J. Math. Kyoto Univ. (JMKYAZ)
28-2 (1988) 173-211

Finite multiplicity theorems for induced representations of semisimple Lie groups I

By

Hiroshi YAMASHITA

Introduction

Let G be a connected semisimple Lie group with finite center, and $G=KA_pN_m$ be its Iwasawa decomposition. In his early work [5, I], Harish-Chandra proved that any irreducible quasi-simple (hence any irreducible unitary) representation π of G is admissible, that is to say, the restriction $\pi|K$ of π to the maximal compact subgroup K is of multiplicity finite. In view of the Frobenius reciprocity law, this theorem means that unitarily ($=L^2$-) or differentiably ($=C^\infty$-) induced representation $\mathrm{Ind}_K^G(\tau)$ has finite multiplicity property for any $\tau\in\hat{K}$, the unitary dual of K. Moreover, he obtained in [5, III, Theorem 4] an estimate of multiplicities in $\pi|K$ crucial for construction of the distribution character of π: there exists a constant $c_\pi>0$ such that, for any $\tau\in\hat{K}$,

$$(0.1)\qquad \dim \mathrm{Hom}_K(\pi, \tau)=\dim \mathrm{Hom}_G(\pi_\infty, C^\infty\text{-}\mathrm{Ind}_K^G(\tau))\leqq c_\pi \dim \tau,$$

where π_∞ is the smooth representation of G associated with π. (Actually $c_\pi=1$, see [5, II].) These theorems are obtained mainly through a careful study of infinite-dimensional representations of the Lie algebra $\mathfrak{g}$ of G from a purely algebraic point of view. Nevertheless, once the differentiability of K-finite vectors for π is established (the finite multiplicity theorem above assures the analyticity of such vectors), one can derive the important estimate (0.1) also by using the theory of (K, K)-spherical functions in [6], which is a purely analytical method.

In 1984, some parts of the latter analytical method were extended by E. P. van den Ban to the (K, H)-spherical functions for any semisimple symmetric pair (G, H). He proved in [1] that the induced representation $\mathrm{Ind}_H^G(1_H)$ has finite multiplicity property, where 1_H denotes the trivial one-dimensional representation of H. In another direction, M. Hashizume [7] studied (K, N_m)-spherical functions of special kind, so-called class one Whittaker functions. One of his results [7, Theorem 3.3], the finite-dimensionality of spaces of such functions, suggests us that the induced representation $\mathrm{Ind}_{N_m}^G(\xi)$ is of multiplicity finite for any one-dimensional representation (=character) ξ of the maximal unipotent subgroup N_m. (This is proved rigorously in §4 of this paper.)

In the present article, we generalize the result of van den Ban, developing

Received August 29, 1986, by Prof. Yoshizawa.

the theory of spherical functions in a quite general setting which includes those of (K, H)- and (K, N_m)-spherical functions above. This generalization enables us to understand finite multiplicity theorems for induced representations of G in a unified manner. To be more precise, let $P_1=LN$ with $L=P_1\cap\theta P_1$ be a Levi decomposition of an arbitrary parabolic subgroup P_1 of G, where θ is a Cartan involution of G such that $K=\{g\in G; \theta(g)=g\}$. We denote by σ an involutive automorphism of L which commutes with $\theta | L$ and coincides with θ on the split component A of L. Let H be a closed subgroup of the fixed subgroup L_σ of σ, containing the identity component of L_σ. For a continuous representation ζ of the semidirect product subgroup $HN=H\ltimes N\subseteq LN=P_1$, we consider the induced representation (L^2- or C^∞-) $\mathrm{Ind}_{HN}^G(\zeta)$. Notice that, if $P_1=G$, then $(G, HN=H)$ is a semisimple symmetric pair, which is the case of Harish-Chandra and van den Ban. We estimate the multiplicities in $\mathrm{Ind}_{HN}^G(\zeta)$ through our theory of (K, HN)-spherical functions, and give good sufficient conditions for ζ that $\mathrm{Ind}_{HN}^G(\zeta)$ has finite multiplicity property. Application of our results to the case of $P_1=G$ reproves the finite multiplicity theorems of Harish-Chandra and van den Ban.

Our emphasis is, however, placed on the point that our criterions can be applied successfully to the representations induced from *infinite-dimensional* ζ's, too. One of such examples is the representation $\mathrm{Ind}_{MN_m}^G(\zeta)\simeq\mathrm{Ind}_{N_m}^G(\xi)$ with $\zeta=\mathrm{Ind}_{N_m}^{MN_m}(\xi)$, where M is the centralizer of A_p in K, and ξ is a unitary character of N_m. (Precisely speaking, G must not be split over $\boldsymbol{R}$ in order that ζ is infinite-dimensional.) This is the case suggested by [7], and contains the case of so-called *Gelfand-Graev representation* (=GGR). Any GGR is of multiplicity free ([14], see also 4.3 in this paper) if G is linear and quasi-split.

Besides, the more interesting examples of such cases are in *generalized* Gelfand-Graev representateons (=GGGRs), more precisely, in a variant of GGGRs called *reduced* GGGRs in [19] and [20]. The GGGR is an important extension of GGR, introduced by N. Kawanaka [8]. In the second part [19], we give finite multiplicity theorems for reduced GGGRs, by applying results of this article. The important cases connect with Whittaker models for (holomorphic) discrete series representations (cf. [20]). In the subsequent paper [20], we prove multiplicity one theorems for some of the above important cases, by generalizing the technique of Shalika [14]. (The method of spherical functions is too rough to prove theorems of such types.)

Now we explain how the theory of spherical functions is used to estimate multiplicities in induced representations. Let $U(\mathfrak{g}_C)$ denote the enveloping algebra of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$, and $Z(\mathfrak{g}_C)$ the center of $U(\mathfrak{g}_C)$. Let (ρ, E) be a compatible $(\mathfrak{g}_C, K)$-module. Then, for any algebra homomorphism $\chi: Z(\mathfrak{g}_C)\rightarrow\boldsymbol{C}$, the joint χ-eigenspace $E(\chi)=\{v\in E; \rho(z)v=\chi(z)v\ (z\in Z(\mathfrak{g}_C))\}$ is clearly a $(\mathfrak{g}_C, K)$-submodule of E. For $\tau\in\hat{K}$, $E(\chi)_\tau$ denotes the τ-isotypic component in $E(\chi)$. Now let $(\pi_K, \mathcal{H}_K)$ be an irreducible admissible $(\mathfrak{g}_C, K)$-module with infinitesimal character $\chi_\pi: Z(\mathfrak{g}_C)\rightarrow\boldsymbol{C}$. Then we have easily an estimate of multiplicities:

(0.2)
$$I_{\mathfrak{g}_C-K}(\pi_K, \rho)\leqq M_{\mathfrak{g}_C-K}(\pi_K, E(\chi_\pi))$$
$$\leqq \min_{\tau\in\hat{K}} [\dim E(\chi_\pi)_\tau\cdot I_K(\tau, \pi_K)^{-1}],$$

where, for X-modules β_1 and β_2, $I_X(\beta_1, \beta_2)$ denotes the intertwining number (see §2) from β_1 to β_2, and $M_{\mathfrak{g}_C-K}(\pi_K, E(\chi_\pi))$ the multiplicity of π_K in $E(\chi_\pi)$ as subquotient. By virtue of (0.2), we have finite multiplicity property for the $(\mathfrak{g}_C, K)$-module (ρ, E) if

(0.3)
$$\dim E(\chi)_\tau<+\infty \qquad \text{for any } \tau\in\hat{K} \text{ and any } \chi.$$

We consider the case where $\rho=C^\infty\text{-Ind}_{HN}^G(\zeta)$, our induced representation in C^∞-context (see 2.1). In this case, each element in $E(\chi)_\tau$ is said to be a (K, HN)-*spherical function* of type $(\tau, \zeta:\chi)$. Thus, the multiplicity of an irreducible admissible $(\mathfrak{g}_C, K)$-submodule π_K of $C^\infty\text{-Ind}_{HN}^G(\zeta)$ is bounded by the minimum of dimensions of the spaces of (K, HN)-spherical functions of type $(\tau, \zeta:\chi_\pi)$, where τ ranges over the elements of $\hat{K}$ occurring in π_K.

Furthermore, we can relate, using the results by Penney [11], the multiplicities in $C^\infty\text{-Ind}_{HN}^G(\zeta)$ with the multiplicity function of unitarily induced representation $L^2\text{-Ind}_{HN}^G(\zeta)$, at least when ζ is a finite-dimensional unitary representation. Thus, we obtain good sufficient conditions for ζ that C^∞- or $L^2\text{-Ind}_{HN}^G(\zeta)$ has finite multiplicity property (Theorems 2.12 and 3.13). These are the main results of this paper.

Now let us explain the results of this article in more detail.

In §1, we give a decomposition theorem (Theorem 1.12) of elements in $U(\mathfrak{g}_C)$ useful to our estimate of multiplicities in §2. This is a variant of the theorem of such type as giving the "radial component" of differential operators $D\in U(\mathfrak{g}_C)$ with respect to (K, HN).

In §2, we estimate multiplicities in induced representations $(\pi_\zeta, C^\infty(G;\zeta))=C^\infty\text{-Ind}_{HN}^G(\zeta)$ in C^∞-context by the method explained above. Here ζ is a continuous representation of the semidirect product subgroup $HN=H\ltimes N$ $(\subseteqq P_1)$ on a Fréchet space F. For this purpose, we study, for any $\tau\in\hat{K}$ and any ideal I of $Z(\mathfrak{g}_C)$ with finite codimension, the subspace $A(G;\zeta:I)_\tau$ of $C^\infty(G;\zeta)$ consisting of τ-isotypic vectors for π_ζ annihilated by $\pi_\zeta(I)$. In case where $I=\operatorname{Ker}\chi$ for some homomorphism $\chi: Z(\mathfrak{g}_C)\to \boldsymbol{C}$, this subspace is nothing but the space of (K, HN)-spherical functions of type $(\tau, \zeta:\chi)$. The point is that any $f\in A(G;\zeta:I)_\tau$ is an F-valued weakly analytic function on G (Lemma 2.5) thanks to the regularity theorem for elliptic differential operators. From this analyticity theorem together with Theorem 1.12, we get an upper bound for $\dim A(G;\zeta:I)_\tau$ (Theorem 2.8 and (2.8)). Accordingly, (0.2) applied to $\rho=\pi_\zeta$ gives an estimate of multiplicities in π_ζ (Theorem 2.10).

To be more precise, let $\mathfrak{p}_1=\mathfrak{l}\oplus\mathfrak{n}$ be the Levi decomposition of the Lie algebra $\mathfrak{p}_1$ of the parabolic subgroup P_1 corresponding to the decomposition $P_1=LN$. The differential of σ gives an involution on $\mathfrak{l}$ denoted again by σ. Let $\mathfrak{l}=\mathfrak{h}\oplus\mathfrak{q}$ be the eigenspace decomposition of $\mathfrak{l}$ with respect to σ, where $\mathfrak{h}$ (resp. $\mathfrak{q}$) is the $+1$ (resp. -1) eigenspace, and let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$

determined by θ. Extend the Lie algebra $\mathfrak{a}$ of the split component A ($\subseteqq L$) to a maximal abelian subspace $\mathfrak{a}_{pq}$ in $\mathfrak{p}\cap\mathfrak{q}$. Denote by $\mathfrak{l}_0$ the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$. Then, $\mathfrak{l}_0$ is, by construction, a reductive Lie subalgebra of $\mathfrak{g}$ contained in $\mathfrak{l}$. Let R_1 and R_2 be the orders of the complex Weyl groups of $\mathfrak{g}_C$ and $(\mathfrak{l}_0)_C$ respectively. We set $M_{kh}=Z_{K\cap H}(\mathfrak{a}_{pq})$, the centralizer of $\mathfrak{a}_{pq}$ in $K\cap H$. Then the multiplicities in π_ζ are estimated as in

Theorem A (see Theorem 2.10). *Let ζ be a continuous representation of HN ($\subseteqq P_1$), and $\pi_\zeta=C^\infty\text{-Ind}_{HN}^G(\zeta)$ the induced representation in C^∞-context. For an algebra homomorphism $\chi'\colon Z(\mathfrak{g}_C)\to C$, let $\pi_{\zeta,\chi'}$ be the subrepresentation of π_ζ on the joint χ'-eigenspace for $\pi_\zeta(Z(\mathfrak{g}_C))$. If $(\pi, \mathcal{H})$ is an irreducible admissible representation of G with infinitesimal character χ, then the multiplicites $I_G(\pi_\infty, \pi_\zeta)$, $I_{\mathfrak{g}_C-K}(\pi_K, (\pi_\zeta)_K)$ and $M_{\mathfrak{g}_C-K}(\pi_K, (\pi_{\zeta,\chi})_K)$ admit an upper bound as follows:*

$$(0.4)\qquad \begin{aligned} I_G(\pi_\infty, \pi_\zeta) &\leqq I_{\mathfrak{g}_C-K}(\pi_K, (\pi_\zeta)_K)\leqq M_{\mathfrak{g}_C-K}(\pi_K, (\pi_{\zeta,\chi})_K) \\ &\leqq R_1R_2^{-1}\cdot \min_{\tau\in\hat{K}}[I_{M_{kh}}(\tau, \zeta)\cdot I_K(\tau, \pi_K)^{-1}], \end{aligned}$$

where π_K, $(\pi_\zeta)_K$ and $(\pi_{\zeta,\chi})_K$ denote respectively the representations of $\mathfrak{g}_C$ and K on the space of K-finite vectors for π, π_ζ and $\pi_{\zeta,\chi}$.

This is the main result in § 2. From this theorem, we obtain a sufficient condition for the finiteness of multiplicities in π_ζ as follows.

Theorem B (see Theorem 2.12). *The induced representation π_ζ of G has finite multiplicity property if so does the restriction $\zeta|M_{kh}$ of ζ to the compact subgroup M_{kh}: $I_{M_{kh}}(\mu, \zeta)=\dim \mathrm{Hom}_{M_{kh}}(\mu, \zeta)<+\infty$ for any irreducible finite-dimensional representation μ of M_{kh}.*

This theorem covers, to a large extent, the finite multiplicity theorems for induced representations of G, especially the case of van den Ban [1], i.e., the case of $P_1=G$ and $\zeta=1_H$.

In § 3, we treat the multiplicity functions for unitarily induced representations $\mathcal{U}_\zeta=L^2\text{-Ind}_{HN}^G(\zeta)$ in connection with those for π_ζ in C^∞-context. First we proceed to a more general situation. Let G be a Lie group of type I. Consider the representation $\mathcal{U}_\zeta=L^2\text{-Ind}_Q^G(\zeta)$ induced from a unitary representation ζ of a closed subgroup Q of G. Let

$$(0.5)\qquad \mathcal{U}_\zeta\simeq\int_{\hat{G}}^{\oplus}\mathcal{U}_\zeta(\pi)d\mu_\zeta(\pi), \qquad \mathcal{U}_\zeta(\pi)\simeq[m_\zeta(\pi)]\cdot\pi$$

be the factor decomposition of $\mathcal{U}_\zeta$ (see 3.4). Here m_ζ is the multiplicity function for $\mathcal{U}_\zeta$ on the unitary dual $\hat{G}$ of G. Using the results by Penney [11] on the disintegration of C^∞-vectors for unitary representations, we can prove:

Proposition C (see Theorem 3.12). *If ζ is finite-dimensional, then the intertwining numbers $I_G(\pi_\infty, \pi_\zeta)=\dim \mathrm{Hom}_G(\pi_\infty, \pi_\zeta)$ ($\pi\in\hat{G}$) from π_∞ to $\pi_\zeta=C^\infty\text{-Ind}_Q^G(\zeta)$ give an upper bound for m_ζ:*

(0.6) $m_\zeta(\pi) \leqq I_G(\pi_\infty, \pi_\zeta)$ *for almost every* $\pi \in \hat{G}$

with repect to the Borel measure μ_ζ *on* $\hat{G}$ *in* (0.5).

We remark that the inequality (0.6) is false for infinite-dimensional ζ in general (Example 3.11).

Now we return to our original objects, and let G be a semisimple Lie group again. From Theorem B combined with Proposition C, we get a finite multiplicity theorem for $\mathcal{U}_\zeta = L^2\text{-Ind}_{HN}^G(\zeta)$, which extends Theorem 3.1 in [1].

Theorem D (see Theorem 3.13). *Let ζ be a finite-dimensional unitary reprentation of the semidirect product subgroup HN ($\subseteq P_1$). Then, the multiplicity function $\hat{G} \ni \pi \mapsto m_\zeta(\pi)$ for $\mathcal{U}_\zeta = L^2\text{-Ind}_{HN}^G(\zeta)$ takes finite values for almost every $\pi \in \hat{G}$ with respect to μ_ζ in* (0.5).

This is the main result in §3.

To establish a general result such as Theorem D, we have been forced to assume ζ to be finite-dimensional. Nevertheless, Theorem A and Proposition C are still applicable to infinite-dimensional ζ to prove finite multiplicity property for some specified $\mathcal{U}_\zeta$.

In §4, we give important examples of such $\mathcal{U}_\zeta$, including Gelfand-Graev representation (=GGR; see Definition A.5). More precisely, let $M = Z_K(A_p)$ as before, and consider the semidirect product subgroup MN_m of the minimal parabolic subgroup MA_pN_m. As a representation ζ of MN_m, we take $\zeta = L^2\text{-Ind}_{N_m}^{MN_m}(\xi)$, the representation induced from a unitary character ξ of the maximal unipotent subgroup N_m. Then ζ is infinite-dimensional if $\dim M > 0$. Consider (L^2- or C^∞-) $\text{Ind}_{MN_m}^G(\zeta)$. The stage theorem for induced representations tells us $\text{Ind}_{MN_m}^G(\zeta) \simeq \text{Ind}_{N_m}^G(\xi)$. First we apply Theorem A to $C^\infty\text{-Ind}_{MN_m}^G(\zeta)$, and then, keeping its result in mind, we apply Proposition C to $L^2\text{-Ind}_{N_m}^G(\xi)$. Thus, we find out that $\text{Ind}_{MN_m}^G(\zeta)$ is of multiplicity finite (Theorems 4.2 and 4.3) even if ζ is infinite-dimensional.

In Appendix, we deal with the problem of decomposing $L^2\text{-Ind}_{N_m}^G(\xi)$ explicitly into irreducibles. On one hand, we have a complete answer (Theorem A.4) in case $\xi = 1_{N_m}$, the trivial character of N_m. On the other hand, we reduce the problem for general ξ mainly to that for non-degenerate ξ's, that is, to decomposition of the GGRs (of Levi subgroups of G).

The author expresses his gratitude to Professor S. Sano for his stimulating lectures on harmonic analysis on semisimple symmetric spaces. The author wishes to thank Professors T. Hirai, N. Tatsuuma and T. Nomura for their kind discussions and constant encouragement.

§1. A decomposition theorem of elements in $U(\mathfrak{g}_C)$

Let G be a connected semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. $U(\mathfrak{g}_C)$ will denote the universal enveloping algebra of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. We generalize in this section the results by Harish-Chandra [6, Lemma 7] and van den Ban [1, Lemma 3.8], and get a decomposition theorem (Theorem 1.12) of elements in $U(\mathfrak{g}_C)$. This theorem will play a crucial role when we estimate in §2 the multiplicities in representations of G induced from those of certain semidirect product subgroups $H\ltimes N$ (see 1.2 for the definition of $H\ltimes N$).

1.1. Preliminaries. First of all, we prepare some notations on a semisimple Lie group G after [17, Chap. 1] and [15, Part II, §6].

Let θ be a Cartan involution of G and K the fixed subgroup of $\theta: K=\{g\in G; \theta(g)=g\}$. Then K is a maximal compact subgroup of G. Denote by $\mathfrak{k}$ the Lie algebra of K. The differential of θ gives an involutive automorphism of $\mathfrak{g}$ denoted again by θ. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to θ.

By a Borel subalgebra of $\mathfrak{g}_C$, we mean a maximal solvable complex subalgebra of $\mathfrak{g}_C$. Borel subalgebras are all conjugate under the adjoint group of $\mathfrak{g}_C$. A subalgebra of $\mathfrak{g}$ is said to be parabolic if its complexification contains a Borel subalgebra of $\mathfrak{g}_C$. For a parabolic subalgebra $\mathfrak{p}_1$ of $\mathfrak{g}$, put $P_1=N_G(\mathfrak{p}_1)$, the normalizer of $\mathfrak{p}_1$ in G. Then P_1 is self-normalizing, $N_G(P_1)=P_1$, and $\mathfrak{p}_1$ coincides with the Lie algebra of P_1. We call P_1 the parabolic subgroup of G corresponding to $\mathfrak{p}_1$.

Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{p}_1$ and N the analytic subgroup of G corresponding to $\mathfrak{n}$. Then P_1 (resp $\mathfrak{p}_1$) is expressed as

$$P_1=L\ltimes N \qquad (\text{resp. } \mathfrak{p}_1=\mathfrak{l}\oplus\mathfrak{n}) \qquad (\text{a Levi decompositon}),$$

where $L=P_1\cap\theta P_1$ (resp. $\mathfrak{l}=\mathfrak{p}_1\cap\theta\mathfrak{p}_1$) normalizes N (resp. $\mathfrak{n}$). L (resp. $\mathfrak{l}$) is called a Levi subgroup (resp. a Levi subalgebra) of P_1 (resp. $\mathfrak{p}_1$). Put $\mathfrak{a}=\mathfrak{z}_{\mathfrak{l}}\cap\mathfrak{p}$ and $A=\exp\mathfrak{a}$, where $\mathfrak{z}_{\mathfrak{l}}$ is the center of $\mathfrak{l}$. A (resp. $\mathfrak{a}$) is said to be a split component of P_1 (resp. $\mathfrak{p}_1$). Then L admits a direct product decomposition $L=MA$ with $M=\cap\operatorname{Ker}\chi$, where χ runs through the continuous group homomorphisms from L to the multiplicative group of positive real numbers. In view of the Levi decompositions above, we have $P_1=MAN$ and $\mathfrak{p}_1=\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$ (Langlands decompositions) with $\mathfrak{m}$ the Lie algebra of M.

