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On the asymptotic behavior of the increments of a Wiener process

By

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§ 1. Introduction

For a Wiener process $\{W(t); 0 \leqq t < +\infty\}$, Erdös-Rényi law [5] implies the following strong limit theorem:

$$\lim_{T\to\infty} \sup_{0\leqq t\leqq T-c\log T} |W(t+c\log T)-W(t)|/\log T=\sqrt{2c} \text{ a.s. } (c>0)$$

After them varieties of such limit theorems are proved (for examples, see [3], [4]). Furthermore, P. Révész [8] has investigated much sharper limit theorems using the notion of upper class or lower class.

Definition 1. Let f and g be two real (not necessarily random functions) defined on the positive half line, and we assume that g is a monotone function. Then g is called an upper-upper function of f(briefly, $g\in UUC(f)$) if and only if there exists $t_0>0$ such that for all $t>t_0, f(t)<g(t)$ holds, and g is called an upper-lower function of f (briefly, $g\in ULC(f)$) if and only if there exists an infinite sequence $t_1<t_2<\cdots\to+\infty$ such that $f(t_n)>g(t_n)$ holds for all n.

In this paper, we will give an integral test which determines whether a given determinstic function belongs to UUC or ULC of the folowing functionals of a Wiener process with probability one:

$$X_0(T)=\sup_{0\leqq s\leqq a_T}\sup_{0\leqq t\leqq T-s}|W(s+t)-W(t)|/\sqrt{a_T},$$

$$X_1(T)=\sup_{0\leqq s\leqq a_T}\sup_{0\leqq t\leqq T-a_T}|W(s+t)-W(t)|/\sqrt{a_T},$$

$$X_2(T)=\sup_{0\leqq s\leqq a_T}\sup_{0\leqq t\leqq T-a_T}(W(s+t)-W(t))/\sqrt{a_T},$$

$$X_3(T)=\sup_{0\leqq t\leqq T-a_T}|W(t+a_T)-W(t)|/\sqrt{a_T},$$

$$X_4(T)=\sup_{0\leqq t\leqq T-a_T}(W(t+a_T)-W(t))/\sqrt{a_T},$$

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where a_T *is a real function of* T *satisfying the following conditions*;
(a-1) $0 < a_T \leqq T$,
(a-2) a_T *is a non-decreasing function and*
(a-3) $T - a_T$ *is also a non-decreasing function.*

We remark that the above assumption (a-3) is a little bit weaker than that in Révész's paper [8]. (He assumed that a_T/T is non-increasing instead of (a-3)).
The case of multi-dimensional Wiener process is also investigated.

Remark 1. We are concerning only with the behavor of functions near at the infinity, so all statements in this paper about functions defined on the half line hold only at some neighborhood of the infinity.

We use letters $c, c_1, c_2, \ldots$ for non-essential positive absolute constants which may be different from line to line. We also use notation $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, $\log_{(1)} t = \log t$ and $\log_{(k)} t = \log(\log_{(k-1)} t)$.

Acknowledgment. I am indebted to Professor P. Révész who kindly sent me K. Grill's pre-print "On the increments of the Wiener process" (to be published in the Annals of Probability) giving a reply to my first draft. A part of his results (Theorem 1 and 2) give exactly the same conclusion as mine, though regularity conditions are somewhat different. The main difference between my first draft and his is that he is investigating $X_0(T)$ instead of $X_1(T)$. Of course $X_1(T) \leqq X_0(T)$. Therefore I gave a new proof of the original Theorem 1 for $X_0(T)$. I have given the complete proofs in the unifying manner which is standard technique after Chung-Erdös-Sirao [2] and Sirao [9]. Unfortunately the proof of his Theorem 2 is omitted saying "there are no new ideas needed", but in my feeling this case is more delicate than the case of his Theorem 1. Taking account of these and the difference of regularity conditions, I believe that this manuscript is also worth publishing.

§2. Main results.

Set $T_k = e^k$, $d_k = a_{T_k}$, $b_k = T_k - d_k$,

$$\gamma_k = \sqrt{\log(T_k/d_k) + \operatorname{loglog} T_k} \quad \text{and} \quad \delta_k = d_k/\gamma_k^2.$$

Theorem 1. If

$$I_U(g) = \sum_k^{\infty} \left(\frac{b_{k+1}}{\delta_k} \vee 1 \right) \left(\frac{d_{k+1} - d_k}{\delta_k} \vee 1 \right) \gamma_k^{-1} e^{-g^2(T_k)/2} < +\infty,$$

then $g \in UUC(X_i)$, $i = 0, 1, 2, 3, 4$.

Theorem 2. If

$$I_L(g) = \sum_k^{\infty} \left(\frac{b_{k-1}}{\delta_k} \vee 1 \right) \left(\frac{d_k - d_{k-1}}{\delta_k} \vee 1 \right) \gamma_k^{-1} e^{-g^2(T_k)/2} = +\infty,$$

then $g \in ULC(X_i)$, $i = 0, 1, 2, 3, 4$.

Corollary 1.

Case I. If $\overline{\lim}_{T\to\infty} \frac{T - a_T}{a_T} \log_{(2)} T < +\infty$, *then*
$g \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if*

$$\int^{+\infty} \frac{(\log_{(2)} t)^{1/2}}{t} e^{-g^2(t)/2}\, dt < +\infty .$$

Example.

$$g(T) = \sqrt{2\log_{(2)} T + 3\log_{(3)} T + 2\log_{(4)} T + \cdots + (2 + \varepsilon)\log_{(k)} T}$$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ if and only if $\varepsilon > 0$.

$a_T = T - cT/(\log_{(2)} T)^{\alpha}$, $1 \leqq \alpha$, $0 \leqq c$.

$a_T = T - c(\log T)^{\beta}$, $0 \leqq c$, $0 \leqq \beta$ (this example is not included in Grill's Theorem 2.)

Case II. If

(i) $\lim_{T\to+\infty} \frac{T - a_T}{a_T} \log_{(2)} T = +\infty$,

(ii) $\lim_{T\to+\infty} a_T/T = 1$ *and*

(iii) $\overline{\lim}_{k\to\infty} b_{k+1}/b_k < +\infty$, *then*

$g \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if*

$$\int^{+\infty} \frac{(\log_{(2)} t)^{3/2}}{t} e^{-h^2(t)/2}\, dt < +\infty, \text{ where}$$

$$h^2(T) = g^2(T) - 2\log\{(T - a_T)/a_T\}.$$

Example.

$$g(T) = \sqrt{2\log_{(2)} T + 5\log_{(3)} T + 2\log\{(T - a_T)/a_T\} + 2\log_{(4)} T + \cdots + (2 + \varepsilon)\log_{(k)} T}$$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$, *if and only if* $\varepsilon > 0$.

(*Here we assume that* g *is non-decrasing.*)

In particular, $a_T = T - cT/(\log_{(2)} T)^{\alpha}$, $0 < \alpha < 1$, $0 < c$, *then*

$$g(T) = \sqrt{2\log_{(2)} T + (5 - 2\alpha)\log_{(3)} T + 2\log_{(4)} T + \cdots + (2 + \varepsilon)\log_{(k)} T}$$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

Remark. *Case I and II are not considered in Révész's paper* [8], *since* a_T/T *can not be non-increasing unless* $a_T = T$.

