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Erdös-Rényi-type laws applied to Gaussian processes

By

Yong K. Choi

1. Introduction

The Erdös-Rényi law of large numbers applied to standard normal random variables is as follows:

Theorem A. *Let* $\{\xi_j; j=1, 2, \ldots\}$ *be independent standard normal random variables with partial sums* $S_0 = 0$ *and* $S_n = \xi_1 + \cdots + \xi_n$. *Then for each* $c > 0$

$$\lim_{n\to\infty} \max_{0\le j\le n-[c\log n]} \frac{S_{j+[c\log n]} - S_j}{[c\log n]} = \sqrt{2/c}, \qquad \text{a.s.}$$

where $[y]$ *denotes the greatest integer not exceeding* y.

Many extensions and developments, in various directions, of Erdös-Rényi-type laws for i.i.d.r.v.'s have been obtained in [2], [9], [14], [25–27] and others.

Recently, Choi [5] has extended the Erdös-Rényi law to stationary Gaussian sequences in dependent situations, under milder conditions than those of Deo [10].

The Erdös-Rényi law for Wiener processes has the following form:

Theorem B. *Let* $W(t)(0 \le t < \infty)$ *be a standard Wiener process. Then for each* $c > 0$

$$\lim_{T\to\infty} \sup_{0\le t\le T-c\log T} \frac{W(t+c\log T) - W(t)}{(2c)^{1/2}\log T} = 1, \qquad \text{a.s.}$$

Csörgő and Révész [7] obtained the following result for the increments of $W(t)$:

Theorem C. *Let* $a_T(0 \le T < \infty)$ *be a nondecreasing function of* T *for which*

(1.1) $$0 < a_T \le T$$

(1.2) $$a_T/T \text{ is nonincreasing}$$

(1.3) $$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = \infty.$$

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Then

(1.4)
$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{|W(t+a_T)-W(t)|}{\{2a_T(\log(T/a_T)+\log\log T)\}^{1/2}}=1, \qquad \text{a.s.}$$

and

(1.5)
$$\lim_{T\to\infty}\sup_{\substack{0\le t\le T-a_T\\ 0\le s\le a_T}}\frac{|W(t+s)-W(t)|}{\{2a_T(\log(T/a_T)+\log\log T)\}^{1/2}}=1, \qquad \text{a.s.}$$

Remark. By virtue of (1.3), $2a_T\{\log(T/a_T)+\log\log T\}$ in the denominators of (1.4) and (1.5) can be replaced by $2a_T\log(T/a_T)$. But we stated them in this form because it is a right form when we do not assume (1.3), cf. Csörgő-Révész [7].

Many authors have investigated the asymptotic behavior on the increments of Wiener processes and related processes. See, for example, [3], [4], [6] and [8].

In section 2, we extend the above Theorem C to Gaussian processes with stationary increments which contain fractional Brownian motions. Our result is as follows: let $\{\xi(t); 0\le t<\infty\}$ be a continuous, centered Gaussian process with $\xi(0)=0$ and stationary increments: $E\{(\xi(t)-\xi(s))^2\}=\sigma^2(|t-s|)$. Assume that $\sigma(t)$, $t>0$, is a nondecreasing continuous, regularly varying function with index $\gamma(0<\gamma<1)$ at ∞ (for details, see Kôno [15–16], Qualls and Watanabe [22–23]), and that $\sigma(t)$ satisfies

$$\int_0^\infty \sigma(e^{-y^2})\,dy<+\infty.$$

Further assume that either

(1.6) $\sigma^2(t)$ is concave

or

(1.7) $\sigma^2(t)$ is twice continuously differentiable which satisfies the condition $|(\sigma^2(t))''|\le c\sigma^2(t)/t^2$ for a constant c.

Let a_T be as in Theorem C. Then we have

(1.8)
$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{\xi(t+a_T)-\xi(t)}{\{2(\log(T/a_T)+\log\log T)\}^{1/2}\sigma(a_T)}=1, \qquad \text{a.s.}$$

and

(1.9)
$$\lim_{T\to\infty}\sup_{\substack{0\le t\le T-a_T\\ 0\le s\le a_T}}\frac{\xi(t+s)-\xi(t)}{\{2(\log(T/a_T)+\log\log T)\}^{1/2}\sigma(a_T)}=1, \qquad \text{a.s.}$$

In section 3 we investigate the analogous of the above (1.8) and (1.9) for stationary Gaussian processes. More precisely, let $\{\xi(t);\ -\infty < t < \infty\}$ be a centered stationary Gaussian process with $E\xi(t)^2 = 1$ and continuous sample functions. Set $\sigma^2(t) = E\{(\xi(t) - \xi(0))^2\}$ and let $r(t) = E\xi(0)\xi(t)$ be continuous. Assume that $r(t) \to 0$ as $t \to \infty$, $\sigma(t)$ is nondecreasing and satisfies

$$\int_0^\infty \sigma(e^{-y^2})\,dy < +\infty.$$

Suppose that $a_T (0 \le T < \infty)$ is an increasing function of T for which

(1.10) $\qquad 0 < a_T \le T^\theta$ for all large $T > 0$ and some $0 < \theta < 1$

(1.11) $\qquad a_T/T$ is nonincreasing.