1.2. The subgroups $H\ltimes N$. Let σ_0 be an involutive (i.e., $\sigma_0^2=1$) automorphism of M which commutes with $\theta|M$. Extend σ_0 to an involution σ of $L=MA$ in such a way that $\sigma(ma)=\sigma_0(m)a^{-1}$ ($m\in M$, $a\in A$). Let H denote a closed subgroup of L such that $(L_\sigma)_0\subseteq H\subseteq L_\sigma$, where L_σ is the fixed subgroup of σ and $(L_\sigma)_0$ the identity component of L_σ. Consider the semidirect product subgroup $H\ltimes N$. We will treat in §§2-3 the representations of G induced from those of $HN=H\ltimes N$ and examine the multiplicities of irreducible constituents of them through (K, HN)-spherical functions.

In a special case $P_1=G$, we have $N=(1)$ and (G, H) is a semisimple symmetric pair. Our arguments will generalize in some aspects the theory of (K, H)-spherical functions developed by Harish-Chandra [6] and van den Ban [1].

1.3. Root space decompositions. By taking the differential of σ, one gets an involution of $\mathfrak{l}$ denoted again by σ. Let $\mathfrak{l}=\mathfrak{h}\oplus\mathfrak{q}$ be the eigenspace decomposition of $\mathfrak{l}$ with respect to σ, where $\mathfrak{h}$ (resp. $\mathfrak{q}$) is the $+1$ (resp. -1) eigenspace of σ. Then $\mathfrak{h}$ is the Lie algebra of H. Since σ commutes with $\theta|L$, we have a direct sum decomposition of $\mathfrak{l}$

$$\mathfrak{l}=(\mathfrak{k}\cap\mathfrak{h})\oplus(\mathfrak{k}\cap\mathfrak{q})\oplus(\mathfrak{p}\cap\mathfrak{h})\oplus(\mathfrak{p}\cap\mathfrak{q}) \qquad \text{(as vector spaces).} \tag{1.1}$$

Let $\mathfrak{a}_{pq}$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. Extend $\mathfrak{a}_{pq}$ to a maximal abelian subspace $\mathfrak{a}_p$ of $\mathfrak{p}\cap\mathfrak{l}$. Then one deduces just as in the case of semisimple symmetric pairs (see [10, Lemma 2.2]) the following lemma.

Lemma 1.1. *The vector spaces* $\mathfrak{a}$, $\mathfrak{a}_{pq}$ *and* $\mathfrak{a}_p$ *satisfy the following relations* (1) *and* (2):

(1) $\mathfrak{a}\subseteqq\mathfrak{a}_{pq}\subseteqq\mathfrak{a}_p$,

(2) $\mathfrak{a}_p=\mathfrak{a}_{pq}\oplus(\mathfrak{a}_p\cap\mathfrak{h})$, *in particular* $\mathfrak{a}_p$ *is* σ*-stable.*

Proof. (1) Since $\sigma(a)=a^{-1}$ for any $a\in A$, one has $\sigma|\mathfrak{a}=-I$ (I the identity operator), whence $\mathfrak{a}\subseteqq\mathfrak{q}$. Thus it holds that $\mathfrak{a}=\mathfrak{z}_{\mathfrak{l}}\cap\mathfrak{p}\cap\mathfrak{q}$, which implies that $\mathfrak{a}+\mathfrak{a}_{pq}$ is an abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$ containing $\mathfrak{a}_{pq}$. By the maximality of $\mathfrak{a}_{pq}$, $\mathfrak{a}$ is contained in $\mathfrak{a}_{pq}$. The second inclusion $\mathfrak{a}_{pq}\subseteqq\mathfrak{a}_p$ is obvious by the definition of $\mathfrak{a}_p$.

(2) For an arbitrary $X\in\mathfrak{a}_p$, express X as $X=Y+Z$ with $Y\in\mathfrak{p}\cap\mathfrak{h}$ and $Z\in\mathfrak{p}\cap\mathfrak{q}$ according as the decomposition $\mathfrak{p}\cap\mathfrak{l}=(\mathfrak{p}\cap\mathfrak{h})\oplus(\mathfrak{p}\cap\mathfrak{q})$. We show that $Y\in\mathfrak{a}_p\cap\mathfrak{h}$ and $Z\in\mathfrak{a}_{pq}$. In fact, for any $W\in\mathfrak{a}_{pq}$, one has $0=[X, W]=[Y, W]+[Z, W]$. On the other hand we have $[Y, W]\in[\mathfrak{h}, \mathfrak{q}]\subseteqq\mathfrak{q}$ and $[Z, W]\in[\mathfrak{q}, \mathfrak{q}]\subseteqq\mathfrak{h}$. Hence $[Z, W]=-[Y, W]\in\mathfrak{h}\cap\mathfrak{q}=(0)$ for any $W\in\mathfrak{a}_{pq}$. Then Z must be contained in $\mathfrak{a}_{pq}$ because $\mathfrak{a}_{pq}$ is a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. We thus get $Y=X-Z\in\mathfrak{a}_p\cap\mathfrak{h}$ by (1). Consequently one obtains $\mathfrak{a}_p\subseteqq\mathfrak{a}_{pq}\oplus(\mathfrak{a}_p\cap\mathfrak{h})$. The converse inclusion is obvious, which completes the proof. Q.E.D.

We need to treat various kinds of root spaces at the same time, so it is convenient to prepare some general notations as follows. Let $\mathfrak{x}$ be a commutative Lie algebra over $F=\boldsymbol{R}$ or $\boldsymbol{C}$ acting on a vector space V over F. For an element $\lambda\in\mathfrak{x}^*$, the dual space of $\mathfrak{x}$, $V(\mathfrak{x};\lambda)$ will denote the space of $v\in V$ such that $Z\cdot v=\lambda(Z)v$ for every $Z\in\mathfrak{x}$. We denote by $\Lambda(V:\mathfrak{x})$ the set of all $\lambda\neq0$ with $V(\mathfrak{x};\lambda)\neq(0)$. In case where $\mathfrak{x}$ is a subalgebra of a Lie algebra $\mathfrak{y}$ and V an $(\mathrm{ad}\,\mathfrak{x})$-invariant subspace of $\mathfrak{y}$, we always consider the adjoint action of $\mathfrak{x}$ on V: $\mathfrak{x}\ni Z\mapsto(\mathrm{ad}\,Z)|V$.

Let $v_1, v_2, \cdots, v_n$ be a basis of a real vector space E. Define a total order $>$ on E^* as follows: for two elements $\lambda, \mu\in E^*$, $\lambda>\mu$ if there exists $1\leqq s\leqq n$ such that

$$\lambda(v_i)=\mu(v_i) \quad \text{for} \quad 1\leqq i\leqq s-1 \quad \text{and} \quad \lambda(v_s)>\mu(v_s).$$

We call this the lexicographic order on E^* with respect to the basis $(v_1, v_2, \cdots, v_n)$. For a subset Ψ of E^*, put $\Psi^+=\{\lambda\in\Psi\,;\,\lambda>0\}$.

Now let $\mathfrak{j}=\mathfrak{j}^+\oplus\mathfrak{a}_p$ with $\mathfrak{j}^+\subseteqq\mathfrak{k}$ be a θ-stable maximally split Cartan subalgebra of $\mathfrak{g}$. We define compatible lexicographic orders on $\mathfrak{a}^*$, $\mathfrak{a}_p^*$ and $\mathfrak{j}_R^*$ ($\mathfrak{j}_R=\sqrt{-1}\mathfrak{j}^+\oplus\mathfrak{a}_p$) as follows. First take a lexicographic order on $\mathfrak{a}^*$ for which the elements in $\Lambda(\mathfrak{n}:\mathfrak{a})$ are all positive. Such an order always exists. Let $(H_1, \cdots, H_{m_1})$ be the basis of $\mathfrak{a}$ which determines this order on $\mathfrak{a}^*$. Secondly extend $(H_1, \cdots, H_{m_1})$ to a basis $(H_1, \cdots, H_{m_1}, H_{m_1+1}, \cdots, H_n)$ of $\mathfrak{j}_R$ in such a way that $(H_k)_{1\leq k\leq m_2}$ (resp. $(H_k)_{1\leq k\leq m_3}$) forms a basis of $\mathfrak{a}_{pq}$ (resp. $\mathfrak{a}_p$), where $m_1\leqq m_2\leqq m_3\leqq n$. Define lexicographic orders on $\mathfrak{a}^*$, $\mathfrak{a}_{pq}^*$, $\mathfrak{a}_p^*$ and $\mathfrak{j}_R^*$ through the above bases.

Let $\mathfrak{l}_0$ be the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$. Then $\mathfrak{l}_0$ is a θ-stable reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{j}$ as a Cartan subalgebra. From Lemma 1.1(1), $\mathfrak{l}_0$ is contained in $\mathfrak{l}$.

Using the above notations, we have joint eigenspace decompositions of $\mathfrak{g}$ with respect to the adjoint actions of $\mathfrak{a}$ and $\mathfrak{a}_{pq}$ as follows:

$$(1.2)\qquad \mathfrak{g}=\theta\mathfrak{n}\oplus\mathfrak{l}\oplus\mathfrak{n}, \quad \mathfrak{n}=\sum_{\lambda\in\Lambda^+(\mathfrak{g}:\mathfrak{c})}\mathfrak{g}(\mathfrak{a}\,;\lambda), \quad \theta\mathfrak{n}=\sum_{\lambda\in\Lambda^+(\mathfrak{g}:\mathfrak{a})}\mathfrak{g}(\mathfrak{a}\,;-\lambda),$$

$$(1.3)\qquad \begin{cases} \mathfrak{g}=\theta\mathfrak{n}(\mathfrak{a}_{pq})\oplus\mathfrak{l}_0\oplus\mathfrak{n}(\mathfrak{a}_{pq}), \\ \mathfrak{n}(\mathfrak{a}_{pq})\equiv\sum_{\lambda\in\Lambda^+(\mathfrak{g}:\mathfrak{a}_{pq})}\mathfrak{g}(\mathfrak{a}_{pq}\,;\lambda), \quad \theta\mathfrak{n}(\mathfrak{a}_{pq})=\sum_{\lambda\in\Lambda^+(\mathfrak{g}:\mathfrak{a}_{pq})}\mathfrak{g}(\mathfrak{a}_{pq}\,;-\lambda). \end{cases}$$

One should note that $\mathfrak{n}(\mathfrak{a}_{pq})=\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\mathfrak{n}$ with $\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})=\mathfrak{l}\cap\mathfrak{n}(\mathfrak{a}_{pq})$ by virtue of our choice of lexicographic orders. Hence $\mathfrak{l}$ is expressed as

$$(1.4)\qquad \begin{cases} \mathfrak{l}=\theta\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\mathfrak{l}_0\oplus\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq}), \\ \mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})=\sum_{\lambda\in\Lambda^+(\mathfrak{l}:\mathfrak{a}_{pq})}\mathfrak{g}(\mathfrak{a}_{pq}\,;\mathfrak{l}), \quad \theta\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})=\sum_{\lambda\in\Lambda^+(\mathfrak{l}:\mathfrak{a}_{pq})}\mathfrak{g}(\mathfrak{a}_{pq}\,;-\lambda). \end{cases}$$

We proceed to the root space decompositions of $\mathfrak{g}_C$ and $\mathfrak{l}_{0C}=(\mathfrak{l}_0)_C$. Put $\Phi=\Lambda(\mathfrak{g}_C:\mathfrak{j}_C)$ and $\Phi_0=\Lambda(\mathfrak{l}_{0C}:\mathfrak{j}_C)$, then $\Phi_0\subseteqq\Phi$. Every element in Φ takes real values on the real form $\mathfrak{j}_R$ of $\mathfrak{j}_C$. So we may consider Φ canonically as a subset of $\mathfrak{j}_R^*$ and denote by Φ^+ (resp. Φ_0^+) the positive system of Φ (resp. Φ_0) with respect to our order on $\mathfrak{j}_R^*$. Then one has root space decompositions of $\mathfrak{g}_C$ and $\mathfrak{l}_{0C}$ with respect to $\mathfrak{j}_C$ as follows:

$$(1.5)\qquad \begin{cases} \mathfrak{g}_C=\mathfrak{u}_C(\Phi)\oplus\mathfrak{j}_C\oplus\mathfrak{n}_C(\Phi), \\ \mathfrak{n}_C(\Phi)=\sum_{\alpha\in\Phi^+}\mathfrak{g}_C(\mathfrak{j}_C\,;\alpha), \quad \mathfrak{u}_C(\Phi)=\sum_{\alpha\in\Phi^+}\mathfrak{g}_C(\mathfrak{j}_C\,;-\alpha), \end{cases}$$

$$(1.6)\qquad \begin{cases} \mathfrak{l}_{0C}=\mathfrak{u}_C(\Phi_0)\oplus\mathfrak{j}_C\oplus\mathfrak{n}_C(\Phi_0), \\ \mathfrak{n}_C(\Phi_0)=\sum_{\alpha\in\Phi_0^+}\mathfrak{g}_C(\mathfrak{j}_C\,;\alpha), \quad \mathfrak{u}_C(\Phi_0)=\sum_{\alpha\in\Phi_0^+}\mathfrak{g}_C(\mathfrak{j}_C\,;-\alpha). \end{cases}$$

It follows from the compatibility of our orders on $\mathfrak{a}_{pq}^*$ and $\mathfrak{j}_R^*$ that $\mathfrak{n}_C(\Phi)=\mathfrak{n}_C(\Phi_0)\oplus\mathfrak{n}(\mathfrak{a}_{pq})_C$ with $\mathfrak{n}(\mathfrak{a}_{pq})_C=\mathfrak{n}(\mathfrak{a}_{pq})\otimes_R\boldsymbol{C}$.

1.4. Structure of $Z(\mathfrak{l}_{0C})$ as a $Z(\mathfrak{g}_C)$-module.

For a Lie algebra $\mathfrak{x}$ over $\boldsymbol{C}$, $Z(\mathfrak{x})$ denotes the center of the enveloping algebra $U(\mathfrak{x})$ of $\mathfrak{x}$. $Z(\mathfrak{l}_{0C})$ has a canonical structue of $Z(\mathfrak{g}_C)$-module through the homomorphism $\mu: Z(\mathfrak{g}_C)\rightarrow Z(\mathfrak{l}_{0C})$ defined below. For the later use, we clarify in this subsection the $Z(\mathfrak{g}_C)$-module structure of $Z(\mathfrak{l}_{0C})$.

By (1.3), (1.5) and (1.6) together with the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g}_C)$ and $U(\mathfrak{l}_{0C})$ are decomposed respectively as

(1.7) $$U(\mathfrak{g}_C)=\{\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\theta\mathfrak{n}(\mathfrak{a}_{pq})\}\oplus U(\mathfrak{l}_{0C}),$$

(1.8) $$U(\mathfrak{g}_C)=\{\mathfrak{n}_C(\Phi)U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\mathfrak{u}_C(\Phi)\}\oplus U(\mathfrak{j}_C),$$

(1.9) $$U(\mathfrak{l}_{0C})=\{\mathfrak{n}_C(\Phi_0)U(\mathfrak{l}_{0C})+U(\mathfrak{l}_{0C})\mathfrak{u}_C(\Phi_0)\}\oplus U(\mathfrak{j}_C).$$

Let $\tilde{\mu}: U(\mathfrak{g}_C)\to U(\mathfrak{l}_{0C})$, $\tilde{\gamma}: U(\mathfrak{g}_C)\to U(\mathfrak{j}_C)$ and $\tilde{\gamma}_0: U(\mathfrak{l}_{0C})\to U(\mathfrak{j}_C)$ be the projections along the decompositions (1.7), (1.8) and (1.9) respectively. Then we see easily the following

Lemma 1.2. (1) *The map $\tilde{\gamma}$ is expressed as $\tilde{\gamma}=\tilde{\gamma}_0\circ\tilde{\mu}$.*

(2) *The restriction of $\tilde{\mu}$ to $Z(\mathfrak{g}_C)$ gives an algebra homomorphism from $Z(\mathfrak{g}_C)$ into $Z(\mathfrak{l}_{0C})$.*

(3) *$Z-\tilde{\mu}(Z)\in\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)\theta\mathfrak{n}(\mathfrak{a}_{pq})$ for every $Z\in Z(\mathfrak{g}_C)$.*

Proof. (1) For an element $D\in U(\mathfrak{g}_C)$, one has

(1.10) $$D-(\tilde{\gamma}_0\circ\tilde{\mu})(D)=\{D-\tilde{\mu}(D)\}+\{\tilde{\mu}(D)-\tilde{\gamma}_0(\tilde{\mu}(D))\}.$$

Note that $\mathfrak{n}(\mathfrak{a}_{pq})$, $\mathfrak{n}_C(\Phi_0)\subseteqq\mathfrak{n}_C(\Phi)$ and that $\theta\mathfrak{n}(\mathfrak{a}_{pq})$, $\mathfrak{u}_C(\Phi_0)\subseteqq\mathfrak{u}_C(\Phi)$. Then, from the definition of $\tilde{\mu}$ and $\tilde{\gamma}_0$, the right hand side of (1.10) is in $\mathfrak{n}_C(\Phi)U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\mathfrak{u}_C(\Phi)$. Therefore we have $\tilde{\gamma}(D)=(\tilde{\gamma}_0\circ\tilde{\mu})(D)$.

(2) and (3). Now assume that $D\in Z(\mathfrak{g}_C)$. First we show that $\tilde{\mu}(D)\in Z(\mathfrak{l}_{0C})$. Indeed, for any $X\in\mathfrak{l}_0$, $[\tilde{\mu}(D), X]=[X, D-\tilde{\mu}(D)]$ is in $\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\theta\mathfrak{n}(\mathfrak{a}_{pq})$ since $\mathfrak{l}_0$ normalizes both $\mathfrak{n}(\mathfrak{a}_{pq})$ and $\theta\mathfrak{n}(\mathfrak{a}_{pq})$. On the other hand, $[\tilde{\mu}(D), X]\in[U(\mathfrak{l}_{0C}), \mathfrak{l}_0]\subseteqq U(\mathfrak{l}_{0C})$. Hence we have

$$[X, \tilde{\mu}(D)]\in U(\mathfrak{l}_{0C})\cap\{\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\theta\mathfrak{n}(\mathfrak{a}_{pq})\}=(0).$$

This means that $\tilde{\mu}(D)\in Z(\mathfrak{l}_{0C})$.

Before proving (2), we show the assertion (3). Let $\alpha_1<\alpha_2<\cdots<\alpha_r$ be the elements of Φ^+. For every $1\leqq i\leqq r$, take a non-zero root vector $X_i^{\pm}\in\mathfrak{g}_C(\mathfrak{j}_C; \pm\alpha_i)$. Let $H_1, \cdots, H_n$ be a basis of $\mathfrak{j}_C$. By the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g}_C)$ has a basis consisting of elements

$$M((s_i), (u_m), (t_i))=(X_r^+)^{s_r}\cdots(X_1^+)^{s_1}H_1^{u_1}\cdots H_n^{u_n}(X_1^-)^{t_1}\cdots(X_r^-)^{t_r},$$

with non-negative integers s_i, t_i $(1\leqq i\leqq r)$, u_m $(1\leqq m\leqq n)$. Then, for a $D\in Z(\mathfrak{g}_C)$, the element $D-\tilde{\mu}(D)\in\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\theta\mathfrak{n}(\mathfrak{a}_{pq})$ has a unique expansion

$$D-\tilde{\mu}(D)=\sum C((s_i), (u_m), (t_i))M((s_i), (u_m), (t_i))$$

with complex coefficients $C((s_i), (u_m), (t_i))$, where the summation is over $((s_i), (u_m), (t_i))$ such that $\sum_{i>r_0}(s_i+t_i)>0$. Here α_{r_0} is the highest root in Φ_0. On the other hand, $[U(\mathfrak{l}_{0C}), D-\tilde{\mu}(D)]=(0)$ since $\tilde{\mu}(D)\in Z(\mathfrak{l}_{0C})$ as proved above. In particular, $[H, D-\tilde{\mu}(D)]=0$ for every $H\in\mathfrak{j}_C$, which means that

$$\sum\left\{\sum_{k=1}^{r}(s_k-t_k)\alpha_k(H)\right\}C((s_i),(u_m),(t_i))M((s_i),(u_m),(t_i))=0$$

for all $H\in\mathfrak{j}_C$. Thus $C((s_i),(u_m),(t_i))=0$ unless $\sum_k(s_k-t_k)\alpha_k=0$. For a triplet $((s_i),(u_m),(t_i))$ such that $\sum_{i>r_0}(s_i+t_i)>0$, the sum $\sum_k(s_k-t_k)\alpha_k$ can not be equal to zero if either $(s_i)_{i>r_0}=(0)$ or $(t_i)_{i>r_0}=(0)$. Therefore we have $D-\tilde{\mu}(D)\in\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)\theta\mathfrak{n}(\mathfrak{a}_{pq})$.