Case III. *If*

(*i*) $\overline{\lim}_{T\to+\infty} a_T/T < 1$ and

(ii) $1 < \underline{\lim}_{k\to+\infty} d_{k+1}/d_k \leqq \overline{\lim}_{k\to+\infty} d_{k+1}/d_k < +\infty$, *then*

$g \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$, *if and only if*

$$\int^{+\infty} \frac{1}{t} e^{-h^2(t)/2}\, dt < +\infty, \text{ where}$$

$h^2(T) = g^2(T) - \{2\log(T/a_T) + 3\log(\log(T/a_T) + \log_{(2)} T)\}$.

Example.

$g(T) = \sqrt{2\log(T/a_T) + 3\log\{\log(T/a_T) + \log_{(2)}T\} + 2\log_{(2)}T + \cdots + (2+\varepsilon)\log_{(k)}T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$, *if and only if* $\varepsilon > 0$.

(*Here we assume that g is non-decreasing.*)

In particular, $a_T = \alpha T$ $0 < \alpha < 1$, *then*

$g(T) = \sqrt{2\log_{(2)}T + 5\log_{(3)}T + 2\log_{(4)}T + \cdots + (2+\varepsilon)\log_{(k)}T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

$a_T = Te^{-(\log_{(2)}T)^\alpha}$, *then for* $0 < \alpha \leqq 1$

$g(T) = \sqrt{2(\log_{(2)}T)^\alpha + 2\log_{(2)}T + 5\log_{(3)}T + 2\log_{(4)}T + \cdots + (2+\varepsilon)\log_{(k)}T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$,

and for $\alpha \geqq 1$

$g(T) = \sqrt{2(\log_{(2)}T)^\alpha + 2\log_{(2)}T + (2+3\alpha)\log_{(3)}T + 2\log_{(4)}T + \cdots + (2+\varepsilon)\log_{(k)}T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

$a_T = T/(\log T)^\alpha$, $\alpha > 0$, *then*

$g(T) = \sqrt{(2\alpha+2)\log_{(2)}T + 5\log_{(3)}T + 2\log_{(4)}T + \cdots + (2+\varepsilon)\log_{(k)}T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

$a_T = \alpha T^\beta$ $0 < \alpha \leqq 1, 0 < \beta < 1$, *then*

$g(T) = \sqrt{2(1-\beta)\log T + 5\log_{(2)}T + \cdots 2\log_{(3)}T + \cdots + (2+\varepsilon)\log_{(k)}T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

Case IV. *if*

(*i*) $\lim_{k\to+\infty} d_{k+1}/d_k = 1$,

$\left(\textit{this implies that } \lim_{k\to+\infty} d_k/T_k = 0 \textit{ and } \lim_{k\to+\infty} \frac{\log(T_k/d_k)}{k} = 1 \right)$,

(*ii*) $\lim_{k\to+\infty} k(d_{k+1} - d_k)/d_k = +\infty$ *and*

(iii) $0 < \varliminf_{k\to+\infty} (d_{k+1} - d_k)/(d_k - d_{k-1}) \leqq \varlimsup_{k\to+\infty} (d_{k+1} - d_k)/(d_k - d_{k-1}) < +\infty$,

then $g \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if*

$$\sum_k^\infty \frac{T_k}{d_k} k^{3/2} \frac{d_{k+1} - d_k}{d_k} e^{-g^2(T_k)/2} < +\infty.$$

Example. $a_T = \exp\left(\int_1^T \varepsilon(t)/t \, dt\right)$, *where* $\varepsilon(t)$ *is a continuous slowly varying function such that* $\varepsilon(t) \downarrow 0$ *as* $t \to +\infty$ *and* $\lim_{T\to+\infty} \varepsilon(T) \log T = +\infty$. *Then* $g \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if*

$$\int^{+\infty} \frac{(\log t)^{3/2}}{t} e^{-h^2(t)/2} \, dt < +\infty, \textit{ where}$$

$h^2(T) = g^2(T) - \{2\log(T/a_T) + 2\log\varepsilon(T)\}$.

$g(T) = \sqrt{2\log(T/a_T) + 2\log \varepsilon(T) + 5\log_{(2)} T + 2\log_{(3)} T + \cdots + (2+\varepsilon)\log_{(k)} T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

(*Here we assume that* g *is non-decreasing.*)

In particular, $a_T = \exp\{\beta(\log T)^\alpha\}$, $0 < \alpha < 1$, $0 < \beta$, *then*

$g(T) = \sqrt{2\log T - 2\beta(\log T)^\alpha + (3 + 2\alpha)\log_{(2)} T + 2\log_{(3)} T + \cdots + (2+\varepsilon)\log_{(k)} T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

Case V. *If*

(*i*) $\lim_{k\to+\infty} d_{k+1}/d_k = 1$ *and*

(*ii*) $\varlimsup_{k\to+\infty} k(d_{k+1} - d_k)/d_k < +\infty$, *then*

$g \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if*

$$\int^{+\infty} \frac{(\log t)^{1/2}}{t} e^{-h^2(t)/2} \, dt < +\infty, \textit{ where}$$

$h^2(T) = g^2(T) - 2\log(T/a_T)$.

Example.

$g(T) = \sqrt{2\log(T/a_T) + 3\log_{(2)} T + 2\log_{(3)} T + \cdots + (2+\varepsilon)\log_{(k)} T}$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

(*Here we assume that g is non-decreasing.*)

$a_T = c(\log T)^\alpha$ $0 \leqq \alpha$, $0 < c$, *then*

$$g(T) = \sqrt{2\log T + (3 - 2\alpha)\log_{(2)} T + 2\log_{(3)} T + \cdots + (2 + \varepsilon)\log_{(k)} T}$$

$\in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if* $\varepsilon > 0$.

Corollary 2 (*K. Grill's Theorem* 1 *and* 2.).

(*i*) *If* $\limsup_{T\to\infty} a_T/T < 1$ *and* $a_T = c_0 \exp\left(\int_c^T \eta(y)/y\,dy\right)$,

where $\eta(y)$ *is slowly varying as* $y \to \infty$, *then*

$g(T) \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if*

$$K(g) = \int_1^\infty (1 + \eta(t)\gamma^2(t))\frac{\gamma(t)}{a_t}e^{-g^2(t)/2}\,dt < +\infty,$$

where $\gamma(t) = (\log(t/a_t) + \log_{(2)} t)^{1/2}$.

(*ii*) *If* $\lim_{T+\infty} a_T/T = 1$ *and* $a_T = T(1 - b_T)$, *where* b_T *is decreasing to* 0 *and slowly varying as* $T \to \infty$, *then*

$g(T) \in UUC(X_i)$ $i = 0, 1, 2, 3, 4$ *if and only if*

$$K(g) = \int_1^\infty (1 + b_t \log_{(2)} t)\frac{(\log_{(2)} t)^{1/2}}{t}e^{-g^2(t)/2}\,dt < +\infty.$$

Remark 2. Corollary 2, (i) corresponds to Case III-V of Corollary 1 and Corollary 2, (ii) corresponds to Case I-II of Corollary 1.

