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On first variation of Green's functions under quasiconformal deformation

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Introduction

The purpose of this note is to give variational formulas for Green's functions on arbitrary Riemann surfaces under quasiconformal deformation, which contains known formulas such as those due to Sontag [7], Guerrero [3] and Maitani [5].

After some preliminary discussion on quasiconformal mappings in §1, we will prove the main formulas in §2 and §3 (Theorems 2 and 3).

§1. A surgery of quasiconformal mappings

Let U be the unit disk $\{|z|<1\}$ and $U'=\{|z|<r<1\}$. Then we can define a surgery of a given quasiconformal mapping f of U onto itself such that $f(0)=0$ as follows.

Let $\mu=\mu_f$ be the complex dilatation of f, and first decompose f as $f_2\circ f_1$ with quasiconformal mappings $f_1=f_1^{\mu}$ and $f_2=f_2^{\mu}$ of U onto itself such that the complex dilatations μ_{f_1} and μ_{f_2} are equal to $\mu|_{(U-U')}$ and $(\mu|_{U'}\cdot(f_1)_z/(\bar{f}_1)_{\bar{z}})\circ(f_1)^{-1}$, respectively, $f_1(0)=0$ (hence $f_2(0)=0$) and $f_2(1)=1$.

Next let H be the upper half plane in C, and set

$$\pi(z)=\exp(2\pi i\cdot z).$$

Then f_2 can be lifted to a quasiconformal mapping $F=F_2^{\mu}$ of H onto itself such that $F(0)=0$ and $\pi\circ F=f_2\circ\pi$. Since we can find a constant $r'<1$ depending only on r and a given $k\,(<1)$ such that $f_1(U')$ is contained in $\{|z|<r'\}$ whenever $\|\mu\|_\infty$ ($=\text{ess.}\sup_U|\mu|$) $\leqq k$, F is conformal on $\{z\in H: 0<y<c\}$ with $c=(-1/2\pi)\cdot\log r'$, where $z=x+iy$. Here we may also assume that $c<1$.

Now set

$$\begin{aligned} F^\circ(z)&=F(z) \qquad \text{on } \{0<y<c/3\},\\ &=\frac{y-(c/3)}{c/3}\cdot z+\frac{(2c/3)-y}{c/3}\cdot F(z) \qquad \text{on } \{c/3\leqq y\leqq 2c/3\},\\ &=z \qquad \text{on } \{2c/3<y\}. \end{aligned}$$

Then clearly $F^\circ(0)=0$ and $F^\circ(z+1)=F^\circ(z)+1$, hence F° can be projected to a self-mapping $S(f_2)$ of $\bar{U}$ fixing 0 and 1. And setting $S(f)=S(f_2)\circ f_1$, we have a reformation of f. Note that $S(f)$ is conformal on U'. Moreover we can show the following

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Theorem 1. i) *There are positive constants* k_0 *and* C_0 *depending only on* r' *such that, for every quasiconformal mapping* f *of* U *onto itself such that* $f(0)=0$ *and* $\|\mu_f\|_\infty \leqq k_0$, $S(f)$ *is quasiconformal and*

$$\|\mu_{S(f)}\|_\infty \leqq C_0 \cdot \|\mu_f\|_\infty . \tag{1}$$

ii) *Moreover, for every such* f *as in* i) *and every meromorphic function* h *on* U *which is holomorphic on* $U-\{0\}$ *and has at most simple pole at* 0,

$$\omega_{f,h} = d(f-S(f)) \wedge h dz \tag{2}$$

is absolutely integrable on U, *and satisfies that*

$$\iint_U \omega_{f,h} = 0 . \tag{3}$$

Remark. Above Theorem 1 is closely related to Ohtake's recent more general result [6. Theorem 1].

Proof of Theorem 1 is elementary and involves no use of the Teichmüller theory.

The second assertion ii) of Theorem 1 is related also to the locally trivial Beltrami differentials. See (8) in §2.