Finally we return to (2). Let us show that $\tilde{\mu}|Z(\mathfrak{g}_C): Z(\mathfrak{g}_C)\to Z(\mathfrak{l}_{0C})$ is a homomorphism. For $D_1, D_2\in Z(\mathfrak{g}_C)$, one has

$$D_1D_2-\tilde{\mu}(D_1)\tilde{\mu}(D_2)=(D_1-\tilde{\mu}(D_1))D_2+\tilde{\mu}(D_1)(D_2-\tilde{\mu}(D_2)).$$

By the assertion (3) proved above, the right hand side is in $\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)+U(\mathfrak{g}_C)\theta\mathfrak{n}(\mathfrak{a}_{pq})$. Hence we have $\tilde{\mu}(D_1D_2)=\tilde{\mu}(D_1)\tilde{\mu}(D_2)$, which completes the proof.

Q. E. D.

Let $W(\Phi)$ (resp. $W(\Phi_0)$) be the Weyl group of Φ (resp. Φ_0). Then $W(\Phi_0)$ is the subgroup of $W(\Phi)$ generated by reflections corresponding to the elements of Φ_0. $W(\Phi)$ acts on $\mathfrak{j}_C$, hence it acts also on $U(\mathfrak{j}_C)$. Let $I(\mathfrak{j}_C)$ (resp. $I_0(\mathfrak{j}_C)$) denote the algebra of $W(\Phi)$-invariant (resp. $W(\Phi_0)$-invariant) elements in $U(\mathfrak{j}_C)$. For $\beta\in\mathfrak{j}_C^*$, we denote by T_β the automorphism of $U(\mathfrak{j}_C)$ such that $T_\beta(H)=H+\beta(H)$ for $H\in\mathfrak{j}_C$.

Put $\gamma=T_\rho\circ\tilde{\gamma}$ (resp. $\gamma_0=T_{\rho_0}\circ\tilde{\gamma}_0$) with $\rho=2^{-1}\sum_{\alpha\in\Phi^+}\alpha$ (resp. $\rho_0=2^{-1}\sum_{\alpha\in\Phi_0^+}\alpha$). We can now state a fundamental lemma on the structure of $Z(\mathfrak{g}_C)$ (resp. $Z(\mathfrak{l}_{0C})$) as follows.

Lemma 1.3 (Harish-Chandra). *The map γ (resp. γ_0) gives an algebra isomorphisms from $Z(\mathfrak{g}_C)$ (resp. $Z(\mathfrak{l}_{0C})$) onto $I(\mathfrak{j}_C)$ (resp. $I_0(\mathfrak{j}_C)$).*

This lemma is well-known. Refer to [3, 7.4.5] for example.

The map $\gamma|Z(\mathfrak{g}_C)$ (resp. $\gamma_0|Z(\mathfrak{l}_{0C})$) is called the Harish-Chandra isomorphism from $Z(\mathfrak{g}_C)$ to $I(\mathfrak{j}_C)$ (resp. from $Z(\mathfrak{l}_{0C})$ to $I_0(\mathfrak{j}_C)$).

Since $I(\mathfrak{j}_C)$ is a subalgebra of $I_0(\mathfrak{j}_C)$, $I_0(\mathfrak{j}_C)$ has a canonical structure of $I(\mathfrak{j}_C)$-module, which is described in the following lemma.

Lemma 1.4. *$I_0(\mathfrak{j}_C)$ is a free $I(\mathfrak{j}_C)$-module of rank $|W(\Phi)/W(\Phi_0)|$, where, for a set Y, $|Y|$ denotes the cardinal number. Moreover, one can choose a module basis consisting of homogeneous elements.*

This lemma follows, as a special case, from [15, Part I, § 4, Cor. 10], where the invariants of finite reflection groups are treated in full generality. But we sketch the proof of Lemma 1.4 in order to clarify our succeeding arguments.

Outline of proof. $S(\mathfrak{j}_C^*)$ denotes the symmetric algebra of $\mathfrak{j}_C^*$. $W(\Phi)$ acts on $\mathfrak{j}_C^*$ by duality, hence it acts also on $S(\mathfrak{j}_C^*)$. Define for any $\alpha\in\mathfrak{j}_R^*$ a differential operator $\partial(\alpha)$ on $\mathfrak{j}_R^*$ by

$$\partial(\alpha)\phi(\lambda)=\frac{d}{dt}\phi(\lambda+t\alpha)|_{t=0}\qquad(\phi\in C^\infty(\mathfrak{j}_R^*),\ \lambda\in\mathfrak{j}_R^*).$$

The assignment $\alpha \to \partial(\alpha)$ extends uniquely to an isomorphism from $S(\mathfrak{j}_C^*)$ onto the algebra of constant coefficient defferential operators on $\mathfrak{j}_R^*$.

Identify canonically $U(\mathfrak{j}_C)$ with the algebra of complex polynomial functions on $\mathfrak{j}_R^*$. An element $p \in U(\mathfrak{j}_C)$ is said to be *$W(\Phi)$-harmonic* if $\partial(u)p=0$ for every $W(\Phi)$-invariant element $u \in S(\mathfrak{j}_C^*)$ without constant term. Denote by $H(\mathfrak{j}_C)$ the space of $W(\Phi)$-harmonic elements in $U(\mathfrak{j}_C)$. Then $H(\mathfrak{j}_C)$ is a $W(\Phi)$-stable subspace of dimension $|W(\Phi)|$, and compatible with the grading on $U(\mathfrak{j}_C)$. Moreover one can show that the map $I(\mathfrak{j}_C) \times H(\mathfrak{j}_C) \ni (p, e) \mapsto pe \in U(\mathfrak{j}_C)$ gives a linear isomorphism from $I(\mathfrak{j}_C) \otimes H(\mathfrak{j}_C)$ onto $U(\mathfrak{j}_C): U(\mathfrak{j}_C) \simeq I(\mathfrak{j}_C) \otimes H(\mathfrak{j}_C)$. From this we see immediately that $I_0(\mathfrak{j}_C) \simeq I(\mathfrak{j}_C) \otimes H_0(\mathfrak{j}_C)$, where $H_0(\mathfrak{j}_C)$ is the space of $W(\Phi_0)$-fixed elements in $H(\mathfrak{j}_C)$. One can choose a basis of $H_0(\mathfrak{j}_C)$ consisting of homogeneous elements, since each homogeneous component of $H(\mathfrak{j}_C)$ is stable under $W(\Phi_0)$. We have thus proved that $I_0(\mathfrak{j}_C)$ is a free $I(\mathfrak{j}_C)$-module of rank $\dim H_0(\mathfrak{j}_C)$ and that it has a module basis consisting of homogeneous elements.

To complete the proof, it is enough to show that $\dim H_0(\mathfrak{j}_C) = |W(\Phi)/W(\Phi_0)|$. This is done as follows. One can show that the representation of $W(\Phi)$ on $H(\mathfrak{j}_C)$ is equivalent to the regular representation of $W(\Phi)$. The space $W(\Phi_0)$-fixed elements in $L^2(W(\Phi))$ is naturally isomorphic to $L^2(W(\Phi)/W(\Phi_0))$, whose dimension is equal to $|W(\Phi)/W(\Phi_0)|$. Consequently we get $\dim H_0(\mathfrak{j}_C) = \dim L^2(W(\Phi)/W(\Phi_0)) = |W(\Phi)/W(\Phi_0)|$ as desired. Q. E. D.

Let κ be an automorphism of $U(\mathfrak{l}_{0C})$ defined by

(1.11) $$\kappa(X) = X + 2^{-1} \operatorname{tr}(\operatorname{ad} X | \mathfrak{n}(\mathfrak{a}_{pq})) \qquad (X \in \mathfrak{l}_0).$$

Denote by ω the inverse of κ. Put $\mu = \kappa \circ \tilde{\mu}$, then by Lemma 1.2(2), μ restricted to $Z(\mathfrak{g}_C)$ gives a homomorphism from $Z(\mathfrak{g}_C)$ into $Z(\mathfrak{l}_{0C})$. So $Z(\mathfrak{l}_{0C})$ has a structure of $Z(\mathfrak{g}_C)$-module through μ.

We can rewrite the equality $\tilde{\gamma} = \tilde{\gamma}_0 \circ \tilde{\mu}$ in Lemma 1.2(1) in terms of γ, γ_0 and μ as follows.

Lemma 1.5. *The map γ is decomposed as $\gamma = \gamma_0 \circ \mu$. In particular, one has the following commutative diagram.*

(1.12) $$\begin{tikzcd} I(\mathfrak{j}_C) \arrow[r, hook, "\mathrm{Id}"] & I_0(\mathfrak{j}_C) \\ Z(\mathfrak{g}_C) \arrow[u, "\gamma\,\wr"] \arrow[r, hook, "\mu"'] & Z(\mathfrak{l}_{0C}) \arrow[u, "\wr\,\gamma_0"'] \end{tikzcd}$$

Proof. For any $D \in U(\mathfrak{l}_{0C})$, one has by the definition of $\tilde{\gamma}_0$

(1.13) $$D - \tilde{\gamma}_0(D) \in \mathfrak{n}_C(\Phi_0) U(\mathfrak{l}_{0C}) + U(\mathfrak{l}_{0C}) \mathfrak{u}_C(\Phi_0).$$

Applying κ to the both sides of (1.13), we get

(1.14) $$\kappa(D) - \kappa(\tilde{\gamma}_0(D)) \in \mathfrak{n}_C(\Phi_0) U(\mathfrak{l}_{0C}) + U(\mathfrak{l}_{0C}) \mathfrak{u}_C(\Phi_0).$$

Here we used the fact $\kappa(X) = X$ for any $X \in \mathfrak{n}_C(\Phi_0)$ or any $X \in \mathfrak{u}_C(\Phi_0)$. (1.14) means that $\tilde{\gamma}_0(\kappa(D))=\kappa(\tilde{\gamma}_0(D))$. Since $\mathfrak{n}(\mathfrak{a}_{pq})_C=\sum_{\alpha\in\Phi^+\setminus\Phi_0^+}\mathfrak{g}_C(\mathfrak{j}_C\,;\,\alpha)$, the restriction of κ to $U(\mathfrak{j}_C)$ coincides with $T_{\rho-\rho_0}$. Hence we have

$$T_{\rho-\rho_0}\circ\tilde{\gamma}_0=\tilde{\gamma}_0\circ\kappa. \tag{1.15}$$

Then it follows from Lemma 1.2(1) together with (1.15) that

$$\gamma_0\circ\mu=T_{\rho_0}\circ\tilde{\gamma}_0\circ\kappa\circ\tilde{\mu}=T_{\rho_0}\circ T_{\rho-\rho_0}\circ\tilde{\gamma}_0\circ\tilde{\mu}=T_\rho\circ\tilde{\gamma}=\gamma.$$

Thanks to Lemma 1.3 and the above equality, one gets (1.12) as desired.

Q. E. D.

By virtue of the commutative diagram (1.12), the $Z(\mathfrak{g}_C)$-module $Z(\mathfrak{l}_{0C})$ is equivalent to the natural $I(\mathfrak{j}_C)$-module $I_0(\mathfrak{j}_C)$ through Harish-Chandra isomorphisms. In Lemma 1.4 we described the structure of $I_0(\mathfrak{j}_C)$ as a $I(\mathfrak{j}_C)$-module. Therefore the $Z(\mathfrak{g}_C)$-module structure on $Z(\mathfrak{l}_{0C})$ is now clear. We summarize this as follows.

Proposition 1.6. *There exist $r=|W(\Phi)/W(\Phi_0)|$ number of elements $\nu_1=1, \nu_2, \cdots, \nu_r$ in $Z(\mathfrak{l}_{0C})$ satisfying the following conditions* (1) *and* (2).

(1) *For every $1\leqq i\leqq r$, $\gamma_0(\nu_i)$ is a homogeneous element in $I_0(\mathfrak{j}_C)$.*

(2) *Every $\nu\in Z(\mathfrak{l}_{0C})$ is expressed uniquely as*

$$\nu=\sum_{i=1}^{r}\mu(Z_i)\nu_i \tag{1.16}$$

with $Z_i\in Z(\mathfrak{g}_C)$. Moreover one has $\deg\nu\geqq\deg Z_i+\deg\nu_i$ *for $1\leqq i\leqq r$, where* $\deg X$ *is the degree of an element $X\in U(\mathfrak{g}_C)$.*

Proof. By Lemmas 1.4 and 1.5, the assertions are clear except the last one in (2). Taking the commutative diagram (1.12) into account, we apply γ_0 to the both sides of (1.16). Then we have an equality in $I_0(\mathfrak{j}_C)$

$$\gamma_0(\nu)=\sum_{i=1}^{r}\gamma(Z_i)\gamma_0(\nu_i).$$

By the uniqueness of the above expansion, one has

$$\deg\gamma(Z_i)+\deg\gamma_0(\nu_i)\leqq\deg\gamma_0(\nu)\qquad\text{for}\quad 1\leqq i\leqq r.$$

On the other hand, it is easily checked that $\deg\gamma(Z)=\deg Z$ (resp. $\deg\gamma_0(Z_0)=\deg Z_0$) for any $Z\in Z(\mathfrak{g}_C)$ (resp. $Z_0\in Z(\mathfrak{l}_{0C})$). Therefore we obtain $\deg Z_i+\deg\nu_i\leqq\deg\nu$ for $1\leqq i\leqq r$.

Q. E. D.

1.5. Direct sum decompositions of $\mathfrak{g}$ and $\mathfrak{l}$.

First we explain after [1] an Iwasawa decomposition and a Cartan decomposition of $\mathfrak{l}$ with respect to $\mathfrak{h}$, $+1$ eigenspace of σ. For this purpose, we need some more notations and a lemma.

The centralizer $\mathfrak{l}_0$ of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ is stable under both θ and σ because $\theta(X)=\sigma(X)=-X$ for any $X\in\mathfrak{a}_{pq}$. So it splits into a direct sum of vector spaces

$$\mathfrak{l}_0=\mathfrak{l}_0^{kh}\oplus\mathfrak{l}_0^{kq}\oplus\mathfrak{l}_0^{ph}\oplus\mathfrak{l}_0^{pq}, \tag{1.17}$$

where $\mathfrak{l}_0^{kh}=\mathfrak{l}_0\cap\mathfrak{k}\cap\mathfrak{h}$, $\mathfrak{l}_0^{kq}=\mathfrak{l}_0\cap\mathfrak{k}\cap\mathfrak{q}$ and so on. Note that $\mathfrak{l}_0^{pq}=\mathfrak{a}_{pq}$.

For any $\lambda\in\Delta(\mathfrak{l}:\mathfrak{a}_{pq})$, the root space $\mathfrak{g}(\mathfrak{a}_{pq};\lambda)$ is stable under the involution $\sigma\theta$ of L because $\sigma\theta$ is identity on $\mathfrak{a}_{pq}$. Let

$$\mathfrak{g}(\mathfrak{a}_{pq};\lambda)=\mathfrak{g}_+(\mathfrak{a}_{pq};\lambda)\oplus\mathfrak{g}_-(\mathfrak{a}_{pq};\lambda)$$

be the eigenspace decomposition of $\mathfrak{g}(\mathfrak{a}_{pq};\lambda)$ with respect to $\sigma\theta$, where $\mathfrak{g}_\pm(\mathfrak{a}_{pq};\lambda)$ is the ±1 eigenspace of $\sigma\theta$ on $\mathfrak{g}(\mathfrak{a}_{pq};\lambda)$. Set

$$\Delta_\pm(\mathfrak{l}:\mathfrak{a}_{pq})=\{\lambda\in\Delta(\mathfrak{l}:\mathfrak{a}_{pq})\,;\ \mathfrak{g}_\pm(\mathfrak{a}_{pq};\lambda)\neq(0)\}\,. \tag{1.18}$$

We define an open dense subset A'_{pq} of $A_{pq}=\exp\mathfrak{a}_{pq}$ by

$$A'_{pq}=\{a\in A_{pq}\,;\ a^\lambda\neq1 \text{ for all } \lambda\in\Delta_+(\mathfrak{l}:\mathfrak{a}_{pq})\}\,.$$

Here $(\exp H)^\lambda=\exp\lambda(H)$ for $\lambda\in(\mathfrak{a}_{pq})^*_c$ and $H\in\mathfrak{a}_{pq}$.

For any $g\in G$ and any $D\in U(\mathfrak{g}_c)$, write ${}^gD$ for $\mathrm{Ad}(g)D$ for simplicity. Then we have

Lemma 1.7. *If $\lambda\in\Delta(\mathfrak{l}:\mathfrak{a}_{pq})$, then every element $X^\pm\in\mathfrak{g}_\pm(\mathfrak{a}_{pq};\lambda)$ is expressed as*

$$X^\pm=(a^\lambda\mp a^{-\lambda})^{-1}\{{}^a(X^\pm+\sigma X^\pm)\mp a^{-\lambda}(X^\pm+\theta X^\pm)\} \tag{1.19}$$

for every $a\in A'_{pq}$.

Proof. For any $Y\in\mathfrak{g}(\mathfrak{a}_{pq};\lambda)$, σY as well as θY is in $\mathfrak{g}(\mathfrak{a}_{pq};-\lambda)$. This implies that ${}^a(Y+\sigma Y)=a^\lambda Y+a^{-\lambda}\sigma Y$ for any $a\in A_{pq}$. If $X^\pm\in\mathfrak{g}_\pm(\mathfrak{a}_{pq};\lambda)$, then $\sigma X=\pm\theta X$. Hence we have

$$\begin{aligned}{}^a(X^\pm+\sigma X^\pm)\mp a^{-\lambda}(X^\pm+\theta X^\pm)&=(a^\lambda X^\pm\pm a^{-\lambda}\theta X^\pm)\mp a^{-\lambda}(X^\pm+\theta X^\pm)\\&=(a^\lambda\mp a^{-\lambda})X^\pm\,.\end{aligned}$$

Noting that $a^\lambda\pm a^{-\lambda}\neq0$ for any $a\in A'_{pq}$, we obtain (1.19) Q.E.D.

Using the above notations one gets the following

Lemma 1.8. *The Lie algebra $\mathfrak{l}$ admits the following two kinds of direct sum decompositions as vector spaces:*

$$\mathfrak{l}=\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\mathfrak{a}_{pq}\oplus\mathfrak{l}_0^{ph}\oplus(\mathfrak{k}\cap\mathfrak{l})\,, \tag{1.20}$$

$$\mathfrak{l}={}^a\mathfrak{h}'\oplus\mathfrak{a}_{pq}\oplus(\mathfrak{k}\cap\mathfrak{l})\quad \textit{for any}\quad a\in A'_{pq}\,, \tag{1.21}$$

where we put $\mathfrak{h}'=\mathfrak{h}\cap\{\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\sigma\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\}\oplus\mathfrak{l}_0^{ph}$.

The decomposition (1.20) (resp. (1.21)) is called an *Iwasawa* (resp. a *Cartan*) *decomposition of $\mathfrak{l}$ with respect to $\mathfrak{h}$.*

Proof of Lemma 1.8. First we prove (1.20). Recall the decomposition (1.3). Since both $\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\theta\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})$ and $\mathfrak{l}_0$ are θ-stable, $\mathfrak{k}\cap\mathfrak{l}$ splits into a direct sum of vector spaces

$$\mathfrak{k}\cap\mathfrak{l}=(\mathfrak{k}\cap\mathfrak{l}_0)\oplus[\mathfrak{k}\cap\{\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\theta\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\}]\,. \tag{1.22}$$

If $X, Y\in\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})$, then one has $X+\theta Y=(X-Y)+(\theta Y+Y)$. From this we see immediately

$$(1.23)\qquad \mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\theta\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})=\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus[\mathfrak{k}\cap\{\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\theta\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\}].$$

By replacing the right hand side of (1.3) by those of (1.17) and (1.23), one gets

$$(1.24)\qquad \mathfrak{l}=\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus[\mathfrak{k}\cap\{\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\theta\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\}]\oplus(\mathfrak{k}\cap\mathfrak{l}_0)\oplus\mathfrak{a}_{pq}\oplus\mathfrak{l}_0^{ph}.$$

By (1.22) and (1.24) we obtain (1.20) as desired.

Secondly we show (1.21) using (1.20) proved above. It is easily verified that the assignment $X\mapsto X+\sigma X$ gives a linear bijection from $\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})$ onto $\{\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\sigma\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\}\cap\mathfrak{h}$. So we get $\dim\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})=\dim[\mathfrak{h}\cap\{\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\oplus\sigma\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})\}]$. Then it follows from (1.20) and this equality that

$$(1.25)\qquad \dim\mathfrak{l}=\dim\mathfrak{h}'+\dim\mathfrak{a}_{pq}+\dim\mathfrak{k}\cap\mathfrak{l}.$$

On the other hand, thanks to Lemma 1.7, $\mathfrak{n}(\mathfrak{l}:\mathfrak{a}_{pq})$ is contained in ${}^a\mathfrak{h}'+(\mathfrak{k}\cap\mathfrak{l})$ for every $a\in A'_{pq}$. Hence we obtain $\mathfrak{l}={}^a\mathfrak{h}'+\mathfrak{a}_{pq}+\mathfrak{k}\cap\mathfrak{l}$ using (1.20) again. This sum must be direct in view of (1.25). Q. E. D.

Taking into account the relation $\mathfrak{n}\oplus\theta\mathfrak{n}=\mathfrak{n}\oplus\{\mathfrak{k}\cap(\mathfrak{n}\oplus\theta\mathfrak{n})\}$, we apply Lemma 1.8 to $\mathfrak{l}$ in the right hand side of the equality $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{l}\oplus\theta\mathfrak{n}$. Then we obtain two kinds of direct sum decompositions of $\mathfrak{g}$ corresponding to those (1.20) and (1.21) of $\mathfrak{l}$ as follows.