Remark 3. When $\eta(t)$ or b_t in Corollary 2 is continuous slowly varying function, then our conditions (a-2) and (a-3) are not required.

Let W^d be a d-dimensional Wiener process. Set

$$X_5(T) = \sup_{0 \leqq s \leqq a_T} \sup_{0 \leqq t \leqq T-s} \| W^d(s+t) - W^d(t) \| / \sqrt{a_T},$$

$$X_6(T) = \sup_{0 \leqq t \leqq T - a_T} \| W^d(t + a_T) - W^d(t) \| / \sqrt{a_T},$$

where $\|\ \|$ denotes the d-dimensional Euclidean metric.

Theorem 3. *If*

$$I_U^d(g) = \sum_k^\infty \left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\left(\frac{d_{k+1} - d_k}{\delta_k}\right) \vee 1\right)\gamma_k^{d-2} e^{-g^2(T_k)/2} < +\infty,$$

then $g \in UUC(X_i)$, $i = 5, 6$.

Theorem 4. *If*

$$I_L^d(g) = \sum_k^\infty \left(\frac{b_{k-1}}{\delta_k} \vee 1\right)\left(\frac{d_k - d_{k-1}}{\delta_k} \vee 1\right)\gamma_k^{d-2} e^{-g^2(T_k)/2} = +\infty,$$

then $g \in ULC(X_i)$, $i = 5, 6$.

§3. Preliminary lemmas.

In order to prove our theorems, we will list several well known lemmas, which are modified for our purpose.

Lemma 1 (*cf.* [2], *lemma* 1). *It is enough to prove Theorem* 1 *and* 2 *for g which satisfies*

(*i*) $\gamma_k/2 \leqq g(T_k) \leqq 3\gamma_k$,

(*ii*) $\lim_{k\to\infty} \left(\frac{b_{k-1}}{\delta_k} \vee 1\right)\left(\frac{d_k - d_{k-1}}{\delta_k} \vee 1\right)\gamma_k^{-1} e^{-g^2(T_k)/2} = 0$ *when* $I_L(g) = +\infty$.

Proof. (*i*) First we assume that $I_U\{g\} < +\infty$. We will show that there exists k_0 such that for $^\forall k > k_0$, $g(T_k) \geqq \gamma_k/2$ holds. Assume the contrary, i.e. $^\exists 2 \leqq k_1 < k_2 <, \cdots \to +\infty$ such that $g(T_{k_n}) < \gamma_{k_n}/2$. Then,

$$I_U(g) \geqq \sum_{n=1}^{k_n} \left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\{-g^2(T_k)/2\} \geqq c\gamma_{k_n}^{-1}$$

$$\exp\{-g^2(T_{k_n})/2\} \sum_{n=1}^{k_n} \left(\frac{b_{k+1}}{\delta_k} \vee 1\right)$$

$$\geqq c\gamma_{k_n}^{-1} \exp\{-g^2(T_{k_n})/2\}\{T_{k_n}/c_{k_n}) \vee k_n\}/2$$

$$\geqq c\gamma_{k_n}^{-1} \exp\{-\log(T_{k_n}/d_{k_n}) + \log k_n)/4\}\{(T_{k_n}/d_{k_n}) \vee k_n\}/2$$

$\to +\infty$ as $k_n \to +\infty$, which yields a contradiction.

By simple calculus we have

$$I_\gamma \equiv \sum_k^\infty \left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{d_{k+1} - d_k}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp(-9\gamma_k^2/2) < +\infty.$$

Therefore choosing a monotone continuous function $g' \leqq g$ such that $g'(T_k) = g(T_k) \wedge (3\gamma_k)$, we have $I_U(g') \leqq I_U(g) + I_\gamma < +\infty$, and $\gamma_k/2 \leqq g'(T_k) \leqq 3\gamma_k$. Obviously if $g' \in UUC$, then $g \in UUC$. It shows that it is enough to prove Theorem 1 under the condition (i).

Next we assume that $I_L(g) = +\infty$. Choose a monotone function g' such that $g'(T_k) = g(T_k) \vee (\gamma_k/2)$ and $g' \geqq g$. If $g(T_k) > \gamma_k/2$ holds for all k, then nothing is to be proved. Otherwise there exists a sequence $k_1 < k_2 \cdots \to +\infty$ such that $g'(T_{k_n})$

$= \gamma_{k_n}/2$. Then we have

$$I_L(g') \geqq \sum_k^{k_n} \left(\frac{b_{k-1}}{\delta_k} \vee 1\right)\left(\frac{d_k - d_{k-1}}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp\{-g'(T_k)/2\}$$

$$\geqq c\gamma_{k_n}^{-1} \exp(-\gamma_{k_n}^2/8) \sum_{k=1}^{k_n} \left(\frac{b_{k-1}}{\delta_k} \vee 1\right)$$

$$\geqq c\gamma_{k_n}^{-1} \exp(-\gamma_{k_n}^2/8)\{(T_{k_n}/d_{k_n}) \vee k_n\}$$

$$\to +\infty \text{ as } k_n \to +\infty.$$

Let g_γ be a monotone continuous function satisfying $g_\gamma(T_k) = 3\gamma_k$. Easily we can check $I_L(g' \wedge g_\gamma) = +\infty$. Since $g_\gamma \in UUC$ by Theorem 1 (Theorem 1 must be proved before Theorem 2), if $g' \wedge g_\gamma \in ULC$, then $g' \in ULC$.

(ii) Since $\lim_{k \to \infty} \left(\frac{b_{k-1}}{\delta_k} \vee 1\right)\left(\frac{d_k - d_{k-1}}{\delta_k} \vee 1\right)\gamma_k^{-1} \exp(-9\gamma_k^2/2) = 0$ and $I_L(\gamma) < +\infty$, where $\gamma(T_k) = 3\gamma_k$, we can replace $g(T_k)$ by a smaller function (say g'') which satisfies (i), (ii) and $I_L(g'') = +\infty$. Clearly $g'' \in ULC$ implies $g \in ULC$. This completes the proof of Lemma 1.

Lemma 2. *Let $\{A_n\}$ be a sequence of events.*

(i) $\sum_{n=1}^{\infty} P(A_n) = +\infty$,

(ii) for each n there exists a finite subsequence $n \leqq n_1 < n_2 < \cdots < n_{i(n)}$ *such that*

(ii-a) $\sum_{k=1}^{i(n)} P(A_n \cap A_{n_k}) \leqq c_1 P(A_n)$,

(ii-b) $P(A_n \cap A_i) \leqq c_2 P(A_i)$ *holds if* $i \neq n_k$ *and* $n < i$,

then $P(\overline{\lim} A_n) \geqq 1/c_2$.

Lemma 3 ([2] *lemma* 3, 4). *Let (X, Y) be a two dimensional random variable with the Gaussian distribution such that* $E[X] = E[Y] = 0$, $E[X^2] = E[Y^2] = 1$ *and* $E[XY] = r$. Set $\phi(x) = P(X \geqq x)$, $x > 0$. *Then,*

(i) $c_1(x+1)^{-1}\exp(-x^2/2) \leqq \phi(x) \leqq P(|X| \geqq x) \leqq c_2 x^{-1}\exp(-x^2/2)$,

(ii) $P(X \geqq x, Y \geqq y) \leqq c_\varepsilon \phi(x)\phi(y)$ *for any* $-1 < r < \varepsilon/(xy)$ *and* $x, y > 0$, *where* $\lim_{\varepsilon \downarrow 0} c_\varepsilon = 1$.