To prove Theorem 1, we need the following

Lemma 1. *Set* $E=\{c/3 \leqq y \leqq 2c/3,\ 0 \leqq x \leqq 1\}$ *and* $E'=\{c/6 \leqq y \leqq 5c/6,\ -1/2 \leqq x \leqq 3/2\}$. *Then there are constants* k_1 *and* C_1 *depending only on* c *such that*

$$|F(z)-z| \leqq C_1 \cdot \|\mu\|_\infty \quad \text{on } E' \text{ and} \tag{4}$$

$$|F'(z)-1| \leqq C_1 \cdot \|\mu\|_\infty \quad \text{on } E \tag{5}$$

for every $F=F_2^\mu$ *as above with* $\|\mu\|_\infty \leqq k_1$.

This lemma is a corollary of a basic result on quasiconformal mappings due to Ahlfors-Bers [2]. But we include a direct proof.

Proof. First extend F to a quasiconformal mapping $\hat{F}$ of $\boldsymbol{C}$ onto itself with the complex dilatation $\hat{\mu}$, the symmetric extension of μ, and consider $\tilde{F}(z)=1/\hat{F}(1/z)$, which fixes again 0, 1 and ∞ and has the complex dilatation $\tilde{\mu}(z)=\hat{\mu}(1/z)\cdot(z^2/\bar{z}^2)$. Since the support of $\tilde{\mu}$ is contained in $\{|z|<1/c\}$, there is a unique quasiconformal mapping $f^{\tilde{\mu}}(z)$ of $\boldsymbol{C}$ onto itself with the complex dilatation $\tilde{\mu}$ such that $f^{\tilde{\mu}}(0)=0$ and $(f^{\tilde{\mu}})_z-1$ belongs to $L^p(\boldsymbol{C})$ whenever $\|\mu\|_\infty$ is sufficiently small, where $p>2$ (cf. [1, 91p, Theorem 1]).

By the standard construction of $f^{\tilde{\mu}}$ (cf. [1, 92p]), we can find k_2 and C_2 depending only on c (and p) such that

$$|f^{\tilde{\mu}}(z)-z| \leqq C_2 \cdot \|\mu\|_\infty \quad \text{on } E''=\{|z|<6/c\} \tag{6}$$

for every μ with $\|\mu\|_\infty \leqq k_2$ (, also see [1, 86p-(3)]). Replacing k_2 by a smaller one if necessary, we may assume by (6) that $|f^{\tilde{\mu}}(1)| \geqq 1/2$ whenever $\|\mu\|_\infty \leqq k_2$. Since $\tilde{F}(z)=$

$f^{\bar\mu}(z)/f^{\bar\mu}(1)$, and hence

(7) $$\tilde{F}(z)-z=(f^{\bar\mu}(z)-z)/f^{\bar\mu}(1)-z\cdot(f^{\bar\mu}(1)-1)/f^{\bar\mu}(1),$$

we have

(8) $$|\tilde{F}(z)-z|\leqq 2(1+6/c)\cdot C_2\cdot\|\mu\|_\infty\equiv C_3\|\mu\|_\infty \quad \text{on } E''.$$

In particular, $|\tilde{F}(1/z)|\geqq 1/(5c/6+3/2)-C_3\cdot\|\mu\|_\infty$ on E', hence we can find desired constants for (4) by using (8). The second assertion (5) can be seen by using Cauchy's integral formula. q.e.d.

Proof of Theorem 1-i). Since $F=F_2^\mu$ is conformal on $\{0<y<c\}$, so does F° outside of $\{c/3\leqq y\leqq 2c/3\}$, where it holds that

(9) $$(F^\circ)_{\bar z}=(3/2ic)\cdot(F(z)-z), \quad \text{and}$$

(10) $$(F^\circ)_z=1+\frac{2c-3y}{c}(F'(z)-1)-\frac{3}{2ic}\cdot(F(z)-z).$$

Set $k_0=\min\left\{k_1, \left(\frac{1}{2+3/c}\right)/C_1\right\}$, where k_1 and C_1 are as in Lemma 1, then for every F with $\|\mu\|_\infty\leqq k_0$, we have

(11) $$|(F^\circ)_{\bar z}|\leqq\frac{3}{2c}\cdot C_1\cdot\|\mu\|_\infty<1/2, \quad \text{and}$$

(12) $$|(F^\circ)_z|\geqq 1-\left(1+\frac{3}{2c}\right)\cdot C_1\cdot\|\mu\|_\infty\geqq 1/2 \quad \text{on } E.$$

Hence it holds that

(13) $$|\mu^\circ|\equiv|(F^\circ)_{\bar z}/(F^\circ)_z|\leqq\frac{3C_1/2c}{1-(1+3/2c)\cdot C_1\cdot k_0}\cdot\|\mu\|_\infty$$
$$\leqq(3C_1/c)\cdot\|\mu\|_\infty<1 \quad \text{on } E,$$

which still holds on $\{c/3\leqq y\leqq 2c/3\}$, for $\mu^\circ(z+1)\equiv\mu^\circ(z)$.