If $r(t)$ is convex for $t > 0$, then we obtain

$$\limsup_{T\to\infty} \sup_{\substack{0 \le t \le T-a_T \\ 0 \le s \le a_T}} \frac{\xi(t+s) - \xi(t)}{(2 \log T)^{1/2}\sigma(a_T)} \le \sqrt{1+\theta}, \qquad \text{a.s.} \tag{1.12}$$

and

$$\liminf_{T\to\infty} \sup_{0 \le t \le T-a_T} \frac{\xi(t+a_T) - \xi(t)}{(2 \log T)^{1/2}\sigma(a_T)} \ge \sqrt{1-\theta}, \qquad \text{a.s.} \tag{1.13}$$

On the other hand, instead of the convexity of $r(t)$, we shall consider the convergence rate of $r(t)$ to zero. Let $r(t)$ be such that

(1.14) $\qquad \lim_{t\to\infty} r'(t)\,t^{1+\nu} = 0 \qquad$ for some $\nu > 0$

where $r'(t) = dr(t)/dt$ and the range of a_T be such that

(1.15) $\qquad 0 < a_T \le T^\theta$ for all large $T > 0$ and some $0 < \theta < 1/2$.

Then we also have (1.12) and (1.13).

In particular, suppose that either

(1.16) $\qquad r(t)$ is convex for $t > 0$

or

(1.17) $\qquad \lim_{t\to\infty} r(t)\,t^\nu = 0 \qquad$ for some $\nu > 0$

and that the range of a_T is such that

(1.18) $\qquad$ for every $0 < \theta < 1$, $0 < a_T < T^\theta \qquad$ for all large $T > 0$.

Then we have

$$\lim_{T\to\infty} \sup_{0 \le t \le T-a_T} \frac{\xi(t+a_T) - \xi(t)}{(2 \log T)^{1/2}\sigma(a_T)} = 1, \qquad \text{a.s.} \tag{1.19}$$

$$\lim_{T\to\infty} \sup_{\substack{0 \le t \le T-a_T \\ 0 \le s \le a_T}} \frac{\xi(t+s) - \xi(t)}{(2 \log T)^{1/2}\sigma(a_T)} = 1, \qquad \text{a.s.} \tag{1.20}$$

Moreover, if $r(t)$ and a_T are such that

(1.21) $$\lim_{t\to\infty} r(t)(\log t)^{1+v} = 0 \qquad \text{for some } v > 0$$

and

(1.22) $$0 < a_T \le (\log T)^{\omega} \text{ for all large } T > 0 \text{ and some } 0 < \omega < v$$

then we also have (1.19) and (1.20).

Thus we see that the convergence rate of $r(t)$ to zero and the range of a_T are correspondingly related in obtaining the results (1.19) and (1.20).

However, the above arguments leave open the following problems: (i) Are there any other conditions on $r(t)$ or a_T so that (1.19) and (1.20) may hold in the wider range of a_T than (1.18)? (ii) Are there any other conditions on a_T or $r(t)$(except the case of the convexity of $r(t)$) so that (1.13) may hold in the wider range on a_T than (1.15)?

In the last section 4 we shall prove the theorems in sections 2 and 3.

2. Gaussian Processes with stationary increments

Let $\{\xi(t); 0 \le t < \infty\}$ be a continuous, centered Gaussian process with $\xi(0) = 0$ and stationary increments: $E\{(\xi(t) - \xi(s))^2\} = \sigma^2(|t - s|)$. Throughout section 2, we shall assume that $\sigma(t)$, $t > 0$, is a nondecreasing continuous, regularly varying function with index γ at ∞ for some $0 < \gamma < 1$ and that $\sigma(t)$ satisfies

(2.1) $$\int_0^\infty \sigma(e^{-y^2})\,dy < +\infty.$$

For $0 < a_T \le T$, denote

$$\mathbf{B}(T, a_T) = \sup_{0 \le t \le T - a_T} \frac{\xi(t + a_T) - \xi(t)}{\{2(\log(T/a_T) + \log\log T)\}^{1/2}\sigma(a_T)},$$

$$\mathbf{B}^*(T, a_T) = \sup_{\substack{0 \le t \le T - a_T \\ 0 \le s \le a_T}} \frac{\xi(t + s) - \xi(t)}{\{2(\log(T/a_T) + \log\log T)\}^{1/2}\sigma(a_T)}.$$