Lemma 1.9. *The Lie algebra $\mathfrak{g}$ splits, in two different manners, into direct sums of vector spaces as*

$$(1.26)\qquad \mathfrak{g}=\mathfrak{n}(\mathfrak{a}_{pq})\oplus\mathfrak{a}_{pq}\oplus\mathfrak{l}_0^{ph}\oplus\mathfrak{k},$$

$$(1.27)\qquad \mathfrak{g}={}^a(\mathfrak{h}'\oplus\mathfrak{n})\oplus\mathfrak{a}_{pq}\oplus\mathfrak{k}\qquad \textit{for every}\quad a\in A'_{pq}.$$

1.6. A decomposition theorem of elements in $U(\mathfrak{g}_C)$.

Let $\mathcal{F}^+$ be the ring of functions on A'_{pq} generated by the following functions:

$$(1.28)\qquad a^{-\lambda}\qquad \text{for}\quad \lambda\in\Lambda(\mathfrak{n}:\mathfrak{a}_{pq}),$$

$$(1.29)\quad (a^{\lambda}-a^{-\lambda})^{-1},\ a^{-\lambda}(a^{\lambda}-a^{-\lambda})^{-1}\qquad \text{for}\quad \lambda\in\Lambda^+(\mathfrak{l}:\mathfrak{a}_{pq})\cap\Lambda_+(\mathfrak{l}:\mathfrak{a}_{pq}),$$

$$(1.30)\quad (a^{\lambda}+a^{-\lambda})^{-1},\ a^{-\lambda}(a^{\lambda}+a^{-\lambda})^{-1}\qquad \text{for}\quad \lambda\in\Lambda^+(\mathfrak{l}:\mathfrak{a}_{pq})\cap\Lambda_-(\mathfrak{l}:\mathfrak{a}_{pq})\quad (\text{cf. (1.18)}).$$

$\mathcal{F}$ denotes the ring generated by $\mathcal{F}^+$ and the constant function 1 on A'_{pq}. For an $a\in A'_{pq}$, consider the linear map

$$\Psi_a:\mathcal{F}\otimes U(\mathfrak{h}_C\oplus\mathfrak{n}_C)\otimes\Big\{\sum_{i=1}^{r}Z(\mathfrak{g}_C)\omega(\nu_i)\Big\}U(\mathfrak{k}_C)\longrightarrow U(\mathfrak{g}_C)$$

defined by

$$(1.31)\qquad \Psi_a(f\otimes\xi\otimes\eta)=f(a)\cdot{}^a\xi\eta\qquad \text{for}\quad f\in\mathcal{F},\ \xi\in U(\mathfrak{h}_C\oplus\mathfrak{n}_C)$$

and $\eta\in\{\sum_{1\le i\le r}Z(\mathfrak{g}_C)\omega(\nu_i)\}U(\mathfrak{k}_C)$.

In this subsection we find, for any given $D\in U(\mathfrak{g}_C)$, an element in the inverse image $\Psi_a^{-1}(D)$, independent of a (Theorem 1.12). This will be achieved in the same line as in the proof of [1, Lemma 3.8].

For an integer $m\geqq 0$, let $U(\mathfrak{g}_C)_m$ be the space of elements $D\in U(\mathfrak{g}_C)$ such that $\deg D\leqq m$. Set $U(\mathfrak{g}_C)_{-m}=(0)$. Then one has a filtration of $U(\mathfrak{g}_C)$:

$$(0)=U(\mathfrak{g}_C)_{-m}=\cdots=U(\mathfrak{g}_C)_{-1}\subseteqq U(\mathfrak{g}_C)_0=\boldsymbol{C}\subseteqq\cdots\subseteqq U(\mathfrak{g}_C)_m\subseteqq U(\mathfrak{g}_C)_{m+1}\subseteqq\cdots$$

For a subspace E of $U(\mathfrak{g}_C)$, set $E_m=E\cap U(\mathfrak{g}_C)_m$. Then we obtain

Lemma 1.10. *For any non-negative integer m, $U(\mathfrak{g}_C)_m$ admits a decomposition*

$$U(\mathfrak{g}_C)_m=\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_{m-1}+\Big[U(\mathfrak{h}_C\cap\mathfrak{l}_{0C})\Big\{\sum_{i=1}^{r}Z(\mathfrak{g}_C)\omega(\nu_i)\Big\}U(\mathfrak{k}_C)\Big]_m. \tag{1.32}$$

Proof. Take a basis $X_1,\cdots,X_s$ of the vector space $\mathfrak{l}_0^{ph}=\mathfrak{l}_0\cap\mathfrak{p}\cap\mathfrak{h}$. Let V denote the subspace of $U(\mathfrak{g}_C)$ generated by the elements $X_1^{t_1}\cdots X_s^{t_s}$ with integers $t_i\geqq 0$ $(1\leqq i\leqq s)$. Consider the decomposition (1.26) of $\mathfrak{g}$. Then, by the Poincaré-Birkhoff-Witt theorem, we have

$$U(\mathfrak{g}_C)_m=\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_{m-1}\oplus[V\cdot U((\mathfrak{a}_{pq})_C)U(\mathfrak{k}_C)]_m. \tag{1.33}$$

Let $D\in[V\cdot U((\mathfrak{a}_{pq})_C)U(\mathfrak{k}_C)]_m$. Write $D=\sum_{1\leq n\leq N}Q_nH_nW_n$, where $Q_n\in V$, $H_n\in U((\mathfrak{a}_{pq})_C)$ and $W_n\in U(\mathfrak{k}_C)$ such that $\deg Q_n+\deg H_n+\deg W_n\leqq m$. Apply Proposition 1.6 to $\kappa(H_n)\in U((\mathfrak{a}_{pq})_C)\subseteqq Z(\mathfrak{l}_{0C})$. Then H_n is expressed uniquely as

$$H_n=\sum_{i=1}^{r}\tilde{\mu}(Z_{n,i})\omega(\nu_i)$$

with $Z_{n,i}\in Z(\mathfrak{g}_C)$. Moreover, $\deg H_n\geqq\deg Z_{n,i}+\deg\nu_i$ $(1\leqq i\leqq r)$. Thus we have

$$D=\sum_{n,i}Q_nZ_{n,i}\omega(\nu_i)W_n+\sum_{n,i}Q_n(\tilde{\mu}(Z_{n,i})-Z_{n,i})\omega(\nu_i)W_n.$$

By Lemma 1.2(3), $\tilde{\mu}(Z_{n,i})-Z_{n,i}\in\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_{v-1}$ with $v=\deg Z_{n,i}$. Since $\mathfrak{l}_0$ normalizes $\mathfrak{n}(\mathfrak{a}_{pq})$, $\sum_{n,i}Q_n(\tilde{\mu}(Z_{n,i})-Z_{n,i})\omega(\nu_i)W_n$ belongs to $\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_{m'-1}$ with $m'=\deg Q_n+\deg\nu_i+\deg W_n+v\leqq m$. Consequently,

$$[VU((\mathfrak{a}_{pq})_C)U(\mathfrak{k}_C)]_m\subseteqq\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_{m-1}+[U(\mathfrak{h}_C\cap\mathfrak{l}_{0C})\{\sum_i Z(\mathfrak{g}_C)\omega(\nu_i)\}U(\mathfrak{k}_C)]_m. \tag{1.34}$$

Combining (1.34) with (1.33), we get the desired decomposition (1.32).

Q.E.D.

The map Ψ_a restricted to $1\otimes U(\mathfrak{h}_C\cap\mathfrak{l}_{0C})\otimes\{\sum_i Z(\mathfrak{g}_C)\omega(\nu_i)\}U(\mathfrak{k}_C)$ does not depend on $a\in A_{pq}$ because $\mathfrak{h}\cap\mathfrak{l}_0$ centralizes A_{pq}. We denote this map by Ψ instead of Ψ_a. Thus, in view of the above Lemma, it suffices to consider the elements in $\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)$ in order to find an element in $\bigcap_{a\in A'_{pq}}\Psi_a^{-1}(D)$ for any $D\in U(\mathfrak{g}_C)$.

Lemma 1.11. *If $D\in\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)$, then there exist finitely many elements $f_n\in\mathcal{F}^+$, $\xi_n\in U(\mathfrak{h}_C\oplus\mathfrak{n}_C)$ and $\eta_n\in\{\sum_i Z(\mathfrak{g}_C)\omega(\nu_i)\}U(\mathfrak{k}_C)$ $(1\leqq n\leqq I)$ such that*

(1) *$\Psi_a(\sum_n f_n\otimes\xi_n\otimes\eta_n)=D$ for any $a\in A'_{pq}$,*

(2) $\deg \xi_n + \deg \eta_n \leqq \deg D$ $(1 \leqq n \leqq I)$.

Proof. We prove the lemma by induction on $\deg D$. For an integer $m \geqq 0$, assume that the assertion is true for any D in $[\mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)]_m = \mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_{m-1}$. Now let $D \in \mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_m$. In view of (1.3), we may assume that $D = XD'$ with $D' \in U(\mathfrak{g}_C)_m$ and $X \in \mathfrak{g}(\mathfrak{a}_{pq}; \lambda)$ for some $\lambda \in \Lambda^+(\mathfrak{g} : \mathfrak{a}_{pq})$.

Case 1: $\lambda \in \Lambda^+(\mathfrak{l} : \mathfrak{a}_{pq})$. It follows from Lemma 1.7 that

$$X = f_1(a) \cdot {}^a(X + \sigma X) + f_2(a)(X + \theta X) \qquad (a \in A'_{pq}),$$

with $f_i \in \mathcal{F}^+$ $(i = 1, 2)$. This implies that

$$D = f_1(a)^a(X + \sigma X)D' + f_2(a)D'(X + \theta X) + f_2(a)D'' \tag{1.35}$$

with $D'' = [X + \theta X, D'] \in U(\mathfrak{g}_C)_m$. According as (1.32), decompose D' (resp. D'') as $D' = D'_0 + D'_1$ (resp. $D'' = D''_0 + D''_1$) with $D'_0, D''_0 \in \mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)_{m-1}$ and $D'_1, D''_1 \in [U(\mathfrak{h}_C \cap \mathfrak{l}_{0C})\{\sum_i Z(\mathfrak{g}_C)\omega(\nu_i)\}U(\mathfrak{k}_C)]_m$. Apply the induction hypothesis to D'_0 and D''_0. Then in view of (1.35), we obtain the desired result for D.

Case 2: $\lambda \in \Lambda(\mathfrak{n} : \mathfrak{a}_{pq})$. In this case, $X = a^{-\lambda} \cdot {}^aX$ holds for $a \in A_{pq}$, whence $D = a^{-\lambda} \cdot {}^aXD'$. Repeating the above argument, we can prove the assertion in this case. This completes the proof. Q. E. D.

By Lemmas 1.10 and 1.11, we obtain immediately a decomposition theorem of an arbitrary element in $U(\mathfrak{g}_C)$ as follows.

Theorem 1.12. *Let $D \in U(\mathfrak{g}_C)$. Then there exist $D_0 \in 1 \otimes U(\mathfrak{h}_C \cap \mathfrak{l}_{0C}) \otimes \{\sum_i Z(\mathfrak{g}_C)\omega(\nu_i)\}U(\mathfrak{k}_C)$ and finitely many elements $f_n \in \mathcal{F}^+$, $\xi_n \in U(\mathfrak{h}_C \oplus \mathfrak{n}_C)$, $\eta_n \in \{\sum_i Z(\mathfrak{g}_C)\omega(\nu_i)\}U(\mathfrak{k}_C)$ $(1 \leqq n \leqq I)$ such that*

(1) $\Psi_a(D_0 + \sum_{1 \leqq n \leqq I} f_n \otimes \xi_n \otimes \eta_n) = D$ *for all* $a \in A'_{pq}$,

(2) $\deg \Psi(D_0) \leqq \deg D$, $\deg \xi_n + \deg \eta_n \leqq \deg D$ $(1 \leqq n \leqq I)$,

(3) $D - \Psi(D_0) \in \mathfrak{n}(\mathfrak{a}_{pq})U(\mathfrak{g}_C)$.

Here $\mathcal{F}^+$ is the ring of functions on A'_{pq} generated by functions (1.28), (1.29) *and* (1.30), *and Ψ_a is the map defined by* (1.31).

§ 2. A finite multiplicity theorem for induced representations in C^∞-context

Let ζ be a continuous representation of the subgroup HN $(\subseteqq G)$ in 1.2. In this section we consider the induced representation $\pi_\zeta = C^\infty\text{-Ind}_{HN}^G(\zeta)$ in C^∞-context, and examine the multiplicities of irreducible constituents of it. For this purpose we study $Z(\mathfrak{g}_C)$-finite, K-finite vectors for π_ζ. For any $\tau \in \hat{K}$ (= the unitary dual of K) and any ideal I of $Z(\mathfrak{g}_C)$ with finite codimension, consider the space of τ-isotypic vectors for π_ζ annihilated by $\pi_\zeta(I)$. We estimate its dimension in Theorem 2.8 making use of Theorem 1.12. Thanks to this estimate, an upper bound of the multiplicity in π_ζ is given for any irreducible representation of G (Theorem 2.10). Especially, we get a sufficient condition for ζ that each irreducible representation occurs in π_ζ with at most finite multiplicity

(Theorem 2.12).

2.1. Induced representations in C^∞-context.

In this subsection, let G be an arbitrary Lie group. First we recall after [17, 4.4] some notations about smooth representations of G.

Let π be a continuous representation of G on a locally convex, complete, Hausdorff, topological vector space E. A vector v in E is said to be *smooth* if the map $\tilde{v}: G \ni g \mapsto \pi(g)v \in E$ is C^∞. The collection E^∞ of all smooth vectors for π forms a $\pi(G)$-stable, dense subspace of E. The assignment $v \mapsto \tilde{v}$ gives a linear embedding from E^∞ onto a closed subspace of $C^\infty(G, E)$. Here $C^\infty(G, E)$ is the space of E-valued smooth functions on G equipped with the topology of uniform convergence on any compact subset of a function and its derivatives. Equip E^∞ with the topology inherited from that of $C^\infty(G, E)$ through the embedding above. The representation (π, E) is called smooth if $E=E^\infty$ with coincidence of topologies. In this case, E has a structure of $U(\mathfrak{g}_C)$-module in such a way that

$$\pi(X)v=\frac{d}{dt}\pi(\exp tX)v|_{t=0} \qquad (X\in\mathfrak{g},\ v\in E).$$

Set $\pi_\infty(g)=\pi(g)|E^\infty$ $(g\in G)$ for any continuous representation (π, E). Then π_∞ defines a smooth representation of G on E^∞, which is called the *smooth representation associated to π.*

Now we define the representations induced in C^∞-context. Let L be a closed subgroup of G. For a continuous representation ζ of L on a (locally convex) Fréchet space F, let $C^\infty(G;\zeta)$ be the space of all $f\in C^\infty(G, F)$ satisfying

$$f(gh)=(\delta_L(h)/\delta_G(h))^{1/2}\zeta(h)^{-1}f(g) \qquad (g\in G,\ h\in L).$$

Here δ_X is the modular function on a Lie group X with respect to a left Haar measure $d_X(x)$: $\delta_X(g)=d_X(hg)/d_X(h)$ $(g\in X)$. Then $C^\infty(G;\zeta)$ is a closed subspace of $C^\infty(G, F)$. Equip $C^\infty(G;\zeta)$ with the topology inherited from that of $C^\infty(G, F)$. Then G acts smoothly on $C^\infty(G;\zeta)$ by left translation:

$$\pi_\zeta(g)f(x)=f(g^{-1}x) \qquad (f\in C^\infty(G;\zeta),\ x,\ g\in G).$$

Thus one gets a smooth representation $(\pi_\zeta, C^\infty(G;\zeta))$ of G. We call π_ζ the *representation induced in C^∞-context* from (ζ, F), and often express this as C^∞-$\mathrm{Ind}_L^G(\zeta)$ instead of π_ζ.

Lemma 2.1. *Let (ζ, F) be as adove. Then one has $C^\infty(G;\zeta)=C^\infty(G;\zeta_\infty)$ with coincidence of topologies. In particular, it holds that C^∞-$\mathrm{Ind}_L^G(\zeta)=C^\infty$-$\mathrm{Ind}_L^G(\zeta_\infty)$.*

Proof. For an arbitrary $f\in C^\infty(G;\zeta)$, we show that $f\in C^\infty(G;\zeta_\infty)$. Indeed, by the definition of $C^\infty(G;\zeta)$ one has the equality

$$\zeta(h)f(g)=(\delta_L(h)/\delta_G(h))^{1/2}f(gh^{-1}) \qquad (g\in G,\ h\in L). \tag{2.1}$$

This implies that $f(g)\in F^\infty$ for all $g\in G$. Moreover, from (2.1) we see easily that the map $G\ni g\mapsto f(g)^\sim\in C^\infty(L, F)$ is C^∞, where $f(g)^\sim(h)=\zeta(h)f(g)$. These two facts mean that $f\in C^\infty(G;\zeta_\infty)$. Hence we obtain $C^\infty(G;\zeta)\subseteq C^\infty(G;\zeta_\infty)$. The reverse inclusion is clear, whence $C^\infty(G;\zeta)=C^\infty(G;\zeta_\infty)$.

The identical map $\iota\colon C^\infty(C;\zeta_\infty)\to C^\infty(G;\zeta)$ is continuous because the topology on F^∞ is, in general, finer than that inherited from F. Notice that both $C^\infty(G;\zeta)$ and $C^\infty(G;\zeta_\infty)$ have structures of Fréchet spaces. Then, by the closed graph theorem, ι must be bicontinuous, which completes the proof. Q. E. D.

For continuous representations (π_i, E_i) $(i=1, 2)$ of G, let $\mathrm{Hom}_G(\pi_1, \pi_2)$ be the space of continuous intertwining operators from E_1 to E_2. Put $I_G(\pi_1, \pi_2)=\dim \mathrm{Hom}_G(\pi_1, \pi_2)$. $I_G(\pi_1, \pi_2)$ is said to be the *intertwining number* from π_1 to π_2. In case where π_1 is irreducible, we call $I_G(\pi_1, \pi_2)$ the *multiplicity* of π_1 in π_2 as subrepresentation.

For the representation $C^\infty\text{-}\mathrm{Ind}_L^G(\zeta)$, we get a reciprocity law on intertwining numbers as follows.

Lemma 2.2. *For a smooth representation (π, E) of G, one has a canonical isomorphism of vector spaces*

$$\mathrm{Hom}_G(\pi, \pi_\zeta)\simeq \mathrm{Hom}_L(\pi, (\delta_G/\delta_L)^{1/2}\zeta). \tag{2.2}$$

The correspondence is given as

$$\begin{aligned} &\mathrm{Hom}_G(\pi, \pi_\zeta)\ni A\longmapsto T\in \mathrm{Hom}_L(\pi, (\delta_G/\delta_L)^{1/2}\zeta),\\ &T(v)=A(v)(1) \qquad (v\in E), \end{aligned} \tag{2.3}$$

where 1 is the unit element of G.

Proof. One can easily check that the assignment (2.3) gives the isomorphism (2.2). Q. E. D.

2.2. Regularity of $Z(\mathfrak{g}_C)$-finite, K-finite vectors for π_ζ.

Now we assume G be a connected semisimple Lie group with finite center again. For a maximal compact subgroup K, let $C^\infty(G;\zeta)_K$ denote the space of K-finite vectors for $\pi_\zeta=C^\infty\text{-}\mathrm{Ind}_L^G(\zeta)$. Then $C^\infty(G;\zeta)_K$ has naturally a structure of compatible $(\mathfrak{g}_C, K)$-module:

$$(\pi_\zeta)_K(X)f(g)=\frac{d}{dt}\pi_\zeta(\exp tX)f(g)|_{t=0},$$
$$(\pi_\zeta)_K(k)f(g)=f(k^{-1}g) \qquad (g\in G),$$

for $X\in\mathfrak{g}$, $k\in K$ and $f\in C^\infty(G;\zeta)_K$. First we give the irreducible decomposition of $C^\infty(G;\zeta)_K$ as a K-module.