(iii) $P(X \geqq x, Y \geqq y) \leqq c \exp\{-(1-r)y^2/4\}\ \phi(x)$ *for any* $y \geqq x \geqq 0$ *and* $r \geqq 0$.

The idea of the following lemma has been used in the proof of Theorem 1 in [9]. After him many people modified it.

Lemma 4 ([1], [7]) *Let $\{X(t); t \in S\}$ be a centered separable Gaussian process with the pseudo-metric* $d_X(s, t) \equiv (E[(X(s) - X(t))^2])^{1/2}$ *satisfying the following conditions;*

(i) (S, d_X) *is a compact space,*

(ii) $0 < \underline{\sigma} \leqq [E(X^2(t))]^{1/2} \leqq \bar{\sigma} < +\infty$ *for* $^\forall t \in S$.

Then for $0 < \varepsilon_j \leqq \sqrt{2}$, $0 < \lambda_{j+1}$, $j = 0, 1, 2, \dots$ *and* $0 < x$,

$$P(\sup_{t\in S} X(t) \geqq \bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1})) \leqq (N_X(S, \underline{\sigma}\varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(S, \underline{\sigma}\varepsilon_j)e^{-\lambda_j^2/2})\phi(x),$$

where $N_X(S, \varepsilon)$ *is the minimal number of balls* $B(t, \varepsilon) \equiv \{s \in S;\ d_X(s, t) \leqq \varepsilon\}$ *whose union covers* S.

§4. Proofs of Theorems.

First, we shall prove two lemmas to prove Theorem 1.

Lemma 5. *For* $\gamma_k/3 \leqq x_k \leqq 3\gamma_k$,

$$P(\sup_{0<s\leqq d_k} \sup_{0\leqq t \leqq T_{k+1}-s} |W(s+t) - W(t)| \geqq \sqrt{d_k}(x_k + 36 x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}))$$
$$\leqq c\left(\frac{b_{k+1}}{\delta_k} \vee 1\right)\left(\frac{d_{k+1} - d_k}{\delta_k} \vee 1\right)\phi(x_k).$$

Proof. Set $\alpha_k = 1 - 1/\gamma_k^2$, $s_n = \delta_k \sum_{i=n+1}^{\infty} \alpha_k^{i-1} = d_k \alpha_k^n$ $(n = 0, 1, 2, \dots)$, $S = \{(s, t);\ 0 < s \leqq d_k,\ 0 \leqq t \leqq T_{k+1} - s\}$, $\Lambda_n = \{(s, t);\ s_{n+1} \leqq s \leqq s_n,\ 0 \leqq t \leqq T_{k+1} - s\}$ and $X_n(s, t) = (W(s+t) - W(t))/\sqrt{d_k}$, $(s, t) \in \Lambda_n$. Then we have $\underline{\sigma} = \sqrt{s_{n+1}/d_k} \leqq (E[X_n^2(s, t)])^{1/2} \leqq \sqrt{s_n/d_k} = \bar{\sigma}$ and $E[X_n(s, t) - X_n(s', t'))^2] \leqq 4(|s - s'| + |t - t'|)/a_k$. Now we can apply Lemma 4 by setting $\varepsilon_j = (\delta_k/s_{n+1})^{1/2}(j+1)^{-2}$, $\lambda_j = 6(\log(j+1))^{1/2}$ and $x = \sqrt{d_k/s_n}\, x_k$. Then we have

$$\bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1}) = x_k + 6\sqrt{(\delta_k s_n)/(d_k s_{n+1})}) \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}$$

(by using $s_n/s_{n+1} = 1/\alpha_k = (1 - 1/\gamma_k^2)^{-1} \leqq 2$)

$$\leqq x_k + 12 \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}/\gamma_k$$
$$\leqq x_k + 36 \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}/x_k,$$

$$s_n - s_{n+1} = \delta_k \alpha_k^n \leqq \delta_k,$$

$$N_X(\Lambda_n, \underline{\sigma}\varepsilon_j) = N_X(\Lambda_n, (\delta_k/d_k)^{1/2}(j+1)^{-2}) \leqq c(T_{k+1}/\delta_k)(j+1)^8$$
$$\leqq c((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} - d_k)/\delta_k) \vee 1)(j+1)^8,$$

$$N_X(\Lambda_n, \underline{\sigma}\varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(\Lambda_n, \underline{\sigma}\varepsilon_j)\exp(-\lambda_j^2)$$
$$\leqq c((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} - d_k)/\delta_k) \vee 1) \sum_{j=1}^{\infty} (j+1)^{-10},$$

$$\sum_{n=0}^{\infty} \phi(\sqrt{d_k/s_n}\, x_k) \leqq c x_k^{-1} \sum_{n=0}^{\infty} \exp\left(-\frac{d_k}{2s_n} x_k^2\right) = c x_k^{-1} \sum_{n=0}^{\infty} \exp(-\alpha_k^{-n} x_k^2/2)$$

$$= c x_k^{-1} \exp(-x_k^2/2)(1 + \sum_{n=1}^{\infty} \exp\{-(\alpha_k^{-n} - 1) x_k^2/2\})$$

(by using $x_k^2(\alpha_k^{-n} - 1) = x_k^2(\alpha_k^{-1} - 1)(\alpha_k^{-n+1} + \cdots + 1) = \alpha_k^{-1}(x_k/\gamma_k)^2(\alpha_k^{-n+1} + \cdots + 1) \geqq cn) \leqq c_1 x_k^{-1} \exp(-x_k^2/2)(1 + \sum_{n=1}^{\infty} \exp(-c_2 n))$.

Combining all these estimations, we have

$$P(\sup_{(s,t)\in S} |W(s+t) - W(t)| \geqq \sqrt{d_k}(x_k + 36\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}))$$

$$\leqq \sum_{n=0}^{\infty} P(\sup_{(s,t)\in \Lambda_n} |W(s+t) - W(t)| \geqq \sqrt{d_k}(x_k + 36\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}))$$

$$\leqq c_1((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} - d_k)/\delta_k) \vee 1) x_k^{-1} \exp(-x_k^2/2)(1 + \sum_{n=1}^{\infty} \exp(-c_2 n)).$$

This completes the proof of Lemma 5.