Our theorems are as follows:

Theorem 2.1. *Let $a_T (T \ge 0)$ be a nondecreasing function of T for which*

(2.2) $$0 < a_T \le T$$

(2.3) $$a_T/T \text{ is nonincreasing.}$$

Then

$$\limsup_{T\to\infty} \mathbf{B}^*(T, a_T) \le 1, \qquad \text{a.s.}$$

Theorem 2.2. *Let a_T satisfy the conditions* (2.2), (2.3) *and*

$$\lim_{T\to\infty}\frac{\log(T/a_T)}{\log\log T}=\infty. \tag{2.4}$$

Assume that for $t>0$, either

(2.5) $\sigma^2(t)$ *is concave*

or

(2.6) $\sigma^2(t)$ *is twice continuously differentiable which satisfies*

$$|(\sigma^2(t))''|\le c\sigma^2(t)/t^2 \qquad \textit{for some } c>0.$$

Then

$$\liminf_{T\to\infty}\mathbf{B}(T,a_T)\ge 1, \qquad \text{a.s.}$$

Combining Theorems 2.1 and 2.2, we immediately obtain

Theorem 2.3. *Under the assumptions of Theorem 2.2,*

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{\xi(t+a_T)-\xi(t)}{\{2(\log(T/a_T)+\log\log T)\}^{1/2}\sigma(a_T)}=1, \qquad \text{a.s.} \tag{2.7}$$

and

$$\lim_{T\to\infty}\sup_{\substack{0\le t\le T-a_T\\ 0\le s\le a_T}}\frac{\xi(t+s)-\xi(t)}{\{2(\log(T/a_T)+\log\log T)\}^{1/2}\sigma(a_T)}=1, \qquad \text{a.s.} \tag{2.8}$$

Example 1. Let $\{\xi(t);0\le t<\infty\}$ be a continuous, centered Gaussian process with $\xi(0)=0$ and $E\{(\xi(t)-\xi(s))^2\}=\sigma^2(|t-s|)$, where $\sigma(t)=t^\alpha(0<\alpha<1)$. This process is a fractional Brownian motion of index α. Now if $0<\alpha\le 1/2$, then $\sigma^2(t)$ is concave and if $0<\alpha<1$, then $\sigma^2(t)$ satisfies the condition (2.6). Therefore, Theorem 2.3 can be applied to every fractional Brownian motion. In particular, in case of $\alpha=1/2$, Theorem 2.3 immediately implies Theorem *C* for the Wiener process. Furthermore, for $0<\alpha<1$ if $a_T=T^\theta(\log T)^\tau$ for all large $T>0$, for some $0<\theta<1$ and $\tau>0$, then Theorem 2.3 (2.7) implies

$$\lim_{T\to\infty}\sup_{0\le t\le T-a_T}\frac{\xi(t+a_T)-\xi(t)}{(2\log T)^{1/2}(a_T)^\alpha}=\sqrt{1-\theta}, \qquad \text{a.s.} \tag{2.9}$$

If $a_T=1$, then it also implies

$$\lim_{T\to\infty}\sup_{0\le t\le T-1}\frac{\xi(t+1)-\xi(t)}{(2\log T)^{1/2}}=1, \qquad \text{a.s.} \tag{2.10}$$

(2.10) contains as a special case the well-known result of Chan [4] for the Wiener process.

3. Stationary Gaussian processes

Let $\{\xi(t);\ -\infty < t < \infty\}$ be a centered stationary Gaussian process with $E\xi(t)^2 = 1$ and continuous sample functions. Set $\sigma^2(t) = E\{\xi(t) - \xi(0)\}^2$ and let the correlation function $r(t) = E\xi(0)\xi(t)$ be continuous. In this section we shall assume that $r(t) \to 0$ as $t \to \infty$, $\sigma(t)$ is nondecreasing and satisfies

$$\int_0^\infty \sigma(e^{-y^2})\,dy < +\infty. \tag{3.1}$$

For $0 < a_T \le T$, denote

$$\mathbf{D}(T, a_T) = \sup_{0 \le t \le T - a_T} \frac{\xi(t + a_T) - \xi(t)}{\{2\log(T/a_T)\}^{1/2}\sigma(a_T)},$$

$$\mathbf{D}^*(T, a_T) = \sup_{\substack{0 \le t \le T - a_T \\ 0 \le s \le a_T}} \frac{\xi(t + s) - \xi(t)}{\{2\log(Ta_T)\}^{1/2}\sigma(a_T)}.$$

Our main results of this section are as follows:

Theorem 3.1. *Let $a_T (T \ge 0)$ be a nondecreasing function of T for which*

$$0 < a_T \le T \tag{3.2}$$

$$a_T/T \text{ is nonincreasing.} \tag{3.3}$$

Then

$$\limsup_{T\to\infty} \mathbf{D}^*(T, a_T) \le 1, \qquad \text{a.s.}$$

Theorem 3.1 clearly implies

Corollary 1. *Instead of the condition* (3.2) *in Theorem* 3.1, *if we impose*

$$0 < a_T \le T^\theta \text{ for all large } T > 0 \text{ and some } 0 < \theta \le 1 \tag{3.2$'$}$$

then we have

$$\limsup_{T\to\infty} \sup_{\substack{0 \le t \le T - a_T \\ 0 \le s \le a_T}} \frac{\xi(t + s) - \xi(t)}{(2\log T)^{1/2}\sigma(a_T)} \le \sqrt{1 + \theta}, \qquad \text{a.s.}$$

Theorem 3.2. *Let $a_T (T \ge 0)$ be an increasing function of T for which*

$$0 < a_T \le T^\theta \text{ for all large } T > 0 \text{ and some } 0 < \theta < 1 \tag{3.4}$$

$$a_T/T \text{ is nonincreasing.} \tag{3.5}$$

Assume that $r(t)$ is convex for $t > 0$. Then

$$\liminf_{T\to\infty} \mathbf{D}(T, a_T) \ge 1, \qquad \text{a.s.} \tag{3.6}$$

Theorem 3.3. *Let $a_T (T \ge 0)$ be an increasing function of T for which*

(3.7) $0 < a_T \leq T^{\theta}$ *for all large* $T > 0$ *and some* $0 < \theta < 1/2$

(3.8) a_T/T *is nonincreasing.*

Assume that

(3.9) $$\lim_{t\to\infty} r'(t)t^{1+\nu} = 0 \qquad \text{for some } \nu > 0$$

where $r'(t) = dr(t)/dt$. *Then we have* (3.6).

Corollary 2. *Under the assumptions of Theorems* 3.2 *or* 3.3, *we have*

$$\liminf_{T\to\infty} \sup_{0\leq t\leq T-a_T} \frac{\xi(t+a_T)-\xi(t)}{(2\log T)^{1/2}\sigma(a_T)} \geq \sqrt{1-\theta}, \qquad \text{a.s.}$$

If instead of (3.7) we impose the stronger conditions (3.10) and (3.17) below, then the condition (3.9) on $r(t)$ can be correspondingly weakened as in the following (3.13) and (3.16).

Theorem 3.4. *Let* $a_T (T \geq 0)$ *be an increasing function of* T *for which we assume that*

(3.10) *for every* $0 < \theta < 1$, $0 < a_T < T^{\theta}$ *for all large* $T > 0$

(3.11) a_T/T *is nonincreasing.*

Assume also that either

(3.12) $r(t)$ *is convex for* $t > 0$

or

(3.13) $$\lim_{t\to\infty} r(t)t^{\nu} = 0 \qquad \text{for some } \nu > 0.$$

Then

(3.14) $$\lim_{T\to\infty} \sup_{0\leq t\leq T-a_T} \frac{\xi(t+a_T)-\xi(t)}{(2\log T)^{1/2}\sigma(a_T)} = 1, \qquad \text{a.s.}$$

(3.15) $$\lim_{T\to\infty} \sup_{\substack{0\leq t\leq T-a_T\\ 0\leq s\leq a_T}} \frac{\xi(t+s)-\xi(t)}{(2\log T)^{1/2}\sigma(a_T)} = 1, \qquad \text{a.s.}$$

Theorem 3.5. *Assume that*

(3.16) $$\lim_{t\to\infty} r(t)(\log t)^{1+\nu} = 0 \qquad \text{for some } \nu > 0.$$

Let $a_T (T \geq 0)$ *be an increasing function of* T *for which we assume that*

(3.17) $0 < a_T \leq (\log T)^{\omega}$ *for all large* $T > 0$ *and some* $0 < \omega < \nu$

(3.18) $a_T/(\log T)^d$ *is nonincreasing for some* $d > 0$.

Then we have (3.14) *and* (3.15).

A simple illustration is the following:

Example 2. Let the correlation function $r(t) = e^{-|t|^\alpha}$ $(0 < \alpha < 2)$ be given. Set $a_T = c \log T (c > 0)$ in Theorems 3.4 and 3.5. Then all assumptions of Theorems 3.4 and 3.5 are satisfied and hence we obtain (3.14) and (3.15).

