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Some Numerical invariants of hyperelliptic fibrations

By

Shigeru MATSUSAKA

0. Introduction

In this paper we shall define certain numerical invariants of hyperelliptic fibrations and study their properties. A proper surjective morphism Π: $X \to C$ is called a *hyperelliptic fibration of genus* g if X is a smooth surface, C is a smooth curve and the general fiber of Π is a hyperelliptic curve of genus g.

Hyperelliptic fibrations of genus 2 have been studied by several mathematicians. Ueno [U] and Xiao [X] proved that the *topological index* $i_{top}(x) = \frac{1}{3}(c_1(X)^2 - 2c_2(X))$ is non-positive if X has a hyperelliptic fibration of genus 2 over a smooth compact curve.

In this paper we will show the following inequalities for every relatively minimal hyperelliptic fibration Π: $X \to C$ over a smooth compact curve:

$$\frac{-g-1}{2g+1}\cdot\sum_{t\in C} e_t(X) \leq i_{top}(X) \leq \frac{g^2-2g-1}{2g+1}\cdot\sum_{t\in C} e_t(X) \qquad \cdots(0.1.1)$$

if g *is even*,

$$\frac{-g-1}{2g+1}\cdot\sum_{t\in C} e_t(X) \leq i_{top}(X) \leq \frac{g^2-2g}{2g+1}\cdot\sum_{t\in C} e_t(X) \qquad \cdots(0.1.2)$$

if g *is odd.*

where $e_t(X) = $ (*Euler number of* $\Pi^{-1}(t)$) $- (2-2g)$. (See Theorem 4.0.1 below).

To prove these inequalities, we need a section D of $(\overset{g}{\Lambda}\Pi_*\omega_{X/C})^{\otimes 4(2g+1)}$.

For every hyperelliptic fibration Π: $X \to C$, there exist a $\mathbf{P}^1$-bundle p: $Y \to C$ and a double covering $\hat{\Pi}$: $\hat{X} \to Y$ such that $\hat{X}$ is birational to X over C. There exists an open set C^0 in C which satisfies the following:

i) $p^{-1}(C^0)$ is isomorphic to $\mathbf{P}^1 \times C^0$;

ii) $\Pi^{-1}(C^0)$ can be identified with the closure of $\{(x, t, y) \in \mathbf{C} \times C \times \mathbf{C};\ y^2 = \varphi(x, t)\}$, where x is an inhomogeneous coodinate of $\mathbf{P}^1$ and φ is a polynomial of x of degree $2g+1$ or $2g+2$ with coefficients in the rational function field of C^0.

The section D of $(\overset{g}{\Lambda}\Pi_*\omega_{X/C})^{\otimes 4(2g+1)}$ is defined to be

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$$D = \triangle(\varphi)^g \cdot \left(\frac{dx}{y} \Lambda\, x\frac{dx}{y} \Lambda \cdots \Lambda\, x^{g-1}\frac{dx}{y} \right)^{\otimes 4(2g+1)}$$
$$\in \Gamma_{rat}(C^0, (\overset{g}{\Lambda}\, \Pi_* \omega_{X/C})^{\otimes 4(2g+1)})$$
$$= \Gamma_{rat}(C, (\overset{g}{\Lambda}\, \Pi_* \omega_{X/C})^{\otimes 4(2g+1)})$$

where $\triangle(\varphi)$ is the discriminant of φ as a polynomial of x (cf. [U]). The definition of D is independent of the choice of Y, C^0, x, y and φ (see Proposition 3.1.2 below). Moreover we shall show that D is a regular section of $(\overset{g}{\Lambda}\, \Pi_* \omega_{X/C})^{\otimes 4(2g+1)}$ on C (see Corollary 4.0.8 below).

The numerical invariant $d_t(X)$ of a hyperelliptic fibratinon Π: $X \to C$ is defined to be

$$d_t(X) = \frac{1}{4(2g+1)} ord_t\, D.$$

Then we have the following:

Theorem 4.0.4. *Let Π: $X \to C$ be a relatively minimal hyperelliptic fibration of genus g. For every closed point t in C, we have*

$$\frac{g}{4(2g+1)} e_t(X) \leq d_t(X) \leq \frac{g^2}{4(2g+1)} e_t(X) \qquad \cdots (0.1.3)$$

if g is even,

$$\frac{g}{4(2g+1)} e_t(X) \leq d_t(X) \leq \frac{g^2+1}{4(2g+1)} e_t(X) \qquad \cdots (0.1.4)$$

if g is odd.

To prove the Main Theorem we shall use the theory of cananical resolutions of double coverings studied by Horikawa and other mathematicians (see [H1], for example).

In chapter 1 we shall summerize the definitions and fundamental facts concerning with canonical resolutuions and hyperelliptic fibration. In (1.3), we shall prove that, for every hyperelliptic fibration Π: $X \to C$, there is a double covering $\hat{X}$ over a $\mathbf{P}^1$-bundle Y such that the covering space $\hat{X}$ is birational to X. Using this double covering, we can express the invariants $d_t(X)$ and $e_t(X)$ in the language of canonical resolutuions (see Chapter 3 below). And this expression and some calculation in Chapter 4 lead to the inequalities (0.1.3) and (0.1.4). In the course of the calculation in Chapter 4, the double covering $\hat{X}$ needs to satisfy some conditions (see Proposition 2.0.1). We devote Chapter 2 to construct such a double covering. In the construction of such a double covering, the auther is inspired by Debarre's paper [D] and Tokunaga's suggestions.

He would like to express his thanks to Hirô Tokunags for his useful suggestions. He would also like to express his thanks to Professor Kenji Ueno and Kazuhiko Kurano for many stimulating discussions.

Notation and Convention. All algebraic schemes are defined over the complex number field $\mathbf{C}$. A *surface* is an algebraic scheme of dimension 2. A *curve* is an algebraic scheme of dimension 1.

A *blowing-up* means a blowing-up at a point if a center of the blowing-up is not mentioned. While f: $Y' \to Y$ is called a *blowing-down* if f^{-1} is a blowing-up of Y.

In the following we shall use the following notation freely.

$\mathscr{F}^{\vee}$	:	dual sheaf of $\mathscr{F}$;
$[\mathscr{F}]$	:	spectrum of the symmetric algebra on the dual sheaf of a locally free sheaf $\mathscr{F}$;
$f^{-1}[B]$	:	union of all the irreducible divisors on a smooth surface X whose images of a proper surjective morphism f: $X \to Y$ are components of a reduced effective divisor B on a smooth surface Y.

For simplicity, we call $f^{-1}[B]$ the *proper transform* of B by f^{-1}. (This coincides with the usual notation when f is a proper surjective birational morphism.)

A curve C on a smooth surface is called a $(-i)$-*curve* if C is isomorphic to $\mathbf{P}^1$ and $C^2 = -i$.

Let Π: $X \to C$ be a proper surjective morphism from a smooth surface X to a smooth curve C. The surface X is called *relatively minimal* if each fiber of Π: $X \to Y$ contains no (-1)-curves.

Let Π: $X \to C$ be a surjective morphism from a smooth surface X to a smooth curve C and let $B = \tilde{B} + V$ be a reduced effective divisor on X where $\tilde{B} \geq 0$ and $V \geq 0$, $\Pi | \tilde{B}$ is finite and V is contained in fibers of Π. Then $\tilde{B}$ is called a *horizontal part* of B and V is called a *vertical part* of B.

1. Preliminary

In this chapter we shall state definitions and fundamental facts concerning with double coverings, canonical resolutions and hyperelliptic fibrations.

(1.1) Let X be a normal surface and Y a smooth surface. The morphism f: $X \to Y$ is called a double covering if f is a finite surjective morphism of degree 2. By purity of branch loci, the branch locus B of f is a divisor on Y.

For every line bundle L and a reduced effective divisor B on a smooth surface Y satisfying $\mathcal{O}_Y(L)^{\otimes 2} \simeq \mathcal{O}_Y(B)$, we obtain a double covering $\operatorname{Spec}(\mathcal{O}_Y \oplus \mathcal{O}_Y(L)^{\vee})$ over Y whose branch locus is B. In this paper this double covering is denoted by $X(Y, L, B)$.

Conversely, for every double covering f: $X \to Y$, there exist a line bundle L and a reduced effective divisor B on Y such that $\mathcal{O}_Y(L)^{\otimes 2} \simeq \mathcal{O}_Y(B)$ and $X \simeq X(Y, L, B)$ (cf. [T] or [12, *Th.* 2. 24]).

Let g': $Y' \to Y$ be a proper surjective birational morphism between smooth surfaces, B a reduced effective divisor on Y and L a line bundle on Y such that $\mathcal{O}_Y(L)^{\otimes 2} \simeq \mathcal{O}_Y(B)$. We define integers a_i's by $g'^*(B) = g'^{-1}[B] + \sum_{i=1}^{n} a_i E_i$ where E_i's are the exceptional curves of g'.

Definition 1.1.1. Under the above notation, a divisor $B_{Y'}$ and a line bundle $L_{Y'}$ on Y' are defined as

$$B_{Y'} = g'^*B - 2\sum_{i=1}^{n}\left[\frac{a_i}{2}\right]E_i,$$

$$\mathcal{O}_{Y'}(L_{Y'}) = g'^*\mathcal{O}_Y(L) \otimes \mathcal{O}_{Y'}\left(-\sum_{i=1}^{n}\left[\frac{a_i}{2}\right]E_i\right),$$

where $\left[\frac{a_i}{2}\right]$ is an greatest integer not exceeding $\frac{a_i}{2}$.

Note that the divisor $B_{Y'}$ is reduced and effecive, $\mathcal{O}_{Y'}(B_{Y'})$ is isomorphic to $\mathcal{O}_{Y'}(L_{Y'})^{\otimes 2}$ and $X(Y', L_{Y'}, B_{Y'})$ is the normalization of $X(Y, L, B) \times_Y Y'$ (see [H1]). Moreover, for every proper surjective birational morphisms $g'\colon Y' \to Y$ and $g''\colon Y'' \to Y'$ between smooth surfaces we have $B_{Y''} = (B_{Y'})_{Y''}$ and $L_{Y''} = (L_{Y'})_{Y''}$.

The following fact is well-known.

Proposition 1.1.2 [H1, Chap. 2]. *Let X be a double covering of a smooth surface Y. Then X is smooth if and only if the branch locus of $X \to Y$ is smooth.*

Proposition 1.1.3 (Canonical resolution and its universality).

