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Logarithmic transformations on elliptic fiber spaces

By

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Introduction.

In [6], Kodaira introduced the notion of *logarithmic transformations* and showed that any elliptic surface possessing multiple singular fibers can be reduced to an elliptic surface free from multiple fibers by means of *logarithmic transformations*. In this paper, we will generalize this logarithmic transformations on an elliptic threefold.

The difficulty is that we cannot perform logarithmic transformations along arbitrary divisors on the base space. So the following problem is fundamental.

Problem. Given an elliptic threefold $f: X \longrightarrow S$ over S and a divisor C on S, define "logarithmic transformations along C" and give necessary and sufficient conditions to perform logarithmic transformations along C.

Such an attempt was first done by Perrson [8] and later developed by Nishiguchi [7], Ueno [9] and the present author. They used logarithmic transformations to construct strange non-Kähler degeneration of surfaces. In [3], the author found the simpler method to construct them.

In §1, we shall review the theory of logarithmic transformations on an elliptic surface. In §2, we shall define logarithmic transformations along divisors and give partial answers to the above problem. In §3, as an application of theorem (2.1), we shall construct *examples of non-Kähler degenerations of elliptic surfaces.*

In §4, we shall consider logarithmic transformations in the case where the divisors have only normal crossings. And we shall construct an elliptic threefold which has a multiple fiber of type mI_0 and a singular fiber of type I_0^* along the divisor which are crossing normally.

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Notation and convention.

By an elliptic fiber space $f: V \longrightarrow W$, we mean that f is a proper surjective

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morphism of a complex manifold V to a complex manifold W, where each fiber is connected and the general fibers are smooth elliptic curves. In particular, when W is a surface and V is a three dimensional complex manifold, we say that V is an elliptic threefold over W. Here, "surface" means a two dimensional (not necessarily compact) complex manifold.

For a compact complex manifold X, we use the following notation.

$b_i(X)$: the i-th Betti number of X.

$\kappa(X)$: the Kodaira dimension of X.

K_X: the canonical bundle of X.

$P_m(X)=\dim_c H^{\circ}(X, 0(mK_X))$.

$N_{V/W}$: the normal bundle of V in W, where V is a submanifold of W

$h^{p,q}=\dim_c H^q(X, \Omega_X^p)$

$q(X)=\dim_c H^1(X, 0_X)$

$e=\exp(2\pi\sqrt{-1}/m)$

If D is a divisor on X, we set

$[D]$: the line bundle on X determined by D

$[D]^*$: the dual bundle of $[D]$

(D, D): the self-intersection number of D

§1. Logarithmic transformations.

In this section, we review the theory of logarithmic transformations on an elliptic surface by Kodaira. (c.f. [7], [8])

Let $f: S \longrightarrow C$ be an elliptic surface. Assume that f is smooth on $f^{-1}(U)$, where U is a neighborhood of the point $a \in C$ with a local coordinate τ. Each fiber is a smooth elliptic curve and we can normalize periods of each fiber in the form $(1, \omega(\tau))$, where $\omega: U \longrightarrow H=\{z \in \mathbf{C}; \operatorname{Im}(z)>0\}$ is a holomorphic mapping. Define an automorphism $g(k, l)$ of $U\times\mathbf{C}$ by

$$\begin{array}{ccc} g(k, l): U\times\mathbf{C} & \longrightarrow & U\times\mathbf{C} \\ \Downarrow & & \Downarrow \\ (\tau, \zeta) & \longmapsto & (\tau, \zeta+k\omega(\tau)+l) \end{array}$$

Put $G=\langle g(k, l); (k, l)\in\mathbf{Z}^2\rangle$ and $X=U\times\mathbf{C}/G$. Then X is smooth and by a natural morphism $\phi: X\longrightarrow U$, $(\tau, \zeta)\longmapsto\tau$, X is an elliptic surface over U. Here, by the symbol $(\tau, [\zeta])$, we denote the point on X corresponding to a point $(\tau, \zeta)\in U\times\mathbf{C}$. We infer readily that $f: S|_{f^{-1}(U)} \longrightarrow U$ and $\phi: X \longrightarrow U$ are isomorphic as elliptic fiber spaces over U. Take the m-sheeted covering $\tilde{U}$ of $U: \tilde{U}=\{|t|<\varepsilon^{1/m}\} \longrightarrow U=\{|\tau|<\varepsilon\}$, $\tau=t^m$.

Define an analytic automorphism $\tilde{g}(k, l)$ of $\tilde{U}\times\mathbf{C}$ by

(1) $$\tilde{g}(k, l) : (t, \zeta) \longmapsto (t, \zeta + k\omega(t^m) + l)$$

and put $\tilde{X} = \tilde{U} \times \mathbf{C}/\langle \tilde{g}(k, l);\ (k, l) \in \mathbf{Z}^2 \rangle$.

Next, define an automorphism g of $\tilde{X}$ as follows.

$$\begin{array}{ccc} g : \tilde{X} & \longrightarrow & \tilde{X} \\ \cup\!\!\!| & & \cup\!\!\!| \\ (t, [\zeta]) & \longmapsto & (e_m t, [\zeta + 1/m]) \end{array}$$

Put $Y = \tilde{X}/\langle g \rangle$. There is a natural holomorphic mapping

$$\begin{array}{ccc} h : Y & \longrightarrow & U \\ \cup\!\!\!| & & \cup\!\!\!| \\ [t, [\zeta]] & \longmapsto & t^m, \end{array}$$

where we denote the point on Y corresponding to a point $(t, [\zeta]) \in \tilde{X}$ by the symbol $[t, [\zeta]]$. Y is an elliptic surface over U.

Since $h^*(\tau) = [mE]$, where E is a divisor on Y defined by $t=0$, Y has a multiple fiber of type $\mathrm{m}I_0$ at the origin. By (1); there is an isomorphism $\triangle$ defined by

$$\begin{array}{ccc} \triangle : Y | h^{-1}(U^*) & \xrightarrow{\sim} & X | \phi^{-1}(U^*), \text{ where } U^* = U/\{a\}. \\ \cup\!\!\!| & & \cup\!\!\!| \\ [t, [\zeta]] & \longmapsto & \left(t^m, \left[\zeta - \frac{\log(t)}{2\pi i}\right]\right) \end{array}$$

Now we patch together $S | f^{-1}(C/\{a\})$ and $Y | U$ by the isomorphism $\triangle$, and obtain a new elliptic surface S^* over C. By our construction, the elliptic surface $S^* \longrightarrow C$ has a multiple fiber of type mI_0 at $a \in C$, and $S^* | C/\{a\}$ is isomorphic to $S | C/\{a\}$. We write $S^* := L_a(S)$ and call L_a the logarithmic transformation.

Theorem (1.1). (Kodaira [5]) *Let $f : S \longrightarrow C$ be an elliptic surface and assume that S has multiple fibers of multiplicity m_i at $P_i \in C$ $(i=1, 2, \ldots, r.)$ i.e. $f^*[P_i] = [m_i E_i]$ where E_i is a reduced divisor $f^{-1}(P_i)$.*

(1) *Then S can be obtained from the basic member $B \longrightarrow C$ by twisting and successive logarithmic transformations, that is,*

$S = L_{P_1} L_{P_2} \cdots L_{P_r}(B^\eta)$, *where B^η is obtained by twisting $B \longrightarrow C$ by* $\eta \in H^1(C, 0(B_0^\sharp))$.

(2) (*The canonical bundle formula*)

The canonical bundle of S has the following form.

$K_S = f^*(K_C - f) + \sum_i (m_i - 1)[E_i]$, *where f is a divisor on C with* $\deg(f) = -\chi(0_S)$.

Let $f : V \longrightarrow W$ be an elliptic fiber space.

Put $\Sigma := \{w \in W \mid f$ is not smooth over $f^{-1}(w)\}$ and let F be an irreducible com ponent of Σ with $\dim F = \dim W - 1$. For a general point x of F, there exists a curve Z in W such that Z meets F transversally and $f^{-1}(Z) \longrightarrow Z$ is non-singular. Then $f^{-1}(x)$ is a singular fiber of the elliptic surface $f : V|_Z \longrightarrow Z$ and the multi plicity of $f^{-1}(x)$ is independent of the choice of Z and x. *So the multiplicity of the multiple fibers along F are well-defined.*

§2. Logarithmic transformations along a divisor.