$\hat{K}$ denotes the set of equivalence classes of all irreducible unitary representations of K. For a $\tau\in\hat{K}$, take an irreducible representation of K of class τ, and denote it again by τ. Let χ_τ be the character of τ. Define a linear operator E_τ on $C^\infty(G;\zeta)_K$ by

$$E_\tau f=\dim\tau\cdot\int_K\overline{\chi_\tau(k)}(\pi_\zeta)_K(k)f\,dk \qquad (f\in C^\infty(G;\zeta)_K),$$

where dk is the normalized Haar measure on K. One should note that the above integral has a meaning because $(\pi_\zeta)_K(k)f$, $k\in K$, span a finite-dimensional vector space. It follows from the orthogonality relations of characters that for any $\tau_1, \tau_2\in\hat{K}$

$$E_{\tau_1}E_{\tau_2}=\begin{cases} 0 & \text{if } \tau_1\neq\tau_2, \\ E_{\tau_1} & \text{if } \tau_1=\tau_2. \end{cases} \tag{2.4}$$

Put $C^\infty(G;\zeta)_\tau=E_\tau C^\infty(G;\zeta)_K$, then K acts on $C^\infty(G;\zeta)_\tau$ according to τ. From (2.4) we see easily that

$$C^\infty(G;\zeta)_K=\sum_{\tau\in\hat{K}}^{\oplus}C^\infty(G;\zeta)_\tau \qquad \text{(a direct sum of vector spaces).} \tag{2.5}$$

Let V_τ be the representation space of τ with a K-invariant inner product $\langle\,,\rangle$. Consider the space $C_\tau^\infty(G;\zeta)$ consisting of C^∞-functions ϕ on G with values in $\mathrm{H}(V_\tau, F)$ which satisfy the following two conditions:

$$\phi(gh)=(\delta_L(h)/\delta_G(h))^{1/2}\zeta(h)^{-1}\phi(g) \quad (g\in G,\ h\in L),$$

$$\phi(k^{-1}g)=\phi(g)\circ\tau(k) \qquad (k\in K).$$

Here $\mathrm{H}(V_\tau, F)=\mathrm{Hom}_C(V_\tau, F)$ is a Fréchet space canonically isomorphic to $V_\tau^*\otimes F$. For any $\phi\in C_\tau^\infty(G;\zeta)$ and any $v\in V_\tau$, put $\phi_v(g)=\phi(g)v$ $(g\in G)$. Clealy ϕ_v is in $C^\infty(G;\zeta)_\tau$. Consider $C_\tau^\infty(G;\zeta)$ as a trivial K-module: $k\cdot\phi=\phi$ $(k\in K)$. Then one has

Lemma 2.3. *The map $\phi\otimes v\mapsto\phi_v$ gives an isomorphism of K-modules between $C_\tau^\infty(G;\zeta)\otimes V_\tau$ and $C^\infty(G;\zeta)_\tau$.*

Proof. The map in question is clearly K-equivariant, so we need only prove that this map is bijective. We show first the injectivity. Let $v_1, v_2, \cdots, v_n$ be an orthonormal basis of V_τ. Suppose that $\phi_{v_1}^1+\phi_{v_2}^2+\cdots+\phi_{v_n}^n=0$ for $\phi^i\in C_\tau^\infty(G;\zeta)$ $(1\leqq i\leqq n)$. Then one has for any i

$$\begin{aligned} 0&=\int_K\overline{\langle\tau(k)v_i, v_i\rangle}\Big\{\sum_{j=1}^n\phi_{v_j}^j(k^{-1}g)\Big\}dk \\ &=\sum_{j=1}^n\phi^j(g)\cdot\int_K\overline{\langle\tau(k)v_i, v_i\rangle}\tau(k)v_j dk \\ &=(\dim\tau)^{-1}\phi^i(g)v_i \qquad (g\in G). \end{aligned}$$

Here we used in the last equality above the following well-known orthogonality relation (see for example [16, 2.9.3]):

$$\dim\tau\cdot\int_K\overline{\langle\tau(k)v, w\rangle}\langle\tau(k)v_1, w_1\rangle dk=\langle v_1, v\rangle\langle w, w_1\rangle$$

for $v, v_1, w, w_1\in V_\tau$. Noting that $\phi^i(kg)v_i=\phi^i(g)\tau(k)^{-1}v_i$, we thus get $\phi^i=0$ for all $1\leqq i\leqq n$, which proves the injectivity.

Let f be an element in $C^\infty(G;\zeta)_\tau$ such that $\pi_\zeta(k)f$, $k\in K$, generate an irreducible K-module V_f of class τ. In order to prove the surjectivity, we have only to show that such an f can be expressed as $f=\phi_v$ for some $\phi\in C_\tau^\infty(G;\zeta)$ and some $v\in V_\tau$. Let $\iota: V_\tau\to V_f$ be an isomorphism of K-modules. Define an $\mathrm{H}(V_\tau, F)$-valued C^∞-function ϕ on G by $(\phi(g))(v)=(\iota(v))(g)$ $(v\in V_\tau,\ g\in G)$. Then we see immediately $\phi\in C_\tau^\infty(G;\zeta)$ and $f=\phi_{\iota^{-1}(f)}$. Q.E.D.

By (2.5) and Lemma 2.3, one obtains the irreducible decomposition of K-module $C^\infty(G;\zeta)_K$ as follows.

Lemma 2.4. *One has an isomorphism of K-modules*

$$C^\infty(G;\zeta)_K\simeq \sum_{\tau\in\hat{K}}^{\oplus} C_\tau^\infty(G;\zeta)\otimes V_\tau. \tag{2.6}$$

Since $Z(\mathfrak{g}_C)$ commutes with K, $C^\infty(G;\zeta)_\tau$ is stable under $(\pi_\zeta)_K(Z(\mathfrak{g}_C))$. Now we prove the analyticity of $Z(\mathfrak{g}_C)$-finite vectors in $C^\infty(G;\zeta)_\tau$ by applying the elliptic regularity theorem.

Let ψ be a function on an analytic manifold M with values in a topological vector space E. Then ψ is said to be *weakly analytic* if $e^*\circ\psi$ is analytic for any $e^*\in E^*$ (= the topological dual space of E).

Lemma 2.5. *If $f\in C^\infty(G;\zeta)_\tau$ is $Z(\mathfrak{g}_C)$-finite (i.e., $\dim(\pi_\zeta)_K(Z(\mathfrak{g}_C))f<+\infty$), then it is wealky analytic.*

Proof. The statement is proved in the same line as in [15, p. 310]. Let $X_1, \cdots, X_r$ (resp. $X_{r+1}, \cdots, X_m$) be a basis of $\mathfrak{k}$ (resp. $\mathfrak{p}$) such that $-B(X_i, \theta X_j)=\delta_{ij}$ $(1\leq i, j\leq m)$, where B is the Killing form of $\mathfrak{g}$. Put $\Omega=-\sum_{1\leq i\leq r}X_i^2+\sum_{r<j\leq m}X_j^2$ (the Casimir operator), then $\Omega\in Z(\mathfrak{g}_C)$ and $\Delta=\sum_{1\leq i\leq m}X_i^2=2\sum_{1\leq i\leq r}X_i^2+\Omega$ is in $Z(\mathfrak{g}_C)U(\mathfrak{k}_C)$.

If $f\in C^\infty(G;\zeta)_\tau$ is $Z(\mathfrak{g}_C)$-finite, then $(\pi_\zeta)_K(ZD)f$ $(Z\in Z(\mathfrak{g}_C),\ D\in U(\mathfrak{k}_C))$ generate a finite-dimensional subspace of $C^\infty(G;\zeta)_\tau$. Hence there exists a polynomial p of one valuable with $\deg p\geq 1$ such that $(\pi_\zeta)_K(p(\Delta))f=0$. This implies that $p(\Delta)(e^*\circ f)=0$ for all $e^*\in F^*$, where $U(\mathfrak{g}_C)$ acts on $C^\infty(G)$ as the algebra of right G-invariant differential operators on G. Since Δ^s $(s\geq 1)$ are elliptic operators, $e^*\circ f$ must be real analytic by the regularity theorem of elliptic operators. Consequently, f is weakly analytic. Q.E.D.

Analogously to the case $C^\infty(G;\zeta)_\tau$, $C_\tau^\infty(G;\zeta)$ has a structure of $Z(\mathfrak{g}_C)$-module: $U(\mathfrak{g}_C)$ acts on $C^\infty(G, \mathrm{H}(V_\tau, F))$ by

$$X\phi(g)=\frac{d}{dt}\phi(\exp(-tX)g)|_{t=0} \qquad (g\in G,\ X\in\mathfrak{g},\ \phi\in C^\infty(G, \mathrm{H}(V_\tau, F))).$$

Then the subspace $C_\tau^\infty(G;\zeta)$ is stable under $Z(\mathfrak{g}_C)$. For any $D\in Z(\mathfrak{g}_C)$, one has

$$(\pi_\zeta)_K(D)\phi_v=(D\phi)_v \qquad (v\in V_\tau,\ \phi\in C_\tau^\infty(G;\zeta)). \tag{2.7}$$

For an ideal I of $Z(\mathfrak{g}_C)$, let $A(G;\zeta:I)_\tau$ (resp. $A_\tau(G;\zeta:I)$) be the space of $f\in C^\infty(G;\zeta)_\tau$ (resp. $\phi\in C_\tau^\infty(G;\zeta)$) such that $(\pi_\zeta)_K(I)f=(0)$ (resp. $I\phi=(0)$). In view of (2.7) and Lemma 2.3, it holds that

$$A(G;\zeta:I)_\tau \simeq A_\tau(G;\zeta:I)\otimes V_\tau \tag{2.8}$$

through the isomorphism in Lemma 2.3. Therefore we can rewrite Lemma 2.5 in terms of $Z(\mathfrak{g}_C)$-finite vectors in $C_\tau^\infty(G;\zeta)$ as follows.

Lemma 2.6. *If $\phi\in C_\tau^\infty(G;\zeta)$ is $Z(\mathfrak{g}_C)$-finite, then ϕ is weakly analytic.*

2.3. An upper bound for $\dim A_\tau(G;\zeta:I)$. We return to our original objects in 1.2 and use the notations in § 1 without any comment. We prepare a lemma. Put $M_{kh}=Z_{K\cap H}(A_{pq})$, then M_{kh} acts on $\mathfrak{l}_{0C}$ through the adjoint representation because $\mathfrak{l}_{0C}$ is the centralizer of $\mathfrak{a}_{pq}$ in $\mathfrak{g}_C$.

Lemma 2.7. *Let* $\mathrm{Int}(\mathfrak{l}_{0C})$ *be the adjoint group of* $\mathfrak{l}_{0C}$. *Then one has an inclusion* $\mathrm{Ad}(M_{kh})|\mathfrak{l}_{0C}\subseteqq \mathrm{Int}(\mathfrak{l}_{0C})$.

Proof. Let L_{0C} be the centralizer of $\mathfrak{a}_{pq}$ in the adjoint group of $\mathfrak{g}_C$. Then L_{0C} is connected, so one has $\{g|\mathfrak{l}_{0C};g\in L_{0C}\}=\mathrm{Int}(\mathfrak{l}_{0C})$. On the other hand, $\mathrm{Ad}(M_{kh})\subseteqq L_{0C}$ by definition. This proves the lemma. Q. E. D.

Let ζ be a continuous representation of HN on a Fréchet space F. Consider the induced representation $\pi_\zeta=C^\infty\text{-}\mathrm{Ind}_{HN}^G(\zeta)$. Now we can derive from the previous results a certain upper bound for $\dim A_\tau(G;\zeta:I)$, which is a key step toward our finite multiplicity theorem in 2.4.

Theorem 2.8. *Let I be an ideal of $Z(\mathfrak{g}_C)$. Then one has for any $\tau\in\hat{K}$*

$$\dim A_\tau(G;\zeta:I)\leqq |W(\Phi)/W(\Phi_0)|\dim(Z(\mathfrak{g}_C)/I)\cdot I_{M_{kh}}(\tau,\zeta). \tag{2.9}$$

Here $W(\Phi)$ and $W(\Phi_0)$ are Weyl groups defined in 1.4, *and $I_{M_{kh}}(\tau,\zeta)$ is the intertwining number from $\tau|M_{kh}$ to $\zeta|M_{kh}$.*

This theorem generalizes [1, Lemma 3.9] obtained by E. P. van den Ban in the special case where $N=(1)$ and $\dim\zeta<+\infty$.

Proof of Theorem 2.8. In order to prove (2.9), we may assume that $0<\dim(Z(\mathfrak{g}_C)/I)=p<+\infty$ without loss of generality. Select elements $z_k\in Z(\mathfrak{g}_C)$ $(1\leqq k\leqq p)$ such that $z_1=1$ and $Z(\mathfrak{g}_C)=\sum_{1\le k\le p}Cz_k\oplus I$ (as vector spaces). Put $r=|W(\Phi)/W(\Phi_0)|$. Let ν_i $(1\leqq i\leqq r)$ be the elements in $Z(\mathfrak{l}_{0C})$ in Proposition 1.6. If $\phi\in C_\tau^\infty(G;\zeta)$, then $(z_k\omega(\nu_i)\phi)(a)$ $(1\leqq k\leqq p,\ 1\leqq i\leqq r)$ belong to $\mathrm{Hom}_{M_{kh}}(\tau,\zeta)$ for all $a\in A_{pq}$, where ω is the automorphism of $U(\mathfrak{l}_{0C})$ defined in 1.4. In fact, for an $m\in M_{kh}$, one has from the definition of $C_\tau^\infty(G;\zeta)$

$$\begin{aligned}\zeta(m)(z_k\omega(\nu_i)\phi)(a)&=(z_k\omega(\nu_i)\phi)(am^{-1})=(z_k\omega(\nu_i)\phi)(m^{-1}a)\\&=({}^m(z_k\omega(\nu_i))\phi)(a)\tau(m)=(z_k\cdot{}^m(\omega(\nu_i))\phi)(a)\tau(m).\end{aligned}$$

On the other hand, we see from Lemma 2.7 that ${}^m(\omega(\nu_i))=\omega(\nu_i)$. Thus

$$\zeta(m)(z_k\omega(\nu_i)\phi)(a)=(z_k\omega(\nu_i)\phi)(a)\tau(m)\qquad(m\in M_{kh}).$$

Now we fix an $a\in A'_{pq}$. Consider a linear map

$$A_\tau(G\,;\zeta:I)\ni\phi\longmapsto((z_k\omega(\nu_i)\phi)(a))_{k,i}\in[\mathrm{Hom}_{M_{kh}}(\tau,\zeta)]^{pr}.$$

For the inequality (2.9), it suffices to show that this map is injective. This is done as follows. Suppose $(z_k\omega(\nu_i)\phi)(a)=0$ for all k and all i. Then, $(Z(\mathfrak{g}_C)\omega(\nu_i)\phi)(a)=0$ holds for any i because $I\phi=(0)$. Let $D\in U(\mathfrak{g}_C)$. By Theorem 1.12, there exist finitely many elements $f_{m,i}\in\mathcal{F}$, $\xi_{m,i}\in U(\mathfrak{h}_C\oplus\mathfrak{n}_C)$, $Z_{m,i}\in Z(\mathfrak{g}_C)$ and $\eta_{m,i}\in U(\mathfrak{k}_C)$ $(1\leqq m\leqq J,\ 1\leqq i\leqq r)$ such that

$$D=\sum_{m,i}f_{m,i}(a')\cdot{}^{a'}\xi_{m,i}Z_{m,i}\omega(\nu_i)\eta_{m,i}$$

for all $a'\in A'_{pq}$. Thus we get

$$D\phi(a)=\sum_{m,i}f_{m,i}(a)\zeta(\xi_{m,i})(Z_{m,i}\omega(\nu_i)\phi)(a)\tau(\eta_{m,i})=0.$$

In particular, it holds for any $e^*\in\mathrm{H}(V_\tau,F)^*$ that $(D(e^*\circ\phi))(a)=0$. Since ϕ is weakly analytic on G by Lemma 2.6, one has $e^*\circ\phi=0$. Hence $\phi=0$ as desired.

Q. E. D.

2.4. A finite multiplicity theorem for C^∞-$\mathrm{Ind}_{HN}^G(\zeta)$.

Using Theorem 2.8, we give in this subsection an estimate of the multiplicities of irreducible constituents of the $(\mathfrak{g}_C, K)$-module $(\pi_\zeta)_K$.

To begin with, we comment briefly about admissible representations of G in order to clarify our terminology. For them, refer to [17, Chap. 4]. Let π be a continuous representation of G on a Hilbert space $\mathcal{H}$ on which K acts unitarily. Such a π is said to be *quasi-simple* if $Z(\mathfrak{g}_C)$ acts on $\mathcal{H}^\infty$ by scalars: $\pi_\infty(D)=\chi_\pi(D)I$ (I the identity operator) for $D\in Z(\mathfrak{g}_C)$. In this case, the algebra homomorphism $\chi_\pi: Z(\mathfrak{g}_C)\to \boldsymbol{C}$ is called the *infinitesimal character* of π. For a $\tau\in\hat{K}$, denote by $\mathcal{H}_\tau$ the τ-isotypic component of $\mathcal{H}$. Then $\mathcal{H}$ is decomposed into a direct sum of Hilbert spaces: $\mathcal{H}=\sum_{\tau\in\hat{K}}^{\oplus}\mathcal{H}_\tau$. We call π *admissible* if $\dim\mathcal{H}_\tau<+\infty$ for every $\tau\in\hat{K}$. Irreducible unitary representations are always quasi-simple. Moreover, an irreducible representation π is admissible if and only if it is quasi-simple.

Now suppose that π be admissible. Then the space $\mathcal{H}_K$ of K-finite vectors for π consists of analytic vectors: the map $G\ni g\mapsto\pi(g)v\in\mathcal{H}$ is real analytic for every $v\in\mathcal{H}_K$. In particular, $\mathcal{H}_K\subseteqq\mathcal{H}^\infty$. Furthermore, $\mathcal{H}_K$ is a dense subspace of $\mathcal{H}^\infty$ stable under $\pi_\infty(\mathfrak{g}_C)$ as well as $\pi(K)$. Thus one gets naturally a compatible $(\mathfrak{g}_C, K)$-module structue on $\mathcal{H}_K$ denoted by π_K.

Concerning the irreducibility of π, π_∞ and π_K, the following lemma is well-known (see [17, p. 254 and p. 324]).

Lemma 2.9. *For an admissible representation $(\pi,\mathcal{H})$ of G, the following three conditions are mutually equivalent.*

(1) *π is irreducible, that is, there are no G-invariant closed subspaces of $\mathcal{H}$ except* (0) *and $\mathcal{H}$,*

(2) *π_∞ is an irreducible representation of G,*

(3) *$\mathcal{H}_K$ is algebraically irreducible as a $\mathfrak{g}_C$-module.*

Note. The above condition (1) is equivalent to (2) for any continuous representation of an arbitrary Lie group.

For $(\mathfrak{g}_C, K)$-modules (ρ_i, V_i) $(i=1, 2)$, $I_{\mathfrak{g}_C-K}(\rho_1, \rho_2)$ denotes, as in the case of group representations, the intertwining number from ρ_1 to ρ_2, that is, the dimension of the space $\mathrm{Hom}_{\mathfrak{g}_C-K}(\rho_1, \rho_2)$ of $(\mathfrak{g}_C, K)$-module homomorphisms from V_1 to V_2. In case where ρ_1 is irreducible, $I_{\mathfrak{g}_C-K}(\rho_1, \rho_2)$ is called the multiplicity of ρ_1 in ρ_2 as submodule. Moreover, the multiplicity $M_{\mathfrak{g}_C-K}(\rho_1, \rho_2)$ of ρ_1 in ρ_2 as subquotient is defined to be the supremum of integers n for which there exists a chain of $(\mathfrak{g}_C, K)$-submodules

$$W_0 \subseteqq W_1 \subseteqq \cdots \subseteqq W_n \subseteqq V_2 \quad \text{with} \quad W_i/W_{i-1} \simeq V_1 \quad (1 \leqq i \leqq n).$$

$I_{\mathfrak{g}_C-K}(\rho_1, \rho_2)$ is smaller than $M_{\mathfrak{g}_C-K}(\rho_1, \rho_2)$ in general.

For any homomorphism $\chi: Z(\mathfrak{g}_C) \to C$, $A(G; \zeta, \chi) = \sum_{\tau\in\hat{K}}^{\oplus} A(G; \zeta : \mathrm{Ker}\,\chi)_\tau \subseteqq C^\infty(G; \zeta)_K$ has a structure of $(\mathfrak{g}_C, K)$-submodule, which will be denoted by $(\pi_{\zeta,\chi})_K$.

Now we establish our main theorems in this section.

Theorem 2.10. *Let ζ be a continuous representation of $HN = H \ltimes N$ on a Fréchet space F. Consider the induced representation $\pi_\zeta = C^\infty\text{-}\mathrm{Ind}_{HN}^G(\zeta)$ in C^∞-context. If $(\pi, \mathcal{H})$ is an irreducible admissible representation of G with infinitesimal character χ, then the multiplicites $I_G(\pi_\infty, \pi_\zeta)$, $I_{\mathfrak{g}_C-K}(\pi_K, (\pi_\zeta)_K)$ and $M_{\mathfrak{g}_C-K}(\pi_K, (\pi_{\zeta,\chi})_K)$ admit an upper bound as*

$$\begin{aligned}(2.10)\qquad I_G(\pi_\infty, \pi_\zeta) &\leqq I_{\mathfrak{g}_C-K}(\pi_K, (\pi_\zeta)_K) \leqq M_{\mathfrak{g}_C-K}(\pi_K, (\pi_{\zeta,\chi})_K) \\ &\leqq |W(\Phi)/W(\Phi_0)| \min_{\tau\in\hat{K}_\pi} [I_{M_{kh}}(\tau, \zeta) \cdot I_K(\tau, \pi)^{-1}],\end{aligned}$$

where $\hat{K}_\pi = \{\tau \in \hat{K}; \mathcal{H}_\tau \neq (0)\}$ and $I_K(\tau, \pi)$ is the multiplicity of $\tau \in \hat{K}$ in $\pi|K$.

Remark 2.11. Contrary to the case of Harish-Chandra and van den Ban, our theorem can be applied effectively even to π_ζ with infinite-dimensional ζ. Actually, we shall show in §4 and in the second part of this series of our articles that some important types of such π_ζ have finite multiplicity property, by using Theorems 2.10 and 2.12.

Proof of Theorem 2.10. If $T \in \mathrm{Hom}_G(\pi_\infty, \pi_\zeta)$, then the restriction T_K of T to $\mathcal{H}_K$ gives a $(\mathfrak{g}_C, K)$-module homomorphism from $\mathcal{H}_K$ into $C^\infty(G; \zeta)_K$. Since $\mathcal{H}_K$ is dense in $\mathcal{H}^\infty$, the linear map

$$\mathrm{Hom}_G(\pi_\infty, \pi_\zeta) \ni T \longmapsto T_K \in \mathrm{Hom}_{\mathfrak{g}_C-K}(\pi_K, (\pi_\zeta)_K)$$

is injective, which proves the first inequality in (2.10). The second one is obvious from the definition of multiplicities.