Let Y be a smooth surface, $\hat{X}$ a double covering over Y, L a line bundle on Y and B a reduced effective divisor on Y such that $\mathcal{O}_Y(L)^{\otimes 2} \simeq \mathcal{O}_Y(B)$ and $\hat{X} \simeq X(Y, L, B)$. Then there exist a unique smooth surface Y_{CR} and a proper surjective birational morphism $g_{CR}\colon Y_{CR} \to Y$ which satisfy that:

i) *the double covering $X_{CR} = X(Y_{CR}, L_{Y_{CR}}, B_{Y_{CR}})$ over Y_{CR} is smooth;*

ii) *if $g'\colon Y' \to Y$ is a proper surjective birational morphism between smooth surfaces and the divisor $B_{Y'}$ is smooth, then there is a unique morphism $g''\colon Y' \to Y_{CR}$ such that $g' = g_{CR} \circ g''$.*

Proof. Let i be an involution on $X(Y, L, B)$ such that $X(Y, L, B)/\langle i \rangle \simeq Y$ and $h_{res}\colon X_{res} \to X(Y, L, B)$ the minimal resolution of $X(Y, L, B)$. There exists a unique involution i' on X_{res} such that $h_{res} \circ i' = i \circ h_{res}$. We define Y_{CR} to be the minimal resolution of $X_{res}/\langle i' \rangle$. Then there exist proper birational surjective morphisms $g_{CR}\colon Y_{CR} \to Y$ and $h\colon X(Y_{CR}, L_{Y_{CR}}, B_{Y_{CR}}) \to X_{res}$. Since the singularities of $X_{res}/\langle i' \rangle$ are of A_1-type, the rational map $h^{-1}\colon X_{res} \cdots \to X(Y_{CR}, L_{Y_{CR}}, B_{Y_{CR}})$ is a blowing-up at isolated fixed points of i'.

If $g'\colon Y' \to Y$ is a proper surjective birational morphism between smooth surfaces and the divisor $B_{Y'}$ is smooth, then $X(Y', L_{Y'}, B_{Y'})$ is smooth and there exists a unique morphism $h'\colon X(Y', L_{Y'}, B_{Y'}) \to X_{res}$. Let i'' be an involution on $X(Y', L_{Y'}, B_{Y'})$ such that $Y' \simeq X(Y', L_{Y'}, B_{Y'})/\langle i'' \rangle$. Then we have $i' \circ h' = h' \circ i''$. Since Y' is smooth, the involution i'' on $X(Y', L_{Y'}, B_{Y'})$ does not have any isolated fixed points. Since h^{-1} is a blowing-up at the isolated fixed points of the involution i' on X_{res}, universality of blowing-ups induces the proper surjective birational morphism $h''\colon X(Y', L_{Y'}, B_{Y'}) \to X(Y_{CR}, L_{Y_{CR}}, B_{Y_{CR}})$ such that $h' = h \circ h''$. Therefore we have a desired morphism $g''\colon Y' \to Y_{CR}$ (see Proposition 1.1.2). $\square$

In the following we shall often use X_{CR} instead of $X(Y_{CR}, L_{Y_{CR}}, B_{Y_{CR}})$. The resolution $X_{CR} \to X(Y, L, B)$ is called a canonical resolution of a double covering $X(Y, L, B)$ over Y(see [H1]).

Remark 1.1.4. Let (Y, L, B) be a triple such that $\mathcal{O}_Y(L)^{\otimes 2} \simeq \mathcal{O}_Y(B)$. Put $Y_0 = Y$ and let Y_i be a blowing-up of Y_{i-1} at a singular point of $B_{Y_{i-1}}$. If B_{Y_n} is non-singular and for every $i \leq n-1$ B_{Y_i} has some sinularities, then Y_n is isomorphic to Y_{CR}(see [H1]).

(1.2) Let Y and Y' be smooth surfaces and $g'\colon Y' \to Y$ a proper surjective birational morphism. In the section (1.1) we constructed a double covering over Y' from a double covering over Y. In this section we will make a double covering over Y from that over Y' and prove that this process is the converse process of (1.1).

Let L' be a line bundle on Y' and B' a reduced effective divisor on Y' such that $\mathcal{O}_{Y'}(L')^{\otimes 2} \simeq \mathcal{O}_{Y'}(B')$. Then $L = g'_*L'$ is a line bundle on Y and $B = g'_*B'$ is a effective reduced divisor on Y such that $\mathcal{O}_Y(L)^{\otimes 2} \simeq \mathcal{O}_Y(B)$. Hence we obtain a double covering $X(Y, L, B)$ over Y from a double covering $X(Y', L', B')$ over Y'.

Lemma 1.2.1. *Under the above notation, L' and $L_{Y'}$, B' and $B_{Y'}$ are both isomorphic. That is, we can identify $X(Y', L', B')$ with $X(Y', L_{Y'}, B_{Y'})$.*

Proof. On a dence open set, the morphism $g'\colon Y' \to Y$ is an isomorphism and L' (or B') is isomorphic to L(or B, resp.). Hence there is a natural birational map $h'\colon X' \cdots \to X$. Put $X = X(Y, L, B)$, $X' = X(Y', L', B')$ and $X'' = X(Y', L_{Y'}, B_{Y'})$. Moreover, let $\varphi''\colon X'' \to Y'$, $h''\colon X'' \to X$ and $\varphi'\colon X' \to Y'$ be natural morphisms.

Then the homomorphism $(\varphi'')^{\#}\colon R(Y') \to R(X'')$ between rational function fields can be identified with $(\varphi')^{\#}\colon R(Y') \to R(X')$ under the identification $(h'^{-1} \circ h'')^{\#}\colon R(X'') \tilde{\to} R(X')$. Therefore the universality of normalization implies that we can identify X' with X''. Under this identification, the equalities $\varphi' = \varphi''$ and $h' = h''$ holds. Consequently, the branch locus B' of $X' \to Y'$ coincides with the branch locus $B_{Y'}$ of $X'' \to Y'$. Since $\mathcal{O}_{Y'}(L_{Y'}) \otimes \mathcal{O}_{Y'}(\sum_i a_i \cdot E_i)$ (E_i's are curves contracted by g'), the equality $B' = B_{Y'}$ induces $L' = L_{Y'}$.

(1.3)

Definition 1.3.1. For every hyperelliptic fibration $\Pi\colon X \to C$, a triple (Y, L, B) is called a triple associated with Π if $p\colon Y \to C$ is a $\mathbf{P}^1$-bundle over C, L is a line bundle on Y and B is a reduced effective divisor on Y such that $X(Y, L, B)$ is birational to X.

Lemma 1.3.2. *For every hyperelliptic fibration $\Pi\colon X \to C$, there is a triple (Y, L, B) associated with Π.*

Proof. If $\tilde{f}\colon X \cdots \to \mathrm{Proj}(\Pi_*\omega_{X/C})$ is a rational map associated with the relative linear system over C of the relative dualizing sheaf $\omega_{X/C}$ of Π, then the general fiber over $\mathrm{Im}(\tilde{f})$ is $\mathbf{P}^1$ and $\tilde{f}$: $X \cdots \rightarrow \mathrm{Im}(\tilde{f})$ is of degree 2. Therefore, there exists a donimant rational map $f{:}X \cdots \rightarrow Y$ of degree 2 from X to a $\mathbf{P}^1$-bundle Y over C. If $\hat{X}$ denotes the normalization of Y in the rational function field $R(X)$, $\hat{X}$ is the double covering over Y which is birational to X. By the argument of (1.1), there exist a line bundle L on Y and a reduced effective divisor B on Y such that $X(Y, L, B)$ is birational to X. □

Since, for every hyperelliptic fibration, the fibers do not contain any (-1)-curves which intersect other (-1)-curves, we obtain the following lemma.

Lemma 1.3.3. *Let Π: $X \rightarrow C$ and Π': $X' \rightarrow C$ be hyperelliptic fibrations such that there is a birational map f': $X' \cdots \rightarrow X$ over C. If Π is relatively minimal, f' can be extended to a unique regular morphism.*

2. Triples associated with relatively minimal hyperelliptic fibrations

(2.0) The purpose of this chapter is to prove the following proposition.

Proposition 2.0.1. *If Π: $X \rightarrow C$ is a relatively minimal hyperelliptic fibration, then there is a triple (Y, L, B) associated with Π satisfying the following conditions.*

Let $X_{CR} = X(Y_{CR}, L_{Y_{CR}}, B_{Y_{CR}})$ be the canonical resolution of the double covering $X(Y, L, B)$ over Y, X_{res} the minimal resolution of $X(Y, L, B)$, and let p, p_{CR}, Π_{CR}, Π_{res}, h, h_{res}, f_{CR}, f_{res}, and g_{CR} in the following diagram 2.0.2 *be natural morphisms (cf.* Proposition 1.1.3 *and* Lemma 1.3.3). *Then the surfaces and morphisms satisfy that:*

(A) *the horizontal part $\tilde{B}$ of B has multipicity $\leq g+1$ at every point in $\tilde{B}$;*

(B) *every curve $F' \subset X_{CR}$ contructed by h: $X_{CR} \rightarrow X$ is a (-1)-curve on X_{CR} and the image of the curve F' to Y_{CR} is a (-2)-curve contained in $B_{Y_{CR}}$;*

(C) *if $\Pi_{res}^{-1}(t)$ contains a curve F contructed by h_{res}, then following conditons are satisfied:*

(C1) *the curve F is the proper transform of $p^{-1}(t)$ to X_{res};*

(C2) *$\Pi^{-1}(t)$ is a fiber of multiplicity 2, that is, if $\Pi^{-1}(t) = \sum_j a_j \cdot F_j$($F_j$'s are irreducible components of $\Pi^{-1}(t)$),* all *a_j's are even numbers.*

$$\begin{tikzcd}
X_{CR} \arrow[r, "h_{CR}"] \arrow[d, "f_{CR}"'] \arrow[ddr, dash, "\Pi_{CR}"] \arrow[rr, bend left, "h"] & X_{res} \arrow[r, "h_{res}"] \arrow[d, "f_{res}"] \arrow[dd, bend left, "\Pi_{res}"] & X \arrow[ddl, "\Pi"] \\
Y_{CR} \arrow[r, "g_{CR}"] \arrow[dr, "p_{CR}"'] & Y \arrow[d, "p"] & \\
& C &
\end{tikzcd}$$

Diagram 2.0.2.

In the following sections of this chapter we are devoted to construct a triple (Y, L, B) satisfying above conditions.

(2.1) Let $\Pi: X \to C$ be a relatively minimal hyperelliptic fibration. Since every hyperelliptic fibration has a triple associated with it (see Lemma 1.3.2), there is a birational map $i: X \cdots \to X$ such that $i \circ i = id$. By the minimality of Π and Lemma 1.3.3 we may regard the rational map i as a morphism.

Let $\hat{Y}$ be a quotient space of X by a group generated by this involution i and Y_{CR} the minimal resolution of $\hat{Y}$. The normal surface $\hat{Y}$ has singularities of A^1-type at the image of isolated fixed points of i. Let X_{CR} be a blowing-up at isolated fixed points of i. By the universality of blowing-ups, there is a morphism $f_{CR}: X_{CR} \to Y_{CR}$. Since f_{CR} is a quasi-finite projective morphism of degree 2, it is finite of degree 2. That is, X_{CR} is a double covering of Y_{CR}. Let L_{CR} be a line bundle on Y_{CR} and B_{CR} a reduced effective divisor on Y_{CR} such that $X_{CR} = X(Y_{CR}, L_{CR}, B_{CR})$ (see (1.1)). Note that, since X_{CR} is smooth, B_{CR} is smooth (see Proposition 1.1.2).

For convenience we summarize the properties of X_{CR} and Y_{CR}.

Lemma 2.1.1.