Now we can see that F° is locally injective. In fact, the assertion is clear outside of $\{c/3\leqq y\leqq 2c/3\}$, and at any point in $\{c/3<y<2c/3\}$, F° is smooth and has a positive Jacobian by (11) and (12), hence is injective on some neighborhood of the point.

Next fix a point z_0 with $\operatorname{Im} z_0=2c/3$, and set

$$\eta(z)=\max\left\{0, \frac{(2c/3)-y}{c/3}\right\}\cdot(F(z)-z),$$

then $F^\circ(z)=z+\eta(z)$ near z_0. Noting that (5) in Lemma 1 holds on $\{c/3\leqq y\leqq 2c/3\}$ (, for $F'(z+1)\equiv F'(z)$), we can find a neighborhood W of z_0 such that

(14) $$|\eta(z_1)-\eta(z_2)|\leqq\frac{1}{2}\cdot|z_1-z_2|, \quad \text{and hence}$$

$$|F^\circ(z_1)-F^\circ(z_2)|\geqq|z_1-z_2|-|\eta(z_1)-\eta(z_2)|\geqq\frac{1}{2}\cdot|z_1-z_2|$$

for every z_1 and z_2 in W. Then F° is injective on W. And similarly we can find a neighborhood of any point z_0 with $\operatorname{Im} z_0=c/3$ on which F° is injective, and we can conclude that F° is locally injective on the whole H.

Thus by the monodromy theorem, F° is a homeomorphism of H onto itself. Finally by (13), F° is quasiconformal and we can find a C_0 as desired. q. e. d.

Proof of Theorem 1-ii). Let f and h be as in ii), and set $w=f-S(f)$. Then $w\equiv 0$ outside of a compact set of U. Also it is well-known that w has L^p-derivarives on U for any given $p>2$. And by assumption, $|h|^{p/(p-1)}$ is locally integrable on U. Hence by Hölder's inequality, we see that $\omega_{f,h}=dw\wedge h dz$ is absolutely integrable on U.

Take a suitable sequence $\{w_n\}_{n=1}^\infty$ of smooth functions on U approximating w in $L^\infty(U)$ so that the derivatives of w_n approximate those of w in $L^p(U)$ (, cf. for instance [4, III §6]), then by Green's formula we have

$$(15)\qquad \iint_U \omega_{f,h}=\lim_{\varepsilon\to 0}\iint_{U_\varepsilon}\omega_{f,h}=\lim_{\varepsilon\to 0}\Big(\lim_{n\to\infty}\iint_{U_\varepsilon} dw_n\wedge h dz\Big)$$

$$=\lim_{\varepsilon\to 0}\Big(\lim_{n\to\infty}\int_{\partial U_\varepsilon} w_n\cdot h dz\Big)=\lim_{\varepsilon\to 0}\int_{\partial U_\varepsilon} w\cdot h dz\,,$$

where $U_\varepsilon=\{\varepsilon<|z|<1-\varepsilon\}$ for every $\varepsilon>0$. Since $\lim_{z\to 0}w(z)=0$ and $zh(z)$ is bounded near $z=0$, we conclude the assertion. q. e. d.

Here as an application of Theorem 1, we prove the following Lemma 2 due to Ahlfors-Bers [2].