To prove the third one, let $\tau \in \hat{K}$. Then we see easily that

$$\dim \mathcal{H}_\tau \cdot M_{\mathfrak{g}_C-K}(\pi_K, (\pi_{\zeta,\chi})_K) \leqq \dim A(G; \zeta : \mathrm{Ker}\,\chi)_\tau.$$

We apply Theorem 2.8 keeping (2.8) in mind. Then it holds that

$$\dim A(G;\zeta:\operatorname{Ker}\chi)_\tau \leqq \dim\tau\cdot|W(\Phi)/W(\Phi_0)|\cdot I_{M_{kh}}(\tau,\zeta).$$

Consequently, we obtain

$$\dim \mathcal{H}_\tau\cdot M_{\mathfrak{g}_C-K}(\pi_K,(\pi_{\zeta,\chi})_K)\leqq \dim\tau\cdot|W(\Phi)/W(\Phi_0)|\cdot I_{M_{kh}}(\tau,\zeta)$$

for all $\tau\in\hat{K}$, which proves the third inequality in (2.10). Q.E.D.

Notice that $M_{\mathfrak{g}_C-K}(\pi_K,(\pi_{\zeta,\chi'})_K)=0$ if $\chi'\neq\chi$, the infinitesimal character of π. Then we deduce immediately from Theorem 2.10 a finite multiplicity theorem as follows.

Theorem 2.12. *Suppose that the restriction of ζ to the compact subgroup M_{kh} has finite multiplicity property: $I_{M_{kh}}(\mu,\zeta)<+\infty$ for all $\mu\in\hat{M}_{kh}$. Then, the multiplicity $M_{\mathfrak{g}_C-K}(\pi_K,(\pi_{\zeta,\chi})_K)$ is finite for any irreducible admissible representation π of G and any homomorphism $\chi: Z(\mathfrak{g}_C)\to C$. In particular, π_K (resp. π_∞) occurs in $(\pi_\zeta)_K$ (resp. in π_ζ) as subrepresentation with at most finite multiplicity.*

In particular, one has

Corollary 2.13. *Let ζ be a finite-dimensional representation of HN. Then, for any π, π_K occurs as a $(\mathfrak{g}_C, K)$-submodule of $C^\infty(G;\zeta)_K$ with at most finite multiplicity.*

Applying Theorem 2.10 to the case $H=K$, $N=(1)$, one obtains a well-known but an important estimate of multiplicities $I_K(\tau,\pi)$ $(\tau\in\hat{K})$ as follows.

Corollary 2.14 [5, III, Theorem 4]. *Let π be as above. Then there exists a positive constant c_π such that any $\tau\in\hat{K}$ occurs in $\mathcal{H}_K$ at most $c_\pi\dim\tau$ times.*

The last corollary assures the existence of the distribution characters of irreducible admissible representations of G. (The above constant c_π can be chosen as 1 by virtue of Harish-Chandra's subquotient theorem [5, II].)

§ 3. A finite multiplicity theorem for unitarily induced representrtions

In this section, let G be a Lie group of type I. For a unitary representation ζ of a closed subgroup L $(\subseteqq G)$, consider the unitarily induced representation $\mathcal{U}_\zeta=L^2\text{-}\mathrm{Ind}_L^G(\zeta)$. Let

$$\mathcal{U}_\zeta\simeq\int_{\hat{G}}^{\oplus}\mathcal{U}_\zeta(\pi)d\mu_\zeta(\pi),\qquad \mathcal{U}_\zeta(\pi)\simeq[m_\zeta(\pi)]\cdot\pi$$

be the factor decomposition (cf. 3.4) of $\mathcal{U}_\zeta$. We treat in this section the multiplicity function m_ζ of $\mathcal{U}_\zeta$ on $\hat{G}$ (= the unitary dual of G).

In more detail, we first collect, to clarify our terminology, basic facts about the direct integral decomposition theory for unitary representations of locally compact groups. Our main reference is Dixmier's text book [4]. Furthermore, we prepare some (versions of) theorems due to Penney [11] and Poulsen

[12] on C^∞-vectors for unitary representations of G.

After that, we relate $m_\zeta(\pi)$ ($\pi\in\hat{G}$) with the intertwining number $I_G(\pi_\infty, \pi_\zeta)$ (see 2.1) using the results by Penney and Poulsen. Under certain assumptions on $\pi\in\hat{G}$ or on ζ, $I_G(\pi_\infty, \pi_\zeta)$ gives an upper bound for $m_\zeta(\pi)$ (Lemma 3.10 and Theorem 3.12).

In 3.8, we assume that G be a semisimple group as in § 1. We apply Theorem 3.12 to $\mathcal{U}_\zeta=L^2\text{-Ind}_{HN}^G(\zeta)$, where ζ is a finite-dimensional unitary representation of the semidirect product subgroup HN in 1.2. Then, by virtue of Corollary 2.13 one deduces that the multiplicity function m_ζ takes finite values for μ_ζ-almost every $\pi\in\hat{G}$ (Theorem 3.13). This is the main theorem of this section which extends [1, Theorem 3.1].

In order to state Theorem 3.13 in a general form, we are forced to assume ζ to be finite-dimensional. Nevertheless, even for certain kinds of $\mathcal{U}_\zeta$ with infinite-dimensional ζ, (variants of) Theorems 2.10 and 3.12 are still useful to prove finite multiplicity property of such $\mathcal{U}_\zeta$. We shall give in § 4 and in the second part [19] important examples of such cases.

3.1. Direct integral of unitary representations.

First, we recall the notion of direct integral of Hilbert spaces. Let $(\mathcal{H}(t))_{t\in Z}$ be a family of Hilbert spaces indexed by a set Z. A mapping $f: Z\to \coprod_{t\in Z}\mathcal{H}(t)$ is said to be a vector field on Z if $f(t)\in H(t)$ for all $t\in Z$. Now assume that Z be a measure space with a positive Borel measure μ on Z. By integrating $t\mapsto\mathcal{H}(t)$ over Z, we construct a Hilbert space $\int_Z^\oplus \mathcal{H}(t)d\mu(t)$, which is reduced to the direct sum of the $\mathcal{H}(t)$ in case where μ is a discrete measure. For this purpose, some measurability of the family $(\mathcal{H}(t))_{t\in Z}$ is required.

Definition 3.1. For a measure space (Z, μ), a *measurable field of Hilbert spaces* is a family $(\mathcal{H}(t))_{t\in Z}$ of Hilbert spaces with a set Γ of vector fields on Z satisfying the following four conditions.

(1) Γ has a structure of complex vector space by pointwise addition and by multiplication with complex numbers.

(2) There exists a countable subset $\{f_1, f_2, \cdots\}\subseteq\Gamma$ such that, for every $t\in Z$, $\{f_n(t)\}_{n=1,2,\dots}$ forms a total subset of $\mathcal{H}(t)$.

(3) The function $t\mapsto\|f(t)\|_{\mathcal{H}(t)}$ is measurable for every $f\in\Gamma$.

(4) If h is a vector field such that $t\mapsto(h(t), f(t))_{\mathcal{H}(t)}$ is measurable for any $f\in\Gamma$, then h belongs to Γ. Here $(\ ,\ )_{\mathcal{H}(t)}$ denotes the inner product on $\mathcal{H}(t)$ and $\|\cdot\|_{\mathcal{H}(t)}$ the corresponding norm on $\mathcal{H}(t)$.

Each $f\in\Gamma$ is called a measurable vector field.

For a measurable field of Hilbert space $((\mathcal{H}(t))_{t\in Z}, \Gamma)$, let $\mathcal{H}$ be the vecor space consisting of $f\in\Gamma$ such that $\int_Z\|f(t)\|^2_{\mathcal{H}(t)}d\mu(t)<+\infty$. Then, after identifying two vector fields which are equal almost everywhere with respect to μ, $\mathcal{H}$ has a structure of Hilbert space with an inner product

(3.1) $$(f_1, f_2)_{\mathcal{H}}=\int_Z (f_1(t), f_2(t))_{\mathcal{H}(t)} d\mu(t) \qquad (f_1, f_2 \in \mathcal{H}).$$

Definition 3.2. One calls the Hilbert space $\mathcal{H}$ constructed above the *direct integral* of the $\mathcal{H}(t)$ over (Z, μ), and denotes it by $\int_Z^{\oplus} \mathcal{H}(t) d\mu(t)$.

Now we proceed to the integration of a measurable field of bounded operators.

Definition 3.3. Under the above notations, an assignment $Z \ni t \mapsto T(t) \in \mathcal{L}(\mathcal{H}(t))$ is said to be a *measurable field of operators*, if the vector field $t \mapsto T(t)f(t)$ is measurable for any $f \in \Gamma$. Here $\mathcal{L}(E)$ is the space of bounded linear operators on a Hilbert space E.

For a measurable field $(T(t))_{t \in Z}$ of operators, if the function $t \mapsto \|T(t)\|$ is essentially bounded on Z, then $(Tf)(t)=T(t)f(t)$ $(t \in Z,\ f \in \mathcal{H})$ gives a bounded linear operator on $\mathcal{H}$. In this case, we express T as $\int_Z^{\oplus} T(t) d\mu(t)$.

Now let G be a separable locally compact group. Assume that all the unitary representations in question are acting on separable Hilbert spaces. Under the notations prepared above, the direct integral of unitary representations of G is defined in the following way.

Let $((\mathcal{H}(t))_{t \in Z}, \Gamma)$ be a measurable field of Hilbert spaces on a measure space (Z, μ). Suppose that a unitary representation $\mathcal{U}(t)$ of G acting on $\mathcal{H}(t)$ is attached for every $t \in T$. The map $t \mapsto \mathcal{U}(t)$ is said to be a measurable field of representations if, for any $g \in G$, the field of operators $t \mapsto \mathcal{U}(t)(g)$ is measurable. In this case, put $\mathcal{U}(g)=\int_Z^{\oplus} \mathcal{U}(t)(g) d\mu(t)$ for each $g \in G$. Then $\mathcal{U}: g \mapsto \mathcal{U}(g)$ gives a unitary representation of G acting on $\int_Z^{\oplus} \mathcal{H}(t) d\mu(t)$.

Definition 3.4. The unitary representation $\mathcal{U}$ is called the direct integral of $\mathcal{U}(t)$ and denoted by $\mathcal{U}=\int_Z^{\oplus} \mathcal{U}(t) d\mu(t)$.

3.2. A Borel structure on $\hat{G}$. Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of G. We equip $\hat{G}$ with a Borel structure as follows.

For every $n \in \boldsymbol{N} \cup \{\infty\}$, take a Hilbert space $\mathcal{H}_n$ of dimension n. Here $\boldsymbol{N}$ is the set of natural numbers, and $\mathcal{H}_\infty$ is a separable infinite-dimensional Hilbert space. Let $\mathrm{Irr}_n(G)$ denote the collection of all concrete irreducible unitary representations of G acting on $\mathcal{H}_n$. Firstly, assign to $\mathrm{Irr}_n(G)$ the coarsest Borel structure for which the functions $\mathrm{Irr}_n(G) \ni \pi \mapsto (\pi(g)v, w)_{\mathcal{H}_n} \in \boldsymbol{C}$ are Borel functions for all $g \in G$ and all $v, w \in \mathcal{H}_n$. Secondly, equip $\mathrm{Irr}(G)=\coprod_n \mathrm{Irr}_n(G)$ with the direct sum Borel structure of those of the $\mathrm{Irr}_n(G)$. Finally, the *Mackey Borel structure* on $\hat{G}$ is the quotient of the structure of $\mathrm{Irr}(G)$ through the canonical surjection $\mathrm{Irr}(G) \to \hat{G}$.

3.3. Representations of type I. For a set S of bounded operators on a Hilbert space $\mathcal{H}$, S' denotes the algebra of all $s' \in \mathcal{L}(\mathcal{H})$ which commute with every $s \in S$. S' is called the commutant of S. A subalgebra M of $\mathcal{L}(\mathcal{H})$ containing the identitiy operator $I_{\mathcal{H}}$ is said to be a *von Neumann algebra* if it is stable under the *-operation, and coincides with the double commutant $M''=(M')'$. We say that a von Neumann algebra M is a *factor* if the center $M \cap M'$ consists only of the scalar multiples of $I_{\mathcal{H}}$. A *type* I von Neumann algebra is an M which is isomorphic, as an involutive *-algebra, to some von Neumann algebra B for which B' is commutative.

For a unitary representation $(\pi, \mathcal{H})$ of G, let M_π ($\subseteq \mathcal{L}(\mathcal{H})$) denote the von Neumann algebra generated by the operators $\pi(g)$, $g \in G$. One says that π is a *factor representation* (resp. a *representation of type* I) if M_π is a factor (resp. of type I). A type I factor representation is just a multiple of an irreducible one.

A (separable) locally compact group G is said to be of type I if any factor representation is necessarily of type I. Abelian groups, compact groups, connected nilpotent Lie groups and connected semisimple Lie groups are all of type I. But solvable Lie groups are not always of type I.

3.4. Factor decompositions. Any type I unitary representation of G may be disintegrated over $\hat{G}$ into factor representations as follows.

Lemma 3.5 [4, 8.4 and 13.9]. *Let $\mathcal{U}$ be a unitary representation of a separable locally compact group G. Assume that $\mathcal{U}$ be of type* I. *Then there exist a unique measure class Δ on $\hat{G}$ and a unique, up to modification on a negligible subset, measurable function $m_{\mathcal{U}}: \hat{G} \to \mathbf{N} \cup \{0, \infty\}$ such that*

$$\mathcal{U} \simeq \int_{\hat{G}}^{\oplus} \mathcal{U}(\pi) d\mu_{\mathcal{U}}(\pi) \qquad (\textit{factor decomposition}) \tag{3.2}$$

and

$$\mathcal{U}(\pi) \simeq [m_{\mathcal{U}}(\pi)] \cdot \pi \qquad \textit{for } \mu_{\mathcal{U}}\textit{-almost every } \pi \in \hat{G}. \tag{3.3}$$

Here $\mu_{\mathcal{U}}$ is a Borel measure on $\hat{G}$ of class Δ, and $\pi \mapsto \mathcal{U}(\pi)$ is a measurable family of factor representations of G.

The function $m_{\mathcal{U}}$ above is called the *multiplicity function* for $\mathcal{U}$.

3.5. Direct integral decompositions of C^∞-vectors.
In this subsection, let G be an arbitrary (not necessarily of type I) Lie group. Keep to the notations in 2.1. Let

$$\mathcal{U} \simeq \int_Z^{\oplus} \mathcal{U}(t) d\mu(t), \qquad \mathcal{H} \simeq \int_Z^{\oplus} \mathcal{H}(t) d\mu(t) \tag{3.4}$$

be a direct integral decomposition of a unitary representation $(\mathcal{U}, \mathcal{H})$ of G over some measure space (Z, μ). We now summarize, in a form convenient for our later use, the results of R. Penney [11] about the disintegration along (3.4) of the space $\mathcal{H}^\infty$ of smooth vectors for $\mathcal{U}$ and of its topological dual space.

Proposition 3.6 [11, Theorem C]. *Under the notations above, the space* $\mathcal{H}^{\infty}$ *is a direct integral of the* $\mathcal{H}(t)^{\infty}$ $(t \in Z)$ *in the following sense.*

(1) *If* $f=(f(t))_{t\in Z}\in\mathcal{H}^{\infty}$, *then* $f(t)\in\mathcal{H}(t)^{\infty}$ *for a.e. (=almost every)* $t\in Z$ *and* $\mathcal{U}_{\infty}(X)f=(\mathcal{U}(t)_{\infty}(X)f(t))_{t\in Z}$ *holds for any* $X\in U(\mathfrak{g}_C)$. *Here* $U(\mathfrak{g}_C)$ *is the enveloping algebra of the complexification* $\mathfrak{g}_C$ *of* $\mathfrak{g}=\mathrm{Lie}\,G$.

(2) *If* $f=(f(t))_{t\in Z}\in\mathcal{H}$ *is such that* $f(t)\in\mathcal{H}(t)^{\infty}$ *for a.e.* $t\in Z$ *and* $(\mathcal{U}(t)_{\infty}(X)f(t))_{t\in Z}\in\mathcal{H}$ *for all* $X\in U(\mathfrak{g}_C)$, *then* $f\in\mathcal{H}^{\infty}$.

Using this proposition, one easily gets the following

Proposition 3.7 (cf. [11, Corollary C. I]). *Let E be a finite-dimensional vector space. Then the space* $\mathrm{Hom}_C(\mathcal{H}^{\infty}, E)$ *of continuous linear mappings from* $\mathcal{H}^{\infty}$ *to* E *is a direct integral of the* $\mathrm{Hom}_C(\mathcal{H}(t)^{\infty}, E)$ *in the followihg sense.*

(1) *If* $T\in\mathrm{Hom}_C(\mathcal{H}^{\infty}, E)$, *then for a.e. t there exist continuous linear mappings* $T(t)\in\mathrm{Hom}_C(\mathcal{H}(t)^{\infty}, E)$ *such that*

$$T(f)=\int_Z T(t)(f(t))d\mu(t) \tag{3.5}$$

for all $f=(f(t))\in\mathcal{H}^{\infty}$. *The integral* (3.5) *of E-valued function is absolutely convergent and $T(t)$ are unique almost everywhere.*

(2) *Let* $T(t)$, $t\in Z$, *be a collection of elements in* $\mathrm{Hom}_C(\mathcal{H}(t)^{\infty}, E)$ *such that the E-valued function* $t\mapsto T(t)(f(t))$ *is μ-integrable for every* $f=(f(t))\in\mathcal{H}^{\infty}$. *Then the map* $f\mapsto\int_Z T(t)(f(t))d\mu(t)$ *defines a continuous linear map from* $\mathcal{H}^{\infty}$ *to E.*

An element T in $\mathrm{Hom}_C(\mathcal{H}^{\infty}, E)$ is said to be a *generalized cyclic map* for $\mathcal{U}$, if, for a $v\in\mathcal{H}^{\infty}$, $T(\mathcal{U}(g)v)=0$ for all $g\in G$ implies that $v=0$.

Proposition 3.8 (cf. [11, Theorem II. 5]). *Let* $T\in\mathrm{Hom}_C(\mathcal{H}^{\infty}, E)$ *be a generalized cyclic map for $\mathcal{U}$. If* $T=\int_Z^{\oplus}T(t)d\mu(t)$ *denotes the disintegration of T in the sense of Proposition* 3.7, *then the maps* $T(t)\in\mathrm{Hom}_C(\mathcal{H}(t)^{\infty}, E)$ *are, for a.e. t, generalized cyclic maps for* $\mathcal{U}(t)$.

Note. R. Penney proved Propositions 3.7 and 3.8 for the case $E=\boldsymbol{C}$. Nevertheless, his proof works for any finite-dimensional E. But, for infinite dimensional E, the assertion of Proposition 3.7 is no longer true.

3.6. Unitarily induced representations (see e.g. [9, Chap. III]).

Let G be a locally compact group again. To a unitary representation $(\zeta, \mathcal{H}(\zeta))$ of a closed subgroup L of G, we attach a representation $(\mathcal{U}_{\zeta}, L^2(G;\zeta))$ of G induced unitarily from ζ.

Let ρ be a strictly positive continuous function on G satisfying $\rho(xh)=(\delta_L(h)/\delta_G(h))\rho(x)$ $(x\in G,\ h\in L)$, where δ_L (resp. δ_G) is the modular function of L (resp. G) defined in 2.1. Such a function, so-called a rho-function, always exists. Moreover, if G is a Lie group, then there exists a rho-function of C^{∞}-class.

One can associate with ρ a unique measure μ_ρ on G/L such that

$$\int_G \phi(x)\rho(x)d_G(x)=\int_{G/L} d\mu_\rho(xL)\int_L \phi(xh)d_L(h) \tag{3.6}$$

for all $\phi\in C_0(G)$ (= the space of all continuous functions on G with compact supports). Here d_Gx (resp. d_Lh) is the left Haar measure on G (resp. L). From the uniqueness of μ_ρ, we see easily for any $g\in G$

$$d\mu_\rho(g\cdot y)=\rho'(g, y)d\mu_\rho(y) \qquad (y\in G/L), \tag{3.7}$$

where $\rho'(g, y)=\rho(gx)/\rho(x)$ $(y=xL)$ is a well-defined continuous function on G/L, and $g\cdot y=gxL$. In particular, μ_ρ is a quasi-invariant measure on G/L. Namely, for any $g\in G$, the measure $\widetilde{L}_g\mu_\rho$ is equivalent to μ_ρ, where $(\widetilde{L}_g\mu_\rho)(E)=\mu_\rho(g^{-1}E)$ for a Borel subset E of G/L.

Now let $L^2(G;\zeta)$ denote the set of $\mathscr{H}(\zeta)$-valued functions f on G satisfying the following conditions (1)-(3).

(1) For every $a\in\mathscr{H}(\zeta)$, the function $x\mapsto(f(x), a)_{\mathscr{H}(\zeta)}$ is a Borel function on G,

(2) $f(xh)=\rho(h)^{1/2}\zeta(h)^{-1}f(x)$ $(x\in G,\ h\in L)$,

(3) $\|f\|^2_{L^2(G;\zeta)}=\int_{G/L}\rho(x)^{-1}\|f(x)\|^2_{\mathscr{H}(\zeta)}d\mu_\rho(xL)<+\infty$.

