Highest weight vectors for the principal series of semisimple Lie groups and embeddings of highest weight modules

By

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Introduction

Let $G$ be a connected semisimple Lie group with finite center and $K$ a maximal compact subgroup of $G$. We denote by $\mathfrak{g}$ the Lie algebra of $G$, and by $\mathfrak{g}_c$ its complexification. By Casselman's subrepresentation theorem, every irreducible admissible $(\mathfrak{g}_c, K)$-module can be embedded into some member of the principal series induced from a minimal parabolic subgroup of $G$. This theorem assures the existence of an embedding, but it does not tell us all the places where the irreducibles in question are embedded. So, it is important to describe explicitly embeddings of irreducibles into the principal series.

This embedding problem is easily settled for finite-dimensional representations $F$ (see e.g., [4, 8.5]), because such an $F$ is a highest weight module with respect to a maximally split (and an arbitrary) Cartan subalgebra of $\mathfrak{g}$. Besides these $F$'s, the group $G$ admits irreducible infinite-dimensional highest weight modules with respect to a compact Cartan subalgebra if $G/K$ is a hermitian symmetric space. These interesting infinite-dimensional representations include the (limit of) holomorphic discrete series and more generally unitary highest weight modules classified by Enright-Howe-Wallach [3].

In this article, we describe completely embeddings of all the irreducible highest weight representations into the principal series, by the method of highest weight vectors in our earlier paper [6, Part II] (see also [7]). Although such a description may be derived from Wallach's result in [5], our method, explained below, is quite different from his and much more direct (see Remark 3.5).

To be more precise, assume that $G$ be simple and of hermitian type. We described in [6] embeddings of highest weight modules, called Whittaker models, into generalized Gelfand-Graev representations (GGGRs), important kinds of representations of $G$ induced from certain unipotent subgroups. This is done by
determining all the $K$-finite highest weight vectors in GGGRs. As shown in this paper, this method can be applied successfully also for the principal series, and gives the most elementary and the simplest way to describe embeddings of highest weight modules.

We now explain the main result of this paper. Let $G = KA_p N_m$ be an Iwasawa decomposition of $G$, and put $M = Z_K(A_p)$, the centralizer of $A_p$ in $K$. Then $P = MA_p N_m$ is a minimal parabolic subgroup of $G$.

**Theorem** (Theorem 3.1). Let $L_\lambda$ be the irreducible admissible $(\frak g, K)$-module with highest weight $\lambda \in (\sqrt{-1}t)^*$, where $t$ is a compact Cartan subalgebra of $\frak g$ contained in $\frak t \equiv \text{Lie} (K)$. Then $L_\lambda$ can be embedded, as a $(\frak g_C, K)$-module, into one and only one principal series $\text{Ind}_{P_m}^G (\sigma_\lambda \otimes e^{\psi_\lambda} \otimes 1_{\text{Nm}})$ with multiplicity one. Here $\sigma_\lambda$ is the irreducible $M$-module with highest weight $\lambda$, and a linear form $\psi_\lambda$ on $A_p \equiv \text{Lie} (A_p)$ is defined by $\psi_\lambda = (\mu \circ (\mu |_{A_p})$ through a Cayley transform $\mu$ of $\frak g_C$ carrying $A_p$ into $\sqrt{-1}t$.

From this theorem, we get in particular the unique embedding property for highest weight modules (Corollary 3.2), due to Collingwood [2]. We have further specified explicitly the place into which $L_\lambda$ is embedded.

This paper is organized as follows. In §1, we recall after [6] some fine structures for simple Lie groups of hermitian type and highest weight modules of such groups. We also give Iwasawa decomposition of root vectors. §2 is devoted to determining explicitly all the $K$-finite highest weight vectors in the principal series, by solving a system of equations characterizing such vectors. This enables us to describe in §3 embeddings of irreducible highest weight modules into the principal series completely.

The results of this paper have been reported in [7].