Lemma 2. *Let $p>2$ be given, then there are constants k_p and C_p depending only on p such that*

$$(16)\qquad \|f_z-1\|_p\leqq C_p\cdot\|\mu_f\|_\infty$$

for every quasiconformal mapping f of U onto itself, fixing 0 *and* 1, *with $\|\mu_f\|_\infty\leqq k_p$, where $\|\cdot\|_p$ is the L^p-norm on U.*

Proof. Set $\tilde{f}(z)=f(z)$ on U and $\tilde{f}(z)=1/\overline{S(f)(1/\bar{z})}$ on $\boldsymbol{C}-U$, where $S(f)$ is as above with $r=1/2$. Then by Theorem 1-i), there are universal constants k (<1) and C such that, for every f as in Lemma 2 with $\|\mu_f\|_\infty\leqq k$, $\tilde{f}$ is quasiconformal and $\|\mu_{\tilde{f}}\|_\infty\leqq C\cdot\|\mu_f\|_\infty$. Recall that $\tilde{f}$ fixes 0, 1 and ∞, and that $\tilde{\mu}=\mu_{\tilde{f}}$ has a compact support in $\boldsymbol{C}$. Let $f^{\tilde{\mu}}$ be as in the proof of Lemma 1, then since $\tilde{f}(z)=f^{\tilde{\mu}}(z)/f^{\tilde{\mu}}(1)$, we have

$$(17)\qquad \tilde{f}_z-1=((f^{\tilde{\mu}})_z-1)/f^{\tilde{\mu}}(1)-(f^{\tilde{\mu}}(1)-1)/f^{\tilde{\mu}}(1)\,.$$

Hence the assertion follows from (6) and the standard construction of $f^{\tilde{\mu}}$ (cf. [1, 92p. (7), (8)]). q. e. d.

§2. Generalized Sontag-Maitani's formulas

First, we generalize Sontag's formula [7, Proposition 4.1] and Maitani's one [5, Formula 3] as follows.

Theorem 2. *Let R_0 be a Riemann surface admitting Green's functions, and $\{f_t\}_{t\geq 0}$ be a one-parameter family of quasiconformal mappings f_t of R_0 onto another R_t with the complex dilatation μ_t.*

Let $g_t(\cdot, q)$ be Green's function on R_t with the pole $q\in R_t$, and set $\phi_t(q)=dg_t(\cdot, q)+i{*dg_t}(\cdot, q)$ for every t ($\geqq 0$).

I) *Assume that $k_t=\|\mu_t\|_\infty=O(t)$ as t tends to 0. Then*

$$g_t(f_t(p_1), f_t(p_2))-g_0(p_1, p_2)=\frac{1}{2\pi}\operatorname{Re}\iint_{R_0}\phi_0(p_1)\cdot\mu_t\wedge{*\phi_0}(p_2)+E(t^2) \tag{1}$$

as t tends to 0 for every mutually distinct point p_1 and p_2 on R_0, where $\limsup_{t\to 0}E(t^2)/t^2$ *is finite.*

II) *Further assume that*

$$\mu=\lim_{t\to 0}\mu_t/t \qquad \text{exsits in } L^\infty(R_0). \tag{*}$$

Then it holds that

$$g_t(f_t(p_1), f_t(p_2))-g_0(p_1, p_2)=\frac{t}{2\pi}\cdot\operatorname{Re}\iint_{R_0}\phi_0(p_1)\cdot\mu\wedge{*\phi_0}(p_2)+\varepsilon(t) \tag{2}$$

as t tends to 0 for every mutually distinct p_1 and p_2 on R_0, where $\lim_{t\to 0}\varepsilon(t)/t=0$.

III) *In* I), *the family $\{E(t^2)/t^2\}$ is locally uniformly bounded with respect to (p_1, p_2) on $R_0\times R_0-\Delta$, where $\Delta=\{(p, p): p\in R_0\}$.*

In II), *$\varepsilon(t)/t$ converges to 0 locally uniformly with respect to (p_1, p_2) on $R_0\times R_0-\Delta$ as t tends to 0.*

Proof of Theorem 2-I). Fix (p_1, p_2) and mutually disjoint simply connected neighhorhoods U_j of p_j ($j=1, 2$). Map U_j onto $\{|z|<2\}$ by a conformal mapping z_j so that $z_j(p_j)=0$, and set $U_j^r=(z_j)^{-1}(\{|z|<r\})$ for every positive r (<2) and j. Fix a positive $r<1$, and reform a quasiconformal mapping $h_t^j=z_{t,j}\circ f_t\circ(z_j)^{-1}$, of $U=z_j(U_j^1)$ onto itself fixing the origin, to $S(h_t^j)$ as in § 1 for every t and j, where $z_{t,j}$ maps $f_t(U_j^1)$ conformally onto U so that $z_{t,j}(f_t(p_j))=0$. And set

$$\begin{aligned} S(f_t)&=(z_{t,j})^{-1}\circ S(h_t^j)\circ z_j \qquad \text{on } U_j^1\ (j=1, 2), \text{ and}\\ &=f_t \qquad \text{on } R_0-(U_1^1\cup U_2^1). \end{aligned}$$

