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# Supplement to the decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators

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In the previous paper [U], we calculated the trace of the Hecke operator $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on the space of cusp forms $S(k+1/2, N, \chi)$ of half-integral weight under the assumption $\chi^2 = 1$. The purpose of this calculation is to find a relation between these traces and those of the Hecke operators of integral weight $2k$. When the 2-order of the level $N$ ($= \mathrm{ord}_2(N)$) is small, we found certain relations between the traces, in [U].

In this paper, we report relations for the remaining cases.

## Notations.

In the following, we keep the notations and the assumptions in the previous paper [U]. For a prime number $p$, $|\ |_p$ denotes the $p$-adic norm such that $|p|_p = p^{-1}$. For a Dirichlet character $\chi$, $f(\chi)$ denotes the conductor of $\chi$. For a finite-dimensional vector space $V$ over $\mathbf{C}$ and a linear operator $T$ on $V$, $\mathrm{tr}(T|V)$ denotes the trace of $T$ on $V$.

For the sake of simplicity, we omit the subscripts of the Hecke operators, i.e., $\tilde{T}(n^2) = \tilde{T}_{k+1/2,N,\chi}(n^2)$, $T(n) = T_{2k,N}(n)$, and $W(N_0) = [W(N_0)]_{2k}$, etc..

## Statement of results

**Theorem.** *Let $k$ be a positive integer and $N$ a positive integer divisible by* 4 *and put $M = 2^{-\mu}N$ with $\mu = \mathrm{ord}_2(N)$. Let $\chi$ be an even character modulo $N$ such that $\chi^2 = 1$. We have the following relations between traces.*

(1) *Suppose that $4 \le \mu \le 6$ and besides $f(\chi_2)$ divides* 4 *if $\mu = 4, 6$. For all natural numbers $n$ with $(n, N) = 1$,*

$$\mathrm{tr}(\tilde{T}(n^2)|S(k+1/2, 2^{\mu}M, \chi)) = 2\Theta[2k, 2^{\mu-2}M, \chi] \quad \textit{if } k \ge 2$$

*and*

$$\mathrm{tr}(\tilde{T}(n^2)|V(2^{\mu}M, \chi)) = 2\Theta[2, 2^{\mu-2}M, \chi].$$

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(2) *Suppose that* $\mu=7$. *Put* $\varepsilon=\chi_2(-1)$. *For all natural numbers n with* $(n, N)=1$,

$$\mathrm{tr}(\tilde{T}(n^2)|S(k+1/2, 2^7M, \chi)) - \mathrm{tr}\left(\tilde{T}(n^2)\middle|S\left(k+1/2, 2^6M, \left(\frac{2\varepsilon}{}\right)\chi_M\right)\right)$$

$$= 2\left(\Theta[2k, 2^5M, \chi] - \Theta\left[2k, 2^4M, \left(\frac{2\varepsilon}{}\right)\chi_M\right]\right) \quad \textit{if } k\geq 2$$

and

$$\mathrm{tr}(\tilde{T}(n^2)|V(2^7M, \chi)) - \mathrm{tr}\left(\tilde{T}(n^2)\middle|V\left(2^6M, \left(\frac{2\varepsilon}{}\right)\chi_M\right)\right)$$

$$= 2\left(\Theta[2, 2^5M, \chi] - \Theta\left[2, 2^4M, \left(\frac{2\varepsilon}{}\right)\chi_M\right]\right).$$

(3) *Suppose that* $\mu\geq 8$. *For all natural numbers n with* $(n, N)=1$,

$$\mathrm{tr}(\tilde{T}(n^2)|S(k+1/2, 2^{\mu}M, \chi)) - \mathrm{tr}\left(\tilde{T}(n^2)\middle|S\left(k+1/2, 2^{\mu-1}M, \chi\left(\frac{2}{}\right)\right)\right)$$

$$= 2\left(\Theta[2k, 2^{\mu-2}M, \chi] - \Theta\left[2k, 2^{\mu-3}M, \chi\left(\frac{2}{}\right)\right]\right) \quad \textit{if } k\geq 2$$

*and*

$$\mathrm{tr}(\tilde{T}(n^2)|V(2^{\mu}M, \chi)) - \mathrm{tr}\left(\tilde{T}(n^2)\middle|V\left(2^{\mu-1}M, \chi\left(\frac{2}{}\right)\right)\right)$$

$$= 2\left(\Theta[2, 2^{\mu-2}M, \chi] - \Theta\left[2, 2^{\mu-3}M, \chi\left(\frac{2}{}\right)\right]\right).$$

The above notations are as follows.

$$\Theta[2k, 2^{\mu-2}M, \chi] = \textstyle\sum_0 A(n, \tilde{L}_0)\,\mathrm{tr}(W(\tilde{L}_0)T(n)|S(2k, \tilde{L}_0L_1)),$$

where $\tilde{L}_0$ in the sum $\sum_0$ runs over all square divisors of $2^{\mu-2}M$ such that $\mathrm{ord}_2(\tilde{L}_0)\neq 2$ and

$$L_1 = 2^{\mu-2}M\prod_{p|\tilde{L}_0}|2^{\mu-2}M|_p.$$

$$A(n, \tilde{L}_0) = \prod_{p|2M}\lambda(p, n; \mathrm{ord}_p(\tilde{L}_0)/2),$$

where for a prime divisor $p$ of $M$,

$$\lambda(p, n; e) = \begin{cases} 1, & \text{if } e=0 \\ 1+\left(\dfrac{-n}{p}\right), & \text{if } 1\leq e\leq [(\nu-1)/2] \\ \chi_p(-n), & \text{if } e=\nu/2 \text{ and } \nu \text{ is even,} \end{cases}$$

with $\nu = \mathrm{ord}_p(M)$, and

$$\lambda(2, n; e) = \begin{cases} 1, & \text{if } e = 0 \\ 0, & \text{if } e = 1 \\ \xi(n)\left(1 + \left(\frac{2}{n}\right)\right), & \text{if } 2 \le e \le [(\mu - 3)/2] \\ \xi(n)\chi_2(-n), & \text{if } e = (\mu/2) - 1 \text{ and } \mu \text{ is even,} \end{cases}$$

with $\xi(n) = \left(1 - \left(\frac{-1}{n}\right)\right)\Big/2$. We decompose the character $\chi = \chi_2 \cdot \chi_M$, where $\chi_2$ is the 2-primary component of $\chi$ and $\chi_M$ is the odd part of $\chi$.

*Proof.* We can deduce these relations, similarly to the proof of the Theorem of [U].

**Remarks.** (1) Combining this Theorem with the Theorem of [U], we get the relations between the traces except for the case of $\mu = 6$ and $f(\chi_2) = 8$.

(2) When the 2-order of the level $N$ is big, the form of the relations differ from those of the Theorem of [U]. In particular, we point out the following. In the case of the Theorem of [U], cusp forms of half-integral weight $k + 1/2$ of level $N$ correspond to cusp forms of integral weight $2k$ of level $N/2$. But, in the case of this paper, cusp forms of half-integral weight $k + 1/2$ of level $N$ correspond to cusp forms of integral weight $2k$ *of level* (*at most*) $N/4$.

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## References

[U] M. Ueda, The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators, J. Math. Kyoto Univ., **28** (1988), 505–555.