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§1. Simple Lie groups of hermitian type and their highest weight modules

Let $G$ be a connected simple Lie group with finite center, and $\frak g$ its Lie algebra. We assume throughout that the corresponding riemannian symmetric space $G/K$ carries a $G$-invariant complex structure. Such a group $G$ is called of hermitian type. The unitary group $\text{SU}(p, q)$ and the symplectic group $\text{Sp}(n, R)$ are the most typical examples of $G$ (see e.g., [6, 5.1] for the complete list of $\frak g$). The group $G$ has a series of (infinite-dimensional, in general) irreducible representations with highest weights, which is the main object of this paper.

**1.1. Fine structures for $\frak g$**. To begin with, we recall after [6, 5.1] some fine structures for $\frak g$ first clarified by Harish-Chandra. Let $\frak g = \frak t \oplus \frak p$ be a Cartan decomposition of $\frak g$. By identifying $\frak p$ with the tangent space of $G/K$ at the origin in the canonical way, the given $G$-invariant complex structure
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on \( G/K \) gives rise to an \( \text{Ad}(K) \)-invariant complex structure \( J \) on \( p \). Extend \( J \) to \( p_c \leq g_c \) by complex linearity. Here, for any real subspace \( q \) of \( g \), \( g_c \) denotes the subspace of the complexification \( g_c = g \otimes \mathbb{R} \mathbb{C} \) spanned by \( q \) over \( \mathbb{C} \). Then one gets a decomposition of \( g_c \) as

\[
g_c = p_- \oplus t_c \oplus p_+ , \quad p_c = p_- \oplus p_+ ,
\]

(1.1)

\[
[t_c, p_\pm] \subseteq p_\pm , \quad [p_+, p_+] = [p_-, p_-] = (0) ,
\]

where \( p_\pm \) denotes respectively \((\pm \sqrt{-1})\)-eigenspace for \( J \) on \( p_c \). Let \( t \subseteq t \) be a maximal abelian subalgebra of \( t \). Then \( t \) is a compact Cartan subalgebra of \( g \). Denote by \( \Delta \) the root system of \((g_c, t_c)\). A root \( \gamma \in \Delta \) is called compact (resp. non-compact) if the corresponding root space \( g_c(\gamma) \leq g_c \) is contained in \( t_c \) (resp. in \( p_c \)). The totality of compact (resp. non-compact) roots is denoted by \( A_k \) (resp. \( A_p \)). For \( \gamma \in \Delta \), we define elements \( H_\gamma, H_\gamma \in t_c \) respectively by

\[
B(H, H_\gamma) = \gamma(H) (H \in t_c),
\]

where \( B \) is the Killing form of \( g_c \). One can choose a non-zero root vector \( X_\gamma \in g_c \) for \( \gamma \) satisfying

\[
X_\gamma - X_{-\gamma} , \quad \sqrt{-1}(X_\gamma + X_{-\gamma}) \in t + \sqrt{-1}p , \quad [X_\beta, X_{-\gamma}] = H'_\gamma.
\]

We select a positive system \( \Delta^+ \) of \( \Delta \) such as \( p_+ = \sum_{\gamma \in \Delta^+} g_c(\gamma) \) with \( \Delta^+_p \equiv \Delta_p \cap \Delta^+ \), and take a lexicographic order on \( \Delta \) giving this \( \Delta^+ \). Construct a sequence \((\gamma_1, \gamma_2, \ldots, \gamma_l)\) of non-compact positive roots inductively as follows: \( \gamma_k \) is the largest element of \( \Delta^+_k \) strongly orthogonal to \( \gamma_m \); \( \gamma_k \pm \gamma_m \notin \Delta, \neq 0 \); for all \( m > k \). Put \( H_k = X_{\gamma_k} + X_{-\gamma_k} \) for \( 1 \leq k \leq l \). Then \( a_p = \sum_{1 \leq k \leq l} R H_k \) is a maximal abelian subspace of \( p \). Let \( \mu \) be the Cayley transform of \( g_c \) defined by

\[
(1.2) \quad \mu = \exp \left( \frac{\pi}{4} \sum_{1 \leq k \leq l} \text{ad} (X_{\gamma_k} - X_{-\gamma_k}) \right).
\]

Then \( \mu \) carries \( a_p \) into \( \sqrt{-1}t \) in such a way that \( \mu(H_k) = H'_k \). So, \( n_m = \mu^{-1}(\sum_{\gamma \in \Delta^+_m} g_c(\gamma)) \cap g \) is a maximal nilpotent Lie subalgebra of \( g \). We obtain an Iwasawa decomposition \( g = t \oplus a_p \oplus n_m \) of \( g \), and the corresponding decomposition \( G = KA_pN_m \) of \( G \) with \( A_p = \exp a_p, N_m = \exp n_m \).