Lemma 6. *For* $\gamma_k/3 \leqq x_k \leqq 3\gamma_k$,

$$P_k \equiv P(\sup_{T_k \leqq T \leqq T_{k+1}} \sup_{d_k < s \leqq a_T} \sup_{0 \leqq t \leqq T-s} |W(s+t) - W(t)|/\sqrt{a_T} \geqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}$$

$$(\log(j+1))^{1/2} \leqq c((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} - d_k)/\delta_k) \vee 1)\phi(x_k).$$

Proof. (i) The case of $d_{k+1} \leqq 3T_{k+1}/4$. Set $S = \{(s, t); d_k \leqq s \leqq d_{k+1}, 0 \leqq t \leqq T_{k+1} - d_k\}$, $X(s, t) = (W(s+t) - W(t))/\sqrt{s}$. Then we have $E[X^2(s, t)] = 1$ and $E[(X(s, t) - X(s', t'))^2] \leqq 4(|s - s'| + |t - t'|)/d_k$. Now we can apply Lemma 4 by setting $\varepsilon_j = (j+1)^{-2}/x_k$, $\lambda_j = 6(\log(j+1))^{1/2}$ and $x = x_k$. Since we have

$$\bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1}) = x_k + 6\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2},$$

$$N_X(T, \underline{\sigma}\varepsilon_j) \leqq c(((T_{k+1} - d_k)x_k^2/d_k) \vee 1)(((d_{k+1} - d_k)x_k^2/d_k) \vee 1)(j+1)^8$$

$$\leqq c((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} - d_k)/\delta_k) \vee 1)(j+1)^8,$$

$$N_X(T, \underline{\sigma}\varepsilon_0) + 3 \sum_{j=1}^{\infty} N_X(T, \underline{\sigma}\varepsilon_j) \exp(-\lambda_j^2/2)$$

$$\leqq c((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} - d_k)/\delta_k) \vee 1) \sum_{j=1}^{\infty} (j+1)^{-10}.$$

This yields the desired inequality.

(ii) The case of $d_{k+1} \geqq 3T_{k+1}/4$.

First under our conditions we recall the following;

$$d_{k+1} - d_k \geqq 3T_{k+1}/4 - T_k = (3e/4 - 1)T_k,$$

$$d_k \geqq T_k - (T_{k+1} - d_{k+1}) \quad \text{(by (a-3))},$$

$$\geqq T_k - T_{k+1}/4 = (1 - e/4)T_k \text{ and}$$

$$\log k \leqq \gamma_k^2 \leqq 2 \log k.$$

Now set

$$T_{k,i} = T_k + i\Delta_k/\log k,\ i = 0, 1, \dots, i_k,$$

$$T_{k,i_k+1} = T_{k+1} \text{ and } d_{k,i} = a_{T_{k,i}},$$

where $\Delta_k = T_{k+1} - T_k$ and $i_k = [\log k]$. Then we have

$$d_{k,i+1} - d_{k,i} \leqq T_{k,i+1} - T_{k,i} \leqq \Delta_k/\log k \leqq c\ d_k/\log k \leqq c\delta_k.$$

Set $\Lambda_i = \{(s, t);\ d_k < s \leqq d_{k,i},\ 0 \leqq t \leqq T_{k,i} - s\}$, then it follows that

$$P_k \leqq \sum_{i=0}^{i_k} P\Big(\sup_{(s,t)\in \Lambda_{i+1} - \Lambda_i} |W(s + t) - W(t)|/\sqrt{d_{k,i} \vee s} \geqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2} (\log(j + 1))^{1/2}\Big).$$

Set $\Lambda_{i,1} = \{(s, t);\ d_{k,i} < s \leqq d_{k,i+1},\ 0 \leqq t \leqq T_{k,i+1} - s\}$ and

$$\Lambda_{i,2} = \{(s, t);\ d_k < s \leqq d_{k,i},\ T_{k,i} \leqq t + s \leqq T_{k,i+1}\}.$$

Obviously we have $\Lambda_{i+1} - \Lambda_i \subset \Lambda_{i,1} \cup \Lambda_{i,2}$.

Step 1.

$$P\Big(\sup_{(s,t)\in \Lambda_{i,1}} |W(s + t) - W(t)|/\sqrt{s} \geqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j + 1))^{1/2}\Big)$$
$$\leqq c((b_{k+1}/\delta_k) \vee 1)\phi(x_k).$$

Proof. Set $X(s, t) = (W(s + t) - W(t))/\sqrt{s}$, then we have $E[X^2(s, t)] = 1$ and $E[(X(s, t) - X(s', t'))^2] \leqq 4(|s - s'| + |t - t'|)/d_k$. Now we can apply Lemma 4 by setting $\varepsilon_j = (j + 1)^{-2}/x_k$, $\lambda_j = 6(\log(j + 1))^{1/2}$, $\underline{\sigma} = \bar{\sigma} = 1$, and $x = x_k$,

Since we have

$$\bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1}) = x_k + 6\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j + 1))^{1/2},$$

$$N_X(T, \underline{\sigma}\varepsilon_j) \leqq c(((T_{k,i+1} - d_{k,i})x_k^2/d_k) \vee 1)(j + 1)^8$$
$$\leqq c(((T_{k,i} - d_{k,i})/\delta_k) \vee 1)(j + 1)^8$$
$$\leqq c((b_{k+1}/\delta_k) \vee 1)(j + 1)^8, \text{ and}$$

$$N_X(T, \sigma\varepsilon_0) + 3 \sum_{j=1}^{\infty} N_X(T, \underline{\sigma}\varepsilon_j) \exp(-\lambda_j^2/2) \leqq c((b_{k+1}/\delta_k) \vee 1) \sum_{j=1}^{\infty} (j + 1)^{-10}.$$

Step 2.

$$P\Big(\sup_{(s,t)\in \Lambda_{i,2}} |W(s + t) - W(t)|/\sqrt{d_{k,i}} \geqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j + 1))^{1/2}\Big)$$
$$\leqq c\, \phi(x_k).$$

Proof. Set $\Lambda_{i,2}^{(n)} = \{(s, t);\ s_{n+1} < s \leqq s_n,\ T_{k,i} \leqq s + t \leqq T_{k,i+1}\}$, where $s_n = (d_{k,i} - n(d_{k+1} - d_k)/\gamma_k^2) \vee d_k$. Then

$$P\Big(\sup_{(s,t)\in \Lambda_{i,2}} |W(s + t) - W(t)|/\sqrt{d_{k,i}} \geqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j + 1))^{1/2}\Big)$$

$$\leqq \sum_{n=0}^{\infty} P(\sup_{(s,t)\in \Lambda_{i,2}^{(n)}} |W(s+t)-W(t)|/\sqrt{d_{k,i}} \geqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}).$$

Set $X(s, t) = (W(s+t) - W(t))/\sqrt{d_{k,i}}$. Then $\underline{\sigma} = \sqrt{s_{n+1}/d_{k,i}} \leqq (E[X^2(s, t)])^{1/2} \leqq \bar{\sigma} = \sqrt{s_n/d_{k,i}}$. Now apply Lemma 4 by setting $\varepsilon_j = (\delta_k/s_{n+1})^{1/2}\ (j+1)^{-2}$, $\lambda_j = 6(\log(j+1))^{1/2}$, $x = x_k\sqrt{d_{k,i}/s_n}$, we have

$$\bar{\sigma}(x + \sum_{j=0}^{\infty} \varepsilon_j \lambda_{j+1}) = x_k + 6\sqrt{(\delta_k s_n)/(d_{k,i}s_{n+1})} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2},$$

$$\leqq x_k + 60\,\gamma_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2},\ (s_n/s_{n+1} \leqq 10)$$

$$\leqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2},$$

$s_{n+1} - s_n \leqq (d_{k+1} - d_k)/\gamma_k^2 \leqq c\,\delta_k$,

$N_X(\Lambda_{i,2}^{(n)}, \underline{\sigma}\varepsilon_j) = N_X(\Lambda_{i,2}^{(n)}, (\delta_k/d_{k,i})^{1/2}(j+1)^{-2}) \leqq c(j+1)^8$,

$N_X(\Lambda_{i,2}^{(n)}, \underline{\sigma}\varepsilon_0) + 3\sum_{j=1}^{\infty} N_X(\Lambda_{i,2}^{(n)}, \underline{\sigma}\varepsilon_j)\exp(-\lambda_j^2) \leqq c\sum_{j=1}^{\infty} (j+1)^{-10}$.