Then by Theorem 1-i), $S(f_t)$ is quasiconformal for every sufficiently small t, and $S(f_t)(p_j)=f_t(p_j)$ ($j=1, 2$). And for such a t, a standard argument (cf. for instance the proofs of [8, Lemmas 2 and 4]) shows that

$$g_t(f_t(p_1), f_t(p_2))-g_0(p_1, p_2)=\frac{1}{2\pi}\operatorname{Re}\iint_{R_0}\phi_t(f_t(p_1))\circ S(f_t)\wedge{*\phi_0}(p_2), \tag{3}$$

and

$$\|\phi_t(f_t(p_1))\circ S(f_t)-\phi_0(p_1)\|_{R_0}\leqq\frac{\sqrt{2}\tilde{k}_t}{1-\tilde{k}_t}\|\phi_0(p_1)\|_{R-V}, \tag{4}$$

where $\phi\circ f$ is the pull-back of ϕ by f, $\tilde{k}_t=\|\mu_{S(f_t)}\|_\infty$, $V=U_1^r\cup U_2^r$ and $\|\cdot\|_E$ is the Dirichlet norm on $E\subset R_0$. Hence we see by Theorem 1-i) and (4) that

$$(5)\qquad \left|\iint_{R_0}\phi_t(f_t(p_1))\circ S(f_t)\wedge{}^*\phi_0(p_2)-\iint_{R_0}\phi_0(p_1)\cdot\mu_{S(f_t)}\wedge{}^*\phi_0(p_2)\right|$$
$$\leqq\tilde{k}_t\cdot\|\phi_t(f_t(p_1))\circ S(f_t)-\phi_0(p_1)\|_{R_0}\cdot\|\phi_0(p_2)\|_{R_0-V}$$
$$=O(t^2)\qquad\text{as } t \text{ tends to } 0.$$

Next fix j and write $\phi_0(p_k)\circ(z_j)^{-1}=a_k^j(z)dz$ $(j,\ k=1,2)$, $\mu_t\circ(z_j)^{-1}=\mu_t^j(z)d\bar{z}/dz$ and $\mu_{S(f_t)}\circ(z_j)^{-1}=\tilde{\mu}_t^j(z)d\bar{z}/dz$ $(j=1,2)$. Then by Theorem 1-ii), we have

$$(6)\qquad \iint_U d(h_t^j-S(h_t^j))\wedge a_1^j(z)a_2^j(z)dz$$
$$=\iint_U(\mu_t^j(z)\cdot(h_t^j)_z-\tilde{\mu}_t^j(z)\cdot S(h_t^j)_z)d\bar{z}\wedge a_1^j(z)a_2^j(z)dz=0\,.$$

Hence by Theorem 1-i), Lemma 2 and Hölder's inequality, we can show that

$$(7)\qquad \left|\iint_U\mu_t^j(z)a_1^j(z)a_2^j(z)dz\wedge d\bar{z}-\iint_U\tilde{\mu}_t^j(z)a_1^j(z)a_2^j(z)dz\wedge d\bar{z}\right|$$
$$\leqq C_p(k_t^2+\tilde{k}_t^2)\|a_1^j\cdot a_2^j\|_{p/(p-1)}=O(t^2)\,,\qquad\text{hence}$$
$$(8)\qquad \left|\iint_{R_0}\phi_0(p_1)\cdot\mu_t\wedge{}^*\phi_0(p_2)-\iint_{R_0}\phi_0(p_1)\cdot\mu_{S(f_t)}\wedge{}^*\phi_0(p_2)\right|$$
$$=O(t^2)\qquad\text{as } t \text{ tends to } 0.$$

Thus the assertion follows by (3), (5) and (8). q.e.d.