Let \( M \) be the centralizer of \( A_p \) in \( K \) and \( m \) the Lie algebra of \( M \). Then \( m_c = m_c \oplus t_c \) is a Levi subalgebra of \( t_c \). Furthermore, \( m_c \) admits a direct sum decomposition

\[
m_c = m_c \oplus t_c , \quad [m_c, t_c] = (0) ,
\]

where \( t_c = \mu(a_p) \otimes \mathbb{R} C = \sum_{1 \leq k \leq l} CH_\gamma \). One gets a triangular decomposition of \( t_c \) as

\[
t_c = t_c \oplus m_c \oplus t_c , \quad [m_c, t_c^\pm] \subseteq t_c^\pm .
\]

Here \( t_c^\pm \) denotes the nilpotent Lie subalgebra of \( t_c \) defined by

\[
t_c^\pm = \sum_{\gamma \in \pm \Gamma} g_c(\gamma) \quad \text{with} \quad \Gamma \equiv \{ \gamma \in \Delta_p^+ : \gamma|t_c^\pm \neq 0 \} .
\]
1.2. Iwasawa decomposition of elements of $\mathfrak{p}_+$. Express an element $X$ of $\mathfrak{g}_C$ as $X = t[X] + a[X] + n[X]$ according as the Iwasawa decomposition $\mathfrak{g}_C = \mathfrak{t}_C \oplus (a_\mathfrak{p}_C \oplus (n_\mathfrak{m}_C \mathfrak{g}_C)$. We described in Proposition 9.3 of [6] the components $t[X]$, $a[X]$, and $n[X]$ of non-compact positive root vectors $X = X_\gamma$ ($\gamma \in A_\mathfrak{p}^+$) in an explicit way, by making use of Moore’s restricted root theorem (cf. [6, Prop. 5.1]). From this description, we immediately deduce the following lemma, which plays an important role in the next section.

Lemma 1.1. (1) For $1 \leq k \leq l$, the $t$- and $a$-components of $X_{y_k} \in \mathfrak{p}_+$ are given respectively by $t[X_{y_k}] = H_{y_k}/2$, $a[X_{y_k}] = H_k/2$.

(2) Put $\mathfrak{p}_+ = \sum_{\gamma \in \theta} \mathfrak{g}_C(\gamma)$ with $\Theta = A_\mathfrak{p}^+ \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$. Then one has $t[X] \in \mathfrak{t}_C$ and $a[X] = 0$ for all $X \in \mathfrak{p}_+$. Moreover, the assignment $\mathfrak{p}_+ \ni X \mapsto t[X] \in \mathfrak{t}_C$ sets up a bijective linear map from $\mathfrak{p}_+$ onto $\mathfrak{t}_C$.

Remark 1.2. In order to determine highest weight vectors in the principal series, one does not need any information on the $n$-component $n[X]$ ($X \in \mathfrak{p}_+$), because the principal series modules are induced from representations of $P_m = MA_pN_m$ trivial on $N_m$ (see 2.1). This differs from the case of generalized Gelfand-Graev representations treated in [6].