We first state our main theorem in this section.

Theorem (2.1.) *Let S be an arbitrary surface (not necessarily compact) and C a compact smooth curve on S such that its tubular neighborhood is analytically isomorphic to a neighborhood of the zero section of $N_{C/S}$. Let m be an arbitrary integer which divides the self-intersection number $(C.C)$. When $\eta\in H^1(S, 0(E))$ is given (E is a smooth elliptic curve), there exists an elliptic threefold X over S which satisfies the following conditions.*

(1) $X|_{S/C} \xrightarrow{\sim} (S\times E)^{\eta}|_{S/C}$.

(2) *X has multiple fibers of multiplicity m along C.*

If we put $\eta=0$ in theorem (2.1), we get the following corollary.

Corollary (2.2). *Under the same conditions as in theorem (2.1), there exists an elliptic threefold X over S which satisfies the following conditions.*

(1) $X|_{S/C}\xrightarrow{\sim}(S/C)\times E$, *where E is a smooth elliptic curve.*

(2) *X has multiple fibers of multiplicity m along C.*

Corollary (2.3.) *If C is a compact smooth curve on S such that $(C.C)<4-4g$ (g is the genus of the curve C), theorem (2.1) holds automatically.*

Proof of Corollary (2.3). By Grauert [4], if $H^1(C, \Theta_C\otimes_{0_C}N_{C/S}^{*\otimes i})=0$ for every $i>0$, the tubular neighborhood of C is analytically somorphic to the neighborhood of the zero section of $N_{C/S}$, where Θ_C is the sheaf of germs of holomorphic vector fields. In particular, when $(C.C)<4-4g$, this is the case. q.e.d.

To prove theorem (2.1), we need the following lemma.

Lemma (2.4.) *We assume the same conditions as in theorem (2.1). Then there exists a line bundle L on C which is a m-th root of $N_{C/S}$.*

Proof of theorem (2.1) By assumption, we can identify the neighborhood of the zero section of $N_{C/S}$ with the tubular neighborhood of C and we call it U. Take an open covering $\{U_i\}_{i\in I}$ of U with local coordinate (z_i, ζ_i) such that $U_i\cap C$ is defined by $\zeta_i=0$, and ζ_i is a fiber coordinate of $N_{C/S}$. Then $\tilde{U}_i=\{(z_i, w_i); w_i^m=\zeta_i\}$ is an m-sheeted cyclic covering of U_i ramified only along $C\cap U_i$. Now take a line bundle L as in lemma (2.4). The transition functioin of $N_{C/S}$ (resp. L) is expressed by $\{f_{ij}\}$ (resp. $\{g_{ij}\}$) and $\eta\in H^1(U, 0(E))$ expressed by a cocycle $\{\eta_{ij}\}$ with respect to the covering $\{U_i\}$ of U. As $f_{ij}=g_{ij}^m$, $\zeta_i=w_i^m$, and $\zeta_i=f_{ij}\zeta_j$, it follows from lemma (2.4) that $w_i=g_{ij}w_j$ and $\tilde{U}=\bigcup_i \tilde{U}_i$ is well patched.
Then $\tilde{U}\longrightarrow U$ is an m-sheeted cyclic covering of U branched along C.

$$\begin{array}{ccc}\tilde{U} & \longrightarrow & U \\ \uplus & & \uplus \\ (z_i, w_i) & \longmapsto & (z_i, w_i^m)\end{array}$$

Next, identify $(z_i, w_i, [\eta_i])\in\tilde{U}_i\times E$ with $(z_j, w_j, [\eta_j])\in\tilde{U}_j\times E$ if and only if

$$(1)\qquad \begin{cases} z_i = z_j \\ w_i = g_{ij} w_j \\ [\eta_i] = \left[\eta_j + \dfrac{k}{2\pi\sqrt{-1}} \log(g_{ij}) + \eta_{ij}\right], \ k \in \mathbf{Z},\ (k, m) = 1 \end{cases}$$

We can easily see that (1) is well-defined. By patching $\tilde{U}_i \times E's$ in this way, we obtain an elliptic threefold M over $\tilde{U}$.

Define $g_i \in \mathrm{Aut}(M|_{\tilde{U}_i})$ as follows.

$$(2)\quad g_i\colon \underset{\text{ⅲ}}{M|_{\tilde{U}_i}} \longrightarrow \underset{\text{ⅲ}}{M|_{\tilde{U}_i}}$$

$$((z_i, w_i, [\eta_i])) \longmapsto ((z_i, e_m w_i, [\eta_i + k/m]))$$

g_i is compatible with the above patching and defines $g \in \mathrm{Aut}(M)$.

Put $\overline{M} = M/_{\langle g \rangle}$. The group $\langle g \rangle$ acts on M freely and properly discontinuously. Hence $\overline{M}$ is smooth. There is a natural holomorphic map

$$\Phi\colon \underset{\text{ⅲ}}{\overline{M}} \longrightarrow \underset{\text{ⅲ}}{U}$$

$$\overline{((z_i, w_i, [\eta_i]))} \longmapsto ((z_i, w_i^m)),$$

where we denote the point of $\overline{M}$ corresponding to a point $((z_i, w_i, [\eta_i])) \in M|_{\tilde{U}_i}$ by $\overline{((z_i, w_i, [\eta_i]))}$. By this morphism, $\overline{M}$ is an elliptic threefold over U.

We have $\Phi^*[C] = [mE_2]$ as divisors on $\overline{M}$, where E_2 is the support of $\Phi^{-1}(C)$, so $\overline{M}$ has multiple fibers of multiplicity m along C.

By (1) and (2), there is an isomorphism

$$\Lambda\colon \underset{\text{ⅲ}}{\overline{M}|_{U/C}} \xrightarrow{\ \sim\ } [U/C) \times E]^{\eta}$$

$$\overline{((z_i, w_i, [\eta_i]))} \longmapsto \left(z_i, w_i^m, \left[\eta_i - \frac{k}{2\pi\sqrt{-1}} \log(w_i)\right]\right)$$

In fact, by (1) an d(2), we have

$$\left[\eta_i - \frac{k}{2\pi\sqrt{-1}} \log(w_i)\right] = \left[\eta_j - \frac{k}{2\pi\sqrt{-1}} \log(w_j) + \eta_{ij}\right].$$

By patching $\overline{M}|_{U/C}$ and $[(S/C) \times E]^{\eta}$ by the above isomorphism Λ, we obtain an elliptic threefold X over S, which satisfies the properties (1) (2).

q.e.d.

Remark (2.5.) In corollary (2.2), if $(C.C) \neq 0$, by theorem (11.9) in Kodaira [6] iii, $b_1(Y)$ is odd. Hence the elliptic surface Y over C is non-Kähler. Consequently, if C is a rational curve (resp. a smooth elliptic curve,) Y is a Hopf surface. (resp. a Kodaira surface.) (cf. Kodaira [5]) *In particular, X is not in the class $\mathscr{C}$ in the sense of Fujiki* [1]. That is, X cannot be bimeromorphic to any compact Kahler manifold.

Remark (2.6.) As pointed out by the referee, the above theorem also holds for the case where there exists a holomorphic retraction $\alpha: U \longrightarrow C$. This can be easily checked by rewriting the proof without coordinates.

Definition (2.7.) In the proof of theorem (2.1), we patch $\overline{M}|_{U/C}$ and $((S/C) \times E)^{\eta}$ by the isomorphism Λ, and obtain an elliptic threefold $X \longrightarrow S$. We call this process a *logarithmic transformation along* C.