Here one should note that the assignment $xL\to\rho(x)^{-1}\|f(x)\|^2_{\mathscr{H}(\zeta)}$ actually defines a function on G/L thanks to the property (2). After identifying two functions which are equal almost everywhere, $L^2(G;\zeta)$ has a structure of Hilbert space with an inner product

$$(f_1, f_2)_{L^2(G;\zeta)}=\int_{G/L}\rho(x)^{-1}(f_1(x), f_2(x))_{\mathscr{H}(\zeta)}d\mu_\rho(xL) \tag{3.8}$$

for $f_1, f_2\in L^2(G;\zeta)$. Through the left translation, G acts unitarily on $L^2(G;\zeta)$. We denote this action by $\mathcal{U}_\zeta$:

$$(\mathcal{U}_\zeta(g)f)(x)=f(g^{-1}x) \qquad (g, x\in G,\ f\in L^2(G;\zeta)). \tag{3.9}$$

$(\mathcal{U}_\zeta, L^2(G;\zeta))$ is called the *representation of G induced unitarily from* $(\zeta, \mathscr{H}(\zeta))$, and we often express this as $L^2\text{-Ind}_L^G(\zeta)$ instead of $\mathcal{U}_\zeta$. This construction of $\mathcal{U}_\zeta$ apparently depends on a choice of ρ, however, the equivalence class does not.

For the later use, we quote here fundamental properties of unitarily induced representations. Firstly, let $H_1\subseteq H_2$ be two closed subgroups of G. If ζ is a unitary representation of H_1, then

$$L^2\text{-Ind}_{H_1}^G(\zeta)\simeq L^2\text{-Ind}_{H_2}^G(L^2\text{-Ind}_{H_1}^{H_2}(\zeta)). \tag{3.10}$$

This is said to be the stage theorem for unitarily induced representations.

Secondly, let $\zeta\simeq\int_Z^\oplus\zeta(t)d\mu(t)$ be a direct integral decomposition of a representation ζ of a closed subgroup L $(\subseteq G)$. Then one has

$$L^2\text{-Ind}_L^G(\zeta)\simeq\int_Z^\oplus L^2\text{-Ind}_L^G(\zeta(t))d\mu(t). \tag{3.11}$$

Finally, assume that G be a Lie group. Then the space $L^2(G;\zeta)^\infty$ of smooth vectors for $\mathcal{U}_\zeta$ is characterized as follows.

Lemma 3.9 [12, Theorem 5.1 and Corollary 5.1]. *Under the above notations, the space $L^2(G;\zeta)^\infty$ is described as*

$$L^2(G;\zeta)^\infty=\{f\in C^\infty(G;\zeta)\,;\ \pi_\zeta(D)f\in L^2(G;\zeta) \text{ for any } D\in U(\mathfrak{g}_C)\},\tag{3.12}$$

where $(\pi_\zeta, C^\infty(G;\zeta))$ is as in 2.1. Moreover, for each fixed $g\in G$, the map $L^2(G;\zeta)_\infty\ni f\mapsto f(g)\in\mathcal{H}(\zeta)$ is a continuous linear mapping from $L^2(G;\zeta)^\infty$ to $\mathcal{H}(\zeta)$.

3.7. An upper bound for the multiplicity function of an induced representation. Now let G be a Lie group of type I. Consider a unitarily induced representation $L^2\text{-Ind}_L^G(\zeta)=(\mathcal{U}_\zeta, L^2(G;\zeta))$. Let

$$\mathcal{U}_\zeta\simeq\int_{\hat{G}}^{\oplus}\mathcal{U}_\zeta(\pi)d\mu_\zeta(\pi),\qquad L^2(G;\zeta)\simeq\int_{\hat{G}}^{\oplus}\mathcal{H}(\zeta,\pi)d\mu_\zeta(\pi)\tag{3.13}$$

be the factor decomposition of $\mathcal{U}_\zeta$ as in Lemma 3.5. Furthermore, decompose each factor representation $\mathcal{U}_\zeta(\pi)$ into irreducibles:

$$\mathcal{U}_\zeta(\pi)\simeq[m_\zeta(\pi)]\cdot\pi,\qquad \mathcal{H}(\zeta,\pi)\simeq[m_\zeta(\pi)]\cdot\mathcal{H}(\pi)=\mathcal{H}(\pi)\oplus\cdots\oplus\mathcal{H}(\pi)\tag{3.14}$$
$$(m_\zeta(\pi)\text{-copies}).$$

Here we take a concrete representation of G in the equivalence class π, and denote it again by $(\pi, \mathcal{H}(\pi))$.

We wish to rewrite, or estimate the multiplicity function m_ζ by some quantity which is rather computable. If G is a compact group, then $\hat{G}$ is discrete, and the Frobenius reciprocity law (see e.g. [16, 5.3.6]) says that

$$m_\zeta(\pi)=I_G(\pi_\infty, C^\infty\text{-Ind}_L^G(\zeta))=I_L(\pi_\infty, (\delta_G/\delta_L)^{1/2}\zeta)\tag{3.15}$$

for any $\pi\in\hat{G}$. (In this case, $\delta_G/\delta_L=1$ because G and L are compact.)

We ask if the first equality holds for a general pair (G, L) and any unitary representation ζ of L. (The second one always holds thanks to Lemma 2.2.) The answer is "no" in general (see Example 3.11 below). Nevertheless, the intertwining number $I_G(\pi_\infty, \pi_\zeta)$, $\pi_\zeta=C^\infty\text{-Ind}_L^G(\zeta)$, gives an upper bound for $m_\zeta(\pi)$ under a certain assumption on π or on ζ.

First, we put an assumption on $\pi\in\hat{G}$.

Lemma 3.10. *Let $\pi\in\hat{G}$ be a discrete series representation for $\mathcal{U}_\zeta$, that is, π may be realized as a subrepresentation of $\mathcal{U}_\zeta$. Then one has an inequality*

$$m_\zeta(\pi)\leqq I_G(\pi_\infty, (\mathcal{U}_\zeta)_\infty)\leqq I_G(\pi_\infty, \pi_\zeta).\tag{3.16}$$

Proof. Every operator $A\in\text{Hom}_G(\pi, \mathcal{U}_\zeta)$ gives, through its restriction to $\mathcal{H}(\pi)^\infty$, a continuous intertwining operator A_∞ from π_∞ to $(\mathcal{U}_\zeta)_\infty$. Since $\mathcal{H}(\pi)^\infty$ is dense in $\mathcal{H}(\pi)$, this map $A\to A_\infty$ is injective, which implies the first inequality because π is a discrete series for $\mathcal{U}_\zeta$.

It follows from Lemma 3.9 that $L^2(G;\zeta)^\infty\subseteqq C^\infty(G;\zeta)$ and that the assignment

$T: L^2(G;\zeta)^\infty \ni f \mapsto f(1) \in \mathcal{H}(\zeta)$ is a continuous linear map. Hence $T \in \mathrm{Hom}_L((\mathcal{U}_\zeta)_\infty, (\delta_G/\delta_L)^{1/2}\zeta)$. In view of Lemma 2.2, the canonical embedding $L^2(G;\zeta)^\infty \hookrightarrow C^\infty(G;\zeta)$ gives a continuous isomorphism of G-modules from $(\mathcal{U}_\zeta)_\infty$ into π_ζ. This proves the second inequality. Q. E. D.

Now we proceed to estimate the multiplicity $m_\zeta(\pi)$ for $\pi \in \hat{G}$ which are not necessarily discrete series for $\mathcal{U}_\zeta$. The following example suggests that we need to put some assumption on ζ or on L in order to get an estimate

$$m_\zeta(\pi) \leqq I_G(\pi_\infty, C^\infty\text{-}\mathrm{Ind}_L^G(\zeta)) \tag{3.17}$$

on the whole $\hat{G}$.

Example 3.11. Let $G=\boldsymbol{R}$ be the additive group of all real numbers. In this case $\hat{G}$ has a structure $\hat{G}=\{e(\lambda); \lambda \in \boldsymbol{R}\} \simeq \boldsymbol{R}$. Here, for a $\lambda \in \boldsymbol{R}$, $e(\lambda)$ denotes a unitary character of $G: e(\lambda)(x)=\exp\sqrt{-1}\lambda x$ $(x \in \boldsymbol{R})$. Take $L=G$ and ζ=the regular representation of G on $L^2(\boldsymbol{R})$. Then, through the Fourier transform, $\mathcal{U}_\zeta=\zeta$ is decomposed into irreducibles as

$$\mathcal{U}_\zeta \simeq \int_{\hat{G}}^{\oplus} e(\lambda)d\lambda, \tag{3.18}$$

where $d\lambda$ is a suitably normalized Lebesgue measure on $\boldsymbol{R}$. This implies that $m_\zeta(e(\lambda))=1$ for a.e. $\lambda \in \boldsymbol{R}$.

On the other hand, π_ζ is clearly equivalent to the smooth representation ζ_∞ corresponding to ζ. Moreover, it is also clear that $I_G(e(\lambda), \pi_\zeta)=0$ for every $\lambda \in \boldsymbol{R}$. One thus gets

$$I_G(e(\lambda), \pi_\zeta)=0<1=m_\zeta(e(\lambda)) \qquad \text{for a.e. } \lambda \in \boldsymbol{R},$$

which means that, in this case, the inequality (3.17) is false for a.e. π.

Now assume that ζ is a finite-dimensional unitary representation of L. For such a ζ, we can show, using Propositions 3.7 and 3.8, that (3.17) holds on the whole $\hat{G}$. This is done as follows.

For any $f \in L^2(G;\zeta)^\infty$, set $T(f)=f(1) \in \mathcal{H}(\zeta)$. Then, as we remaked in the proof of Lemma 3.10, $T: L^2(G;\zeta)^\infty \to \mathcal{H}(\zeta)$ gives a continuous linear map satisfying

$$T((\mathcal{U}_\zeta)_\infty(h)f)=(\delta_G(h)/\delta_L(h))^{1/2}\zeta(h)T(f) \qquad (f \in L^2(G;\zeta),\ h \in L).$$

This means that $T \in \mathrm{Hom}_L((\mathcal{U}_\zeta)_\infty, (\delta_G/\delta_L)^{1/2}\zeta)$. Moreover, T is clearly a generalized cyclic map from $L^2(G;\zeta)^\infty$ to $\mathcal{H}(\zeta)$.

From Proposition 3.7, for a.e. $\pi \in \hat{G}$, there exist unique continuous linear maps $T(\pi): \mathcal{H}(\zeta, \pi)^\infty \to \mathcal{H}(\zeta)$ such that

$$T(f)=\int_{\hat{G}} T(\pi)(f(\pi))d\mu_\zeta(\pi) \qquad \text{for all } f=(f(\pi)) \in L^2(G;\zeta)^\infty. \tag{3.19}$$

From the uniqueness of $T(\pi)$, we see that $T(\pi) \in \mathrm{Hom}_L(\mathcal{U}_\zeta(\pi)_\infty, (\delta_G/\delta_L)^{1/2}\zeta)$. Moreover, by Proposition 3.8, $T(\pi)$ are, for a.e. π, generalized cyclic maps from $\mathcal{H}(\zeta, \pi)^\infty$ to $\mathcal{H}(\zeta)$. For such a $\pi \in \hat{G}$, define a linear operator $A(\pi): \mathcal{H}(\zeta, \pi)^\infty \to$

$C^\infty(G, \mathcal{H}(\zeta))$ by

$$A(\pi)f(g)=T(\pi)(\mathcal{U}_\zeta(\pi)(g^{-1})f) \qquad (g\in G,\ f\in\mathcal{H}(\zeta, \pi)^\infty). \tag{3.20}$$

Then $A(\pi)$ gives a continuous embedding of a G-module $\mathcal{H}(\zeta, \pi)^\infty$ into π_ζ. Keeping Lemma 2.2 in mind, we thus get the following

Theorem 3.12. *Let G be a Lie group of type* I. *For a finite-dimensional unitary representation ζ of a closed subgroup L of G, consider the unitarily induced representation $\mathcal{U}_\zeta=L^2\text{-Ind}_L^G(\zeta)$. Then, for a.e. $\pi\in\hat{G}$, the multiplicity $m_\zeta(\pi)$ of π in $\mathcal{U}_\zeta$ has an upper bound as*

$$m_\zeta(\pi)\leqq I_G(\pi_\infty, C^\infty\text{-Ind}_L^G(\zeta))=I_L(\pi_\infty, (\delta_G/\delta_L)^{1/2}\zeta). \tag{3.21}$$

Here, for a group X and two representations S_1 and S_2 of X, $I_X(S_1, S_2)$ denotes the intertwining number from S_1 to S_2 as in 2.1.

3.8. Application to induced representations $L^2\text{-Ind}_{HN}^G(\zeta)$ of semisimple groups G. Hereafter, we assume that G be a connected semisimple Lie group with finite center. Apply Theorem 3.12 to $\mathcal{U}_\zeta=L^2\text{-Ind}_{HN}^G(\zeta)$, where $HN=H\ltimes N$ is a semidierct product subgroup of G as in 1.2. Then, by Theorem 2.10 (or by Corollary 2.13) we get the following finite multiplicity theorem for induced representation $\mathcal{U}_\zeta$.

Theorem 3.13. *Under the above notations, let ζ be a finite-dimensional unitary representation of HN, and $\mathcal{U}_\zeta=L^2\text{-Ind}_{HN}^G(\zeta)$. Let*

$$\mathcal{U}_\zeta\simeq\int_{\hat{G}}^{\oplus}\mathcal{U}_\zeta(\pi)d\mu_\zeta(\pi), \qquad \mathcal{U}_\zeta(\pi)\simeq[m_\zeta(\pi)]\cdot\pi$$

denote the factor decomposition of $\mathcal{U}_\zeta$ as in Lemma 3.5. *Then, the multiplicity function m_ζ takes finite values for a.e. $\pi\in\hat{G}$ with respect to μ_ζ.*

This the main result of this section, which generalizes the result [1, Theorem 3.1] for the case $N=(1)$ and ζ=the trivial character of H.

3.9. In order to establish our finite multiplicity theorem in a general form as in Theorem 3.13, we have had to assume ζ to be finite-dimensional. Nevertheless, even for infinite-dimensional ζ, there are interesting examples of $\mathcal{U}_\zeta$ which have finite multiplicity property. Among such examples, the ones we think most interesting are "reduced generalized Gelfand-Graev representations" of certain types, which will be studied in the second part of this series of articles. In the next section, we present other important examples of such $\mathcal{U}_\zeta$.

§4. The case of representations induced from infinite-dimensional ones

In the previous sections, by generalizing the theory of spherical functions in [1] and [6], we gave sufficient conditions for a representation ζ of a subgroup HN $(\subseteqq G)$ as in 1.2 that the induced representation $\text{Ind}_{HN}^G(\zeta)$ is of multiplicity finite. Here Ind means either C^∞-Ind (see 2.1) or L^2-Ind (see 3.6). We emphasized there (Remark 2.11 and 3.9) that our criterions may be applied successfully also to case of infinite-dimensional ζ. It differs from the cases in [1] and [6].

We close the present article with some examples (including Gelfand-Graev representations) of such cases, important in connection with the second part [19] of this paper.

4.1. Representations $\mathrm{Ind}_{MN_m}^G(\zeta)$ with $\zeta=\mathrm{Ind}_{N_m}^{MN_m}(\xi)$.

Let $G=KA_pN_m$ be an Iwasawa decomposition of a connected semisipmle Lie group G with finite center. Set $M=Z_K(A_p)$. Then the semidirect product subgroup $M\ltimes N_m$ satisfies the assumption for $H\ltimes N$ in 1.2. Take a unitary character ξ of the maximal unipotent subgroup N_m and put $\zeta=L^2\text{-}\mathrm{Ind}_{N_m}^{MN_m}(\xi)$. Since M is compact, $C^\infty\text{-}\mathrm{Ind}_{N_m}^{MN_m}(\xi)$ is equivalent to ζ_∞, the smooth representation of MN_m associated with ζ. Moreover, if the Lie algebra $\mathfrak{m}$ of M does not reduce to (0), then ζ is actually infinite-dimensional. We deal with the induced representation $\mathrm{Ind}_{MN_m}^G(\zeta)$.

The stage theorem for unitarily induced representations tells us

$$\mathcal{U}_\zeta=L^2\text{-}\mathrm{Ind}_{MN_m}^G(\zeta)\simeq L^2\text{-}\mathrm{Ind}_{N_m}^G(\xi)\quad\text{(unitary equivalence).}\tag{4.1}$$

For the C^∞-induced representation also, an equivalence similar to (4.1) holds thanks to Lemma 2.1 and the compactness of M:

$$\pi_\zeta=C^\infty\text{-}\mathrm{Ind}_{MN_m}^G(\zeta)\simeq C^\infty\text{-}\mathrm{Ind}_{MN_m}^G(\zeta_\infty)\simeq C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\xi).\tag{4.2}$$

4.2. A finite multiplicity theorem for $\mathrm{Ind}_{MN_m}^G(\zeta)\simeq\mathrm{Ind}_{N_m}^G(\xi)$.

Now we apply our results in §§ 2 and 3 to $\mathrm{Ind}_{MN_m}^G(\zeta)$.

4.2.1. First, we consider the induced representation π_ζ in C^∞-context. The restriction of ζ to M is equivalent to the left regular representation of M. By virtue of the Peter-Weyl theorem, for any $\sigma\in\hat{M}$, the multiplicity $I_M(\sigma,\zeta)$ of σ in ζ is equal to $\dim\sigma$. Theorem 2.10 together with this fact implies the following

Theorem 4.1. *Put $\pi_\zeta=C^\infty\text{-}\mathrm{Ind}_{MN_m}^G(\zeta)$. For any irreducible admissible representation π of G with infinitesimal character χ, the multiplicities $I_G(\pi_\infty,\pi_\zeta)$, $I_{\mathfrak{g}_C-K}(\pi_K,(\pi_\zeta)_K)$ and $M_{\mathfrak{g}_C-K}(\pi_K,(\pi_{\zeta,\chi})_K)$ (see § 2) are estimated as*

$$\begin{aligned}I_G(\pi_\infty,\pi_\zeta)&\leqq I_{\mathfrak{g}_C-K}(\pi_K,(\pi_\zeta)_K)\leqq M_{\mathfrak{g}_C-K}(\pi_K,(\pi_{\zeta,\chi})_K)\\&\leqq R_1R_2^{-1}\min_{\tau\in\hat{K}}\{\dim\tau\cdot I_K(\tau,\pi)^{-1}\}<+\infty.\end{aligned}\tag{4.3}$$

Here R_1 (resp. R_2) denotes the order of the complex Weyl group of $\mathfrak{g}_C$ (resp. $\mathfrak{m}_C$), and $I_K(\tau,\pi)$ is the multiplicity of $\tau\in\hat{K}$ in $\pi|K$.

In wiew of (4.2), one deduces from Theorem 4.1 the following

Theorem 4.2. *The induced representation $C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\xi)$ has finite multiplicity* *property for any one-dimensional representation* ξ *of* N_m, *that is,* $I_{\mathfrak{g}_C-K}(\pi_K, (C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\xi))_K)<+\infty$ *for every irreducible admissible representation* π *of* G.

4.2.2. Secondly, we are concerned with the unitarily induced representation $\mathcal{U}_\zeta$. Taking Theorem 4.2 into account, we apply Theorem 3.12 to $L^2\text{-}\mathrm{Ind}_{N_m}^G(\xi)$. We thus obtain the finite multiplicity theorem for $\mathcal{U}_\zeta \simeq L^2\text{-}\mathrm{Ind}_{N_m}^G(\xi)$ as follows.

Theorem 4.3. *Let* $\mathcal{U}_\zeta \simeq \int_{\hat{G}}^{\oplus}[m_\zeta(\pi)]\cdot\pi d\mu_\zeta(\pi)$ *be the factor decomposition of* $\mathcal{U}_\zeta = L^2\text{-}\mathrm{Ind}_{MN_m}^G(\zeta)$ *as in Lemma* 3.5. *Then the multiplicity function* m_ζ *takes finite values for* μ_ζ*-a.e.* $\pi \in \hat{G}$.

Consequently, the induced representation $\mathrm{Ind}_{MN_m}^G(\zeta)$ has finite multiplicity property, although ζ is infinite-dimensional in general.

4.3. The case of Gelfand-Graev representations.

As is suggested in Appendix, the study of $\mathrm{Ind}_{N_m}^G(\xi)$ for an arbitrary ξ is reduced, in a certain sense, to that for non-degenerate (see A.2) ξ's. Accordingly, we concentrate on such a case, that is, on the case of Gelfand-Graev representation (=GGR) (Definition A.5). The GGRs are of multiplicity finite thanks to Theorems 4.2 and 4.3. Moreover, it is well-known that, under some additional assumptions on G, the GGRs have multiplicity free property in the following sense.

Proposition 4.4 (cf. [14, Theorem 3.1 and Appendix]). *Suppose that* G *be quasi-split* (*i.e.,* $\mathfrak{m}$ *is abelian*) *and linear. Consider the GGR* $\pi_\xi = C^\infty\text{-}\mathrm{Ind}_{N_m}^G(\xi)$ (*in* C^∞*-context*) *with a non-degenerate character* ξ *of* N_m. *Then, for any irreducible unitary representation* $\mathcal{U}$ *of* G, *the intertwining number from* $\mathcal{U}_\infty$ *to* π_ξ *is at most one:* $I_G(\mathcal{U}_\infty, \pi_\xi) \leqq 1$.

By virtue of our Theorem 3.12, we can deduce immediately from Proposition 4.4 the multiplicity one theorem, originally due to Ramakrishnan [13], for the unitary GGRs $\mathcal{U}_\xi = L^2\text{-}\mathrm{Ind}_{N_m}^G(\xi)$.