1.3. Highest weight modules (cf. [6, §7]). For $\lambda \in \mathfrak{t}_C$, the dual space of $\mathfrak{t}_C$, let $L_\lambda$ be the irreducible $\mathfrak{g}_C$-module with $A^+$-highest weight $\lambda$. In other words, $L_\lambda$ is the unique (up to equivalence) irreducible $\mathfrak{g}_C$-module containing a non-zero $\lambda$-highest weight vector $w_\lambda \in L_\lambda$:

\[ H \cdot w_\lambda = \lambda(H)w_\lambda \quad (H \in \mathfrak{t}_C), \quad X_\gamma \cdot w_\lambda = 0 \quad (\gamma \in A^+). \]

By [6, Prop. 7.1], $L_\lambda$ has further an admissible $(\mathfrak{g}_C, K)$-module structure if and only if $\lambda$ satisfies the following two conditions:

\[ \lambda(H_\gamma) \geq 0 \quad \text{for all } \gamma \in A_\mathfrak{t}_k^+ \equiv A_\mathfrak{t}_k \cap A^+, \text{ i.e., } \lambda \text{ is } A_\mathfrak{t}_k^+-\text{dominant}, \]

\[ \exp H \mapsto \exp \lambda(H) \quad (H \in \mathfrak{t}) \text{ gives a unitary character of the maximal torus } \exp \mathfrak{t} \text{ of } K, \text{ i.e., } \lambda \text{ is } K\text{-integral}. \]

Assume that $\lambda$ be $A_\mathfrak{t}_k^+$-dominant and $K$-integral. Then $L_\lambda$ globalizes to a continuous representation $\pi_\lambda$ of $G$ acting on a Hilbert space and $L_\lambda$ is isomorphic to the $(\mathfrak{g}_C, K)$-module of $K$-finite vectors associated to $\pi_\lambda$. This series of irreducible representations $\pi_\lambda$ contains both the holomorphic discrete series and finite-dimensional representations.

§2. Determination of highest weight vectors in the principal series $H_\sigma^0$

In this section, we determine all the $K$-finite highest weight vectors in the principal series by solving equations characterizing such vectors.

2.1. The principal series. Let $P_m = MA_pN_m$ be a minimal parabolic subgroup of $G$. For an irreducible unitary representation $(\sigma, E)$ of $M$ and a linear
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form $\psi \in (a_p)^K$, we put

$$\mathcal{F}_{a,\psi} = \{ f: G \to \mathbb{C} \mid f(g\man) = a^{-\psi}\sigma(m)^{-1}f(g), \quad g \in G, \quad \man \in MA_pN_m \}.$$  

Here $a^{-\psi} = \exp(-\psi(\log a))$ for $a \in A_p$. Define an inner product $(\ , \ )$ on $\mathcal{F}_{a,\psi}$ by

$$(f_1, f_2) = \int_K (f_1(k), f_2(k))_E dk$$

where $dk$ is the normalized Haar measure on $K$ and $(\ , \ )_E$ the $\sigma(M)$-invariant inner product on $E$. Let $H_{a,\psi} \supseteq \mathcal{F}_{a,\psi}$ denote the completion of the pre-Hilbert space $(\mathcal{F}_{a,\psi}, (\ , \ ))$. The group $G$ acts on $\mathcal{F}_{a,\psi}$ and so on $H_{a,\psi}$ by left translation. The resulting representation of $G$ on $H_{a,\psi}$ is called a principal series induced from $P_m$, and is denoted by $n^\infty = (o_\circ e \circ 0, n^\infty,)$.

Define an inner product $(\ , \ )$ on $g^\infty$, $g^\infty$ by

$$(f_1, f_2) = \int_K (f_1(k), f_2(k))_{N_m} dk$$

where $dk$ is the normalized Haar measure on $K$ and $(\ , \ )_{N_m}$ the $N_m$-invariant inner product on $E$. Let $H^0_{a,\psi}$ denote the completion of the pre-Hilbert space $(\mathcal{F}_{a,\psi}, (\ , \ ))$. The group $G$ acts on $\mathcal{F}_{a,\psi}$ and so on $H_{a,\psi}$ by left translation. The resulting representation of $G$ on $H_{a,\psi}$ is called a principal series induced from $P_m$, and is denoted by $n^\infty = (o_\circ e \circ 0, n^\infty,)$.