Finaly we have

$$\sum_{n=0}^{\infty} P(\sup_{(s,t)\in \Lambda_{i,2}^{(n)}} |W(s+t)-W(t)|/\sqrt{d_{k,i}} \geqq x_k + 180\, x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1)^{1/2})$$

$$\leqq c\sum_{j=1}^{\infty} (j+1)^{-10} \sum_{n=0} \phi(x_{k'}\sqrt{d_{k,i}/s_n})$$

$$\leqq c\, x_k^{-1} \sum_{n=0}^{\infty} \exp(-d_{k,i}x_k^2/(2s_n))$$

$$\leqq c\, x_k^{-1} \exp(-x_k^2/2)(1 + \sum_{n=0}^{\infty} \exp(-\{(d_{k,i} - s_n)/(2s_n)\})x_k^2)$$

$$\leqq c_1\phi(x_k)(1 + \sum_{n=0}^{\infty} \exp(-c_2 n)).$$

Step 3, *Combining setp* 1 *and step* 2, *we have*

$$P_k \leqq c(\log k)((b_{k+1}/\delta_k) \vee 1)\phi(x_k) \leqq c((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} - d_k)/\delta_k) \vee 1)\phi(x_k).$$

Proof of Theorem 1. There exists k_0 such that $g^2(T_k) \geqq 2B$ for all $k \geqq k_0$, where $B = 180\sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2}$. Set $x_k = g(T_k) - B/g(T_k)$. Then $x_k + 180x_k^{-1} \sum_{j=1}^{\infty} j^{-2}(\log(j+1))^{1/2} \leqq g(T_k)$ and $\phi(x_k) \leqq c\phi(g(T_k)) \leqq c\gamma_k^{-1}\exp(-g^2(T_k)/2)$ (by Lemma 1). Applying Lemma 5 and 6, we have

$$P(\sup_{T_k \leqq T \leqq T_{k+1}} \sup_{0 \leqq s \leqq a_T} \sup_{0 \leqq t \leqq T-s} |W(s+t) - W(t)|/\sqrt{a_T} \geqq g(T_k))$$

$$\leqq c((b_{k+1}/\delta_k) \vee 1)(((d_{k+1} \dot{-} d_k)/\delta_k) \vee 1)\gamma_k^{-1}\exp(-g^2(T_k)/2).$$

The first Borel-Centelli lemma implies that there exists k_1 with probability one such that

$$\sup_{T_k \leqq T \leqq T_{k+1}} \sup_{0 \leqq s \leqq a_T 0 \leqq t \leqq T-s} |W(s+t) - W(t)|/\sqrt{a_T} < g(T_k)$$

holds for all $k \geqq k_1$. Then it follows for $T_k \leqq T \leqq T_{k+1}$ that

$$\sup_{0 \leqq s \leqq a_T 0 \leqq t \leqq T-s} |W(s+t) - W(t)|/\sqrt{a_T} \leqq \sup_{T_k \leqq T \leqq T_{k+1}} \sup_{0 \leqq s \leqq a_T 0 \leqq t \leqq T-s}$$

$$|W(s+t) - W(t)|/\sqrt{a_T} < g(T_k) \leqq g(T),$$

which implies $g \in UUC(X_0)$.

Proof of Theorem 2. Set $i_k = [[d_k - d_{k-1})/(\delta_k M)] \vee 1$, where $M \geqq 1$ chosen later big enough and $[x]$ means the integral part of x. Then there exists a sequence $e^{k-1} = T_{k-1} \equiv T_{k,1} \leqq \cdots \leqq T_{k,i_k} \leqq T_k = e^k$ such that $d_{k,i} \equiv a_{T_{k,i}} = d_{k-1} + (i-1)\delta_k M$, $i = 1, 2, \ldots, i_k$.
For $k \geqq 2$ let $\tau_k = \max\{p \geqq 0;\ T_{k-1} - d_{k-1} > 2(T_p - d_p)\}, = -1$ if $\{\ \} = \phi$ and $t_{1,1} = 0$, $t_{k,1} = T_{\tau_k} - d_{\tau_k}$ if $\tau_k \geqq 0$, $t_{k,1} = 0$ if $\tau_k = -1$ $(k \geqq 2)$. Easily we can check that

$$(4.1) \qquad T_{k-1} - d_{k-1} - t_{k,1} \geqq (T_{k-1} - d_{k-1})/2 \quad (k = 1, 2, \ldots) \text{ and}$$

$$(4.2) \qquad t_{k,1} \leqq t_{k',1} \text{ if } k < k'.$$

Set $j_k = [(T_{k-1} - d_{k-1} - t_{k,1})/(\delta_k M)] \vee 1$, $t_{k,j} = t_{k,1} + (j-1)\delta_k M$, $j = 1, 2, \ldots, j_k$, $B_{i,j,k} = \{W(t_{k,j} + d_{k,i}) - W(t_{k,j}) > \sqrt{d_{k,i}},\ x_k\}$, where $x_k = g(T_k)$ and $B_k = \bigcup_{1 \leqq i \leqq i_k} \bigcup_{1 \leqq j \leqq j_k} B_{i,j,k}$. Then in order to show $g \in ULC(X_4)$, it is enough to prove that for any $\varepsilon > 0$, there exists $M > 0$ such that

$$(4.3) \quad P(B_k\ i.o) \geqq 1 - \varepsilon.$$

Now we start proving (4.3) *appling Lemma* 2.

Step 1. *For any* $\varepsilon > 0$ *there exist* k_ε *and* $M = M_\varepsilon$ *such that*

$$(4.4) \qquad P(B_k) \geqq (1-\varepsilon)\sum_{i,j} P(B_{i,j,k})$$

$$\geqq c((b_{k-1}/\delta_k) \vee 1)(((d_k - d_{k-1})/\delta_k) \vee 1)\gamma_k^{-1}\exp(-x_k^2).$$

holds for all $k \geqq k_\varepsilon$.

Proof.

$$(4.5) \qquad P(B_k) \geqq \sum_{i,j} P(B_{i,j,k}) - \sum_{(i,j) \neq (i',j')} P(B_{i,j,k} \cap B_{i',j',k}).$$

Set $X_{i,j,k} = (W(t_{k,j} + d_{k,i}) - W(t_{k,j}))/\sqrt{d_{k,i}}$ and $r \equiv E[X_{i,j,k}X_{i',j',k}] = |I_{i,j,k} \cap I_{i',j',k}|/$

$\sqrt{d_{k,i}d_{k,i'}}$, where $I_{i,j,k} = \{t;\ t_{k,j} \leqq t \leqq t_{k,j} + d_{k,i}\}$ and $|I|$ means the length of an interval I.