(i) *$h: X_{CR} \to X$ is a contraction of all (-1)-curves on X_{CR}.*
(ii) *$f_{CR}: X_{CR} \to Y_{CR}$ maps every curve contracted by h onto a (-2)-curve contained in B_{CR}.*

Proof. The assertions are clear by the construction of X_{CR} and Y_{CR}. □

Since the general fiber of $p_{CR}: Y_{CR} \to C$ is isomorphic to $\mathbf{P}^1$, after applying a succession of blowing-downs, we obtain a $\mathbf{P}^1$-bundle Y' over C. We may assume that the sequence of blowing-downs

$$Y_{CR} = Y'_n \xrightarrow[g'_n]{} Y'_{n-1} \xrightarrow[g'_{n-1}]{} Y'_{n-2} \longrightarrow \cdots \longrightarrow Y'_1 \xrightarrow[g'_1]{} Y'_0 = Y' \qquad \cdots(2.1.2)$$

satisfies the following condition (cf.[H2]).

Condition 2.1.3. Let $E'_i \subset Y'_i$ be the (-1)-curve contracted by g'_i and $\tilde{B}'_i$ the horizontal part of $B'_i = (g'_{i+1} \circ g'_{i+2} \circ \cdots \circ g'_n)_*(B_{CR})$. Then for every $i (1 \leq i \leq n)$ we have $E'_i \cdot \tilde{B}'_i \leq g + 1$.

In fact, the general fiber of the structure morphism $p_{CR}: Y_{CR} \to C$ is isomorphic to $\mathbf{P}^1$. Hence every irreducible component E''_j of a singular fiber $p_{CR}^{-1}(t)$ is isomorphic to $\mathbf{P}^1$ and the invertible sheaf $\omega_{Y_{CR}/C} | E''_j$ has degree ≥ -1. The equality of this inequality holds if and only if E''_j is a (-1)-curve. Therefore, since $p_{CR}^{-1}(t) \cdot \omega_{Y_{CR}} = -2$, the singular fiber $p_{CR}^{-1}(t)$ contains either a (-1)-curve along which $p_{CR}^{-1}(t)$ has (geometric) multiplicity ≥ 2 or two (-1)-curves. On the other hand, the intersection number of $p_{CR}^{-1}(t)$ against the horizontal part $\tilde{B}_{CR}$ of B_{CR} is $2g + 2$. There exists a (-1)-curve $E'_n \subset Y'_n$ contained in $p_{CR}^{-1}(t)$ such that $E'_n \cdot \tilde{B}_n \leq g + 1$. If $g'_n: Y_{CR} = Y'_{n-1}$ is a blowing-down contracting E'_n to a point in

Y'_{n-1}, the condition 2.1.4 holds when $i = n$.

After applying a succession of above processes, we obtain a sequence of blowing-downs satisfying the condition 2.1.4 for all i. □

In the following we assume that the sequence of blowing-downs (2.1.2) satisfies the condition 2.1.3.

Lemma 2.1.4. *The horizontal part* $\tilde{B}'$ *of* $B = (g'_{CR})_* B_{CR}$ *has multiplicity* $\leq g + 1$ *at every point in* $\tilde{B}'$.

Proof. We will inductively show that, for each $i (0 \leq i \leq n)$, the reduced effective divisor $\tilde{B}'_i = (g_{i+1} \circ g_{i+2} \circ \cdots \circ g_n)_* \tilde{B}_{CR}$ has multiplicity $\leq g + 1$ at every point in $\tilde{B}'_i$. Since $\tilde{B}' = \tilde{B}'_0$, it is sufficient to prove this assertion.

If $i = n$, the assertion follows from the smoothness of $B'_n = B_{CR}$ (see Proposition 1.1.2).

Assume that $\tilde{B}'_i$ satisfies the assertion. Since $g_i'^{-1}[\tilde{B}'_{i-1}] = \tilde{B}'_i$, we have $mult_{a'_{i-1}} (\tilde{B}'_{i-1}) = \tilde{B}'_i \cdot E'_i$ where $a'_{i-1} = g'_i(E'_i)$. Hence the condition 2.1.3 implies that $mult_{a'_{i-1}} (\tilde{B}'_{i-1}) \leq g + 1$. On the other hand, by the assumption of induction, $\tilde{B}'_{i-1}$ has multiplicity $\leq g + 1$ at every point in $\tilde{B}'_{i-1}$ except the point a'_{i-1}. Therefore $\tilde{B}'_{i-1}$ satisfies the assertion.

Consequently, for every $i (0 \leq i \leq n)$, $\tilde{B}'_i$ has multiplicity $\leq g + 1$ at every point in $\tilde{B}'_i$. □

Let L' be a line bundle $(g'_{CR})_* L_{CR}$ on Y' and B' a reduced effective divisor $(g'_{CR})_* B_{CR}$ on Y'. By Lemma 1.2.1, L_{CR} and $(L')_{Y_{CR}}$, B_{CR} and $(B')_{Y_{CR}}$ are both isomorphic to each other.

$$\begin{tikzcd}
X_{CR} \arrow[r, "h'_{CR}"'] \arrow[rr, bend left, "h"] \arrow[d, "f_{CR}"'] \arrow[ddr, "\Pi_{CR}"] & X'_{res} \arrow[r, "h'_{res}"'] \arrow[d, "f'_{res}"'] \arrow[dd, bend left, "\Pi'_{res}"] & X \arrow[ddl, "\Pi"] \\
Y_{CR} \arrow[r, "g'_{CR}"] \arrow[dr, "p_{CR}"'] & Y' \arrow[d, "p'"'] & \\
& C &
\end{tikzcd}$$

Diagram 2.1.5.

Let X'_{res} be a minimal resolution of $X(Y', L', B')$, $\Pi'_{res}: X'_{res} \to C$ its structure morphism and $f'_{res}: X'_{res} \to Y'_{res}$ a composite morphism of the resolution $X'_{res} \to X(Y', L', B')$ and the covering $X(Y', L', B') \to Y'$. By the minimality of X'_{res} we have $h'_{CR}: X_{CR} \to X'_{res}$, and by Lemma 1.3.3, there is a morphism $h'_{res}: X'_{res} \to X$.

Lemma 2.1.6. *Under the above notation, we have the following*:

(i) (Y', L', B') *is a triple associated with* Π;

(ii) *X_{CR} is the canonical resolution of the double covering $X(Y', L', B')$ over Y';*

(iii) *$(f'_{res})^{-1}[p'^{-1}(t)]$ is a unique curve which is contained in $(\Pi'_{res})^{-1}(t)$ and contructed by h'_{res}.*

Proof. (i) By the construction of Y', the assertion is clear. (ii) Let $\hat{X}_{CR} = X(\hat{Y}_{CR}, L_{\hat{Y}_{CR}}, B_{\hat{Y}_{CR}})$ be a canonical resolution of the double covering $X(Y', L', B')$ over Y'. By the universality of canonical resolutions (see Proposition 1.1.2), there exist morphisms $\hat{g}$: $Y_{CR} \to \hat{Y}_{CR}$ and $\hat{h}$: $X_{CR} \to \hat{X}_{CR}$. By lemma 1.3.3 we obtain a morhism $\hat{h}'$: $\hat{X}_{CR} \to X$ such that $h = \hat{h}' \circ \hat{h}$. Therefore, $\hat{h}$ is a morphism contructiong some (-1)-curves on X_{CR} (see Lemma 2.1.1 (i)). Assume that F is a (-1)-curve contructed by $\hat{h}$. Then $f_{CR}(F)$ is contructed by $\hat{g}$: $Y_{CR} \to \hat{Y}_{CR}$. By Lemma 2.1.1 (ii), $f_{CR}(F)$ is a (-2)-curve, and hence $\hat{Y}_{CR}$ has a singularity at the image of F. This is a contradiction to the smoothness of $\hat{Y}_{CR}$. Therefore X_{CR} contains no curve which is contructed by $\hat{h}$. Consequently, we have $X_{CR} = \hat{X}_{CR}$ and $Y_{CR} = \hat{Y}_{CR}$. (iii) Assume that h'_{res} contructs a curve F in $\Pi'^{-1}_{res}(t)$ other than $f'^{r-1}_{res}[p'^{-1}(t)]$. By Lemma 2.1.1 (i), F is a (-1)-curve in $\Pi'^{-1}_{res}(t)$. This is a contruction to the minimality of X_{res}. □

(2.2) By Lemma 2.1.1, Lemma 2.1.4 and Lemma 2.1.6, the triple (Y', L', B') constructed in (2.1) satisfies the conditions (A), (B) and $(C1)$ in Proposition 2.0.1. But the condition $(C2)$ is not always satisfed. In this section we shall construct a triple (Y, L, B) satisfying $(C2)$ as well as (A), (B) and $(C1)$ by changing the sequence of blowing-downs (2.1.2) partially. Throughout this section we continue using the same notation in (2.1).

Let p'_i be the structure morphism of Y'_i, $E'^{(j)}_i (i < j)$ the proper transform of the exceptional curve E'_i to Y'_j and $(p'^{-1}(t))^{(j)}$ the proper transform of the fiber $p'^{-1}(t)$ to Y'_j. Moreover we put
$\{\Pi'^{-1}_{res}(t_j)\}_{j \in J} = \{$fibers of Π'_{res}: $X'_{res} \to C$ which contain some curves contructed by $h'_{res}\}$. $\cdots$(2.2.1)

Lemma 2.2.2. *Under the above notation we have the following for every t_j $(j \in J)$.*

(i) *$g'^{-1}_{CR}[p'^{-1} t_j)]$ is a (-2)-curve contained in B_{CR}.*

(ii) *Two of the blowing-ups $g'^{-1}_1, g'^{-1}_2, \cdots, g'^{-1}_n$ are the blowing-ups at points in the proper transforms of $p'^{-1}(t_j)$.*

(iii) *Let $g'^{-1}_{k_j}$ and $g'^{-1}_{\ell_j}$ $(k_j > \ell_j)$ be the two blowing-ups in* (ii) *above. Then the fiber $p'^{-1}_{k_j}(t_j)$ has a configuration in* Figure 2.2.3 *or* 2.2.4.

(iv) *If $p'^{-1}_{k_j}(t_j)$ has a configuration in* 2.2.3, *we have $E'_{k_j} \not\subset B'_{k_j}$, $E'^{(k_j)}_{\ell_j} \not\subset B'_{k_j}$ and $(p'^{-1}(t_j))^{(k_j)} \cdot \tilde{B}'_{k_j} = 0$.*

And if $p'^{-1}_{k_j}(t_j)$ has a configuration in 2.2.4, *we have $E'_{k_j} \not\subset B'_{k_j}$ and $(p'^{-1}(t_j))^{(k_j)})^{(k_j)} \cdot \tilde{B}'_{k_j} = 0$.*

Figure 2.2.3.

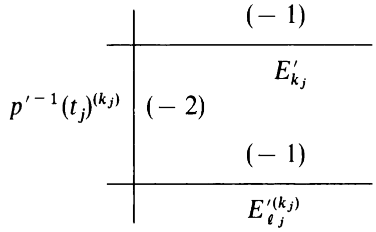
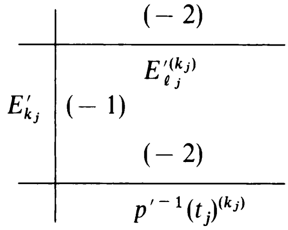


Figure 2.2.4.