Theorem (2.8) *Let $f: X \longrightarrow S$ be an elliptic threefold over S. (S is not necessarily compact.) Assume that X has multiple fibers of multiplicity m along C, where C is a compact smooth curve on S. Then*
(1) *we have the following exact sequence.*

$$\pi_1(S/C) \longrightarrow \mathbf{Z}/\mathrm{m} \longrightarrow 0.$$

Moreover, if we put $(C.C)=d$, we have $m|d$.
(2) *The elliptic threefold X can be obtained by performing logarithmic transformations along C to an elliptic bundle over S.*

Proof. Take a sufficiently fine open covering $\{U_\lambda\}_\lambda$ of S and C is locally defined by $\phi_\lambda=0$. Put $E=\mathrm{Supp}\ (f^{-1}(C))$, $f^{-1}(U_\lambda)=\bigcup_\nu U_{\lambda\nu}$, and E is locally defined by $\{\psi=\psi_{\lambda\nu}\}$.
As we have $f^*[C]=[mE]$, we can take an m-sheeted cyclic covering $\tilde{X}$ of X defined by $\tilde{X}=U\{\zeta_\lambda^m=f^*\phi_\lambda=(\psi_{\lambda\nu})^m\}$, where $\{\zeta_\lambda\}$ is a fiber coordinate of the line bundle $[E]$. Take the normalization $\tilde{X}^*$ of $\tilde{X}$. Clearly $\tilde{X}^* \longrightarrow X$ is an m-sheeted unramified covering of X.
By our construction, the number of the connected components of each fiber of $\tilde{X}^* \longrightarrow S$ is m outside C. So take the Stein factorization of

$$\tilde{X}^* \longrightarrow S, \qquad \begin{tikzcd} \tilde{X}^* \arrow[rr] \arrow[dr] & & S \\ & \tilde{S} \arrow[ur] & \end{tikzcd}$$

From our construction, it is clear that $\tilde{S} \longrightarrow S$ is an m-sheeted cyclic covering ramified only along C and $\tilde{S}$ is irreducible.
Therefore we have the following exact sequence.

$$\pi_1(S/C) \longrightarrow \mathbf{Z}/\mathrm{m} \longrightarrow 0.$$

Combining this with the following lemma (2.9), we can find an m-sheeted cyclic covering of the tubular neighborhood of C branched along C.
The arguments of Kodaira [6], p. 571, 572 implies the theorem.

Lemma (2.9.) (Wavrik [11]) *Let $M \longrightarrow W$ be an k-sheeted cyclic covering of W branched along C. Then we can find a line bundle F on W so that, over a suitably fine covering $\{V_i\}$ of W, M may be identified with the submanifold of F defined by the equations $\xi_i^k=\phi_i$, where ξ_i is the fiber coordinate of F over V_i, and $\phi_i=0$ is the equation of C in V_i.*

q.e.d.

Corollary (2.10) *Let $f: X \longrightarrow \boldsymbol{P}^2$ be an elliptic threefold over $\boldsymbol{P}^2$ with constant moduli. Assume that f is smooth over f^{-1} $(\boldsymbol{P}^2/C)$, and X has multiple fibers of multiplicity m along C, where C is a smooth curve of degree d in $\boldsymbol{P}^2$.*

Then we have $m|d$. In particular, there is no elliptic threefold that has multiple fibers along a line in $\boldsymbol{P}^2$.

Proof. This follows from theorem (2.8) and the fact that $\pi_1(\mathbf{P}^2/C)\simeq\mathbf{Z}/d$.

Corollary (2.11.) *Let C be a smooth curve of degree $d(\geq 2)$ in $\boldsymbol{P}^2$. Then any tubular neighborhood of C is not isomorphic to that of the zero section of N_{C/P^2}.*

Proof. Assume the contrary. We have $(C.C)=d^2$, so from theorem (2.1), there exists an elliptic threefold over $\mathbf{P}^2$ that has multiple fibers of multiplicity d^2 along C. But corollary (2.10) implies $d^2|d$.
This contradicts $d\geq 2$.

q.e.d.

§3. Degeneration of elliptic surfaces.

Let $f: X\longrightarrow\triangle$ be a surjective, proper morphism from a three dimensional complex manifold X to the unit disc $\triangle=\{t\in\mathbf{C};\ |t|<1\}$. Assume that each fiber is connected and f is of maximal rank on $f^{-1}(\triangle^*)$, where $\triangle^*=\triangle/0$. We call $f: X\longrightarrow\triangle$ *a degeneration of surfaces*, $X_t:=f^{-1}(t)$, $t\in\triangle^*$, the general fiber, and the divisor X_0 on X defined by $f=0$ the *singular fiber*. We decompose X_0 into irreducible components and put $X_0=\sum_i r_iS_i$. By Iitaka [5], we have $\kappa(X_t)=\kappa(X_{t'})$ for any t, $t'\in\triangle^*$. The following conjecture is well-known.

Conjecture. $\kappa(S_i)\leq\kappa(X_t)$ for all i, $t\in\triangle^*$.

By Ueno [12], the conjecture holds when X is bimeromorphic to a Kähler manifold, and when $\kappa(S_t)\geq 1$, it holds if we do not assume that f is a $\mathscr{C}$-morphism. On the other hand, it does not necessarily hold if we consider the non-Kähler degenerations. (See Nishiguchi [9].)

In this section, we will give another counter examples to the above conjecture by means of logarithmic transformations defined in §2.

Example (3.1.) There exists a degeneration of elliptic surfaces $\Phi: X\longrightarrow C$, (that is, there exists a proper surjective morphism from a three dimensional compact complex manifold X to a curve C) which satisfies the following conditions.
(1) $X^*=\Phi^{-1}(C/O)\longrightarrow C/O$ is isomorphic to $S\times(C/O)$, where S is an elliptic surface with $\kappa=-\infty$ and O is a point on C.
(2) The singular fiber of ϕ over the point O has the following form. $X_0=Z_0+2\sum_i Z_i+\sum_i\overline{Z}_i$, where Z_0 is an elliptic surface with $\kappa=0$, or 1, and $Z_i's$ and $\overline{Z}_i's$ are Hopf surfaces.

Construction.

Step 1. Let C be an arbitrary smooth curve and let E be a smooth elliptic curve with the period $(1,\tau)$, Im $(\tau)>0$. For any element $\eta\in H^1(\mathbf{P}^1, 0(E))$ we put $S=(\mathbf{P}^1\times E)^\eta$. That is, S is obtained by twisting $\mathbf{P}^1\times E\longrightarrow\mathbf{P}^1$ by $\eta=\{\eta_{ij}\}$. By Kodaira [5], [6], if η is an element of finite (resp. infinite) order of $H^1(\mathbf{P}^1, 0(E))$,

then S is an elliptic ruled surface. (resp. Hopf surface)
Clearly $\kappa(S)=-\infty$.

Step 2. Next, by a natural projection $p:\mathbf{P}^1\times C\longrightarrow C$, $\mathbf{P}^1\times C$ is a $\mathbf{P}^1$-bundle over C. Take any fiber of p (say, f) and n points on f. (say, p_1, $p_2,\cdots p_n$) We blow up $\mathbf{P}^1\times C$ at the points $p_i(i=1, 2,\ldots, n)$.
And we obtain exceptional curves of the first kind $E_i(i=1, 2,\ldots, n)$.
Secondly, take an arbitrary point Q_i on each E_i, again blow up at each Q_i. $(i=1, 2,\ldots., n)$, and we obtain exceptional curves of the first kind $\triangle_i$. $(i=1, 2,\ldots., n)$. Finally we thus obtain a new surface Y.
i.e. $Y=Q_{q_1}Q_{q_2}\cdots Q_{q_n}Q_{p_1}Q_{p_2}\cdots Q_{p_n}(\mathbf{P}^1\times C)$. (See Figure 1.)
Here we denote by $\bar{f}$ (resp, $\bar{E}_i$) the strict transform of f (resp, E_i).

Step 3. Now, we shall try to perform logarithmic transformations along each

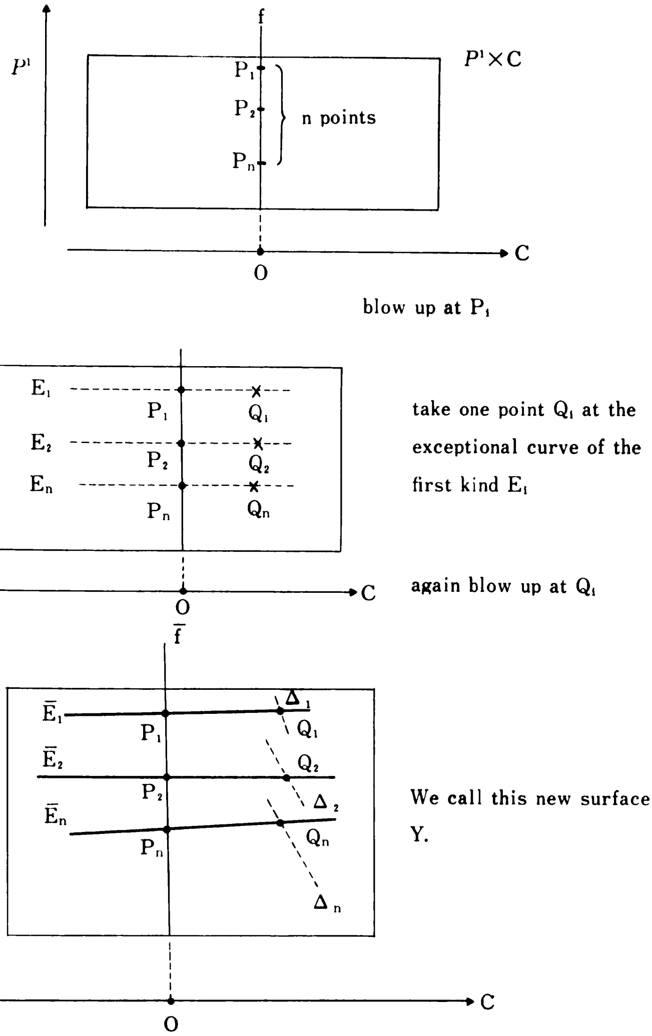

Figure 1.