Case 1. If $|I_{i,j,k} \cap I_{i',j',k}| = 0$, then

$$P(B_{i,j,k} \cap B_{i',j',k}) = P(B_{i,j,k}) P(B_{i',j',k}). \tag{4.6}$$

Case 2. If $t_{k,j} \leqq t_{k,j'}$ and $|I_{i,j,k} \cap I_{i',j',k}| > 0$, then

$$0 \leqq j' - j \leqq i_k^* \equiv [d_k/(\delta_k M)] + 1, \text{ and} \tag{4.7}$$

$$1 - r = (\sqrt{d_{k,i}d_{k,i'}} - ((t_{k,j} + d_{k,i} - t_{k,j'}) \vee 0) \wedge d_{k,i'})/\sqrt{d_{k,i}d_{k,i'}}$$

$$= ((t_{k,j'} - t_{k,j} - d_{k,i} + \sqrt{d_{k,i}d_{k,i'}} \wedge \sqrt{d_{k,i}d_{k,i'}} \vee (\sqrt{d_{k,i}d_{k,i'}} - d_{k,i'}))/\sqrt{d_{k,i}d_{k,i'}}$$

$$\geqq 1 \wedge \{(t_{k,j'} - t_{k,j} - \sqrt{d_{k,i}}(d_{k,i} - d_{k,i'})/(\sqrt{d_{k,i}} + \sqrt{d_{k,i'}})) \vee \sqrt{d_{k,i'}}(d_{k,i} - d_{k,i'})/(\sqrt{d_{k,i}}$$

$$+ \sqrt{d_{k,i'}})\}/\sqrt{d_{k,i}d_{k,i'}}$$

Therefore

$$1 - r \geqq 1 \wedge \{((j' - j) - (i - i')) \vee (i - i')\}\delta_k M/(2d_k). \tag{4.8}$$

Case 3. If $t_{k,j'} < t_{k,j}$ and $|I_{i,j} \cap I_{i',j'}| > 0$, then

$$0 \leqq j - j' \leqq i_k^*, \text{ and} \tag{4.9}$$

$$1 - r = (\sqrt{d_{k,i}d_{k,i'}} - ((t_{k,j'} + d_{k,i'} - t_{k,j}) \vee 0) \wedge d_{k,i})/\sqrt{d_{k,i}d_{k,i'}}$$

$$\geqq 1 \wedge \{((j - j') - (i' - i)) \vee (i' - i)\}\delta_k M/(2d_k). \tag{4.10}$$

If follows from Lemma 1, Lemma 3, (iii), and (4.6)-(4.10) that for fixed (i, j, k)

$$\sum_{(i,j) \neq (i',j')} P(B_{i,j,k} \cap B_{i',j',k}) = \sum_{|I_{i,j,k} \cap I_{i',j',k}| = 0} + \sum_{|I_{i,j,k} \cap I_{i',j',k}| > 0} \equiv I + II,$$

$$I \leqq P(B_{i,j,k}) i_k j_k \phi(x_k)$$

$$\leqq P(B_{i,j,k}) \left(\frac{b_{k-1}}{M\delta_k} \vee 1\right)\left(\frac{d_k - d_{k-1}}{M\delta_k} \vee 1\right)\phi(x_k)$$

$\leqq \varepsilon P(B_{i,j,k})/2$ for $k \geqq k_\varepsilon$ (independent of $M \geqq 1$ by Lemma 1, (ii)) and

$$\mathrm{II} \leqq cP(B_{i,j,k}) \sum_{j'=0}^{i_k^*} \sum_{i'=-k}^{k} \exp\{-\{1 \wedge ((j' - i') \vee i')\delta_k M/(8d_k)\} x_k^2\}$$

$$\leqq cP(B_{i,j,k}) \sum_{j'=0}^{\infty} \sum_{i'=-\infty}^{\infty} \exp\{-((j' - i') \vee i')M/32\}$$

$$\leqq cp(B_{i,j,k}) \sum_{n=1}^{\infty} 4ne^{-nM/32} \leqq \varepsilon P(B_{i,j,k})/2$$

(here we have chosen a sufficently large $M = M_\varepsilon$ such that the last inequality holds). Taking account of Lemma 1, (4.1) and (4.5) we have obtained (4.4).

Step 2. *If* $t_{k',1} \geqq T_k$, *then obviously* $|I_{i,j,k} \cap I_{i',j',k'}| = 0$ *and*

(4.11) $$P(B_k \cap B_{k'}) = P(B_k)P(B_{k'}).$$

Step 3. *If* $t_{k',1} < T_k$, $\tau_{k'} \leqq k' - 3$ and $k + 4\log k \leqq k'$, *then there exists* c_ε *such that*

(4.12) $$P(B_k \cap B_{k'}) \leqq c_\varepsilon(1-\varepsilon)^{-2}P(B_k)P(B_{k'}) \text{ holds, where } \lim_{\varepsilon\downarrow 0} c_\varepsilon = 1.$$

Proof. Set $r' = E[X_{i,j,k}X_{i',j',k'}]$. Then we have

(4.13) $$r' = |I_{i,j,k} \cap I_{i',j',k'}|/\sqrt{d_{k,i}d_{k',i'}} \leqq \sqrt{d_k/d_{k'-1}}.$$

By definition, $\tau_{k'} \leqq k' - 3$ implies that

$$T_{k'-2} \geqq T_{\tau_{k'}+1} - d_{\tau_{k'}+1} \geqq (T_{k'-1} - d_{k'-1})/2 \text{ and}$$

(4.14) $$d_{k'-1} \geqq T_{k'-1} - 2T_{k'-2} = (1 - 2e^{-1})T_{k'-1}.$$

By Lemma 1, (4.13) and (4.14), we have

$$r'x_kx_{k'} \leqq 9\sqrt{d_k/d_{k'-1}}\,\gamma_{k'}^2 \leqq ce^{-(k'-k)/2}\log k'$$

$$(f(x) = e^{-x}\log x \text{ is decreasing at the neighborhood of } +\infty)$$

$$\leqq c\log k\, e^{-\log k} \longrightarrow 0 \text{ as } k \longrightarrow +\infty.$$

This means that by Lemma 3, (ii) and (4.4) there exists k_ε such that for all $k \geqq k_\varepsilon$, $t_{k',1} < T_k$, $\tau_{k'} \leqq k - 3$ and $k' \geqq k + 4\log k$, we have.

$$P(B_k \cap B_{k'}) = \sum_{i,j}\sum_{i',j'} P(B_{i,j,k} \cap B_{i',j',k'})$$

$$\leqq c_\varepsilon\sum_{i,j}\sum_{i',j'} P(B_{i,j,k})P(B_{i',j',k'})$$

$$\leqq c_\varepsilon(1-\varepsilon)^{-2}P(B_k)P(B_{k'}).$$

Step 4. *If* $t_{k',1} < T_k$, $\tau_{k'} \leqq k' - 3$ *and* $k + 5 \leqq k' \leqq k + 4\log k$, *then*

(4.15) $$\sum_{k'} P(B_k \cap B_{k'}) \leqq cp(B_k).$$

Proof. By (4.13) and (4.14)