Proof. (i) By Lemma 2.1.6 (iii), $f'^{-1}_{res}[p'^{-1}(t_j)]$ is the unique curve which is contained in $\Pi'^{-1}_{res}(t_j)$ and contructed by h'_{res}. Since $h = h'_{res} \circ h'_{CR}$, h contructs $(g'_{CR} \circ f_{CR})^{-1}[p'^{-1}(t_j)]$. Hence Lemma 2.1.1 (ii) implies the assertion.
(ii) By (i) above we have $g'^{-1}_{CR}[p'^{-1}(t_j)]^2 - p'^{-1}(t_j)^2 = -2$. Therefore the assertion follows from the fact that each blowing-up at a point on a curve reduces the self-intersection of the curve by one.
(iii) If $g'^{-1}_{k_j}$ is a blowing-up at $(p'^{-1}(t_j))^{(k_j-1)} \cap E'^{(k_j-1)}_{\ell_j}$, the configuration of $p'^{-1}_{k_j}(t_j)$ is that in Figure 2.2.4. Otherwise, it is in Figure 2.2.3.
(iv) By (i) above, $(p'^{-1}(t_j))^{(k_j)}$ is contained in $B'_{k_j} = (g_{k^j+1} \circ g_{k^j+2} \circ \cdots \circ g_n)_* B_{CR}$. On the other hand, B_{CR} is smooth and $g_{k^j+1} \circ g_{k^j+2} \circ \cdots \circ g_n$ is isomorphic in a neiborhood of $g^{-1}_{CR}[p'^{-1}(t_j)]$ (see (ii) above). Therefore we obtain the assertion (cf. Figures 2.2.3 and 2.2.4). □

We shall inductively construct a sequence of blowing-downs

$$Y_{CR} = Y_n \xrightarrow[g_n]{} Y_{n-1} \xrightarrow[g_{n-1}]{} Y_{n-2} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow[g_1]{} Y_0 = Y. \qquad \cdots(2.2.5)$$

Assume that we constructed Y_i. Then we define the morphism $g_i \colon Y_i \longrightarrow Y_{i-1}$ to be a contraction of the following curve $E_i \subset Y_i$ to a point.

Definition of E_i. If $i \neq \ell_j$ for all $j \in J$ (for the definition of ℓ_j and J, see (2.2.1) and Lemma 2.2.2 (iii)), then we define E_i to be a direct image of $E'^{(n)}_i$ to Y_i. If $i = \ell_j$ for some $j \in J$, we define E_i to be a direct image of $g^{-1}_{CR}[p'^{-1}(t_j)]$ on Y_i.

By the following Lemma 2.2.6 (i), the sequence (2.2.5) is well-defined.

Lemma 2.2.6.

(i) *Each E_i is a (-1)-curve.*
(ii) *Y is a $\mathbf{P}^1$-bundle over C.*

Proof. Considering fiberwise, we have the assertions (cf. Figure 2.2.3 and 2.2.4). □

Let $E^{(j)}_i (i < j)$ be the proper transform of E_i to Y_j, $(p^{-1}(t))^{(j)}$ the proper transform of $p^{-1}(t)$ to Y_j, p_i the structure morphism of Y_i, B_i the direct image of

B_{CR} to Y_i and $\tilde{B}_i$ the horizontal part of B_i. Then we have the following Lemmas.

Lemma 2.2.7. *Assume that $\Pi_{res}^{-1}(t)$ contains a curve contracted by h_{res}. Then t equals t_j for some $j \in J$. And we have the followning.*

(i) *$g_{CR}^{-1}(p^{-1}(t))$ is a (-2)-curve contained in B_{CR}.*

(ii) *In a neiborhood of $p_{k_j}^{-1}(t_j)$ (for the definition of k_j, see* Lemma 2.2.2 (iii)), *Y_{k_j} is naturally isomorphic to Y'_{k_j}. Under this isomorphism, $E_{\ell_j}^{(k_j)}$ (E_{k_j} or $p^{-1}(t_j)^{(k_j)}$) corresponds to $p'^{-1}(t_j)^{(k_j)}$ (E'_{k_j} or $E'^{(k_j)}_{\ell_j}$, resp.).*

(iii) *$p'^{-1}(t)$ has a configuration in* Figure 2.2.4. *And we have $(p^{-1}(t_j))^{(k_j)} \cdot \tilde{B}_{k_j} = 0$ and $E_{\ell_j}^{(k_j)} \subset B_{k_j}$.*

(iv) *The intersection number of $(p^{-1}(t_j))^{(k_j)}$ and $\tilde{B}'_{k_j}$ vanishes.*

Proof. By the definition of the sequence (2.2.5) and t_j, if $\Pi_{res}^{-1}(t)$ contains a curve contructed by h_{res}, then we have $t = t_j$ for some $j \in J$.

(i) We have the assertion in the same way in Lemma 2.2.2 (i).

(ii) By the definition of the sequence (2.2.5) we can show the assertion easily.

(iii) By (i) and (ii) above, B'_{k_j} contains the curve $E'^{(k_j)}_{\ell_j}$. Hence $p'^{-1}_{k_j}(t_j)$ has a configuration in Figure 2.2.4 (see Lemma 2.2.2 (iv)). Therefore an equality $(p^{-1}(t_j))^{(k_j)} \cdot \tilde{B}_{k_j} = 0$ follows from Lemma 2.2.2 (iv) and the correspondence in (ii) above. And by the correspondence and Lemma 2.2.2 (i) we have an inclusion $E_{\ell_j}^{(k_j)} \subset B_{k_j}$.

(iv) In the same way in Lemma 2.2.2 (iv) we ave a desired equality. □

Lemma 2.2.8. *For every $i (1 \leq i \leq n)$ we have $E_i \cdot B_i \leq g + 1$.*

Proof. By the definition of the sequence (2.2.5) and the condition 2.1.3, a desired inequality holds if i is not equal to any $\ell_j (j \in J)$. Hence it it sufficient to prove the inequality when $i = \ell_j$.

Since $g_{k_j}^{-1}$ is a blowing-up at point on $E_{\ell_j}^{(k_j-1)}$, we have

$$\begin{aligned} E_{\ell_j}^{(k_j-1)} \cdot \tilde{B}_{k_j-1} &= E_{\ell_j}^{(k_j-1)} \cdot ((g_{k_j})_* \tilde{B}_{k_j}) \\ &= (E_{\ell_j}^{(k_j)} + E_{k_j}) \cdot \tilde{B}_{k_j}. \end{aligned}$$

By the correspondence in Lemma 2.2.7 (ii) we have

$$(E_{\ell_j}^{(k_j)} + E_{k_j}) \cdot \tilde{B}_{k_j} = \{p'^{-1}(t_j)^{(k_j)} + E'_{k_j}\} \cdot \tilde{B}'_{k_j}.$$

Therefore, when $i = \ell_j$, a desired inequality follows from Lemma 2.2.2 (iv) and the condition 2.1.3. □

Proof of Proposition 2.0.1. We will show that the triple (Y, L, B) constructed above satisfies the conditions in Proposition 2.0.1.

Since the double covering $X(Y, L, B)$ is birational to X, the triple (Y, L, B) is associated with Π. In Lemma 2.1.6 (iv) we showed X_{CR} is the canonical resolution of the double covering $X(Y, L, B)$. Note that, by Lemma 1.2.1 and the definitions of L and B, we have the equalities $L_{CR} = L_{Y_{CR}}$ and $B_{CR} = B_{Y_{CR}}$.

(A) In the same way in Lemma 2.1.4, the divisor $\tilde{B}$ satisfies the condition (A).

(B) We have aleady shown the assertion in Lemma 2.1.1.

(C1) We can show (C1) in the same way in Lemma 2.1.6 (iii).

(C2) By Lemma 2.2.7, if $\Pi_{res}^{-1}(t)$ contains a curve contructed by h_{res}, then there is a number $j \in J$ such that $t = t_j$. Hence it is sufficient to prove the assertion when $t = t_j$.

By Lemma 2.2.7 (ii) and (iii), $g_{k_j}^{-1}$ is a blowing-up at the intersection $E_{\ell j}^{(k_j-1)} \cap p^{-1}(t_j)^{(k_j-1)}$ (see Figure 2.2.4). Hence we have a equality

$$p_{k_j}^{-1}(t_j) = p^{-1}(t_j)^{(k_j)} + 2 \cdot E_{k_j} + E_{\ell j}^{(k_j)}.$$

By Lemma 2.2.7 (iii) and (iv), both $E_{\ell j}^{(k_j)}$ and $p^{-1}(t_j)^{(k_j)}$ does not intersect $\tilde{B}_{k_j}$. Hence we obtain

$$p_{CR}^{-1}(t_j) = p^{-1}(t_j)^{(n)} + 2 \cdot (g_{k_j+1} \circ g_{k_j+2} \circ \cdots \circ g_n)^* E_{k_j} + E_{\ell j}^{(n)}.$$

By Lemma 2.2.7 (i) and (iii), $E_{\ell j}^{(n)}$ and $p^{-1}(t_j)^{(n)} (= g_{CR}^{-1}(t_j))$ are contained in B_{CR}. Hence we obtain $f_{CR}^*(p^{-1}(t_j)^{(n)}) = 2 \cdot f_{CR}^{-1}[p^{-1}(t_j)^{(n)}]$ and $f_{CR}^*(E_{\ell j}^{(n)}) = 2 \cdot f_{CR}^{-1}[E_{\ell j}^{(n)}]$. Therefore we have

$$\Pi_{CR}^{-1}(t_j) = 2 \cdot (f_{CR}^{-1}[p^{-1}(t_j)^{(n)}] + (g_{k_j+1} \circ g_{k_j+2} \circ \cdots \circ g_n \circ f_{CR})^* E_{k_j} + f_{CR}^{-1}(E_{\ell j}^{(n)}]).$$

That is, $\Pi_{CR}^{-1}(t_j)$ is a fiber of multiplicity 2. Therefore $\Pi^{-1}(t_j) = h_*(\Pi_{CR}^{-1}(t_j))$ is a fiber of multiplicity 2. □

3. Local canonical degrees and local Euler numbers

(3.1) First we will introduce a local canonical degree $d_t(X)$ and a local Euler number $e_t(X)$ for every hyperelliptic fibration $\Pi: X \to C$ and point $t \in C$.

Let $\Pi: X \to C$ be a hyperelliptic fibration of genus g, (Y, L, B) a triple associated with Π and $p: Y \to C$ the structure morphism. Then there exists an open set C^0 in C such that:

I) X^0 is isomorphic to a double covering $X(Y^0, L^0, B^0)$ and the structure morphism $\Pi^0: X^0 \to C^0$ is smooth;

II) Y^0 is isomorphic to $\mathbf{P}^1 \times C^0$ over C^0;

III) $\mathcal{O}_{Y^0}(L^0) \simeq p_1^*(\mathcal{O}_{\mathbf{P}^1}(g+1))$ where p_1 is the first projection of $\mathbf{P}^1 \times C^0$.

Here $X^0 = \Pi^{-1}(C^0)$, $\Pi^0 = \Pi | X^0$, $Y^0 = p^{-1}(C^0)$, $L^0 = L | Y^0$ and $B^0 = B \cap Y^0$. And we denote the isomorphism from Y^0 to $\mathbf{P}^1 \times C^0$ by j^0.