Remark. Our above results hold on the shorter interval a_T than those of Pickands [21] and Nisio [20]: Let $\{X(t);\ -\infty < t < \infty\}$ be a separable, centered stationary Gaussian process with $EX(t)^2 = 1$ and $\sigma^2(t) = E\{X(t) - X(0)\}^2$. Let $\sigma(t)$ be nondecreasing and satisfy

$$\int_0^\infty \sigma(e^{-y^2})\,dy < +\infty.$$

Let

$$\lim_{T\to\infty} \sup_{t>T} r(t) \leq 0.$$

Then

$$\lim_{T\to\infty} \sup_{0\leq t\leq T} \frac{X(t)}{(2 \log T)^{1/2}} = 1, \qquad \text{a.s.}$$

4. Proofs

Hereafter we shall let c denote an absolute constant changing in lines if necessary. For our proofs, we shall first give the Fernique lemma [13] in a form similar to those appearing in Kôno [15], Marcus and Shepp [18], Marcus [19] and Berman [1]. Let $\mathbb{D} = \{\mathfrak{t};\ \mathfrak{t} = (t_1, \ldots, t_N),\ a_i \leq t_i \leq b_i,\ i = 1, \ldots, N\}$ be a real N-dimensional parameter space. We assume that the space $\mathbb{D}$ has the usual Euclidean metric

$$\|\mathfrak{t} - \mathfrak{s}\|^2 = \textstyle\sum_{i=1}^N (t_i - s_i)^2.$$

Let $\{X(\mathfrak{t});\ \mathfrak{t} \in \mathbb{D}\}$ be a real valued, separable Gaussian process with $EX(\mathfrak{t}) = 0$. Suppose that

$$0 < \sup_{\mathfrak{t}\in\mathbb{D}} EX(\mathfrak{t})^2 = \Gamma^2 < \infty$$

and

$$E\{X(\mathfrak{t}) - X(\mathfrak{s})\}^2 \leq \varphi^2(\|\mathfrak{t} - \mathfrak{s}\|),$$

where $\varphi(\cdot)$ is a nondecreasing continuous function that satisfies

$$\int_0^\infty \varphi(e^{-y^2})\,dy < +\infty. \tag{4.1}$$

This condition (4.1) guarantees that the process $X(\mathfrak{t})$ has continuous sample functions.

Lemma 4.1. *Let $\{X(\mathfrak{t});\ \mathfrak{t} \in \mathbb{D}\}$ be given as the above statements. Then for λ*

> 0, $x \geq 1$ *and* $A > (2\sqrt{N \log 2} \vee (1/\sqrt{2}))$, *we have*

$$P\left\{\sup_{\mathbb{t}\in\mathbb{D}} X(\mathbb{t}) \geq x\left\{\Gamma + (2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{N}\lambda 2^{-y^2})\,dy\right\}\right\}$$
$$\leq (4^N + \psi)\prod_{i=1}^N \left(\frac{b_i - a_i}{\lambda} \vee 1\right) e^{-x^2/2}$$

where $m \vee n = \max\{m, n\}$ *and*

$$\psi = \sum_{n=1}^\infty \exp\left\{\frac{1}{2} - 2^n\left(\frac{A^2}{2} - 2N\log 2\right)\right\} < +\infty.$$

The proofs of our theorems are mainly based on the following lemmas:

Lemma 4.2. (Slepian, [24]) *Let* $\{\xi_i; i = 1, 2,\dots, n\}$ *and* $\{\eta_i; i = 1, 2,\dots, n\}$ *be jointly standardized normal random variables with covariance* $(\xi_i, \xi_j) \leq$ *covariance* (η_i, η_j), $i \neq j$. *Then for any real* $u_1,\dots,u_n$,

$$P\{\xi_j \leq u_j; j = 1, 2,\dots, n\} \leq P\{\eta_j \leq u_j; j = 1, 2,\dots, n\}.$$

Lemma 4.3. (Leadbetter et al. [17]) *Let* $\{\xi_j; j = 1, 2,\dots, n\}$ *be jointly standardized normal random variables with covariance* $(\xi_i, \xi_j) = \Lambda_{ij}$ *such that*

$$\delta = \max_{i\neq j}|\Lambda_{ij}| < 1.$$

Then for any real numbers u *and integers* $1 \leq \ell_1 < \ell_2 < \cdots < \ell_k \leq n$ *with* $k \leq n$,

$$P\{\max_{1\leq j\leq k}\xi_{\ell_j} \leq u\} \leq \Phi(u)^k + c\sum_{1\leq i<j\leq k}|r_{ij}|\exp\left(-\frac{u^2}{1+|r_{ij}|}\right)$$

where $r_{ij} = \Lambda_{\ell_i\ell_j}$, $c = c(\delta)$ *is a constant independent of* n, u *and* k *and* $\Phi(\cdot)$ *denotes the standard normal distribution function.*

Lemma 4.4. *Let* $\xi_j (j = 1, 2,\dots, n)$, δ, k *and* r_{ij} *be as in Lemma* 4.3. *Assume that* $|r_{ij}| \leq \rho_{|i-j|} < 1$ $(i \neq j)$.