$\bar{E}_i(i=1, 2,\dots, n)$ on Y. $\eta\in H^1(\mathbf{P}^1, 0(E))$ can be naurally textended to an element $\eta\in H^1(S, 0(E))$ by taking pull-back. Each $\bar{E}_i$ is a smooth rational curve with $(\bar{E}_i)^2=-2$, so the assumptions of corollary (2.3) are satisfied. Therefore, by corollary (2.3), there exists an elliptic threefold $g: X \longrightarrow Y$ over Y which satisfies the following properties.

(1) $X|_{(Y/\bar{E}_1/\cdots/\bar{E}_n)} \xrightarrow{\sim} \{(Y/\bar{E}_1/\cdots/\bar{E}_n)\times E\}^\eta$ $\eta\in H^1(Y, 0(\bar{E}))$

(2) X has multiple fibers of multiplicity 2 along $\bar{E}_i's$.

Step 4. By the natural morphism $X \xrightarrow{g} Y \xrightarrow{p} C$, we consider X as a fiber space over the curve C. Put $\phi=p\circ g$.
We have $S=(\mathbf{P}^1\times E)^\eta$, $\eta\in H^1(\mathbf{P}^1, 0(E))$ in step 1.
And from (1) in step 3, it follows that

$$X|_{\phi^{-1}(C/O)} \xrightarrow{\sim} S\times(C/O)$$

where $f=\phi^{-1}(o)$, $o\in C$.

Therefore the general fiber $X_t(t\neq 0)$ of ϕ is isomorphic to S and we have $\kappa(X_t)=\kappa(S)=-\infty$ for $t\neq 0$.
On the other hand, the singular fiber X_0 can be written as

$$X_0=Z_0+2\sum_i Z_i+\sum_i \bar{Z}_i,$$

where Z_0 (resp. Z_i, $\bar{Z}_i$) is an elliptic surface over $\bar{f}$(resp. $\bar{E}_i$, $\triangle_i$).

On Y, the curve $\bar{f}$ intersects each $\bar{E}_i$ transversally and in view of (2) in Step 3, the elliptic surface Z_0 over $\bar{f}(\simeq\mathbf{P}^1)$ has multiple fibers of multiplicity 2 at each $P_i(i=1, 2,\dots, n)$.
Therefore by the canonical bundle formula in theorem (1.1), the Kodaira dimension of Z_0 can be calculated as follows.
If $n>4$, $\kappa(Z_0)=1$.
If $n=4$, we have $\kappa(Z_0)=0$.
By remark (2.5), the elliptic surface Z_i over $\bar{E}_i(i=1, 2,\dots, n)$ is a Hopr surface.

Remark (3.2). In Step 2, it does not matter *how many times we blow up.* In this respect, *we can construct infinitely many examples like this.*

Example (3.4).

(1) Let E_1, E_2 be any smooth elliptic curves. When $\eta\in H^1(E_2, 0(E_1))$ is given arbitrarily, we twist an elliptic surface $E_1\times E_2 \longrightarrow E_2$ by $\eta=\{\eta_{ij}\}$ and get a new elliptic surface S. i.e. $S=(E_1\times E_2)^\eta$. By Kodaira [7], [8], if η is of finite (resp. infinite) order in $H^1(E_2, 0(E_1))$, S is an abelian surface. (resp. a Kodaira surface.) The same method as in example (3.1) can be applied to this situation. Then an elliptic surface with $\kappa=0$ degenerates into an elliptic surface with $\kappa=1$.

(2) Similarly we can easily construct examples such that a hyperelliptic surface degenerates into an elliptic surface with $\kappa=1$.

(3) In the same way, we get a degeneration of an elliptic $K3$ surface or an Enriques surface.

§4. Some examples.

In this section, we consider an elliptic threefold which has multiple fibers along simple normal crossing divisors. First, we work in a local situation. We consider generalized logarithmic transformations on the polydisc.

Proposition (4.1.) *For an arbitrary integer $\lambda\geq 2$, let $(m_1, m_2,\dots, m_\lambda)$ be a λ-tuple of positive integers with $m_i\geq 2$ for all i, and assume that any two of them are relatively prime. Put $D_i=\{z_i\in\mathbf{C} : |z_i|<\varepsilon\}$, $i=1, 2,\dots, \lambda$. Then there exists an elliptic fiber space X over $D:=D_1\times D_2\times\cdots\times D_\lambda$ which satisfies the following conditions.*

(1) $X|_{D_1^*\times D_2^*\times\cdots\times D_\lambda^*}\xrightarrow{\sim} D_1^*\times D_2^*\times\cdots\times D_\lambda^*\times E$,

where $D_i^*=\{z_i\in\mathbf{C};\ 0<|z_i|<\varepsilon\}$ and E is a smooth elliptic curve with the period $(1, \tau)$, $\mathrm{Im}\ (\tau)>0$.

(2) *X has multiple fibers of multiplicity m_i along $\{z_i=0\}$ for each i.*

Moreover, $f : X\longrightarrow D$ is flat.

Proof. Let
$$\begin{array}{ccc} \tilde{D}_i=\{t_i\in\mathbf{C};\ |t_i|<\varepsilon^{1/m_i}\} & \longrightarrow & D_i=\{z_i\in\mathbf{C};\ |z_i|<\varepsilon\} \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ t_i & \longrightarrow & t_i^{m_i} \end{array}$$

be an m_i-sheeted cyclic covering of D_i. $(1\leq i\leq\lambda)$

Put $\tilde{D}_1:=\tilde{D}_1\times\tilde{D}_2\times\cdots\times\tilde{D}_\lambda$. Then from the assumption,
$$\begin{array}{ccc} \tilde{D}=\tilde{D}_1\times\tilde{D}_2\times\cdots\times\tilde{D}_\lambda & \longrightarrow & D=D_1\times D_2\times\cdots\times D_\lambda \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (t_1, t_2,\dots, t_\lambda) & \longrightarrow & (t_1^{m_1}, t_2^{m_2},\dots, t_\lambda^{m_\lambda}) \end{array}$$

is the $m_1m_2\cdots m_\lambda$-sheeted cyclic covering of D, and we have
$$\mathrm{Gal}\ (\tilde{D}/D)\simeq\mathbf{Z}/m_1\oplus\mathbf{Z}/m_2\oplus\cdots\oplus\mathbf{Z}/m_\lambda.$$

Now, let us consider an analytic automorphism g of $\tilde{D}\times E$ defined by
$$\begin{array}{ccc} g : \tilde{D}_1\times\tilde{D}_2\times\cdots\times\tilde{D}_\lambda\times E & \longrightarrow & \tilde{D}_1\times\tilde{D}_2\times\cdots\times\tilde{D}_\lambda\times E \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ (t_1, t_2,\dots\dots, t_\lambda, [\zeta]) & \longmapsto & \left(e_{m_1}t_1, e_{m_2}t_2,\dots\dots, e_{m_\lambda}t_\lambda, \left[\zeta+\dfrac{1}{m_1m_2\cdots m_\lambda}\right]\right), \end{array}$$

where e_{m_i} is a primitive m_i-th root of unity.