Let x be an inhomogeneous coordinate of $\mathbf{P}^1$ and W an open set $\{(x, t) \in \mathbf{P}^1 \times C^0; x \neq \infty\}$ in $\mathbf{P}^1 \times C^0$. Take trivializations $L^0 | W \simeq \{(x, t, y) \in \mathbf{C} \times C^0 \times \mathbf{C}\}$ and $[\mathcal{O}_{Y^0}(B^0)] | W \simeq \{(x, t, v) \in \mathbf{C} \times C^0 \times \mathbf{C}\}$ such that $y^2 = v$. Let φ is a regular section of $\mathcal{O}_{Y^0}(B^0) | W$ such that $\mathrm{div}(\varphi) = B^0 \cap W$. Then φ is a polynomial of x of degree $2g+2$ or $2g+1$ with coefficients in the rational function field of C^0. We regard X as a closure of $\{(x, t, y) \in \mathbf{C} \times C \times \mathbf{C}; y^2 = \varphi(x, t)\}$ in L^0. Then the sections $\left\{x^i \dfrac{dx}{y} \in \Gamma(\Pi_* \omega_{X^0/C^0}); i = 0, 1, \cdots, g-1\right\}$ spans the vector bundle $\Pi_* \omega_{X^0/C^0}$.

Definition 3.1.1. Under above notation, We put

$$D = \triangle^g \cdot \left(\frac{dx}{y} \Lambda\, x \frac{dx}{y} \Lambda \cdots \Lambda\, x^{g-1} \frac{dx}{y} \right)^{\otimes 4(2g+1)}$$
$$\in \Gamma_{rat}(C^0, (\overset{g}{\Lambda} \Pi_* \omega_{X/C})^{\otimes 4(2g+1)})$$
$$= \Gamma_{rat}(C, (\overset{g}{\Lambda} \Pi_* \omega_{X/C})^{\otimes 4(2g+1)})$$

where $\triangle = \triangle(\varphi)$ is the discriminant of φ as a polynomial of x and $\Gamma_{rat}(C, \mathscr{F})$ is a set of rational sections of a sheaf $\mathscr{F}$ on C.

Proposition 3.1.2 (Ueno, see [U2]). *The rational section D is independent of the choice of (Y, L, B), C^0, $Y^0 \xrightarrow[j^0]{\sim} \mathbf{P}^1 \times C^0$, x, y and φ.*

Proof. We can show the assertion in a similar way to [U]. □

Definition 3.1.3. *For every point t in C we put*

$$d_t(X) = \frac{1}{4(2g+1)} \operatorname{ord}_t D,$$
$$e_t(X) = \chi_{top}(\Pi^{-1}(t)) - (2 - 2g).$$

We call the numerical invariant $d_t(X)$ a local canonical degree of the fibration $\Pi: X \to C$ at a point t in C and $e_t(X)$ a local Euler number of $\Pi: X \to C$ at t.

By Proposition 3.1.2, $d_t(X)$ is independent of the choice of triples associated with the hyperelliptic fibration $\Pi: X \to C$.

Proposition 3.1.4. *Let $\Pi: X \to C$ be a hyperelliptic fibration of genus g where C is a complete smooth curve of genus b. Then we have*

$$\deg \Pi^* \omega_{X/C} = \sum_{t \in C} d_t(X),$$

$$\chi_{top}(X) = (2 - 2g)(2 - 2b) + \sum_{t \in C} e_t(X).$$

Proof. The first equality follows from the definition of $d_t(X)$. For the second one we can refer to [*B-P-V*, Proposition III.11.4]. □

(3.2) In order to estimate local canonical degrees by local Euler numbers, we need two numerical invariants determined by the singularities of the divisor B of a triple (Y, L, B) associated with a hyperelliptic fibration.

Let $p: Z \to D$ be a smooth morphism from a smooth surface Z to a smooth curve C and let H be a reduced effective divisor on Z such that $p|H$ is a finite morphism. We define a homomorphism $\iota_{H/D}: p^*\omega_D | H \to \omega_H$ by

$$(\iota_{H/D} | U \cap H)(h \cdot \mu) = h \cdot \operatorname{Res}_H \left(\frac{p^*(\mu) \wedge d\varphi}{\varphi} \right) \qquad \cdots (3.2.1)$$

where U is an open set in Z, φ is a regular function whose zero locus is $B \cap U$, h is a section in $\Gamma((U \cap H), \mathcal{O}_H)$ and μ is a section in $\Gamma(U, p^*\omega_D)$. Here $p^*(\mu)$ is the pull-back of μ as a 1-form from $p^*\Omega_D^1(U)$ to $\Omega_Z^1(U)$ and Res_H is the residue map on H (see [*B-P-V*, II. 4]).

Lemma 3.2.2. *The homomorphism* $\iota_{H/D}\colon p^*\omega_D|H \to \omega_H$ *is well defined and it is injective.*

Proof. Take another regular function φ' on U whose zero locus is $B \cap U$. There exists a unit element u in $\Gamma(U, \mathcal{O}_Z)$ such that $\varphi' = u\cdot\varphi$. Thus we can make calculation as follows:

$$\begin{aligned} h\cdot \mathrm{Res}_H\left(\frac{p^*(\mu)\wedge d\varphi'}{\varphi'}\right) &= \frac{h}{u}\cdot \mathrm{Res}_H\left(\frac{p^*(\mu)\wedge d(u\cdot\varphi)}{\varphi}\right) \\ &= h\cdot \mathrm{Res}_H\left(\frac{p^*(\mu)\wedge d\varphi}{\varphi}\right) + \frac{\varphi\cdot h}{u}\cdot \mathrm{Res}_H\left(\frac{p^*(\mu)\wedge du}{\varphi}\right) \\ &= h\cdot \mathrm{Res}_H\left(\frac{p^*(\mu)\wedge d\varphi}{\varphi}\right). \end{aligned}$$

Hence $\iota_{H/D}$ is well-defined. For every closed point q in H such that $(p|H)_q$ is étale, $(\iota_{H/c})_q$ an isomorphism. Therefore the injectivity of $\iota_{H/C}$ follows from the fact that, for every reduced curve K, the homomorphism $\varphi\colon \mathcal{L} \to \mathcal{F}$ between invertible sheaves on K is injective if φ is isomorphic in a dence open set on K. □

Remark 3.2.3. If $\eta\colon Hn \to H$ is the normalization of H, the homomorphism $\eta^*(\iota_{H/D})\colon (p\circ\eta)^*\omega_D \to \eta^*\omega_H$ is the composite of natural homomorphisms $(p\circ\eta)^*\omega_D \to \omega_{Hn}$ and $\omega_{Hn} \to \eta^*\omega_H$ (cf. [*B-P-V*, II. 1]).

Let $\Pi\colon X \to C$ be a hyperelliptic fibration of genus g, (Y, L, B) a triple associated with Π and $\tilde{B}$ is a horizontal part of B. By Lemma 3.2.2 we have an exact sequence of sheaves

$$0 \longrightarrow p^*\omega_C|\tilde{B} \xrightarrow[\iota_{\tilde{B}/D}]{} \omega|\tilde{B} \longrightarrow \mathcal{M} \longrightarrow 0. \qquad \cdots(3.2.4)$$

Note that the support of $\mathcal{M}$ is empty or 0-dimensional.

Definition 3.2.5. For every point t in C, we define an integer $\delta_t(B)$ to be

$$\delta_t(B) = \begin{cases} \mathit{length}_{\mathcal{O}_{C,t}}(p_*\mathcal{M})_t & \text{if } p^{-1}(t) \subset B, \\ \mathit{length}_{\mathcal{O}_{C,t}}(p_*\mathcal{M})_t + (4g+2) & \text{if } p^{-1}(t) \not\subset B, \end{cases}$$

where $(p_*\mathcal{M})_t$ is the stalk of the sheaf $p_*\mathcal{M}$ at t.

Let X_{CR} be a canonical resolution of the double covering $X(Y, L, B)$ over Y and $g_{CR}\colon Y_{CR} \to Y$ a morphism between the base spaces of X_{CR} and $X(Y, L, B)$ (see Proposition 1.1.2). Since $g_{CR}\colon Y_{CR} \to Y$ is a proper surjective birational morphism, g_{CR} is decomposed into a succession of blowing-downs

$$Y_{CR} = Y_n \xrightarrow[g_n]{} Y_{n-1} \xrightarrow[g_{n-1}]{} \cdots \xrightarrow[g_2]{} Y_1 \xrightarrow[g_1]{} Y. \qquad \cdots(3.2.6)$$

Definition 3.2.7. For the seqence of blowing-downs (3.2.6), the divisor E_i is the proper transform of the exceptional curve of g_i to Y_{CR} and the positive integer m_i is the multiplicity of $B_{Y_{i-1}}$ at the point q_i where Y_i is obtained from Y_{i-1} blown-up at q_i.

(3.3) Let $\Pi: X \to C$ be a hyperelliptic fibration of genus g. Then $\Pi_{CR}: X_{CR} \to C$ is a hyperelliptic fibration of genus g and we can define the local canonical degree $d_t(X_{CR})$ and the local Euler number $e_t(X_{CR})$. The numerical invariants $d_t(X_{CR})$ and $e_t(X_{CR})$ can be expressed by $\delta_t(B)$ and m_i in the following forms.

Proposition 3.3.1 (cf. [H1, Lemma 6]). *Let $\Pi: X \to C$ be a hyperelliptic fibration of genus g, (Y, L, B) a triple associated with Π and X_{CR} the canonical resolution of the double covering $X(Y, L, B)$ over Y. Then, for every point t in C we have*

$$d_t(X_{CR}) = \frac{g}{4(2g+1)} \cdot \delta_t(B) - \frac{1}{2} \cdot \sum_{p_{CR}(E_i)=t} \left\{ \left[\frac{m_i}{2}\right] \cdot \left(\left[\frac{m_i}{2}\right] - 1 \right) \right\},$$

$$e_t(X_{CR}) = \delta_t(B) - 2 \cdot \sum_{p_{CR}(E_i)=t} \left(2 \cdot \left[\frac{m_i}{2}\right]^2 - \left[\frac{m_i}{2}\right] - 1 \right),$$

where p_{CR} is the structure morphism of Y_{CR} and $\left[\frac{m_i}{2}\right]$ is the greatest integer not exceeding $\frac{m_i}{2}$.

(3.4) *Proof of the first equality of Proposition* 3.3.1.

Throughout this proof we use the notation in (3.1) freely. Let C^0 be an open set defined in (3.1). Since $\Pi^0: X^0 \to C^0$ is smooth, $B^0 = B \cap Y^0$ is smooth (see the condition (I) in (3.1) and Proposition 1.1.2). Hence, in the sequence (3.2.6), no rational map g_i^{-1} $(1 \le i \le n)$ is a blowing-up at a point in the fiber on $t \in C^0$. Therefore, if $t \in C^0$, the right side of the first equation vanishes. On the other hand, since the section D in Definition 3.1.1 is regular at $t \in C^0$, the left hand of the equation also vanishes for every $t \in C^0$. Thus we have the first equality when $t \in C^0$. In the following we will prove the first equality when $t \notin C^0$.

Assume that t is not contained in C^0. Since $d_t(X)$ equals $d_t(\Pi^{-1}(V))$ for every neiborhood V of t in C, we may replace C by any neiborhood of t in C. Therefore we may assume the following conditions $(a) \sim (f)$ (cf. (3.1)).