Let $H_{a,\psi}$ denote the space of $K$-finite vectors for $H_{a,\psi}$. Through differentiation, $H^0_{a,\psi}$ has a $(g_E, K)$-module structure. One gets the following irreducible decomposition of $H^0_{a,\psi}$ as a $K$-module:

$$(2.1) \quad H^0_{a,\psi} \simeq \bigoplus_{\tau \in \hat{K}} \Hom_M(\tau|M, \sigma) \otimes V_\tau,$$

where, for $\tau \in \hat{K}$, the unitary dual of $K$, we take an irreducible $K$-module $V_\tau$ of class $\tau$. Moreover $\Hom_M(\tau|M, \sigma)$ denotes the space of $M$-module homomorphisms from $\tau|M$ to $\sigma$, equipped with the trivial $K$-module structure. The isomorphism (2.1) is given as follows. For $T \in \Hom_M(\tau|M, \sigma)$ and $v \in V_\tau$, we define an element $f_{T, T, v} \in H^0_{a,\psi}$ by

$$(2.2) \quad f_{T, T, v}(kan) = T(\tau(k)^{-1}v)a^{-\psi} \quad \text{for} \quad (k, a, n) \in K \times A_p \times N_m.$$  

This assignment $T \otimes v \mapsto f_{T, T, v}$ induces the isomorphism (2.1).

### 2.2. Equations characterizing highest weight vectors.  

Let $\lambda$ be a $\Delta^+$-dominant, $K$-integral linear form on $t_E$. Denote by $(\tau_\lambda, V_\lambda)$ the irreducible $K$-module with highest weight $\lambda$. Through differentiation, we regard $V_\lambda$ as a $t_E$-module.

Let $\sigma \in \hat{M}$ and $\psi \in (a_p)^K$. Suppose $f \in H^0_{a,\psi}$ be a non-zero $\lambda$-highest weight vector for $\pi_{a,\psi}$ with respect to $\Delta^+ = \Delta^+_+ \cup \Delta^+_\sigma$. Necessarily $f$ lies in the $\tau_\lambda$-component of $H^0_{a,\psi}$, and is of the form

$$(2.3) \quad f = f_{T, T, v_\lambda} \quad \text{for some} \quad T \in \Hom_M(\tau_\lambda|M, \sigma), \quad \neq 0,$$

where $v_\lambda$ denotes a non-zero highest weight vector of $V_\lambda$. Further, $f$ should satisfy $\pi_{a,\psi}(X)f = 0$ for all $X \in p_+$. Since $\text{Ad}(K)p_+ = p_+$, this is equivalent to $(\pi_{a,\psi}(\text{Ad}(k)X)f)(kan) = 0$. Hence it follows that

$$(2.4) \quad T(\tau_\lambda([X])\tau_\lambda(k)^{-1}v_\lambda) + \psi(a[X])T(\tau_\lambda(k)^{-1}v_\lambda) = 0$$

for $X = [X] + a[X] + n[X] \in p_+$ and $k \in K$. Here we used the facts $f(an) = f(a)$ and $\text{Ad}(a) \cdot n[X] \in (n_m)_E$. 

Conversely, any \( f \in H_{\sigma, \psi}^0 \) satisfying (2.3) and (2.4) is actually a \( \lambda \)-highest weight vector.

In view of Lemma 1.1 and (2.2), we immediately get the following

**Lemma 2.1.** An element \( f = f_{v, \tau, v_0} \in H_{\sigma, \psi}^0 \) satisfies (2.4) if and only if

\[
T(\tau_\lambda(I_\mathbb{C})V_\lambda) = (0),
\]

\[
T(\tau_\lambda(H^\kappa_\lambda)v) + \psi(H_\kappa)T(v) = 0 \quad \text{for} \quad v \in V_\lambda \quad \text{and} \quad 1 \leq k \leq l.
\]

One thus obtains

**Proposition 2.2.** K-finite, \( \lambda \)-highest weight vectors \( f \) in \( H_{\sigma, \psi}^0 \) correspond bijectively to elements \( T \in \text{Hom}_M(\tau_\lambda | M, \sigma) \) satisfying (2.5) and (2.6). The correspondence is given by (2.3).

We determine in the following subsections 2.3 and 2.4 all \( (T, \sigma, \psi) \)'s satisfying (2.5) and (2.6).