Put $X:=\tilde{D}\times E/\langle g\rangle$. The automorphism g acts on $\tilde{D}\times E$ freely and properly discontinuously, so X is smooth. There is a natural holomorphic map
$$\begin{array}{ccc} f : X & \longrightarrow & D=D_1\times D_2\times\cdots\times D_\lambda \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \overline{(t_1, t_2,\dots, t_\lambda, [\zeta])} & \longmapsto & (t_1^{m_1}, t_2^{m_2},\dots, t_\lambda^{m_\lambda}), \end{array}$$

where by $\overline{(t_1, t_2,\dots, t_\lambda, [\zeta])}$ we denote the point of X corresponding to a point $(t_1, t_2,\dots, t_\lambda, [\zeta])\in\tilde{D}\times E$. By this morphism, X is an elliptic fiber space over D. Clearly X has multiple fibers of multiplicity m_i along $\{z_i=0\}$. There is an isomorphism

$$\Lambda : X|_{D_1^*\times D_2^*\times\cdots\times D_\lambda^*} \xrightarrow{\sim} D_1^*\times D_2^*\times\cdots\times D_\lambda^*\times E$$

$$\overline{(t_1, t_2,\ldots, t_\lambda, [\zeta])} \longmapsto \left(t_1^{m_1}, t_2^{m_2},\ldots, t_\lambda^{m_\lambda}, \left[\zeta-\sum_{i=1}^{\lambda}\frac{\alpha_i}{2\pi\sqrt{-1}}\log(t_i)\right]\right)$$

where $\alpha_i (1\leq i\leq\lambda)\in\mathbf{Z}$ are defined as follows.
By the assumption, there exists $\alpha_i\in\mathbf{Z}(1\leq i\leq\lambda)$ such that
(*) $\alpha_i m_1 m_2\cdots\check{m}_i\cdots m_\lambda\equiv 1 \mod m_i$ for each i.
Take such α_i's and fix them.

q.e.d.

Next, we shall give a generalization of proposition (4.1).

Proposition (4.2.) *Let $\lambda\geq 2$ be an arbitrary positive integer and let $(m_1, m_2,\ldots, m_\lambda)$ be a λ-tuple of positive integers with $m_i\geq 2$ for all i, and assume that any two of them are relatively prime.* Put $D_i=\{z_i\in\mathbf{C};\ |z_i|<\varepsilon\}$ $i=1, 2$, *and take λ affine lines (say, l_i, $1\leq i\leq\lambda$) on $D_1\times D_2$, where the arrangement of lines is arbitrary. Then there exists an elliptic threefold X over $D=D_1\times D_2$ which satisfies the following conditions.*
(Here line denotes the divisor defined by linear forms.)

(1) $X|_{D/\bigcup_{i=1}^{\lambda} l_i} \xrightarrow{\sim} (D/\bigcup_{i=1}^{\lambda} l_i)\times E$, *where E is a smooth elliptic curve.*

(2) *X has multiple fibers of multiplicity m_i along each l_i.*
Moreover, if the lines are in a general position, (that is, if they have no multiple points of multiplicity more than two,) $X\longrightarrow D$ is flat.

Proof. We may assume that l_1, l_2 are defined by $z_1=0$, $z_2=0$, respectively. Besides D_1 and D_2, we take (λ-2) discs.
(say, $D_3=\{z_3\in\mathbf{C};\ |z_3|<\varepsilon\},\ldots,\ D_\lambda=\{z_\lambda\in\mathbf{C};\ |z_\lambda|<\varepsilon\}$.
By proposition (4.1), there exists an elliptic threefold Y over $\prod_{i=1}^{\lambda} D_i$ which satisfies the following conditions.
(1) $Y|_{D_1^*\times D_2^*\times\cdots\times D_\lambda^*} \xrightarrow{\sim} D_1^*\times D_2^*\times\cdots\times D_\lambda^*\times E$
(2) Y has multiple fibers of multiplicity m_i along $z_i=0$ for each i.
Now, let l_i's$(3\leq i\leq\lambda)$ be defined by $f_i(z_1, z_2)=0$ respectively.
Define a submanifold V of $D_1\times D_2\times\cdots\times D_\lambda$ by

$$V=\{z_3=f_3(z_1, z_2),\ldots, z_\lambda=f_\lambda(z_1, z_2)\}.$$

By a natural projection onto $D_1\times D_2$, V is isomorphic to $D:=D_1\times D_2$. We restrict the elliptic fiber space $Y\longrightarrow D_1\times D_2\times\cdots\times D_\lambda$ over V, and by projecting over $D_1\times D_2$, we finally obtain an elliptic threefold $Y|_V$ over $D_1\times D_2$. If the lines are in a general position, $Y|_V$ is smooth. However, if they have multiple points of multiplicity more than two, $Y|_V$ has one-dimensional singularities along a fiber. So, take a non-singular model X of $Y|_V$, then $X\longrightarrow D_1\times D_2$ is the desired elliptic threefold.

(1) By (1) in proposition (4.1), X is trivial over $D/\bigcup_{i=1}^{\lambda} l_i$.

(2) By (2) in proposition (4.1), $X \longrightarrow D$ has multiple fibers of multiplicity m_1, m_2, $m_3, \ldots, m_\lambda$ along $z_1=0$, $z_2=0$,

$$l_3=\{f_3(z_1, z_2)=0\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$l_\lambda=\{f_\lambda(z_1, z_2)=0\} \qquad \text{respectively.}$$

q.e.d.

Remaek (4.3.) If $X \longrightarrow D$ is flat, the relative canonical bundle is given as follows. $K_{X/D}=\sum_{i=1}^{\lambda}(m_i-1)E_i$, where $f^*[l_i]=[m_iE_i]$.

Remark (4.4.) Proposition (4.2) is still true if we replace lines by any divisors with normal crossings as is clear from the proof.

Proposition (4.5.) *Let D_1, D_2 be discs, $D_1=\{x\in\mathbf{C};\ |x|<\varepsilon\}$, $D_2=\{y\in\mathbf{C};\ |y|<\varepsilon\}$. Let m, $n\geq 2$ be arbitrary integers. Then there exists an elliptic threefold X over $D_1\times D_2$, which has regular fibers over $D_1^*\times D_2^*$, multiple fibers of multiplicity m along $x=0$ and those of multiplicity n along $y=0$. Moreover, if m and n are relatively prime, $f: X \longrightarrow D_1\times D_2$ is flat. Otherwise it is not flat.*

Proof. (1) If m and n are relatively prime, the assertion follows from *Proposition* (4.1)

(2) Next, we consider the case when $m=n$.

Define an automorphism g of $\mathbf{C}^2$ by $g: (z_1, z_2) \longmapsto (e_m z_1, e_m^{-1}z_2)$. Then, $\mathbf{C}^2/\langle g\rangle$ has a cyclic quotient singularity of type $(m, m-1)$ at the origin, and $\mathbf{C}^2/\langle g\rangle$ is isomorphic to $\{(w, x, y)\in\mathbf{C}^3;\ w^m-xy=0\}$ The minimal resolution of this singularity is given as follows.

Let $U_i(0\leq i\leq m-1)$ be m copies of $\mathbf{C}^2$ with coordinates (u_i, v_i) respectively.

We construct a complex manifold $\hat{U}=\bigcup_{i=1}^{m-1}U_i$ by patching $U_i's$ in the following way.

$$(u_1, v_1)=(u_0^{-1}, u_0^2v_0) \text{ on } U_0\cap U_1$$

$$(u_2, v_2)=(u_1v_1^2, v_1^{-1}) \text{ on } U_1\cap U_2$$

$$(u_3, v_3)=(u_2^{-1}, u_2^2v_2) \text{ on } U_2\cap U_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$\hat{U}$ is a minimal resolution of $\mathbf{C}^2/\langle g\rangle$ with $(m-1)$ exceptional curves defined by $v_0=v_1=0$, $u_1=u_2=0, \cdots\cdots$.

$$(1) \qquad (w, x, y)=(u_0v_0,\ u_0^m v_0^{m-1},\ v_0)$$

$$=(u_1v_1,\ u_1^{m-2}v_1^{m-1},\ u_1^2v_1)$$

$$=(u_2v_2,\ u_2^{m-2}v_2^{m-3},\ u_2^2v_2^3)$$

$$=(u_3v_3,\ u_3^{m-4}v_3^{m-3},\ u_3^4v_3^3)$$

....

$$=(u_{m-1}v_{m-1},\ v_{m-1},\ u_{m-1}^{m}v_{m-1}^{m}) \text{ if } m \text{ is even}$$

$$(u_{m-1}v_{m-1},\ u_{m-1},\ u_{m-1}^{m-1}v_{m-1}^{m}) \text{ if } m \text{ is odd}$$

Now, put $U=\{w^m=xy\}$. By the natural holomorphic mapping

$$U \longrightarrow D_1\times D_2,\ (w,\ x,\ y) \longmapsto (x,\ y),$$

U is an m-sheeted cyclic covering of $D_1\times D_2$ branching along $\{x=0\}$ and $\{y=0\}$. Let E be a smooth elliptic curve with the period $(1,\ \tau)$ $\mathrm{Im}(\tau)>0$ and define an automorphism of $U\times E$ as follows.

$$g: \underset{\Downarrow}{U\times E} \longrightarrow \underset{\Downarrow}{U\times E}$$

$$(w,\ x,\ y,\ [\zeta]) \longmapsto (e_m w,\ x,\ y,\ [\zeta+1/m])$$

U has a cyclic quotient singularity at the origin, and the minimal resolution $\hat{U}$ of U is given above.