Proof of Theorem 2-II). The assumption (*) implies that

$$(9)\qquad \left|\iint_{R_1}\phi_0(p_1)\cdot\mu_f\wedge{}^*\phi_0(p_2)-t\cdot\iint_{R_0}\phi_0(p_1)\cdot\mu\wedge{}^*\phi_0(p_2)\right|$$
$$\leqq\|\mu_t-t\mu\|_\infty\cdot\iint_{R_0}|\phi_0(p_1)\wedge\overline{\phi_0(p_2)}|=o(t)\,.$$

Hence the assertion follows by Theorem 2-I). q.e.d.

To prove Theorem 2-III), we need the following

Lemma 3. *For every positive $q<2$, there is a constant C_q depending only on q and r such that*

$$(10)\qquad \|a_1^j\cdot a_2^j\|_q\leqq C_q\cdot\|\phi_0(p_1)\|_{R_0-V}\cdot\|\phi_0(p_2)\|_{R_0-V}\quad(j=1,2)\,.$$

Proof. Fix j. Since $z\cdot a_1^j(z)\cdot a_2^j(z)$ is holomorphic on $\{|z|<2\}$, we have

$$(11)\qquad |z\cdot a_1^j(z)\cdot a_2^j(z)|\leqq\max_{(|s|=1)}|a_1^j(s)\cdot a_2^j(s)|$$
$$\leqq\max_{(|s|=1)}A_r\cdot\left(\iint_{\{|w-s|<1-r\}}|a_1^j(w)|^2dxdy\right)^{1/2}\left(\iint_{\{|w-s|<1-r\}}|a_2^j(w)|^2dxdy\right)^{1/2}$$
$$\leqq A_r\cdot\|\phi_0(p_1)\|_{R_0-V}\cdot\|\phi_0(p_2)\|_{R_0-V}\,,$$

where A_r is a constant depending only on r. Hence the assertion follows by a simple computation. q.e.d.

Proof of Theorem 2-III). We have shown actually in (4), (5) and (7) that $E(t^2)$ in I) is bounded by

$$t^2\cdot M\cdot(\|\phi_0(p_1)\|_{R_0-V}\cdot\|\phi_0(p_2)\|_{R_0-V}+\|a_1^j\cdot a_2^j\|_{p/(p-1)}) \tag{12}$$

with a constant M independent of p_1 and p_2. Using Lemma 3, we see that boundedness of $\|\phi_0(p_1)\|_{R_0-V}\cdot\|\phi_0(p_2)\|_{R_0-V}$ on some neighborhood of (p_1, p_2) in $R_0\times R_0-\varDelta$ implies the assertion for $E(t^2)$. This also implies the assertion for $\varepsilon(t)$ in II), by (9) and Lemma 3.

Now to clearify the dependence to p_1 and p_2, we write $U_j^r=U_{p_j}^r$ $(j=1, 2)$ and $V=V_{p_1,p_2}$. Then it is clear that there are neighborhood W_j of p_j $(j=1, 2)$ such that V_{q_1,q_2} contains $V'=U_{p_1}^{r/2}\cup U_{p_2}^{r/2}$ for every q_j in W_j $(j=1, 2)$ (, where we use the same U_j as the neighborhood of q_j). Since $\|\phi_0(q_j)\|_{R_0-V'}$ is bounded in a neighborhood of p_j (cf. [8, Theorem 1]), we conclude the desired boundedness, q.e.d.

Remark. We can use the same argument to loose the assumption in other formulas such as [9, Theorems 3 and 4].

§ 3. Generalized Sontag-Guerrero's formulas

By Theorem 2, it is easy to generalize Sontag-Guererro's formulas ([7] and [3, § 3]) as follows.

Theorem 3. *Let S_0 be a subsurface of a given Riemann surface R, and $\{f_t\}$ be a one-parameter family of quasiconformal mappings f_t of S_0 into R with the complex dilatation μ_t.*

Suppose that S_0 admits Green's functions. Let $g_t(\cdot, q)$ be Green's function on $S_t=f_t(S_0)$ with the pole $q\in S_t$, and set $\phi_t(q)=dg_t(\cdot, q)+i^*dg_t(\cdot, q)$ for every t $(\geqq 0)$.