(i) If $\rho_m < \mu m^{-1-\nu}$ *and* $u = ((2-\eta)\log k)^{1/2}$

where μ *and* ν *are positive constants such that* $0 < \eta < \nu$ *and* $0 < \eta + \mu < 1$, *then*

$$\sum := \sum_{1\leq i<j\leq k}|r_{ij}|\exp\left(-\frac{u^2}{1+|r_{ij}|}\right) \leq ck^{-1+\delta_0}$$

where $\delta_0 = (\eta + 2\mu)/(1+\mu)$ *and the constant* c *is independent of* n *and* k.

(ii) If $\rho_m < m^{-\nu}$ *and* $u = ((2-\eta)\log k)^{1/2}$

where ν *and* η *are positive constants such that* $0 < \eta < (1-\delta)\nu/(1+\nu+\delta)$, *then*

$$\sum \leq ck^{-\delta_0}$$

where $\delta_0 = \{\nu(1-\delta) - \eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$ *and the constant* c *is* *independent of* n *and* k.

$$\text{(iii)} \qquad \text{If } \rho_m < c(\log m)^{-1-\nu}, \qquad m = 2, 3, \dots, k-1,$$
$$< \delta, \qquad m = 1,$$

and $u = \{2\log k - (1+\rho)\log\log k\}^{1/2}$ *where* ν *and* ρ *are constants such that* $0 < \rho < \nu$, *then*

$$\sum \le c(\log k)^{-\delta_0}$$

where $\delta_0 = \nu - \rho > 0$ *and* c *is independent of* n *and* k.

Proof. For $0 < \alpha < 1$, we split $\sum$ into two parts:

$$\sum = \sum_{\substack{1 \le i < j \le k \\ |i-j| \le [k^\alpha]}} |r_{ij}| \exp\left(-\frac{u^2}{1+|r_{ij}|}\right)$$
$$+ \sum_{\substack{1 \le i < j \le k \\ |i-j| > [k^\alpha]}} |r_{ij}| \exp\left(-\frac{u^2}{1+|r_{ij}|}\right)$$
$$= \sum\nolimits^{(1)} + \sum\nolimits^{(2)}, \text{ say.}$$

We first prove (i). Let $\alpha = \eta/\nu < 1$. Then

$$\sum\nolimits^{(1)} \le \mu \sum_{\substack{1 \le i < j \le k \\ |i-j| \le [k^\alpha]}} |i-j|^{-1-\nu} \exp\left\{-\frac{(2-\eta)\log k}{1+\mu}\right\}$$
$$\le ck^{1-\frac{2-\eta}{1+\mu}} \sum\nolimits_m m^{-1-\nu}$$
$$\le ck^{1-\frac{2-\eta}{1+\mu}} = ck^{-1+\delta_0}$$

where $\delta_0 = (\eta + 2\mu)/(1+\mu)$. Next,

$$\sum\nolimits^{(2)} \le c \sum_{\substack{1 \le i < j \le k \\ |i-j| > [k^\alpha]}} |i-j|^{-1-\nu} \exp(-u^2) \exp\left(\frac{u^2 |r_{ij}|}{1+|r_{ij}|}\right)$$
$$\le ck \sum\nolimits_{m=[k^\alpha]}^{\infty} m^{-1-\nu} e^{-(2-\eta)\log k} \exp\{c(2-\eta)(\log k) k^{-\alpha(1+\nu)}\}$$
$$\le ck^{-1+\eta} \int_{[k^\alpha]}^{\infty} \chi^{-1-\nu} d\chi \le ck^{-1+\eta-\alpha\nu} = ck^{-1}.$$

(ii) Let $0 < \alpha = (1 + \eta\delta - \delta)/\{(1+\nu)(1+\delta)\} < 1$. Then

$$\sum\nolimits^{(1)} \le ck^{1+\alpha} \exp\left(-\frac{2-\eta}{1+\delta}\log k\right)$$
$$= ck^{\{\eta(1+\delta+\nu) - \nu(1-\delta)\}/\{(1+\nu)(1+\delta)\}}$$

and

$$\sum\nolimits^{(2)} \le \sum_{\substack{1 \le i < j \le k \\ |i-j| > [k^\alpha]}} |i-j|^{-\nu} \exp\{-u^2 + (2-\eta)(\log k)|i-j|^{-\nu}\}$$

$$\leq ck^{2-\alpha\nu}\exp\{-u^2 + k^{-\alpha\nu}(2-\eta)\log k\}$$
$$\leq ck^{2-\alpha\nu}\exp(-u^2) = ck^{-\alpha\nu+\eta}$$
$$= ck^{\{\eta(1+\delta+\nu)-\nu(1-\delta)\}/\{(1+\nu)(1+\delta)\}}.$$