Theorem 4.5. *Let* G *be as in Proposition* 4.4. *Then the unitarily induced GGRs* $\mathcal{U}_\xi$ *are of multiplicity one. In particular, the von Neumann algebras* $\mathrm{H}_G(\mathcal{U}_\xi) = \mathrm{Hom}_G(\mathcal{U}_\xi, \mathcal{U}_\xi)$ *of intertwining operators for* $\mathcal{U}_\xi$ *are commutative.*

Remark 4.6. Ramakrishnan proved in [13] Theorem 4.5 by generalizing the idea of Shalika [14] in the proof of Proposition 4.4 quoted above. More precisely, in order to show that $\mathrm{H}_G(\mathcal{U}_\xi)$ is commutative, he found out an antiautomorphism of $\mathrm{H}_G(\mathcal{U}_\xi)$ which fixes every element in it.

However, as we saw above, his procedure can be replaced by Shalika's result (Proposition 4.4) and our Theorem 3.12.

4.4. Toward the continuation [19], [20] of this paper: Application to generalized Gelfand-Graev representations. N. Kawanaka [8] introduced, by generalizing the construction of GGRs, a series of induced representations of reductive groups $G(F)$ over various fields F. Such an induced representation is called a *generalized Gelfand-Graev representation* (=GGGR). The GGGRs are parametrized by the set of nilpotent classes of the Lie algebra $\mathfrak{g}(F)$ of $G(F)$, and the GGGRs corresponding to regular nilpotent classes are the original GGRs.

Contrary to the case of GGRs, the GGGRs of a semisimple Lie group G are, in general, far from to be of multiplicity finite. Here is a difficulty of the study on GGGRs. In order to reduce the infinite multiplicities of irreducible constituents of them to be finite or to be free (if possible), we will introduce in the second part [19] a version of GGGRs, called *reduced* GGGRs. And then, we shall apply our results in this article to the reduced GGGRs. The important cases connect with "*generalized Whittaker models*" of (holomorphic) discrete series representations (cf. [20]). And we find out that the reduced GGGRs are of multiplicity finite in these cases.

Furthermore, in more restricted cases, we have multiplicity free property, which will be proved in detail in the subsequent paper [20].

Appendix. On the irreducible decomposition of $L^2\text{-Ind}_{N_m}^G(\xi)$: Reduction to the case of Gelfand-Graev representations

Let $G=KA_pN_m$ be an Iwasawa decomposition of a connected semisimple Lie group G with finite center. For a unitary character ξ of N_m, consider $\mathcal{U}_\xi= L^2\text{-Ind}_{N_m}^G(\xi)$ (see 4.1). We have proved in Theorem 4.3 that $\mathcal{U}_\xi$ is of multiplicity finite. Then there arises a natural question: How can $\mathcal{U}_\xi$ be decomposed explicitly into irreducibles?

In this Appendix, we treat this problem and give a complete answer (Theorem A.4) for the case $\xi=1_{N_m}$, the trivial character of N_m. Moreover, we show in A.3 that our problem for arbitrary ξ is reduced mainly to that for non-degenerate characters ξ, that is, to decompose so-called Galfand-Graev representations.

A.1. The irreducible decomposition of $L^2\text{-Ind}_{N_m}^G(1_{N_m})$.

We give in this subsection the explicit irreducible decomposition of $\mathcal{U}_1= L^2\text{-Ind}_{N_m}^G(1_{N_m})$. Firstly, we see from the stage theorem for induced representations that

$$\mathcal{U}_1\simeq L^2\text{-Ind}_P^G(L^2\text{-Ind}_{N_m}^P(1_{N_m})), \tag{A.1}$$

where $P=MA_pN_m$ with $M=Z_K(A_p)$ is a minimal parabolic subgroup of G. Keeping (A.1) in mind, let us decompose $L^2\text{-Ind}_{N_m}^P(1_{N_m})$ into irreducibles.

Lemma A.1. *One has an isomorphism of unitary representations*

$$L^2\text{-Ind}_{N_m}^P(1_{N_m})\simeq \sum_{\sigma\in \hat{M}}^{\oplus}[\dim\sigma]\cdot\int_{\mathfrak{a}_p^*}\sigma\otimes e(\nu)\otimes 1_{N_m}d\nu\,. \tag{A.2}$$

Here $d\nu$ denotes a suitably normalized Lebesgue measure on the dual space $\mathfrak{a}_p^*$ of $\mathfrak{a}_p=\log A_p$, and, for $\nu\in\mathfrak{a}_p^*$, $e(\nu)$ is a unitary character of A_p defined by $e(\nu)(\exp H)$

$=\exp\sqrt{-1}\nu(H)$ $(H\in\mathfrak{a}_p)$.

Proof. The restriction of $L^2\text{-Ind}_{N_m}^P(1_{N_m})$ to the subgroup N_m is a multiple of 1_{N_m} because N_m is normal in P and the character 1_{N_m} is fixed under the adjoint action of P on N_m. Moreover, we see easily that the restriction $L^2\text{-Ind}_{N_m}^P(1_{N_m})|MA_p$ is equivalent to the regular representation $\lambda(MA_p)$ of MA_p. Thus one gets

$$(\text{A.3}) \qquad L^2\text{-Ind}_{N_m}^P(1_{N_m})\simeq\lambda(MA_p)\otimes 1_{N_m}.$$

Notice that MA_p is the direct product of a compact group M and a vector group A_p. Taking into account the Peter-Weyl theorem for compact groups and the Plancherel theorem for vector groups, we obtain

$$(\text{A.4}) \qquad \lambda(MA_p)\simeq \sum_{\sigma\in\hat{M}}^{\oplus}[\dim\sigma]\cdot\int_{\mathfrak{a}_p^*}\sigma\otimes e(\nu)d\nu.$$

(A.3) and (A.4) imply the desired (A.2). Q. E. D.

From (A.1) and (A.2), the representation $\mathcal{U}_1$ is disintegrated as follows:

$$(\text{A.5}) \qquad L^2\text{-Ind}_{N_m}^G(1_{N_m})\simeq \sum_{\sigma\in\hat{M}}^{\oplus}[\dim\sigma]\cdot\int_{\mathfrak{a}_p^*}^{\oplus}\mathcal{U}_{\sigma,\nu}d\nu,$$

where, for $(\sigma,\nu)\in\hat{M}\times\mathfrak{a}_p^*$, $\mathcal{U}_{\sigma,\nu}=L^2\text{-Ind}_P^G(\sigma\otimes e(\nu)\otimes 1_{N_m})$ is a unitary principal series representation of G.

Let $W=N_K(A_p)/M$ be the Weyl group of (G, A_p). Then W acts on M and on $\mathfrak{a}_p^*$ in the following way. Let $w\in W$, $\nu\in\mathfrak{a}_p^*$ and $\sigma\in\hat{M}$. Take a representative $w^*\in N_K(A_p)$ of w and a concrete irreducible representation σ_0 of M of class σ. Define $w^*\nu\in\mathfrak{a}_p^*$ (resp. a representation $w^*\sigma_0$ of M) by $w^*\nu(H)=$ $=\nu(\text{Ad}(w^*)^{-1}H)$ (resp. $w^*\sigma_0(m)=\sigma_0(w^{*-1}mw^*)$) for $H\in\mathfrak{a}_p$ (resp. $m\in M$). Then $w^*\nu$ (resp. the equivalence class of $w^*\sigma_0$) does not depend on a choice of w^* (resp. w^* and σ_0). So we may denote it by $w\nu$ (resp. $w\sigma$). Thus W acts on $\mathfrak{a}_p^*$ (resp. on $\hat{M}$) through $(w,\nu)\mapsto w\nu$ (resp. $(w,\sigma)\mapsto w\sigma$).

In order to show that (A.5) actually gives the irreducible decomposition of $\mathcal{U}_1$, we now quote a fundamental theorem by F. Bruhat about the unitary equivalence and irreducibility of the principal series representations.

Proposition A.2 [2]. (1) *$\mathcal{U}_{w\sigma,w\nu}\simeq\mathcal{U}_{\sigma,\nu}$ for any $w\in W$ and any $(\sigma,\nu)\in$ $\hat{M}\times\mathfrak{a}_p^*$.*

(2) *For $(\sigma_i,\nu_i)\in\hat{M}\times\mathfrak{a}_p^*$ $(i=1, 2)$, the intertwining number $I_G(\mathcal{U}_{\sigma_1,\nu_1},\mathcal{U}_{\sigma_2,\nu_2})$ from $\mathcal{U}_{\sigma_1,\nu_1}$ to $\mathcal{U}_{\sigma_2,\nu_2}$ has an upper bound as*

$$(\text{A.6}) \qquad I_G(\mathcal{U}_{\sigma_1,\nu_1},\mathcal{U}_{\sigma_2,\nu_2})\leq|\{w\in W\,;\, w\sigma_1\simeq\sigma_2 \text{ and } w\nu_1=\nu_2\}|.$$

In particular, if $(\sigma,\nu)\in\hat{M}\times\mathfrak{a}_p^*$ satisfies the condition

$$(\text{A.7}) \qquad \text{either } w\sigma\not\simeq\sigma \text{ or } w\nu\neq\nu \text{ for any } w\in W\setminus\{1\},$$

then $\mathcal{U}_{\sigma,\nu}$ is irreducible.

Let $(\mathfrak{a}_p^*)'$ be the open dense subset of $\mathfrak{a}_p^*$ consisting all $\nu\in\mathfrak{a}_p^*$ such that $w\nu\neq\nu$

for any $w \in W \setminus \{1\}$. A connected component of $(\mathfrak{a}_p^*)'$ is called a Weyl chamber. W acts simply transitively on the set of Weyl chambers. Taking this into account, one deduces immediately from Proposition A.2 the following

Corollary A.3. *Let $(\mathfrak{a}_p^*)^+$ be a Weyl chamber in $\mathfrak{a}_p^*$. Then one has*

(1) *for any $(\sigma, \nu) \in \hat{M} \times (\mathfrak{a}_p^*)^+$, $\mathcal{U}_{\sigma,\nu}$ is irreducible.*

(2) *If $(\sigma, \nu) \neq (\sigma', \nu') \in \hat{M} \times (\mathfrak{a}_p^*)^+$, then $\mathcal{U}_{\sigma,\nu}$ and $\mathcal{U}_{\sigma',\nu'}$ are mutually inequivalent.*

Thanks to Proposition A.2 and Corollary A.3, we can rewrite the right hand side of (A.5), and get the irreducible decomposition of $\mathcal{U}_1$ as follows.

Theorem A.4. *The representation $L^2\text{-Ind}_{N_m}^G(1_{N_m})$ admits the following direct integral decomposition into the irreducible principal series representations $\mathcal{U}_{\sigma,\nu}$ $(\sigma \in \hat{M}, \nu \in (\mathfrak{a}_p^*)^+)$:*

$$\text{(A.8)} \qquad L^2\text{-Ind}_{N_m}^G(1_{N_m}) \simeq \sum_{\sigma \in \hat{M}}^{\oplus} \int_{(\mathfrak{a}_p^*)^+}^{\oplus} [|W| \dim \sigma] \cdot \mathcal{U}_{\sigma,\nu} d\nu .$$

Especially, $|W| \dim \sigma$ is the multiplicity of $\mathcal{U}_{\sigma,\nu}$ in $L^2\text{-Ind}_{N_m}^G(1_{N_m})$.

Proof. W acts simply transitively on Weyl chambers in $\mathfrak{a}_p^*$. So one gets

$$\int_{\mathfrak{a}_p^*}^{\oplus} \mathcal{U}_{\sigma,\nu} d\nu \simeq \sum_{w \in W}^{\oplus} \int_{(\mathfrak{a}_p^*)^+}^{\oplus} \mathcal{U}_{\sigma, w\nu} d\nu \simeq \sum_{w \in W}^{\oplus} \int_{(\mathfrak{a}_p^*)^+}^{\oplus} \mathcal{U}_{w^{-1}\sigma,\nu} d\nu .$$

For the second isomorphism above, we uesd Proposition A.2(1). Accordingly, the right hand side of (A.5) can be rewritten to that of (A.8) by noting that $\dim w^{-1}\sigma = \dim \sigma$.

A.2. Unitary characters of N_m. Before proceeding to decomposition of $\mathcal{U}_\xi$ for general ξ, we now clarify the structure of the group $N_m^\dagger$ of unitary characters of N_m.

Let $\mathfrak{n}_m$ be the Lie algebra of N_m, and let $\Delta(\mathfrak{g} : \mathfrak{a}_p)$ (see 1.1) denote the collection of all roots of $\mathfrak{g}$ with respect to $\mathfrak{a}_p$. Choose a positive system $\Delta^+(\mathfrak{g} : \mathfrak{a}_p)$ of $\Delta(\mathfrak{g} : \mathfrak{a}_p)$ so that

$$\text{(A.9)} \qquad \mathfrak{n}_m = \sum\nolimits_{\lambda \in \Delta^+(\mathfrak{g}:\mathfrak{a}_p)} \mathfrak{g}(\mathfrak{a}_p ; \lambda),$$

where $\mathfrak{g}(\mathfrak{a}_p ; \lambda)$ is the root space of λ. Denote by Π the set of simple roots in $\Delta^+(\mathfrak{g} : \mathfrak{a}_p)$. Set $\mathfrak{g}(\Pi) = \sum_{\lambda \in \Pi} \mathfrak{g}(\mathfrak{a}_p ; \lambda)$. Then, by [18, Lemma 3.2] $\mathfrak{n}_m$ has a structure

$$\text{(A.10)} \qquad \mathfrak{n}_m = \mathfrak{g}(\Pi) \oplus [\mathfrak{n}_m, \mathfrak{n}_m] \qquad \text{(as vector spaces)}.$$

Let $\eta_0 \in \mathfrak{g}(\Pi)^*$, the dual space of $\mathfrak{g}(\Pi)$. Thanks to (A.10), we can, and do, extend η_0 uniquely to a Lie algebra homomorphism $\eta : \mathfrak{n}_m \to \boldsymbol{R}$. Through the exponential mapping, define a unitary character $\xi = \xi(\eta_0) \in N_m^\dagger$ by

$$\xi(\exp X) = \exp \sqrt{-1}\, \eta(X) \qquad (X \in \mathfrak{n}_m).$$

Clearly, the map $\eta_0 \mapsto \xi(\eta_0)$ gives a bijective correspondence between $\mathfrak{g}(\Pi)^*$ and $N_m^\dagger$.

For $\xi=\xi(\eta_0)\in N_m^\dagger$, set $F(\xi)=\{\lambda\in\Pi\,;\,\eta_0|\mathfrak{g}(\mathfrak{a}_p\,;\,\lambda)\neq 0\}$. We call ξ *non-degenerate* if $F(\xi)=\Pi$, namely $\eta_0|\mathfrak{g}(\mathfrak{a}_p\,;\,\lambda)\neq 0$ for every $\lambda\in\Pi$.

Definition A.5. The induced representation $\mathcal{U}_\xi=L^2\text{-Ind}_{N_m}^G(\xi)$ (or $\pi_\xi=C^\infty\text{-Ind}_{N_m}^G(\xi)$) is said to be a *Gelfand-Graev representation* (=GGR) of G if $\xi\in N_m^\dagger$ is non-degenerate.

To any subest $F\subseteqq\Pi$, one can associate canonically a parabolic subgroup P_F $(\supseteqq P)$ such that $\langle F\rangle$ coincides with the restricted root system of its Levi subgroup $L_F=P_F\cap\theta P_F$ (see [18, 1.2]). Here $\langle F\rangle$ is the sub-root-system of $\Lambda(\mathfrak{g}:\mathfrak{a}_p)$ generated by elements of F, and θ is a Cartan involution of G (see 1.1).

For $\xi\in N_m^\dagger$, let $P_\xi=L(\xi)N_\xi$ be a Levi decomposition of $P_\xi=P_{F(\xi)}$ such that $L(\xi)=L_{F(\xi)}$. Then $N(\xi)=L(\xi)\cap N_m$ is a maximal unipotent subgroup of the reductive group $L(\xi)$, and the restriction $\xi'=\xi|N(\xi)$ defines a non-degenerate character of $N(\xi)$. Thus we have associated to each $\xi\in N_m^\dagger$ a non-degenerate character $\xi'\in N(\xi)^\dagger$ in a canonical way.

A.3. Reduction to the case of GGRs. Now we consider $\mathcal{U}_\xi$ for general $\xi\in N_m^\dagger$. In order to give the irreducible decomposition of $\mathcal{U}_\xi$, we generalize our argument in A.1 for $\xi=1_{N_m}$ to arbitrary ξ so far as possible.

Suggested by the isomorphism of representations

$$L^2\text{-Ind}_{N_m}^G(\xi)\simeq L^2\text{-Ind}_{P_\xi}^G(L^2\text{-Ind}_{N_m}^{P_\xi}(\xi)), \tag{A.11}$$

we try to decompose $L^2\text{-Ind}_{N_m}^{P_\xi}(\xi)$ into irreducibles. Then one gets as in the proof of Lemma A.1

$$L^2\text{-Ind}_{N_m}^{P_\xi}(\xi)\simeq L^2\text{-Ind}_{N(\xi)}^{L(\xi)}(\xi')\otimes 1_{N_\xi}. \tag{A.12}$$

Here $L^2\text{-Ind}_{N(\xi)}^{L(\xi)}(\xi')$ is a GGR of the reductive group $L(\xi)$ (defined analogously to that of a semisimple group). Now let

$$L^2\text{-Ind}_{N(\xi)}^{L(\xi)}(\xi')\simeq\int_{\widehat{L(\xi)}}^{\oplus}[m_{\xi'}(\omega)]\cdot\omega\, d\mu_{\xi'}(\omega) \tag{A.13}$$

be the factor decomposition as in Lemma 3.5. Then it follows from (A.11), (A.12) and (A.13) that

$$\mathcal{U}_\xi=L^2\text{-Ind}_{N_m}^G(\xi)\simeq\int_{\widehat{L(\xi)}}^{\oplus}[m_{\xi'}(\omega)]\cdot L^2\text{-Ind}_{P_\xi}^G(\omega\otimes 1_{N_\xi})d\mu_{\xi'}(\omega). \tag{A.14}$$

The induced representations $\mathcal{U}_\omega(P_\xi)=L^2\text{-Ind}_{P_\xi}^G(\omega\otimes 1_{N_\xi})$ $(\omega\in\widehat{L(\xi)})$ are called the generalized (unitary) principal series representations of G along P_ξ. Every $\mathcal{U}_\omega(P_\xi)$ is expressed in general as a direct sum of finitely many irreducible unitary representations of G.

Accordingly, the main step toward the explicit irreducible decomposition of $\mathcal{U}_\xi$ is now reduced to the problem of disintegrating the GGR $L^2\text{-Ind}_{N(\xi)}^{L(\xi)}(\xi')$ into irreducibles as in (A.13).

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] E. P. van den Ban, Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula, Report PM-R8409, Center for Mathemetics and Computer Science, Amsterdam, 1984; Ark. för Mat., **25** (1987), 175-187.

[2] F. Bruhat, Sur les representations induites des groupes de Lie, Bull. Soc. Math. France, **84** (1956), 97-205.

[3] J. Dixmier, Enveloping algebras, North-Holland, Amsterdam, 1977.

[4] J. Dixmier, C^*-algebras, North-Holland, Amsterdam, 1982.

[5] Harish-Chandra, Representations of semisimple Lie groups, I, Trans. Amer. Math. Soc. **75** (1953), 185-243: II, ibid., **76** (1954), 26-65: III, ibid., **76** (1954), 234-253.

[6] Harish-Chandra, Differential equations and semisimple Lie groups, unpublished manuscript (1960), in: Collected Papers Vol. III, Springer-Verlag, New York, 1984, 57-120.

[7] M. Hashizume, Whittaker functions on semisimple Lie groups, Hiroshima Math. J., **12** (1982), 259-293.

[8] N. Kawanaka, Generalized Gelfand-Graev representations and Ennola duality, Advanced Studies in Pure Math. Vol. 6, Kinokuniya, Tokyo, 1985, 175-206.

[9] G. Mackey, The theory of group representations, Univ. of Chicago, 1955.

[10] T. Oshima and J. Sekiguchi, The restricted root system of a semisimple symmetric pair, Advanced Studies in Pure Math. Vol. 4, Kinokuniya, Tokyo, 1984, 433-497.

[11] R. Penney, Abstract Plancherel theorems and a Frobenius reciprocity theorem, J. Funct. Anal., **18** (1975), 177-190.

[12] N. S. Poulsen, On C^∞-vectors and intertwining bilinear forms for representations of Lie groups, J. Funct. Anal., **9** (1972), 87-120.

[13] D. Ramakrishnan, Multiplicity one for the Gelfand-Graev representation of a linear group, Compositio Math., **45** (1982), 3-14.

[14] J. A. Shalika, The multiplicity one theorem for $GL(n)$, Ann. of Math., **100** (1974), 171-193.

[15] V. S. Varadarajan, Harmonic analysis on real reductive groups, Lec. Notes in Math., **576**, Spriner-Verlag, Berlin-Heidelberg, 1977.

[16] N. R. Wallach, Harmonic analysis on homogeneous spaces, Dekker, New York, 1973.

[17] G. Warner, Harmonic analysis on semisimple Lie groups I, Springer-Verlag, Berlin, 1973.

[18] H. Yamashita, On Wittaker vectors for generalized Gelfand-Graev representations of semisimple Lie groups, J. Math. Kyoto Univ., **26** (1986), 263-298.

[19] H. Yamashita, Finite multiplicity theorems for induced representations of semisimple Lie groups II: Applications to generalized Gelfand-Graev representations, to appear in J. Math. Kyoto Univ., **28**-3 (1988).

[20] H. Yamashita, Multiplicity one theorems for generalized Gelfand-Graev representations of semisimple Lie groups and Whittaker models for the discrete series, to appear in Advanced Studies in Pure Math. Vol. 14.