{m-1}e^Q P_1 u_1 = 0. \tag{3.6}$$

Note that P_1 is an operator of Fuchs type.

Lemma 3.1. *Let P_1 be analytic (or hyperbolic). Put $\lambda_1(y) = -c_1(0, y)$. Then for any analytic (C^∞ resp.) function $g(y)$, there is an analytic (C^∞ resp.) function $f(x, \xi, y)$ of $(x, \xi, y) \in \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}^d$ defined in a neighborhood of the origin such that $f(0, 0, y) = g(y)$ and*

$$P_1\{x^{\lambda_1(y)} f(x, x\log x, y)\} = 0, \quad x \neq 0. \tag{3.7}$$

This lemma will be proved later. Put

$$u_1(x, y) = x^{\lambda_1(y)} f(x, x\log x, y).$$

Then $e^{Q(x,y)}u_1(x, y)$ satisfies the equation (3.6) for $x>0$. Under the condition (1.6), $e^Q u_1$ tends to 0 with infinite order as x tends to $+0$. Therefore if we take $g(0) \neq 0$, $\{e^Q u_1\}_{x>0}$ becomes a C^∞ null solution of (3.6). We have thus completed the proof of the first part 1).

We prove next the second part 2). Take δ small and put $\sigma = \sup_{|y|<\delta} \operatorname{Re}\lambda_1(y)$. Then it follows that $u_1 \in A^\sigma$ and $(u_1)_y \in \bigcap_{\sigma'<\sigma} A^{\sigma'}$. Besides, $Q(x, y)$ satisfies the condition (Φ), and $Q_y \in A^{-m+1}$; particularly the existence of Q_x^{-1} is assured by the assumption $c(0, 0) \neq 0$. Thus all the conditions required in the preceding section are satisfied. Using Theorem E, we see that the pseudo-function distribution $\text{Pf.}\,(e^Q u_1)_{x>0}$ satisfies the equation (3.6) in the distribution sense.

Now we prove the last part 3). Let u be a distribution solution of (3.6) vanishing identically on $x<0$. Applying Lemma 3.1, we see that there is a function $h(x, \xi, y)$ such that $h(0, 0, 0) \neq 0$ and if we put $\theta = x^{-\lambda_1(y)}h(x, x\log x, y)$ with $\lambda_1(y) = -c_1(0, y)$, then $\{x\partial_x + xb\partial_y - c_1\}\theta = 0$. We denote

$$q = \{x^{-m}e^{-Q(x,y)}x^{-\lambda_1(y)}h(x, x\log x, y)\}_{x>0}$$

$$r = \{e^{-Q(x,y)}x^{-\lambda_1(y)}h(x, x\log x, y)\}_{x>0}.$$

Then because of the condition (1.8), q and r are C^∞ functions, and further $qP = P_0 r$ as a differential operator with C^∞ coefficients. Here $P_0 = \partial_x + b\partial_y$. Since $Pu=0$, it follows that $P_0(ru)=0$. By the uniqueness theorem of the non-characteristic analytic (hyperbolic) Cauchy problem, we see that $ru=0$ and consequently $u=0$ for $x\neq 0$.

Take a function $\chi(y)\in C_0^\infty\{|y|<\delta/2\}$ which is equal to 1 for $|y|\leqq\delta/4$. Note that $\operatorname{supp}[\chi u]\subset\{x=0,\ |y|<\delta/2\}$. We see easily that there is an integer $n\geqq 0$ such that

$$\langle\chi u,\ \varphi\rangle=\left\langle\chi u,\ \sum_{k=0}^{n}\frac{x^k}{k!}(\partial_x^k\varphi)(0,\ y)\right\rangle,\qquad \text{for every}\quad \varphi\in\mathscr{D}$$

cf. [2]. We define $u_k\in\mathscr{E}'_y\{|y|<\delta/2\}$ by

$$\left\langle\chi u,\ \frac{x^k}{k!}\psi(y)\right\rangle=\langle u_k,\ \psi\rangle_y,\quad k=0,\ 1,\dots,\ n.$$

Then for every $\varphi\in\mathscr{D}\{|x|<\delta,\ |y|<\delta/4\}$

$$0=\langle\chi u,\ {}^tP\varphi\rangle=\sum_{k=0}^{n}\langle u_k,\ \{\partial_x^k({}^tP\varphi)\}(0,\ y)\rangle_y.$$

If we take such φ that $\partial_x^k\varphi(0,\ y)=0$ for every $k<n$, we have

$$\langle u_n,\ c(0,\ y)(\partial_x^n\varphi)(0,\ y)\rangle_y=0.$$

If we take δ small, $c(0,\ y)\neq 0$ for $|y|\leqq\delta$ because of the condition $c(0,\ 0)\neq 0$. It follows therefore that $u_n=0$ for $|y|\leqq\delta/4$. Repeating the same arguments, we see that all u_k vanish for $|y|\leqq\delta/4$. Therefore $u=0$ in a neighborhood of the origin.