(iii) Let $0 < \alpha < (1-\delta)/(1+\delta)$. Then

$$\sum{}^{(1)} \leq \delta k^{1+\alpha}\exp(-\{2\log k - (1+\rho)\log\log k\}/(1+\delta))$$
$$\leq ck^{-\{1-\delta-\alpha(1+\delta)\}/(1+\delta)}(\log k)^{\rho/(1+\delta)}$$

and

$$\sum{}^{(2)} \leq ck^2(\log k)^{-1-\nu}\exp(-u^2)$$
$$\cdot\exp\{c(2\log k - (1+\rho)\log\log k)(\log k)^{-1-\nu}\}$$
$$\leq ck^2(\log k)^{-1-\nu}\exp\{-2\log k + (1+\rho)\log\log k\}$$
$$= c(\log k)^{-\nu+\rho}.$$

4.1. *Proofs of Theorems* 2.1 *and* 2.2. From Lemma 4.1, we get a useful lemma for proving our Theorem 2.1:

Lemma 4.5. *Let $\mathscr{D} = \{(t, s); 0 \leq t \leq T - a_T, 0 \leq s \leq a_T\}$ be a 2-dimensional space. Set*

$$X(t, s) = \frac{\xi(t+s) - \xi(t)}{\sigma(a_T)}, \quad (t, s) \in \mathscr{D}$$

and

$$\varphi(z) = \frac{2\sigma(2z)}{\sigma(a_T)}, \quad z > 0$$

where $\varphi(z)$ is a nondecreasing continuous, regularly varying function with index γ at ∞ for some $0 < \gamma < 1$ and $\varphi(z)$ satisfies (4.1). *Then for any $\varepsilon' > 0$ there exists a constant $C_{\varepsilon'}$ depending only on ε' such that for all $u \geq 0$*

$$P\{\sup_{(t,s)\in\mathscr{D}} X(t, s) \geq u\} \leq C_{\varepsilon'}\frac{T}{a_T}e^{-u^2/(2+\varepsilon')}.$$

Proof. Clearly it follows that $EX(t, s) = 0$,

$$\Gamma = \sup_{(t,s)\in\mathscr{D}} EX(t, s)^2 = 1$$

and

$$E\{X(t_1, s_1) - X(t_2, s_2)\}^2 \leq 2\frac{\sigma^2(|t_1 - t_2| + |s_1 - s_2|) + \sigma^2(|t_1 - t_2|)}{\sigma^2(a_T)}$$
$$\leq \frac{4}{\sigma^2(a_T)}\sigma^2(2\{(t_1 - t_2)^2 + (s_1 - s_2)^2\}^{1/2})$$
$$= \varphi^2(\{(t_1 - t_2)^2 + (s_1 - s_2)^2\}^{1/2}).$$

Since $\sigma(\cdot)$ is a regularly varying function, we can show that for every $\varepsilon' > 0$ we can find a small $c > 0$ such that

$$(2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{2}\,ca_T 2^{-y^2})\,dy \le \varepsilon'/2 \tag{4.2}$$

for all large T, where A is a constant such that $A > \{2\sqrt{N\log 2} \vee (1/\sqrt{2})\}$. Indeed, if a_T is bounded, then (4.2) is obvious from (2.1) by taking $c > 0$ small enough. When a_T is not bounded (i.e. $a_T \to \infty$ as $T \to \infty$), let $M < 2\sqrt{2}ca_T$. Now

$$(2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{2}ca_T 2^{-y^2})\,dy = (2\sqrt{2}+2)\frac{2A}{\sigma(a_T)}\int_0^\infty \sigma(2\sqrt{2}c2^{-y^2}a_T)\,dy$$

$$= (2\sqrt{2}+2)\frac{2A}{\sigma(a_T)}\int_0^{\{\log_2(2\sqrt{2}ca_T/M)\}^{1/2}} \sigma(2\sqrt{2}c2^{-y^2}a_T)\,dy$$

$$+ (2\sqrt{2}+2)\frac{2A}{\sigma(a_T)}\int_{\{\log_2(2\sqrt{2}ca_T/M)\}^{1/2}}^\infty \sigma(2\sqrt{2}c2^{-y^2}a_T)\,dy$$

$$= I_1 + I_2, \text{ say.}$$

As for I_1

$$I_1 \le \text{const.}\int_0^{\{\log_2(2\sqrt{2}ca_T/M)\}^{1/2}} \frac{\sigma(2\sqrt{2}c2^{-y^2}a_T)}{\sigma(a_T)}\,dy$$

and we can choose M sufficiently large so that

$$I_1 \le \text{const.}\int_0^\infty \{2\sqrt{2}c2^{-y^2}\}^\beta\,dy$$

for some $\beta > 0$. The last inequality can be easily obtained if we express the regularly varying function in its canonical form (cf. Feller [12], p. 282). As for I_2, using the condition (2.1), we can easily obtain