**2.3. The condition (2.5).** Let \( E_\lambda \) be the \( \tilde{m}_\mathbb{C} \)-submodule of \( V_\lambda \) generated by the highest weight vector \( v_\lambda \). Then \( E_\lambda \) is irreducible. Notice that \( \tau_\lambda(I_\mathbb{C})V_\lambda \) in (2.5) is also an \( \tilde{m}_\mathbb{C} \)-submodule of \( V_\lambda \) because \([\tilde{m}_\mathbb{C}, I_\mathbb{C}] \subseteq I_\mathbb{C}\).

**Lemma 2.3.** One has a direct sum decomposition \( V_\lambda = E_\lambda \oplus \tau_\lambda(I_\mathbb{C})V_\lambda \) as \( \tilde{m}_\mathbb{C} \)-modules.

**Proof.** For a Lie algebra \( \mathfrak{x} \), let \( U(\mathfrak{x}) \) denote the enveloping algebra of \( \mathfrak{x} \). We extend a given \( \mathfrak{x} \)-module to a \( U(\mathfrak{x}) \)-module in the canonical way. By the Poincaré-Birkhoff-Witt theorem, we get from the triangular decomposition (1.3) of \( I_\mathbb{C} \) the following decomposition of \( U(I_\mathbb{C}) \):

\[
U(I_\mathbb{C}) = I_\mathbb{C}U(I_\mathbb{C}) + U(\tilde{m}_\mathbb{C}) + U(I_\mathbb{C})I_\mathbb{C}^+.
\]

This implies that

\[
V_\lambda = \tau_\lambda(U(I_\mathbb{C})v_\lambda) = \tau_\lambda(I_\mathbb{C})V_\lambda + E_\lambda,
\]

since \( E_\lambda = \tau_\lambda(U(\tilde{m}_\mathbb{C}))v_\lambda \) and \( \tau_\lambda(I_\mathbb{C})v_\lambda = (0) \).

On the other hand, \( \tau_\lambda(I_\mathbb{C})V_\lambda \cap E_\lambda \) is an \( \tilde{m}_\mathbb{C} \)-submodule of \( E_\lambda \). The irreducibility of \( E_\lambda \) implies either \( \tau_\lambda(I_\mathbb{C})V_\lambda \cap E_\lambda = (0) \) or \( E_\lambda \subseteq \tau_\lambda(I_\mathbb{C})V_\lambda \). But the latter case never happens because \( v_\lambda \in E_\lambda \) and \( v_\lambda \notin \tau_\lambda(I_\mathbb{C})V_\lambda \). This shows that the sum in (2.7) is direct. Q.E.D.

Notice that both \( E_\lambda \) and \( \tau_\lambda(I_\mathbb{C})V_\lambda \) are stable under \( \tau_\lambda(M) \). So, putting \( \sigma_\lambda(m) = \tau_\lambda(m)|E_\lambda \) \( (m \in M) \), we obtain an \( M \)-module \( (\sigma_\lambda, E_\lambda) \), which is irreducible because \( \tilde{m}_\mathbb{C} = m_\mathbb{C} + I_\mathbb{C} \). \([I_\mathbb{C}, \tilde{m}_\mathbb{C}] = (0)\). In view of Lemma 2.3, one deduces immediately

**Proposition 2.4.** Let \( \sigma \) be an irreducible \( M \)-module. Then there exists a non-zero element \( T \in \text{Hom}_M(\tau_\lambda | M, \sigma) \) satisfying (2.5) if and only if \( \sigma \cong \sigma_\lambda \). In this case, such an intertwining operator \( T \) is unique up to scalar multiples.
2.4. The condition (2.6). Let us examine (2.6) for $M$-module homomorphisms $T$ satisfying (2.5). By Proposition 2.4, we may and do assume that $T$ is the projection from $V_{\lambda}$ onto $E_{\lambda}$ along the direct sum decomposition in Lemma 2.3. Notice that $t_C$ acts on $E_{\lambda}$ by scalars: $t_C \in H \mapsto \lambda(H)$. This implies

$$T(\tau_\lambda(H_k)v) = \lambda(H_k)T(v) \quad \text{for} \quad v \in V_{\lambda} \quad \text{and} \quad 1 \leq k \leq l.$$  