Q. E. D.

Proof of Lemma 3.1. Analytic case: There exists a solution of the form

$$x^{\lambda_1(y)}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{x^j}{j!}\frac{(\log x)^k}{k!}u_{jk}(y)$$

with an arbitrary analytic function $u_{00}(y)$; the series converges in a neighborhood of the origin, cf. [4].

Hyperbolic case: We use the same change of variables as in the paragraph 3.1. The general solution of the transformed equation $\tilde{P}_1 v=\{\xi\partial_\xi+\tilde{c}_1\}v=0$ is given by

$$v=\varphi(\eta)\exp\left\{-\int(\tilde{c}_1/\xi)d\xi\right\}$$

with an arbitrary C^∞ function $\varphi(\eta)$. We can write

$$\begin{aligned}\exp\{-\tilde{c}_1(0,\ \eta)\log\xi\}&=\exp\{-c_1(0,\ \eta(x,\ y))\log x\}\\&=x^{\lambda_1(y)}\exp\{h(x,\ y)x\log x\},\end{aligned}$$

with a certain C^∞ function $h(x,\ y)$. Thus we obtain the claim. Q. E. D.

3.3. Proof of Theorem C. Let p be an integer $\geqq\max\{1,\ n\}$. We have

$$\begin{aligned}&P[\exp\{x^{-p}f(x,\ y)\}]\\&=x^{n-p}\exp\{x^{-p}f\}\{bf_y+ax^{m-n}f_x-pax^{m-n-1}f+cx^{p-n}\}.\end{aligned}$$

By the assumption, $m>n$ and $b_j(0, 0)\neq 0$ for some j. Therefore there is a solution $f(x, y)$ of the equation

$$bf_y+ax^{m-n}f_x-pax^{m-n-1}f+cx^{p-n}=0$$

with $f(0, 0)=-1$. Then, $\{\exp\{x^{-p}f(x, y)\}\}_{x>0}$, which is a C^∞ function in a neighborhood of the origin, is a null solution of the equation (1.2) Q. E. D.

3.4. Proof of Proposition D. For simplicity we consider only the case $m=n=1$. If $a=0$, the proof is evident, so we suppose $a\neq 0$. We show first the sufficiency. Let $z=b\log x-a\log y$ and $\pi_1=\{0<x<1, 0<y<1\}$. Since a/b is not real positive, there is a constant $\lambda\in \boldsymbol{C}$ such that $-\pi/2<-\arg(-\lambda b)=\arg(\lambda a)<\pi/2$. Let $\alpha=|\arg(\lambda a)|$, $0\leqq\alpha<\pi/2$, and let $A(-\alpha, \alpha)=\{z\in \boldsymbol{C};\ -\alpha<\arg z<\alpha\}$. Let σ be a constant >1 such that $0\leqq\sigma\alpha<\pi/2$. We see easily that the function $\exp\{-(\lambda b\log x-\lambda a\log y)^\sigma\}$ is a solution of (1.12) in π_1. We want to prolong it to a full neighborhood of the origin.

If $(x, y)\in\pi_1$, then $(\lambda z)^\sigma\in A(-\sigma\alpha, \sigma\alpha)$. Let $K>0$ be an arbitrary constant. If $|\lambda z|^{\sigma-1}\cos\sigma\alpha\geqq K$, we have

$$\mathrm{Re}(\lambda z)^\sigma>|\lambda z|^\sigma\cos\sigma\alpha\geqq K|\lambda z|\geqq K\,\mathrm{Re}(\lambda z).$$

And therefore

$$|\exp\{-(\lambda z)^\sigma\}|<\exp\{-K\,\mathrm{Re}(\lambda z)\}=x^{-K\mathrm{Re}(\lambda b)}y^{K\mathrm{Re}(\lambda a)}.$$

Since $-\mathrm{Re}(\lambda b)>0$, $\mathrm{Re}(\lambda a)>0$, $\sigma>1$ and $\cos\sigma\alpha>0$, we see that for any constant $N>0$ there is a constant $\delta>0$ such that

$$|\exp\{-(\lambda b\log x-\lambda a\log y)^\sigma\}|\leqq x^N y^N,$$

for $(x, y)\in\pi_1\cap\{0<xy<\delta\}$. Therefore

$$\{\exp\{-(\lambda b\log x-\lambda a\log y)^\sigma\}\}_{x>0,y>0}$$

is infinitely differentiable in a neighborhood of the origin and satisfies the equation (1.12). Here $\{f\}_{x>0,y>0}$ stands for the function which is equal to f for $x>0$, $y>0$ but equal to 0 otherwise.

Now we prove the necessity. We may suppose both a and b are real positive. In the first quadrant, the characteristic lines are given by

$$b\log x-a\log y=\mathit{constant}.$$

Every point (x, y) is connected with the origin $(0, 0)$ by some characteristic line. Let u be a continuous null solution. Since every continuous solution must be constant on each characteristic line, it follows that $u(x, y)=u(0, 0)=0$ for every (x, y) of the first quadrant. By changing y with $-y$, we have the same conclusion for the second quadrant. Q.E.D.

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