By an easy calculation, g can be extended to an automorphism g^* of $\hat{U}\times E$ in the following way.

$$g^*: \underset{\Downarrow}{\hat{U}\times E} \longrightarrow \underset{\Downarrow}{\hat{U}\times E} \tag{2}$$

$$(u_0,\ v_0,\ [\zeta]) \longmapsto (e_m u_0,\ v_0,\ [\zeta+1/m]) \text{ on } U_0\times E$$

$$(u_1,\ v_1,\ [\zeta]) \longmapsto (e_m^{-1}u_0,\ e_m^2 v_1,\ [\zeta+1/m]) \text{ on } U_0\times E$$

$$(u_2,\ v_2,\ [\zeta]) \longmapsto (e_m^3 u_2,\ e_m^{-2} v_2,\ [\zeta+1/m]) \text{ on } U_2\times E$$

$$(u_3,\ v_3,\ [\zeta]) \longmapsto (e_m^{-3} u_3,\ e_m^4 v_3,\ [\zeta+1/m]) \text{ on } U_3\times E$$

....

$$\left.\begin{array}{l}\text{if } m \text{ is even, } (u_{m-1},\ v_{m-1},\ [\zeta]) \longmapsto (e_m u_{m-1},\ v_{m-1},\ [\zeta+1/m]) \\ \text{if } m \text{ is odd, } (u_{m-1},\ v_{m-1},\ [\zeta]) \longmapsto (u_{m-1},\ e_m v_{m-1},\ [\zeta+1/m])\end{array}\right\} \text{ on } U_{m-1}\times E.$$

$\langle g^*\rangle$ acts on $\hat{U}\times E$ freely and properly discontinuously, so the quotient space $X:=\hat{U}\times E/\langle g^*\rangle$ is smooth. Then we have the following commutative diagram.

$$\begin{tikzcd}
\hat{U}\times E \arrow[r, "g^*\text{ acts}"] \arrow[d, "\text{projection}"] & X=\hat{U}\times E/\langle g^*\rangle \arrow[d] \arrow[dd, bend left, "f"] \\
\hat{U} \arrow[r, "g^*\text{ acts}"] \arrow[d, "\text{minimal resolution}"] & \hat{U}/\langle g^*\rangle \arrow[d] \\
U \arrow[r, "g\text{ acts}"'] & U/\langle g\rangle \xrightarrow{\sim} D_1\times D_2
\end{tikzcd}$$

By the natural holomorphic mapping $f: X \longrightarrow D_1\times D_2$, X is an elliptic threefold over $D_1\times D_2$. Let E_1 (resp. E_2) denote the divisor on X defined by $u_0=0$ (resp.

$u_{m-1}=0$ or $v_{m-1}=0$) and $S_1, S_2, \ldots, S_{m-1}$ denote the divisors on X defined by $v_0=v_1=0$, $u_1=u_2=0, \ldots$ respectively.
Then Supp $(f^{-1}(0))=S_1 \cup S_2 \cup \cdots \cup S_{m-1}$, so $f: X \longrightarrow D_1 \times D_2$ is not flat at the origin.
From (1),

$$(f^*x)=mE_1+(m-1)S_1+(m-2)S_2+\cdots+S_{m-1}$$

$$(f^*y)=S_1+2S_2+\cdots+(m-1)S_{m-1}+mE_2$$

$$(f^*(xy))=m(E_1+E_2+\sum_{i=1}^{m-1} S_i)$$

Let $\{x_i=0\}$ (resp. $\{y_i=0\}$) be the local defining equation of the y-axis (resp. x-axis) with the origin deleted. Then we have $(f^*x_i)=mD$ (resp. $(f^*y_i)=mD$) as the divisor on X.
In this sense, $X \longrightarrow D_1 \times D_2$ has multiple fibers of multiplicity m along the x-axis and the y-axis. g^* acts on $(m-1)$ exceptional curves and accordingly we get $(m-1)$ rational curves, $\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_{m-1}$. And
in (2), put $u_0=0$, $v_0=v_1=0, \cdots$. Each S_i is an elliptic surface over $\hat{C}_i$, and has multiple fibers at the two points where $\hat{C}_i$ intersects $\hat{C}_{i-1}$ and $\hat{C}_{i+1}$. (See Figure 2.)

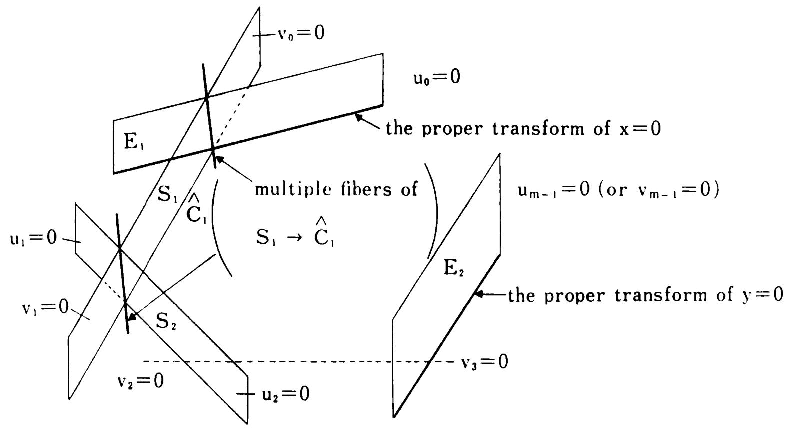

Figure 2.

(3) We treat the case when m and $n(\geq 2)$ are arbitrary integers.
Let d be the greatest common divisor of m and n, and put $m=m'd$, $n=n'd$, where m' and n' are relatively prime. Let

$$\tilde{D}_1=\{s \in \mathbf{C};\ |s|<\varepsilon^{1/m'}\} \longrightarrow D_1=\{x \in \mathbf{C};\ |x|<\varepsilon\},\ x=s^{m'}$$

$$\tilde{D}_2=\{t \in \mathbf{C};\ |t|<\varepsilon^{1/n'}\} \longrightarrow D_2=\{y \in \mathbf{C};\ |y|<\varepsilon\},\ y=t^{n'}$$

be an m'-sheeted covering of D_1 and an n'-sheeted covering of D_2 respectively. Then $\tilde{D}_1 \times \tilde{D}_2 \longrightarrow D_1 \times D_2 : (s, t) \longmapsto (s^{m'}, t^{n'})$ is an $m'n'$-sheeted cyclic covering with the Galois group $\simeq \mathbf{Z}/m' \oplus \mathbf{Z}/n'$.

Now, put $U=\{w^d=st\}$. By the natural morphism $U \longrightarrow \tilde{D}_1 \times \tilde{D}_2$, $(w, s, t) \longmapsto (s, t)$, U is a d-sheeted cyclic covering of $\tilde{D}_1 \times \tilde{D}_2$ branching along $s=0$ and $t=0$.

Define an automorphism g fo $U\times E$ as follows.

$$g: \underset{\Psi}{U\times E} \longrightarrow \underset{\Psi}{U\times E}$$

$$(w,\ s,\ t,\ [\zeta]) \longmapsto \left(\rho w,\ e_{m'}s,\ e_{n'}t,\ \left[\zeta+\frac{1}{m'n'd}\right]\right)$$

Here, $\rho=\exp\left(\frac{2\pi\sqrt{-1}}{m'n'd}\right)$ and from the assumption, there exists a primitive m'-th (resp. n'-th) root of unity $e_{m'}$ (resp. $e_{n'}$) such that $\rho^d=e_{m'}e_{m'}$.