We thus obtain the following

**Proposition 2.5.** Let $T \in \text{Hom}_M(\tau_\lambda | M, \sigma_\lambda)$ be the projection from $V_{\lambda}$ to $E_{\lambda}$ along $V_{\lambda} = E_{\lambda} \oplus \tau_\lambda(t_C)V_{\lambda}$. Then there exists a unique element $\psi = \psi_{\lambda}$ of $(a_p)^*_{\mathfrak{t}}$, satisfying (2.6). This $\psi_{\lambda}$ is given as

$$\psi_{\lambda}(H) = -\lambda(\mu(H)) \quad \text{for} \quad H \in a_p,$$

where $\mu$ is the Cayley transform in (1.2).

We have examined the conditions (2.5) and (2.6) completely.

2.5. Determination of highest weight vectors. Gathering Propositions 2.2, 2.4 and 2.5, we can now describe all the $K$-finite highest weight vectors in the principal series.

**Theorem 2.6.** Let $\lambda \in \mathfrak{t}^*_K$ be a $\delta^*_K$-dominant and $K$-integral, and $(\tau_\lambda, V_{\lambda})$ the irreducible $K$-module with highest weight $\lambda$. Denote by $(\sigma_\lambda, E_{\lambda})$ the $M$-submodule of $V_{\lambda}$ generated by the non-zero highest weight vector $v_{\lambda}$ of $V_{\lambda}$. Then, the principal series representation $\pi_{a, \psi}$ with $a \in \bar{M}$, $\psi \in (a_p)^*_{\mathfrak{t}}$, has a non-zero, $K$-finite $\lambda$-highest weight vector with respect to $\Lambda^+$ if and only if $\sigma \simeq \sigma_\lambda$ and $\psi = \psi_{\lambda} = (\lambda) \circ (\mu|a_p)$. In this case, such a highest weight vector $f$ is unique up to scalar multiples, and is given as

$$f(kan) = T(\tau_\lambda(k^{-1}v_{\lambda})a^{-\psi_{\lambda}}) \quad \text{for} \quad (k, a, n) \in K \times A_p \times N_m,$$

where $T: V_{\lambda} \rightarrow E_{\lambda}$ is the projection in Proposition 2.5.

This is the main result of this section.

§3. Embeddings into the principal series

We now give a complete description of embeddings of irreducible highest weight $(g_C, K)$-modules $L_{\lambda}$ into the principal series $H^0_{a, \psi}$.

3.1. Main result. By virtue of Theorem 2.6, we obtain the following theorem, which is the main result of this paper.

**Theorem 3.1.** Let $\lambda$ be a $\delta^*_K$-dominant, $K$-integral linear form on $t_C$. Then the irreducible admissible highest weight module $L_{\lambda}$ is embedded into one and only one principal series $(g_C, K)$-module $H^0_{a, \psi}$ with multiplicity one. Here $\sigma_\lambda$ is the irreducible $M$-module with highest weight $\lambda$ and we put $\psi_{\lambda} = (\lambda) \circ (\mu|a_p)$ through the Cayley transform $\mu$ in (1.2).
From this theorem, one gets in particular the unique embedding property for highest weight modules, due to Collingwood.

**Corollary 3.2** [2, Prop. 5.15]. Each irreducible highest weight \((\mathfrak{g}_C, K)\)-module is embedded into one and only one principal series with multiplicity one.