(a) $C^0 = C - \{t\}$.
(b) There is an isomorphism $\jmath: Y \longrightarrow \mathbf{P}^1 \times C$ such that $\jmath | Y^0 = \jmath^0$.

In the following we identify Y with $\mathbf{P}^1 \times C$.

(c) $L \simeq p_1{}^*\mathcal{O}_{\mathbf{P}^1}(g+1)$ where p_1 is the first projection of $\mathbf{P}^1 \times C$.
(d) $\{(x, t) \in \mathbf{P}^1 \times C : x \neq \infty\} \cap \tilde{B} = \Phi$ where x is an inhomogeneous coodinate of $\mathbf{P}^1$ and $\tilde{B}$ is a horizontal part of B.
(e) The dualizing sheaf ω_C is generated by a section μ_0.

(f) There is a function τ on C such that $\mathrm{div}(\tau)=\{t\}$.

The proof is separated into two parts (for the definitions of φ and y, see (3.1)).

Part 1. $\mathrm{ord}_t\triangle(\varphi)=\delta_t$.

Part 2. $\mathrm{ord}_t\left(\frac{dx}{y}\Lambda x\frac{dx}{y}\Lambda\cdots\Lambda x^{g-1}\frac{dx}{y}\right)=-\frac{1}{2}\sum_{p_{CR}(E_i)=t}\left[\frac{m_i}{2}\right]\cdot\left(\left[\frac{m_i}{2}\right]-1\right)$.

Since X_{CR} is birational to X, a triple (Y, L, B) is associated with $\Pi_{CR}\colon X_{CR}\to C$ as well as $\Pi\colon X\to C$. Hence we have $d_t(X_{CR})=d_t(X)$. Therefore, if we show Part 1 and Part 2 above, we have a desired equality (see Definitions 3.1.1 and 3.1.3).

Proof of Part 1. Let $\tilde{\varphi}$ be a function on $U=\{(x,t)\in\mathbf{C}\times C\}\subset\mathbf{P}^1\times C$ whose zero locus is $\tilde{B}$. We defined the homomorphism $\iota_{\tilde{B}/C}\colon p^*\omega_C|\tilde{B}\to\omega_{\tilde{B}}$ to be $\iota_{\tilde{B}/C}(h\cdot\mu)=h\cdot\mathrm{Res}_{\tilde{B}}\left(\frac{p^*(\mu)\Lambda d\tilde{\varphi}}{\varphi}\right)$ for every $h\in\Gamma(\tilde{B}\cap U,\mathcal{O}_{\tilde{B}})$ and $\mu\in\Gamma(C,\omega_C)$. If μ_0 is a generator of ω_C (see the condition (*e*) above), then the section $\mathrm{Res}_{\tilde{B}}\left(\frac{p^*(\mu_0)\Lambda dx}{\tilde{\varphi}}\right)$ generates $\omega_{\tilde{B}}$. Since $\iota_{\tilde{B}/C}(\mu_0)=\frac{\partial\tilde{\varphi}}{\partial x}\cdot\mathrm{Res}_{\tilde{B}}\left(\frac{p^*(\mu_0)\Lambda dx}{\tilde{\varphi}}\right)$, we have

$$length_{\mathcal{O}_{C,t}}(p_*\mathcal{M})_t=\mathrm{ord}_t\left(r\left(\frac{\partial\tilde{\varphi}}{\partial x},\tilde{\varphi}\right)\right)=\mathrm{ord}_t\triangle(\tilde{\varphi})\qquad\cdots(3.4.1)$$

where $r(f,g)$ is a resultant of f and g as polynomials of x.

On the other hand, if we put

$$\sigma=\begin{cases}0 & \text{if } p^{-1}(t)\not\subset B,\\ 1 & \text{if } p^{-1}(t)\subset B,\end{cases}$$

then we have $\varphi=u\cdot\tau^\sigma\cdot\tilde{\varphi}$ for some unit element u of $\Gamma(C,\mathcal{O}_C)$ (for the definition τ, see the consition (*f*) above). Therefore the assertion follows from Definiton 3.2.5 and the equation (3.4.1).

Proof of Part 2. Let $E_i^{(j)}$ be the proper transform of the exceptional curve of g_i to Y_j for $i\leq j$ (see Definition 3.2.7). And let F be a divisor on Y_{CR} satisfying

$$\omega_{Y_{CR}/C}\otimes\mathcal{O}_{Y_{CR}}(L_{Y_{CR}})\simeq g_{CR}^*(\omega_{Y/C}\otimes\mathcal{O}_Y(L))\otimes\mathcal{O}_{Y_{CR}}(-F).$$

Then the divisor F is expressed in the form

$$F=\sum_{i=1}^{n}(g_{i+1}\circ\cdots\circ g_n)^*\left(\left(\left[\frac{m_i}{2}\right]-1\right)\cdot E_i^{(i)}\right).\qquad\cdots(3.4.2)$$

By the minimality of Y_{CR}, m_i is greater than 1, hence F is effective or empty. Therefore we have an exact sequence

$$0\longrightarrow\omega_{Y_{CR}/C}\otimes\mathcal{O}_{Y_{CR}}(L_{Y_{CR}})\longrightarrow g_{CR}^*(\omega_{Y/C}\otimes\mathcal{O}_Y(L))\longrightarrow\mathcal{O}_F\longrightarrow 0.\qquad\cdots(3.4.3)$$

Lemma 3.4.4. *Under the above notation, we have*

$$R^1 p_{CR*}(\omega_{Y_{CR}/C} \otimes \mathcal{O}_{Y_{CR}}(L_{Y_{CR}})) = R^1 p_{CR*}(g_{CR}^*(\omega_{Y/C} \otimes \mathcal{O}_Y(L))) = 0.$$

Proof. By the relative duality we have isomorphisms

$$\begin{aligned} R^1 p_{CR*}(g_{CR}^*(\omega_{Y/C} \otimes \mathcal{O}_Y(L))) &\simeq R^1 p_*(\omega_{Y/C} \otimes \mathcal{O}_Y(L)) \\ &\simeq (p_*(\mathcal{O}_Y(L)^\vee))^\vee, \\ R^1 p_{CR}(\omega_{Y_{CR}/C} \otimes \mathcal{O}_{Y_{CR}}(L_{Y_{CR}})) &\simeq (p_{CR*}(\mathcal{O}_{Y_{CR}}(L_{Y_{CR}})^\vee))^\vee \\ &\simeq (p_*(\mathcal{O}_Y(L)^\vee))^\vee \end{aligned}$$

where the last isomorphism is given by the fact that

$$\mathcal{O}_{Y_{CR}}(L_{Y_{CR}}) \simeq g_{CR}^*(\mathcal{O}_Y(L)) \otimes \mathcal{O}_{Y_{CR}}\left(\sum_{i=1}^{n} (g_{i+1} \circ \cdots \circ g_n)^* \left(\left(-\left[\frac{m_i}{2}\right]\right) \cdot E_i^{(i)}\right)\right).$$

Since the intersection number $p^{-1}(t) \cdot L > 0$, we have $p_*(\mathcal{O}_Y(L)^\vee) \simeq 0$. Therefore we obtain the assertion. □

By Lemma 3.4.4 and the exact sequence (3.4.3) we have $H^1(F, \mathcal{O}_F) = 0$ and an exact sequence

$$\begin{aligned} 0 \longrightarrow p_{CR*}(\omega_{Y_{CR}/C} \otimes \mathcal{O}_{Y_{CR}}(L_{Y_{CR}})) &\longrightarrow p_{CR*}(g_{CR}^*(\omega_{Y/C} \otimes \mathcal{O}_Y(L))) \\ &\longrightarrow p_{CR*}\mathcal{O}_F \longrightarrow 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} length_{\mathcal{O}_{C,t}}(p_{CR*}\mathcal{O}_F)_t &= \dim\ H^0(F, \mathcal{O}_F) \\ &= \chi(\mathcal{O}_F) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left[\frac{m_i}{2}\right] \cdot \left(\left[\frac{m_i}{2}\right] - 1\right) \end{aligned}$$

where the last equality follows from the equation (3.4.2) and the following two equalities:

$$\omega_{Y_{CR}} = g_{CR}^* \omega_Y \otimes \mathcal{O}_{Y_{CR}}\left(\sum_{i=1}^{n} (g_{i+1} \circ \cdots \circ g_n)^* E_i^{(i)}\right),$$

$$\chi(\mathcal{O}_F) = \frac{F \cdot (F + \omega_{Y_{CR}})}{2}.$$

Let $\mathscr{C}$ be the cokernel of the injection

$$\overset{g}{\Lambda}\, p_{CR*}(\omega_{Y_{CR}/C} \otimes \mathcal{O}_{Y_{CR}}(L_{Y_{CR}})) \longrightarrow \overset{g}{\Lambda}\, p_*(\omega_{Y/C} \otimes \mathcal{O}_Y(L)).$$

Since $length_{\mathcal{O}_{C,t}}(p_{CR*}\mathcal{O}_F)_t = length_{\mathcal{O}_{C,t}}(\mathscr{C})_t$ by [Ful. Lemma A.2.6], the proof of Part 2 will be completed if we shall show the following equation:

Lemma 3.4.5.

$$length_{\mathcal{O}_{C,t}}(\mathscr{C})_t = -\operatorname{ord}_t\left(\frac{dx}{y}\Lambda x\frac{dx}{y}\Lambda\cdots\Lambda x^{g-1}\frac{dx}{y}\right).$$

Proof. The composite morphism of

$$\Pi_*\omega_{X/C} = \Pi_{CR*}\omega_{X_{CR}/C} \simeq \Pi_{CR*}(f^*_{CR}(\mathcal{O}_{Y_{CR}}(L_{Y_{CR}}) \otimes \omega_{Y_{CR}/C}))$$
$$\simeq p_{CR*}(L_{Y_{CR}} \otimes \omega_{Y_{CR}/C})$$

and

$$p_{CR*}(L_{Y_{CR}} \otimes \omega_{Y_{CR}/C}) \longrightarrow p_*(\mathcal{O}_Y(L) \otimes \omega_{Y/C})$$

maps the rational section $\dfrac{\rho\cdot\zeta}{y}$ to $\rho\cdot\zeta$ for every rational sections ρ in $\mathcal{O}_Y(L)|Y^0$ and ζ in $\omega_{Y/C}|Y^0$. Since the regular sections $\{x^i dx \in \Gamma(p_*(\mathcal{O}_Y(L) \otimes \omega_{Y/C}))\,;\, i = 0, 1, \cdots, g-1\}$ span the locally free sheaf $p_*(\mathcal{O}_Y(L) \otimes \omega_{Y/C})$ on C, we have

$$length_{\mathcal{O}_{C,t}}(\mathscr{C})_t = length_{\mathcal{O}_{C,t}}(Coker(\Pi_*\omega_{X/C} \longrightarrow p_*(\mathcal{O}_Y(L) \otimes \omega_{Y/C}))_t$$
$$= -\operatorname{ord}_t\left(\frac{dx}{y}\Lambda x\frac{dx}{y}\Lambda\cdots\Lambda x^{g-1}\frac{dx}{y}\right). \qquad \square$$

(3.5) *Proof of the second equality in Proposition* 3.3.1. We shall use the same notation in (3.4). Since $e_t(X)$ is a local invariant, in the following we assume $p^{-1}(X)$ contains all the singulalities of the divisor B. For the sequence of blowing-downs (3.2.6) we put $G_i = g_1 \circ g_2 \circ \cdots \circ g_i$. Let V_i be a vertical part of B_{Y_i} and $\mathscr{M}_i$ the cokernel of $\iota_{G_i^{-1}[\tilde{B}]/C}$ (see the exact sequence below).

$$0 \longrightarrow (p\circ G_i)^*\omega_C \xrightarrow[\iota_{G_i^{-1}[\tilde{B}]/C}]{} \omega_{G_i^{-1}[\tilde{B}]} \longrightarrow \mathscr{M}_i \longrightarrow 0$$

If we put

$$\mu_i' = length_{\mathcal{O}_{C,t}}((p\circ G_i)_*\mathscr{M}_i)_t + \deg\omega_{V_i} + 2G_i^{-1}[\tilde{B}]\cdot V_i,$$

then we have the following lemma.