The minimal resolution $\tilde{U}$ of U is given in (2). g can be extended to an automorphism g^* of $\tilde{U}\times E$ as follows.

$$g^*: (u_0,\ v_0,\ [\zeta]) \longrightarrow \left(\frac{\rho}{e_{n'}}u_0,\ e_{n'}v_0,\ [\zeta+1/m'n'd]\right) \text{ on } U_0\times E$$

$$(u_1,\ v_1,\ [\zeta]) \longrightarrow \left(\frac{e_{n'}}{\rho}u_1,\ \frac{\rho^2}{e_{n'}}v_1,\ [\cdots]\right) \text{ on } U_1\times E$$

$$(u_2,\ v_2\ ,[\zeta]) \longrightarrow \left(\frac{\rho^3}{e_{n'}}u_2,\ \frac{e_{n'}}{\rho^2}v_2,\ [\cdots]\right) \text{ on } U_2\times E$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

if d is even, $(u_{d-1},\ v_{d-1},\ [\zeta]) \longrightarrow \left(\frac{\rho}{e_{m'}}u_{d-1},\ e_{m'}v_{d-1},\ [\cdots]\right)$ On $U_{d-1}\times E$

if d is odd, $(u_{d-1},\ v_{d-1},\ [\zeta]) \longrightarrow \left(e_{m'}u_{d-1},\ \frac{\rho}{e_{m'}}v_{d-1},\ [\cdots]\right)$

g^* acts on $\tilde{U}\times E$ freely and properly discontinuously, so the quotient space X is smooth. There is a following diagram.

$$\begin{tikzcd}
\tilde{U}\times E \arrow[r, "g^* \text{ acts}"] \arrow[d, "\text{projection}"] & X \arrow[d] \\
\tilde{U} \arrow[r, "g \text{ acts}"] \arrow[d, "\text{minimal resolution}"] & \tilde{U}/\langle g^*\rangle \arrow[d] \\
U \arrow[r] & U/\langle g\rangle \xrightarrow{\sim} D_1\times D_2
\end{tikzcd}$$

By the natural morphism $f: X\longrightarrow D_1\times D_2$, X is an elliptic threefold over $D_1\times D_2$ which has the desired properties. q.e.d.

Remaek (**4.6.**) The relative canonical bundle is given as follows.

In case (2), we have $K_{X/D_1\times D_2}\xrightarrow{\sim}(m-1)[\sum_{i=1}^{m-1}S_i+E_1+E_2]$

In case (3), we have $K_{X/D}=(m-1)E_1+(n-1)E_2+\sum_{i=1}^{d-1}\{m-1-i(m'-n')\}S_i$.

, where we use the same notation as in case (2). In both cases, we have

$$K_{X/D}=f^*\left(\frac{m-1}{m}x+\frac{n-1}{n}y\right)$$

Proposition (**4.7.**) *Let C_1, C_2 be smooth curves on S crossing normally. Assume that there exist line bundles on S such that $[C_1]\simeq L_1^{\otimes m}$, $[C_2]\simeq L_2^{\otimes n}$. Then there exists an elliptic threefold X over S which has multiple fibers of multiplicity m (resp. n) along C_1 (resp. C_2).*

Proof. Let d be the greatest common divisor of m and n, and put $m=m'd$, $n=n'd$, where m' and n' are relatively prime. We have $[n'C_1+m'C_2]=(L_1L_2)^{\otimes m'n'd}$. So there exists the $m'n'd$-sheeted cyclic covering of S, branching along C_1 (resp. C_2) with multiplicity m (resp. n). So the problem can be reduced to the local case, and proposition (4.5) implies proposition (4.7).

Now, we shall perform logarithmic transformations along a divisor with singularities.

Example (**4.8.**) Let $D_1=\{x\in\mathbf{C};\ |x|<\varepsilon\}$, $D_2=\{y\in\mathbf{C};\ |y|<\varepsilon\}$ be discs, and let E be a smooth elliptic curve. Let $M:=\{z^6=-(x^3+y^2)\}$ be a six-sheeted cyclic covering of $D_1\times D_2$ branching along $C=\{(x,y)\in D_1\times D_2;\ y^2+x^3=0\}$.
Define an automorphism g of $M\times E$ as follows.

$$\begin{array}{ccc} g: M\times E & \longrightarrow & M\times E \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x,\ y,\ z,\ [\zeta]) & \longrightarrow & (x,\ y,\ e_6 z,\ [\zeta+1/6]) \end{array}$$

M has a simple elliptic singularity at the origin and the minimal resolution $\tilde{M}$ of M is given as follows.
Let ∞ be the infinite point on the elliptic curve $X^3+Y^2+1=0$.
Take the line bundle $[-\infty]$, the dual bundle of $[\infty]$, and let M be obtained by contracting the zero section of it. Then we have $(x,y,z)=(t^2X,\ t^3Y,\ t)$, where t is a fiber coordinate of $[-\infty]$. $g\in\mathrm{Aut}\ (M\times E)$ can be extended to an automorphism g^* of $\tilde{M}\times E$ as below.

$$\begin{array}{ccc} (x,\ y,\ z,\ [\zeta])=(t^2X,\ t^3Y,\ t,\ [\zeta]) & \longmapsto & (t,\ X,\ Y,\ [\zeta]) \\ \downarrow g & & \downarrow g^* \\ (x,\ y,\ e_6z,\ [\zeta+1/6])=(t^2X,\ t^3Y,\ e_6t,\ [\zeta+1/6]) & \longmapsto & (e_6t,\ e_3^2X,\ e_2Y,\ [\zeta+1/6]) \end{array}$$

g^* acts on $\tilde{M}\times E$ freely and properly discontinuously, so $X:=\tilde{M}\times E/\langle g\rangle$ is smooth.
We have the following diagram.

$$\begin{array}{ccccc} \tilde{M}\times E & \xrightarrow{g^*\ \text{acts}} & \tilde{M}\times E/\langle g^*\rangle=X & & \\ \downarrow \text{projection} & & \downarrow & & \\ \tilde{M} & \xrightarrow{g^*\ \text{acts}} & \tilde{M}/\langle g^*\rangle & & \downarrow f \\ \downarrow \begin{array}{l}\text{minimal}\\ \text{resolution}\end{array} & & \downarrow & & \\ M & \xrightarrow{g\ \text{acts}} & M/\langle g\rangle \xrightarrow{\sim} D_1\times D_2 & & \end{array}$$

By the natural morphism $f: X \longmapsto D_1 \times D_2$, X is an elliptic threefold over $D_1 \times D_2$ and has multiple fibers of multiplicity 6 along C. And it is not flat at the origin. The special fiber $X_0 = f^{-1}(0) = [t=0]$ is a hyperelliptic surface. X is an elliptic surface over $\mathbf{P}^1$ and has 3 multiple fibers of multiplicity 2, 3, and 6 respectively.

Next, we shall construct an elliptic threefold which has a Kodaira singular fiber (c.f. [8]) and a multiple fiber of type mI_0 along the divisors that are crossing normally.

Example (4.9.) Let $D_1 = \{x \in \mathbf{C};\ |x| < \varepsilon\}$, $D_2 = \{y \in \mathbf{C};\ |y| < \varepsilon\}$ be small discs. Let $\tilde{D}_1 = \{s \in \mathbf{C};\ |s| < \varepsilon^{1/2}\}$, $\tilde{D}_2 = \{t \in \mathbf{C};\ |t| < \varepsilon^{1/2}\}$ be double coverings of D_1 and D_2 respectively. i.e. $x = s^2$, $y = t^2$.
Let E be a smooth elliptic curve with the period $(1, \tau)$, $\mathrm{Im}(\tau) > 0$.
We define analytic automorphisms h, g of $\tilde{D}_1 \times \tilde{D}_2 \times E$ as follows.

$$\begin{array}{ccc} \tilde{D}_1 \times \tilde{D}_2 \times E & \longrightarrow & \tilde{D}_1 \times \tilde{D}_2 \times E \\ \Cup & & \Cup \end{array}$$

$$h : (s,\ t,\ [\zeta]) \longmapsto (-s,\ t,\ [-\zeta])$$

$$g : (s,\ t,\ [\zeta]) \longmapsto (s,\ -t,\ [\zeta + 1/2]).$$

Let G be the finite group generated by h and g, and form the quotient space $X := \tilde{D}_1 \times \tilde{D}_2 \times E/G$. G acts on X properly discontinuously but not freely, so X is a normal complex space with singularities. f has four fixed loci $(s, t, [\zeta]) = (0, t, 0)$ $(0, t, 1/2)$ $(0, t, \tau/2)$ $(0, t, 1/2 + \tau/2)$ and $gh = hg : (s, t, [\zeta]) \longmapsto (-s, -t, [-\zeta + 1/2])$ has four fixed points $(s, t, [\zeta]) = (0, 0, 1/4)$ $(0, 0, 3/4)$ $(0, 0, 1/4 + \tau/2)$ $(0, 0, 3/4 + \tau/2)$.
Next, we construct a resolution of X.