(4.15) $1 - r' \geqq 1 - \sqrt{d_k/d_{k'-1}} \geqq 1 - \sqrt{T_k/T_{k'-2}}\,(1 - e^{-1})^{-1/2} = 1 - e^{-1}(1 - e^{-1})^{-1/2} \geqq c > 0.$

(4.16) $j_{k'} \leqq c(T_{k'-1}/d_{k'-1})\gamma_{k'}^2 \leqq c\log k$ and

(4.17) $i_{k'} \leqq c\gamma_{k'}^2 \leqq c\log k.$

It follows from Lemma 3(iii), (4.4) and (4.15)-(4.17) that

(4.18)
$$\sum_{k'} P(B_k \cap B_{k'}) \leqq \sum_{i,j} \sum_{i',j',k'} P(B_{i,j,k} \cap B_{i',j',k'})$$
$$\leqq (1-\varepsilon)^{-1} P(B_k) \sum_{k'} (\log k)^2 e^{-c\log k}$$
$$\leqq (1-\varepsilon)^{-1} (\log k)^3 e^{-c\log k} P(B_k).$$

Step 5. *If* $t_{k',1} < T_k$, $\tau_{k'} = k' - 2$(by definition $\tau_{k'} \leqq k' - 2$) *and* $k + 4\log k \leqq k'$, *then there exists* c_ε *such that*

(4.19) $P(B_k \cap B_{k'}) \leqq c_\varepsilon (1-\varepsilon)^{-2} P(B_k) P(B_{k'})$ *holds, where* $\lim_{\varepsilon \downarrow 0} c_\varepsilon = 1$.

Proof. By the assumption, $t_{k',1} = T_{\tau_{k'}} - d_{\tau_{k'}} = T_{k'-2} - d_{k'-2} < T_k$ means

(4.20)
$$(1 - e^{-1}) T_{k'-2} < d_{k'-2} \leqq d_{k'-1}.$$

From Lemma 1, (4.13) and (4.20), we have

$$r' x_k x_{k'} \leqq 9\sqrt{T_k/d_{k'-1}}\, \gamma_{k'}^2 \leqq c e^{-(k'-k)/2} \log k' \leqq c(\log k)/k \to 0 \text{ as } k \to +\infty.$$

Therefore by just the same way as Step 3, we can obtain (4.19).

Step 6. *If* $t_{k',1} < T_k$, $\tau_{k'} = k' - 2$ *and* $k + 4 \leqq k' \leqq k + 2\log k$, *then*

(4.21)
$$\sum_{k'} P(B_k \cap B_{k'}) \leqq c P(B_k).$$

Proof. By (4.13) and (4.20)

(4.22)
$$1 - r' \geqq 1 - \sqrt{d_k/d_{k'-1}} \geqq 1 - \sqrt{T_k/T_{k'-2}}\,(1 - e^{-1})^{-1/2}$$
$$= 1 - e^{-1}(1 - e^{-1})^{-1/2} \geqq c > 0.$$

(4.23)
$$j_{k'} \leqq c(T_{k'-1}/d_{k'-1})\gamma_{k'}^2 \leqq c\log k \text{ and}$$

(4.24)
$$i_{k'} \leqq c\gamma_{k'}^2 \leqq c\log k.$$

It follows from Lemma 3(iii), (4.4) and (4.22)–(4.24) that

(4.25)
$$\sum_{k'} P(B_k \cap B_{k'}) \leqq \sum_{i,j} \sum_{i',j',k',} P(B_{i,j,k} \cap B_{i',j',k'})$$
$$\leqq (1-\varepsilon)^{-1} P(B_k) \sum_{k'} (\log k)^2 e^{-c\log k}$$
$$\leqq (1-\varepsilon)^{-1} (\log k)^3 e^{-c\log k} P(B_k).$$

Now we have arrived at the final step for the proof of Theorem 2.

Step 7. *Assume that* $I_L(g) = +\infty$. *Then one of*

$$\sum_{k=1}^{\infty} I_L(g(T_{5k+i})) \quad \textit{for } i = 0, 1, 2, 3, 4$$

is divergent. Taking account of all steps from 1 *to* 6 *and appling Lemma* 2 *for the divergent sub-sequence, we can obtain* (4.3).

The proofs of Theorem 3 and 4 are just the same line as those of Theorem 1 and 2 using lemmas corresponding to Lemma 3 and 4. Lemma 3 for d-dimensional case is well known ([6] Lemma 2.8–2.10), so we only mension the following lemma corresponding to Lemma 4.

Let $Z=(X, Y)$ be a two dimensional centered Gaussian random variable with $E[X^2]=E[Y^2]=1$ and $E[XY]=r$. Set $\varepsilon^2=E[(X-Y)^2]=2-2r$. $\mathfrak{X}_d=(X_1,\dots,X_d)$, $\mathfrak{Y}_d=(Y_1,\dots,Y_d)$, *where* (X_i, Y_i) are independent copies of Z. Denote by $\|\ \|$ the usual d-dimensinal Euclidean norm. Set $\phi_d(x)=P(\|\mathfrak{X}_d\|\geqq x)$ and $I(x, y;\varepsilon)=P(\|\mathfrak{X}_d\|\geqq x+\varepsilon y,\ \|\mathfrak{Y}_d\|\leqq x)$.

Lemma 7. *There exist positive absolute constants* A_d *and* B_d *only depending on* d *such that*

$$I(x, y;\varepsilon)\leqq A_d e^{-y^2/4}\phi_d(x)$$

holds for all $0\leqq\varepsilon\leqq 1/\sqrt{2}$, $y\geqq B_d$ *and* $x>0$.

Using Lemma 7 we can obtain the following lemma.

Lemma 8. *Let* $\{X(t);t\in T\}$ *be a centered separable Gaussian process with the pseudo-metric* $d_x(s, t)\equiv(E[(X(s)-X(t))^2])^{1/2}$ *satisfying the following conditions;*
(i) (T, d_x) *is a compact space,*
(ii) $0<\underline{\sigma}\leqq(E[X^2(t))^{1/2}\leqq\bar{\sigma}<+\infty$ *for* $^\forall t\in T$.
Set $\mathfrak{X}_d(t)=(X_1(t),\dots,X_d(t))$, *where* $\{X_i(t)\}$ $i=1,\dots,d$ *are independent copies of* $\{X(t)\}$. *Then for* $0<\varepsilon_j\leqq 1/\sqrt{2}$, $0<\lambda_{j+1}$, $j=0, 1, 2,\dots$ *and* $0<x$,

$$P(\sup_{t\in T}\|\mathfrak{X}_d(t)\|\geqq\bar{\sigma}(x+\sum_{j=0}^{\infty}\varepsilon_j\lambda_{j+1})\leqq(N_x(T, \underline{\sigma}\varepsilon_0)+A_d\sum_{j=1}^{\infty}N_X(T, \underline{\sigma}\varepsilon_j)e^{-\lambda_j^2/4})\phi_d(x),$$

where $N_X(T, \varepsilon)$ *is the minimal number of balls* $B(t, \varepsilon)\equiv\{s\in T;\ d_X(s, t)\leqq\varepsilon\}$ *whose union covers* T.

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