Lemma 3.5.1. *For every i $(1 \le i \le n)$, we have*

$$\mu_i' = \mu_{i-1}' - 2\left[\frac{m_i}{2}\right]\cdot\left(2\left[\frac{m_i}{2}\right] - 1\right).$$

Proof. Put $n_i = mult_{q_i}((G_{i-1})^{-1}[\tilde{B}])$ and let q_i be the point on Y_{i-1} such that Y_i is obtained from a blowing-up of Y_{i-1} at q_i. By straightforward calculation we obtain the following equalities:

$$\deg\omega_{V_i} - \deg\omega_{V_{i-1}} = -(m_i - n_i)(m_i - n_i - 1) + 2\left(m_i - 2\left[\frac{m_i}{2}\right]\right)(m_i - n_i)$$
$$-\left(m_i - 2\left[\frac{m_i}{2}\right]\right)\left(m_i - 2\left[\frac{m_i}{2}\right] + 1\right),$$

$$G_i^{-1}[\tilde{B}]\cdot V_i - (G_{i-1})^{-1}[\tilde{B}]\cdot V_i = n_i\left(n_i - 2\left[\frac{m_i}{2}\right]\right),$$

$$length_{\mathcal{O}_{C,t}}((p\circ G_i)_*\mathcal{M}_i)_t - length_{\mathcal{O}_{C,t}}((p\circ G_{i-1})_*\mathcal{M}_{i-1})_t$$
$$= -n_i(n_i - 1).$$

By these three equations above we have a desired equality. □

Considering the cell decomposition of $\Pi_{CR}^{-1}(t)$, we have

$$\chi_{top}(\Pi_{CR}^{-1}(t)) = 2\cdot\chi_{top}(p_{CR}^{-1}(t)) - \chi_{top}(B_{Y_{CR}}\cap p_{CR}^{-1}(t)).$$

Since $B_{Y_{CR}}$ intersects $p_{CR}^{-1}(t)$ at $V_n\cup(G_n^{-1}[\tilde{B}]\cap p_{CR}^{-1}(t))$, we obtain the following:

$$\begin{aligned} e_t(X_{CR}) &= \chi_{top}(\Pi_{CR}^{-1}(t)) - (2-2g)\\ &= 2\cdot(2+n) - \{\chi_{top}(V_n) + (2g+2 - length_{\mathcal{O}_{C,t}}(p_{CR*}\mathcal{M}_n)_t)\} - (2-2g)\\ &= 2n + length_{\mathcal{O}_{C,t}}(p_{CR*}\mathcal{M}_n)_t) - \chi_{top}(V_n)\\ &= u_n' + 2n\\ &= u_0' - 2\sum_{i=1}^{n}\left(2\left[\frac{m_i}{2}\right]^2 - \left[\frac{m_i}{2}\right] - 1\right)\\ &= \delta_t(B) - 2\sum_{i=1}^{n}\left(2\left[\frac{m_i}{2}\right]^2 - \left[\frac{m_i}{2}\right] - 1\right). \end{aligned}$$

Therefore a desired equality is proved. □

4. Main theorems

(4.0) The purpose of this chapter is to prove the following two theorems.

Theorem 4.0.1. *Let $\Pi\colon X\to C$ be a hyperelliptic fibration of genus g over a smooth compact curve C. Then we have*

$$\frac{-g-1}{2g+1}\cdot\sum_{t\in C}e_t(X) \le i_{top}(X) \le \frac{g^2-2g-1}{2g+1}\cdot\sum_{t\in C}e_t(X) \qquad \cdots(4.0.2)$$

if g is even,

$$\frac{-g-1}{2g+1}\cdot\sum_{t\in C}e_t(X) \le i_{top}(X) \le \frac{g^2-2g}{2g+1}\cdot\sum_{t\in C}e_t(X) \qquad \cdots(4.0.3)$$

if g is odd.

Theorem 4.0.4. *Let $\Pi\colon X\to C$ be a hyperelliptic fibration of genus g. Then, for every point t in C, we have*

$$\frac{g}{4(2g+1)}\cdot e_t(X) \le d_t(X) \le \frac{g^2}{4(2g+1)}\cdot e_t(X) \qquad \cdots(4.0.5)$$

if g is even,

$$\frac{g}{4(2g+1)}\cdot e_t(X) \leq d_t(X) \leq \frac{g^2+1}{4(2g+1)}\cdot e_t(X) \qquad \cdots(4.0.6)$$

if g is odd.

If $\Pi: X \to C$ is a hyperelliptic fibration of genus 2 over a smooth compact curve C, then the coefficient of $\sum_{t \in C} e_t(X)$ in the left term of (4.0.2) is negative. Hence we have the following collorary.

Collorary 4.0.7. *If a smooth compact surface X is a family of genus 2 curves over a smooth compact curve, then the topological index of X is non-positive.*

Proof. If $\Pi: X \to C$ is a fibration of genus 2, then Π is a hyperelliptic fibration of genus 2. And it is well known that the integer $e_t(X)$ is non-negative for every (hyperelliptic) fibration $\Pi: X \to C$ and point t in C. Therefore the assertion follows from the inequality (4.0.2) in Theorem 4.0.1. □

Since the in teger $e_t(X)$ is non-negative for every (hyperelliptic) fibration $\Pi: X \to C$ and point t in C, the second inequalities of (4.0.5) and (4.0.6) in Theorem 4.0.4 imply that:

Collotrary 4.0.8. *for every t in C, the integer $d_t(X)$ is non-negative.*

By this collorary, the section D in Definition 3.1.1 is a regular section (see Definition 3.1.3).

(4.1) Theorem 4.0.1 follows from Theorem 4.0.4. In fact, by Leray's spectral sequence we have $\chi(\mathcal{O}_X) = (1-g)(1-b) + \deg\omega_{X/C}$ where b is the genus of C. Hence we obtain

$$\begin{aligned} i_{top}(X) &= \frac{1}{3}\cdot(c_1(X)^2 - 2\cdot c_2(X)) \\ &= 4\cdot\deg\omega_{X/C} - \{c_2(X) - (2-2g)(2-2b)\} \\ &= \sum_{t \in C}\{4\cdot d_t(X) - e_t(X)\} \end{aligned}$$

where the last equality follows from Proposition 3.1.4. Apply Theorem 4.0.4 to the last equation. We obtain inequalities in Theorem 4.0.1.

In the following sections we will prove Theorem 4.0.4.

(4.2) In order to prove Theorem 4.0.4, we need some lemmas. In this section we state notation used in these lemmas.

Let $\Pi: X \to C$ be a relatively minimal hyperelliptic fibration and (Y, L, B) a triple mentioned in Proposition 2.0.1. We use the notations in Diagram 2.0.3 freely. Moreover we define

n_t: the number of irreducible curves which is contained in a fiber $p_{CR}^{-1}(t)$ and contructed by g_{CR}.

For every proper surjective birational morphism $\varphi: Z \to W$ between smooth surfaces over C, an integer $\beta_t(\varphi)$ is defined as

$\beta_t(\varphi) =$ (Euler number of the fiber of $Z \to C$ on t)

$-$ (Euler number of the fiber of $W \to C$ on t).

Let

$$Y_{CR} = Y_n \xrightarrow[g_n]{} Y_{n-1} \xrightarrow[g_{n-1}]{} \cdots \xrightarrow[g_2]{} Y_1 \xrightarrow[g_1]{} Y_0 = Y \qquad \cdots(4.2.1)$$

be a decomposition of g_{CR} into a succession of blowing-downs and let $E_i^{(j)}$'s $(1 \le i \le j \le n)$ be curves defined in Definition 3.2.7 and m_i's $(1 \le i \le n)$ integers defined in Definition 3.2.7. Moreover we denote B_{Y_i} (see definition 1.1.1) by B_i.

(4.3)

Lemma 4.3.1. *Under notation in* (4.2), *the following equality holds at every point t in C*:

$$4(2g+1)\cdot d_t(X) - g\cdot e_t(X_{CR}) = -2\cdot \sum_{p_{CR}(E_i^{(n)})=t} \left(\left[\frac{m_i}{2}\right]-1\right)\cdot\left(\left[\frac{m_i}{2}\right]-g\right).$$

Proof. Since (Y, L, B) is a triple associated with $\Pi_{CR}\colon X_{CR} \to C$ as well as $\Pi\colon X \to C$, we have $d_t(X) = d_t(X_{CR})$(see Definition 3.1.1 and 3.1.3). Therefore a desired equality follows from Proposition 3.3.1. □

Lemma 4.3.2. *For every point t in C we have* $\beta_t(h_{CR}) \le \dfrac{n_t}{2}$.

Proof. $\beta_t(h_{CR})$ and n_t depends only on Euler number of $\Pi_{CR}^{-1}(t)$, $\Pi_{res}^{-1}(t)$. Hence, if necessary, replacing C by a neiborhood of t in C, we may assume that the reduced effective divisor B in Y(see (4.2)) has singularities only in the fiber $p^{-1}(t)$. Note that, under this assumption, the integer n_t agrees with n $=$ (the number of blowing-downs of the sequence (4.2.1)).

Let $\tilde{B}_{CR}$ be the horizontal part of $B_{CR} = B_{Y_{CR}}$ and put $p_{CR}^{-1}(t) \cap \tilde{B}_{CR}$ $= \{a_1, a_2, \cdots, a_m\}$. If necessary, changing the order of blowing-downs, we may assume that the sequence of blowing-downs (4.2.1) satisfies the following conditions (cf. Remark 1.1.4).