(1) On the neighborhoods of the four fixed locus of h, blow up $D_1 \times D_2 \times E$ at the center of t-axis, so that the action of f can be extended there.

$$(s,\ t,\ \zeta) = (uv,\ t,\ v) = (v',\ t,\ u'v') \text{ where } uu' = 1,\ v' = uv$$

$$\begin{array}{ccccc} (v',\ t,\ u') & \longrightarrow & (s,\ t,\ \zeta) & \longleftarrow & (u,\ t,\ v) \\ \downarrow & & \downarrow h & & \downarrow \\ (-v',\ t,\ u') & \longrightarrow & (-s,\ t,\ -\zeta) & \longleftarrow & (u,\ t,\ -v) \end{array}$$

Put $w = v^2$, $w' = v^2$, then we have $w' = u^2 w$, $uu' = 1$, and this is a minimal resolution of X.

(2) At the 4 fixed points of $hg = gh$, in terms of local coordinate (x, t, ζ), $hg = gh$ is represented in the form

$$i : (x,\ t,\ \zeta) \longmapsto (-x,\ -t,\ -\zeta).$$

So X has cyclic quotint singularities.
Let G be the cyclic group of analytic automorphism of $\mathbf{C}^3$ generated by $g : (z_1, z_2, z_3) \longmapsto (-z_1, -z_2, -z_3)$. The resolution of singularities of X can be given as follows. Let U_i, $i = 1, 2, 3$, be 3 copies of $\mathbf{C}^3$ with coordinates (w_i^1, w_i^2, w_i^3). We construct a complex manifold $M = \bigcup_{i=1}^{3} U_i$ by patching U_i's as follows.

$$w_i^k = w_{i-1}^k / w_{i-1}^i,\ k \neq i-1,\ i$$

$$w_i^{i-1} = 1 / w_{i-1}^i$$

$$w_i^i = (w_{i-1}^i)^2 w_{i-1}^{i-1}$$

Meromorphic mappings $T_{U_i} : \mathbf{C}^3 \longrightarrow U_i$ $i=1, 2, 3$

$$(z_1, z_2, z_3) \longmapsto \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, (z_i)^2, \frac{z_{i+1}}{z_i}, \ldots \frac{z_3}{z_i}\right)$$

induce a meromorphic mapping $T : \mathbf{C}^3/G \longrightarrow M$.

Put $E = \bigcup_{i=1}^{3} \{w_i^i = 0 \text{ in } U_i\}$, then E is isomorphic to $\mathbf{P}^2$ and the meromorphic mapping T gives an isomorphism $T : \mathbf{C}^3/G \setminus \{0\} \xrightarrow{\sim} M/E$.

Therefore M is a non-singular model of $\mathbf{C}^3/G$. q.e.d.

From the above remark, we obtain a non-singular model $\tilde{X}$ of X.

By a natural holomorphic mapping $f : \tilde{X} \longrightarrow D_1 \times D_2$, $\tilde{X}$ is an elliptic threefold over $D_1 \times D_2$. In terms of local coordinates, f is given by

$$(x, y) = (s^2, t^2) = \begin{cases} (u^2 w, t^2) \\ (w^2, t^2) \end{cases} \qquad \text{——(1)}$$

$$(x, y) = (s^2, t^2) = \begin{cases} (w_1^1, w_1^1 (w_1^2)^2) \text{ on } U_1 \\ ((w_2^1)^2 w_2^2, w_2^2) \text{ on } U_2 \\ ((w_3^1)^2 w_3^3, (w_3^2)^2 w_3^3) \text{ on } U_3 \end{cases} \qquad \text{——(2)}$$

We denote by S_i and E_i $(1 \leq i \leq 4)$ the exceptional divisors in (1) and (2) respectively. E_i is analytically isomorphic to $\mathbf{P}^2$ and is defined by $w_i^i = 0$. Let Z_1 and Z_2 be the strict transform of $\{s=0\}$ and $\{t=0\}$ respectively. Then we have, as divisors on X,

$$(f^* x) = 2Z_1 + \sum_{i=1}^{4} S_i + \sum_{i=1}^{4} E_i,\ (f^* y) = 2Z_2 + \sum_{i=1}^{4} E_i.$$

The singular fiber X_0 over the origin $(x, y) = (o, o)$ can be written as

$$X_0 \xrightarrow{\sim} 2(s=t=0) + \sum_{1}^{4} (w=t=0) + \sum_{i=1}^{4} E_i,$$

so $f : \tilde{X} \longrightarrow D_1 \times D_2$ is not flat over the origin.

By the construction, $f : \tilde{X} \longrightarrow D_1 \times D_2$ has multiple fibers of multiplicity 2 along the x-axis and singular fibers of type I_0^* along the y-axis. The relative canonical bundle of $f : \tilde{X} \longrightarrow D_1 \times D_2$ is as follows (c.f. [10]). (See Figure 3.)

When m is even, we have $mK_{\tilde{X}/D_1 \times D_2} = mZ_1 + \sum_{i=1}^{4} \frac{m}{2} E_i$.

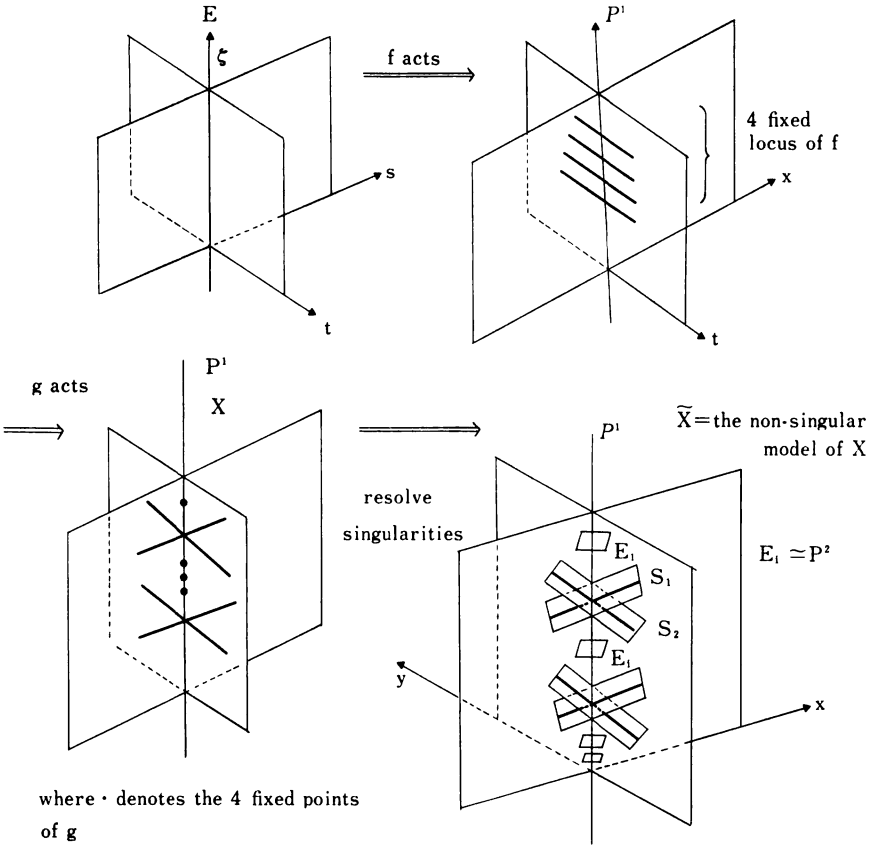

Figure 3.

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