I) *Assume that $\|\mu_t\|_\infty=O(t)$ as t tends to 0, and that*
(#) *$|z\circ f_t-z|/t$ is locally uniformly bounded on W for every local chart (W, z) (, i.e. every pair (W, z) of a simply connected subdomain W of S_0 and a conformal mapping z of W onto U).*
Then it holds that

$$g_t(p_1, p_2)-g_0(p_1, p_2)=\frac{1}{2\pi}\operatorname{Re}\iint_{S_0}\phi_0(p_1)\cdot\mu_t\wedge{}^*\phi_0(p_2) \tag{1}$$
$$-\operatorname{Re}[a_2^1(z_1(p_1))\cdot(z_1(f_t(p_1))-z_1(p_1))+a_1^2(z_2(p_2))\cdot(z_2(f_t(p_2))-z_2(p_2))]+E(t^2)$$

as t tends to 0 for every mutually distinct p_1 and p_2 on S_0, where (W_j, z_j) is a local chart such that $W_j\ni p_j$ for each j, and we set $\phi_0(p_k)\circ(z_j)^{-1}=a_k^j(z)dz$ on U $(=z_j(W_j))$ for each j and k.

Here $\{E(t^2)/t^2\}$ is locally uniformly bounded with respect to (p_1, p_2) on $S_0\times S_0-\varDelta$, $\varDelta=\{(p, p): p\in S_0\}$.

II) *Further assume that*
(*) *$\mu=\lim_{t\to 0}\mu_t/t$ exists in $L^\infty(S_0)$, and that*
(#′) *$(z\circ f_t-z)/t$ converges to some (continuous) function f_W locally uniformly on W for every local chart (W, z) of S_0. Then*

(2) $$g_t(p_1, p_2)-g_0(p_1, p_2)=\frac{t}{2\pi}\operatorname{Re}\iint_{S_0}\phi_0(p_1)\cdot\mu\wedge{}^*\phi_0(p_2)$$
$$-t\cdot\operatorname{Re}(a_2^1(z_1(p_1))\cdot f_{w_1}(p_1)+a_1^2(z_2(p_2))\cdot f_{w_2}(p_2))+\varepsilon(t)$$

as t tends to 0 for every mutually distincts p_1 and p_2 on S_0.

Here $\varepsilon(t)/t$ converges to 0 locally uniformly with respect to (p_1, p_2) on $S_0\times S_0-\Delta$.

Remark. Since

(3) $$\operatorname{Re} a_{3-j}^j(z_j(p_j))\cdot f_{w_j}(p_j)=\frac{\partial}{\partial t}g_0(f_t(p_j), p_{3-j})|_{t=0}$$

for each j, every term of (2) does not depend on the choice of local charts.

To prove Theorem 3, we first note some consequences of Theorem 2-I). Since $\|\mu_t\|_\infty=O(t)$, we have

(4) $$|g_t(f_t(p_1), f_t(p_2))-g_0(p_1, p_2)|$$
$$\leqq\|\mu_t\|_\infty\cdot\left|\iint_{S_0}\phi_0(p_1)\wedge\overline{\phi_0(p_2)}\right|+O(t^2)\equiv E'(t).$$

And similarly as in the proof of Theorem 2-III) we can see that $E'(t)/t$ is locally uniformly bounded on $S_0\times S_0-\Delta$ for $E'(t)$ in (4).

Let $\{U_j\}_{j=1}^2$ and $\{z_j\}_{j=1}^2$ be as in § 2, and set

$$G_t(z, \zeta)=g_t(z_1^{-1}(z), z_2^{-1}(\zeta)), \quad \text{and}$$
$$f_{t,j}(z)=z_j\circ f_t\circ(z_j)^{-1}(z) \quad \text{for each } j.$$

Then by the assumption (#), we may assume, without loss of generality, that $f_{t,j}(z)$ is well-defined on, say $U^0=\{|z|<3/2\}$ and that $f_{t,j}(U^0)$ contains $\bar{U}$ $(=\{|z|\leqq 1\})$ for every t and j. Then (4) implies that there is a constant M_1 such that

(5) $$|G_t(f_{t,1}(z), f_{t,2}(\zeta))-G_0(z, \zeta)|\leqq M_1\cdot t$$

for every t and every z, ζ in U^0.