There are integers $n_0, n_1, \cdots, n_{m+1}$ such that

(I) $0 = n_0 \le n_1 \le n_2 \le \cdots \le n_{m+1} = n_t = n$;

(II) *For every i, j such that $1 \le i \le m$ and $n_{i-1} + 1 \le j \le n_i$, g_j^{-1} is a blowing-up at the image of a_i on Y_{j-1}*;

(III) *For every $i(1, \le i \le m)$, $g_{n_i+1} \circ g_{n_i+2} \circ \cdots \circ g_{n_{m+1}}$ is isomorphic in a neiborhood of a_i.*

By the conditions (B) and ($C1$) in Proposition 2.0.1, an irreducible component F_k of $\Pi_{CR}^{-1}(t)$ is contracted by h_{CR} if and only if the direct image of F_k to Y_{CR} is contained in

$$\mathbf{D} = \{E_j^{(n)}; j = 1, 2, \cdots, n,\ E_j^{(n)} \subset B_{CR} \text{ and } (E_j^{(n)})^2 = -2\}.$$

Since the inverse image of $E_j^{(n)} \in \mathbf{D}$ is irreducible, the number of curves in $\Pi_{CR}^{-1}(t)$ which is contructed by h_{CR} coincides with the number of the element of $\mathbf{D}$. That is, we have $\beta_t(h_{CR}) =$ (the number of elements of $\mathbf{D}$). Therefore the assertion follows from the following Lemma 4.3.3. In fact, Lemma 4.3.3 implies that the number of elements of $\mathbf{D}$ is smaller than or equal to a half of $n_{m+1} = n_t$.

Lemma 4.3.3.

(i) *For every* $i = 1, 2, \cdots, m$, *we have* $E_{n_i}^{(n)} \notin \mathbf{D}$.

(ii) *For every* i, j *such that* $i = 1, 2, \cdots, m$ *and* $n_{i-1} + 1 \leq j \leq n_i - 1$, $E_j^{(n)} \notin \mathbf{D}$ *implies* $E_{j+1}^{(n)} \in \mathbf{D}$.

(iii) *If* $n_m + 1 \leq j \leq n_{m+1}$, *we have* $E_j^{(n)} \notin \mathbf{D}$.

Proof. (i) Since B_{CR} is non-singular and the curve $E_{n_i}^{(n)}$ intersect $\tilde{B}_{CR}$ at a_i, $E_{n_i}^{(n)}$ is not contained in B_{CR}. Therefore the assertion follows.

(ii) Assume a contrary to $E_{j+1}^{(n)} \notin \mathbf{D}$. Then both $E_j^{(n)}$ and $E_{j+1}^{(n)}$ are contained in $B_n = B_{CR}$. Hence the smoothmess of B_{CR} implies that $E_j^{(n)}$ does not intersect $E_{j+1}^{(n)}$. Therefore one of the blowing-ups g_ℓ^{-1}'s $(j + 2 \leq \ell \leq n)$ is a blowing-up at the point $a_j^{(\ell-1} = (g_{j+2} \circ g_{j+3} \circ \cdots \circ g_{\ell-1})^{-1}(E_j^{(j+1)} \cap E_{j+1}^{(j+1)})$. Hence we obtain $(E_j^{(n)})^2 \leq (E_j^{(j+1)})^2 - 1 = -3$. This is a contradiction with $E_j^{(n)} \in \mathbf{D}$.

(iii) By the condition (II) above, the divisor $B_{n_m} = (g_{n_m+1} \circ g_{n_m+2} \circ \cdots \circ g_{n_{m+1}})_* B_{CR}$ is non-singular at the intersection points $\tilde{B}_{n_m} \cap p_{n_m}^{-1}(t)$ where $\tilde{B}_{n_m}$ is a horizontal part of B_{n_m} and $p_{n_m}\colon Y_{n_m} \to C$ is the structure morphism. Hence each singularity of B_{n_m} is a singularity of $p_{n_m}^{-1}(t)$ and it is not contained in $\tilde{B}_{n_m}$. Since no three curves in $p_{n_m}^{-1}(t)$ meet at one point and every irreducible curve in $p_{n_m}^{-1}(t)$ is non-singular, B_{n_m} has multiplicity 2 at the point q_i where $g_{n_m}^{-1}$ is a blowing-up of Y_{n_m} at q_i. Hence we have $(g_{n_m+1})^* B_{n_m} = (g_{n_m+1})^{-1}[B_{n_m}] + 2 \cdot E_{n_m+1}$. Therefore the divisor $B_{n_m+1} = B_{Y_{n_m+1}}$ does not contain E_{n_m+1} (see Definition 1.1.1). Consequently, E_{n_m} is not contained in $\mathbf{D}$. In a similar way, we can also show $E_j \notin \mathbf{D}$ for every j such that $n_m + 2 \leq j \leq n_{m+1}$. □

Lemma 4.3.4. *For every* t *in* C, *we have* $e_t(X_{CR}) \geq n_t + (2g - b_1(\Pi^{-1}(t)))$.

Proof. Since Π_{CR} is a proper surjective morphism, Π_{CR} can be decomposed into a succession of blowing-downs. Since the blowing-downs does not affect the first Betti number of fibers, we obtain $b_1(\Pi_{CR}^{-1}(t)) = b_1(\Pi^{-1}(t))$. On the other hand, we have $b_2(\Pi_{CR}^{-1}(t)) \geq b_2(p_{CR}^{-1}(t)) = n_t + 1$. Therefore a desired inequality follows from the equality $e_t(X_{CR}) = \chi_{top}(\Pi_{CR}^{-1}(t)) - (2 - 2g)$. □

Lemma 4.3.5. *For every* $t \in C$, *we have* $e_t(X_{CR}) \leq 2 \cdot e_t(X)$.

Proof. By Lemma 4.3.2 and Lemma 4.3.4, we have $\beta_t(h_{CR}) \leq \frac{1}{2} \cdot \{e_t(X_{CR}) - (2g - b_1(\Pi^{-1}(t))\}$. Since $e_t(X_{res}) = e_t(X_{CR}) - \beta_t(h_{CR})$, we obtain

$$e_t(X_{CR}) \leq 2 \cdot e_t(X_{res}) - (2g - b_1(\Pi^{-1}(t)). \qquad \cdots (4.3.6)$$

Assume that h_{res} contructs a curve contained in $\Pi_{res}^{-1}(t)$. Then, by the condition $(C1)$ in Proposition 2.0.1, we obtain $e_t(X) = e_t(X_{res}) - 1$. By the condition $(C2)$, there exists a divisor $F \subset X$ such that $\Pi^{-1}(t) = 2 \cdot F$. The dualizing sheaf ω_F is of degree $g - 1$. Since $\deg \omega_F$ is even, g is odd. Hence we have $g \geq 3$. On the other hand, since $b_1(\Pi^{-1}(t)) \leq$ (the arithmetic genus of F), we obtain $b_1(\Pi^{-1}(t)) \leq \dfrac{g+1}{2}$. Thus we obtain $2g - b_1(\Pi^{-1}(t)) \geq 2$. Therefore a desired inequality follows from (4.3.6).

Assume that h_{res} contracts no curves contained in $\Pi_{res}^{-1}(t)$. Then we have $e_t(X_{CR}) = e_t(X)$. Since $2g \geq b_1(\Pi^{-1}(t))$, (4.3.6) also lead us to obtain a desired inequality. □

Lemma 4.3.7. *For every* i $(1 \leq i \leq n)$, we have $2 \leq m_i \leq g + 1$.

Proof. By the mnimality of Y_{CR} (cf. Remark 1.1.4) every g_i^{-1} is a blowing-up at a singular point of B_i. Therefore we obtain $m_i \geq 2$.

By the condition (A) in Proposition 2.0.1, $\tilde{B}_i = (g_1 \circ g_2 \circ \cdots \circ g_i)^{-1}[\tilde{B}]$ has multiplicity $\leq g + 1$ at every point in $\tilde{B}_i$. Since no three curves meet at one point, we have $\mathrm{mult}_a(B_i) \leq \mathrm{mult}_a(\tilde{B}_i) + 2$ for every point a in B_i. Therefore the second inequality follows. □

(4.4) *Proof of Theorem* 4.0.2. First we will prove the first inequalities of (4.0.5) and (4.0.6).

By Lemma 4.3.7, for every i $(1 \leq i \leq n)$ we otain $1 \leq \left[\dfrac{m_i}{2}\right] \leq g$. Therefore the first inequalities follows from Lemma 4.3.1.

Next we will prove the second inequalities of (4.0.5) and (4.0.6).

Since $\left[\dfrac{m_i}{2}\right]$'s are integer, the theory of quadratic equation leads us to obtain

$$-2 \cdot \sum_{pCR(E_i^{(n)})=t} \left(\left[\frac{m_i}{2}\right] - 1\right) \cdot \left(\left[\frac{m_i}{2}\right] - g\right) \leq 2 \cdot n_t \cdot \left[\frac{g-1}{2}\right] \cdot \left(g - 1 - \left[\frac{g-1}{2}\right]\right).$$

Hence Lemma 4.3.1 implies that

$$4 \cdot (2g+1) \cdot d_t(X) - g \cdot e_t(X_{CR}) \leq 2 \cdot n_t \cdot \left[\frac{g-1}{2}\right] \cdot \left(g - 1 - \left[\frac{g-1}{2}\right]\right).$$

On the other hand, since $2g \geq b_1(\Pi^{-1}(t))$, Lemma 4.3.4 implies $e_t(X_{CR}) \geq n_t$. Thus we obtain

$$4 \cdot (2g+1) \cdot d_t(X) \leq 2 \cdot \left(\left[\frac{g-1}{2}\right] \cdot \left(g - 1 - \left[\frac{g-1}{2}\right]\right) + g\right) \cdot e_t(X_{CR}).$$

Therefore desired inequalites follows from Lemma 4.3.5 and some calculation.

(4.5) Examples and Problems. First we will construct an example such that g

is even and the equality $d_t(X) = \dfrac{g^2}{4(2g+1)} e_t(X)$ holds.

Example 4.5.1. Let Y be $\mathbf{P}^1 \times \mathbf{C}$ and g an even number. Put

$$B = \{(x, y) \in \mathbf{P}^1 \times \mathbf{C};\ y = 0\} \cup \left(\bigcup_{i=1}^{g+1} \{(x, y);\ x = a_i y^2\}\right) \cup \left(\bigcup_{i=1}^{g+1} \{(x, y);\ \frac{1}{x} = a_i y^2\}\right)$$

where a_i's are complex numbers such that $a_i \neq a_j$ if $i \neq j$. Then we make calculation and obtain $\delta_0 = 4g(g+1)$, $e_0(X) = 2$ and $d_0(X) = \dfrac{g^2}{2(2g+1)}$.

If g is odd, I cannot find any examples which satisfy the equality $d_t(X) = \dfrac{g^2+1}{4(2g+1)} e_t(X)$ if g is odd.

Problem 4.5.2. Find the upper bound of $d_t(X)/e_t(X)$ when g is odd.

Miyaoka proved $c_1^2(X) \leq 3c_2(X)$ if X is a projective general type surface (see [M]).

Problem 4.5.3 (cf. [C]). Are there any hyperelliptric fibrations which satisfy $(c_1(X))^2 = 3c_2(X)$ and $\kappa(X) = 2$?

The local Euler number $e_t(X)$ can be defined even if $\Pi: X \to C$ is not hyperelliptric.

Problem 4.5.4 (cf. [$S1$] and [W]). Define the local canonical degree $d_t(X)$ for every fibration $\Pi: X \to C$. And find the upper bound of $d_t(X)/e_t(X)$ when $e_t(X) \neq 0$.

Problem 4.5.5. Find the relation between $d_t(X)$ and the number of fixed points in a fiber on t of the relative canonical map.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

Current Address
SYSTEM DEVELŌPMENT DEPARTMENT
ISHIKAWAJIMA SYSTEM
TECHNŌLOGY Co. Lim.

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