**Proof of Theorem 3.1.** Let \(H^0_{\sigma', \psi}(\lambda)\) denote the space of \(K\)-finite, \(\lambda\)-highest weight vectors for the representation \(\pi_{\sigma, \psi}\). To each embedding \(i: L_\lambda \hookrightarrow H^0_{\sigma', \psi}\) as \((\mathfrak{g}_C, K)\)-modules, we can associate a non-zero element \(i(w_\lambda) \in H^0_{\sigma', \psi}(\lambda)\), where \(w_\lambda\) is a fixed non-zero highest weight vector of \(L_\lambda\). This assignment gives an injective linear map:

\[
\text{Hom}_{\mathfrak{g}_C-K}(L_\lambda, H^0_{\sigma', \psi}) \hookrightarrow H^0_{\sigma', \psi}(\lambda) \quad \text{(as vector spaces)}.
\]

Theorem 2.6 on highest weight vectors in \(H^0_{\sigma', \psi}\), combined with (3.1), implies immediately that \(H^0_{\sigma', \psi}\) is the only possible candidate into which \(L_\lambda\) can be embedded and that the multiplicity of \(L_\lambda\) in \(H^0_{\sigma', \psi}\) as submodules is at most one. So, to complete the proof, it suffices to show that \(L_\lambda\) is actually embedded to \(H^0_{\sigma', \psi}\). We show this in the following two cases separately.

**CASE 1.** If \(L_\lambda\) corresponds to the holomorphic discrete series or its limit, the isomorphism (3.1) is surjective thanks to [6, Prop. 12.2]. Hence one deduces from Theorem 2.6,

\[
\text{Hom}_{\mathfrak{g}_C-K}(L_\lambda, H^0_{\sigma', \psi}) \simeq H^0_{\sigma', \psi}(\lambda) \neq (0).
\]

We thus obtain the theorem for this case.

**CASE 2.** For highest weight modules \(L_\lambda\) not necessarily in the (limit of) discrete series, we appeal to Casselman’s subrepresentation theorem [1, Th.8.21], which assures that \(L_\lambda\) can be embedded into at least one place of the principal series. This together with Theorem 2.6 and (3.1) yields our theorem for general \(L_\lambda\)’s.

The theorem is now completely proved. Q.E.D.

**3.2. Concluding remarks.** At last, we comment on our method and results.

**Remark 3.3.** In the proof of Theorem 3.1, **CASE 2** includes **CASE 1** as a special case. So one may say that the discussion in **CASE 1** is unnecessary for the proof. But we did not use the subrepresentation theorem in **CASE 1**, and therefore, we have obtained Theorem 3.1 for the holomorphic discrete series or its limit, without Casselman’s theorem.

**Remark 3.4.** (1) Let \(f_{\tau, \tau, v, s}\) with \(T \in \text{Hom}_{\mathfrak{m}}(\tau_{\lambda}, M, \sigma_{\lambda}), \neq 0\), be a \(\lambda\)-highest weight vector in \(H^0_{\sigma', \psi}\) (see (2.3)). Then the map

\[
i_\lambda: L_\lambda = U(\mathfrak{g}_C)w_\lambda \ni D \cdot w_\lambda \mapsto D \cdot f_{\tau, \tau, v, s} \in H^0_{\sigma, \psi}(\lambda) \quad (D \in U(\mathfrak{g}_C)),
\]

gives an embedding of \(L_\lambda\) into \(H^0_{\sigma, \psi}\). In this way, our theorem describes embeddings of highest weight modules completely.
(2) Let $i_\lambda(L_\lambda) \subseteq H_{\sigma, \psi}$ be the closure of the image of $i_\lambda$ in $H_{\sigma, \psi}$. Then $G$ acts on $\mu_\lambda(L_\lambda)$ irreducibly. Thus one finds that the $(\mathfrak{g}_C, K)$-module $L_\lambda$ admits a globalization $\mu_\lambda(L_\lambda)$, which is a $G$-submodule of $H_{\sigma, \psi}$.

Remark 3.5. Wallach [5] described Jacquet modules associated with $L_\lambda$'s by making use of the notion of “opposite parabolic”, and clarified the $(m + a_p)$-module structure of zero-th $n_m$-homology group $L_\lambda/n_mL_\lambda$. Combining the latter result with the Frobenius reciprocity (see e.g., [2, (1.14)]), one may deduce Theorem 3.1.

Nevertheless, our method of highest weight vectors is, we think, the most direct and the most elementary way to describe embeddings of highest weight modules. Furthermore, as shown in [6], this method is applicable also for embeddings into other types of important representations, called generalized Gelfand-Graev representations.

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References