In particular, $\{G_t(z, \zeta)\}_{\zeta\in U, t\geq 0}$ is a family of uniformly bounded harmonic functions of $z\in U$. Hence by Poisson's formula, there is a constant M_2 such that

(6) $$|G_t(z, \zeta)-G_t(z', \zeta)|\leqq M_2\cdot|z-z'|$$

for every t and every z, z', ζ in $U'=\{|z|<1/2\}$.

Here by the assumytion (#), we may assume, without loss of generality, that $f_{t,j}(z)\in U'$ for every t, j and $z\in U''=\{|z|<1/4\}$. Hence by (5), (6) and (#), we can conclude that

(7) $$|G_t(z, \zeta)-G_0(z, \zeta)|$$
$$\leqq|G_t(z, \zeta)-G_t(z, f_{t,2}(\zeta))|+|G_t(z, f_{t,2}(\zeta))-G_t(f_{t,1}(z), f_{t,2}(\zeta)|$$
$$+|G_t(f_{t,1}(z), f_{t,2}(\zeta))-G_0(z, \zeta)|\leqq M_3\cdot t$$

for every t and z, ζ in U'' with some constant M_3.

Again by Poisson's formula, (7) implies that there is a constant M_4 such that

$$|(G_t)_z(z, \zeta)-(G_0)_z(z, \zeta)| \leqq M_4 \cdot t \tag{8}$$

for every t and z, ζ in $U^* = \{|z|<1/8\}$.

Proof of Theorem 3. Under the assumptions in Theorem 3-I), we have by (8) and the assumption (#)

$$\begin{aligned}
&g_t(f_t(q_1), f_t(q_2))-g_t(q_1, q_2) \\
&=(G_t(f_{t,1}(z_1^0), f_{t,2}(z_2^0))-G_t(z_1^0, f_{t,2}(z_2^0)))+(G_t(z_1^0, f_{t,2}(z_2^0))-G_t(z_1^0, z_2^0)) \\
&=2\cdot \mathrm{Re}\left[\int_{z_1^0}^{f_{t,1}(z_1^0)}(G_t)_z(z, f_{t,2}(z_2^0))dz+\int_{z_2^0}^{f_{t,2}(z_2^0)}(G_t)_z(z, z_1^0)dz\right] \\
&=2\cdot \mathrm{Re}\left[\int_{z_1^0}^{f_{t,1}(z_1^0)}(G_0)_z(z, f_{t,2}(z_2^0))dz+\int_{z_2^0}^{f_{t,2}(z_2^0)}(G_0)_z(z, z_1^0)dz\right]+O(t^2)
\end{aligned} \tag{9}$$

for every (q_1, q_2) in a suitable neighborhood of (p_1, p_2), where we set $z_j^0=z_j(q_j)$ $(j=1,2)$ and the paths in integration are the segments between z_j^0 and $f_{t,j}(z_j^0)$.

Here since $\{G_0(z, \zeta)\}_{\zeta\in U}$ is a family of uniformly bounded harmonic functions of $z\in U$, we can show as before that

$$|(G_0)_z(z, \zeta)-(G_0)_z(z', \zeta')| \leqq M_5\cdot(|z-z'|+|\zeta-\zeta'|) \tag{10}$$

for every z, z', ζ and ζ' in U'' with some constant M_5. Hence by (9) and the assumption (#), we conclude that

$$\begin{aligned}
&g_t(f_t(q_1), f_t(q_2))-g_t(q_1, q_2) \\
&\quad =2\cdot \mathrm{Re}\,((G_0)_z(z_1^0, z_2^0)\cdot(f_{t,1}(z_1^0)-z_1^0)+(G_0)_z(z_2^0, z_1^0)\cdot(f_{t,2}(z_2^0)-z_2^0)) \\
&\qquad +O(t^2) \qquad \text{as } t \text{ tends to } 0.
\end{aligned} \tag{11}$$

Also the above argument shows that $\{E(t^2)/t^2\}$ is locally uniformly bounded on $S_0\times S_0-\Delta$ for $E(t^2)$ in (1). And since $a_k^j(z)=2\cdot(G_0)_z(z, z_k^0)$ on $U=z_j(U_j^1)$, we conclude the assertion I).

The assertion II) follows by Theorem 3-I) and the assumptions. q.e.d.

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