$$I_2 \le \text{const.}\int_0^\infty \frac{\sigma(M2^{-t^2})}{\sigma(a_T)}\,dt < \varepsilon'/2$$

for all large T. Let $u = x(1+\varepsilon'/2)$, $x \ge 1$. From (4.2) and Lemma 4.1 we have

$$P\{\sup_{(t,s)\in\mathscr{D}} X(t,s) \ge u\}$$

$$\le P\left\{\sup_{(t,s)\in\mathscr{D}} X(t,s) \ge x\{1 + (2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{2}ca_T 2^{-y^2})\,dy\}\right\}$$

$$\le K\left(\frac{T-a_T}{ca_T} \vee 1\right)\left(\frac{a_T}{ca_T} \vee 1\right)e^{-x^2/2}$$

$$\le C_{\varepsilon'}\frac{T}{a_T}e^{-u^2/(2+\varepsilon')}$$

where K is a constant.

Proof of Theorem 2.1. As in Lemma 4.5, set

$$X(t, s) = \frac{\xi(t+s) - \xi(t)}{\xi(a_T)}, \quad (t, s) \in \mathscr{D}$$

and $\varphi(z) = 2\sigma(2z)/\sigma(a_T)$, $z > 0$. Applying Lemma 4.5, we have for any $0 < \varepsilon < 1$

$$\begin{aligned} &P\{\mathbf{B}^*(T, a_T) > \sqrt{1+\varepsilon}\} \\ (4.3) \qquad &= P\left\{\sup_{(t,s)\in\mathscr{D}} \frac{\xi(t+s) - \xi(t)}{\sigma(a_T)} > \{2(1+\varepsilon)(\log(T/a_T) + \log\log T)\}^{1/2}\right\} \\ &\leq C_{\varepsilon'} \frac{T}{a_T} \exp\left\{-\frac{2(1+\varepsilon)}{2+\varepsilon'}(\log(T/a_T) + \log\log T)\right\}. \end{aligned}$$

Taking $\varepsilon' < 2\varepsilon$ for given ε and setting $\delta_1 = (2\varepsilon - \varepsilon')/(2+\varepsilon')$, the last term of (4.3) is less than or equal to $C_{\varepsilon'}(\log T)^{-1-\delta_1}$. For given $k \in \mathbb{N}$ (set of positive integers), let $T_k = \exp(k^\alpha)$ where $\alpha = 1 - (\delta_1/2)$. Then we have

$$P\{\mathbf{B}^*(T_k, a_{T_k}) > \sqrt{1+\varepsilon}\} \leq C_{\varepsilon'} k^{-(1+\{\delta_1(1-\delta_1)/2\})}.$$

Applying the Borel-Cantelli lemma, we have

$$\limsup_{k\to\infty} \mathbf{B}^*(T_k, a_{T_k}) \leq \sqrt{1+\varepsilon}, \qquad \text{a.s.}$$

The remainder of the proof is to show that

$$(4.4) \qquad \limsup_{T\to\infty} \mathbf{B}^*(T, a_T) \leq \limsup_{k\to\infty} \mathbf{B}^*(T_k, a_{T_k}).$$

Let T be in $T_{k-1} \leq T \leq T_k$. By (2.3), $T_k - a_{T_k} \geq T - a_T$. Thus we have

$$\begin{aligned} \mathbf{B}^*(T_k, a_{T_k}) &\geq \sup_{(t,s)\in\mathscr{D}} \frac{\xi(t+s) - \xi(t)}{\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}\sigma(a_{T_k})} \\ &\geq \left\{\frac{\log(T_{k-1}/a_{T_{k-1}}) + \log\log T_{k-1}}{\log(T_k/a_{T_k}) + \log\log T_k}\right\}^{1/2} \frac{\sigma(a_{T_{k-1}})}{\sigma(a_{T_k})} \mathbf{B}^*(T, a_T). \end{aligned}$$

By using the condition (2.3) again, we have

$$\begin{aligned} 1 &\geq \frac{\log(T_{k-1}/a_{T_{k-1}}) + \log\log T_{k-1}}{\log(T_k/a_{T_k}) + \log\log T_k} \geq \frac{(T_{k-1}/a_{T_{k-1}})\log T_{k-1}}{(T_k/a_{T_k})\log T_k} \\ (4.5) \qquad &\geq \frac{(T_{k-1}/a_{T_{k-1}})T_{k-1}}{(T_k/a_{T_k})T_k} \geq \left(\frac{T_{k-1}}{T_k}\right)^2 = \exp\{2((k-1)^\alpha - k^\alpha)\} \\ &\geq \exp\{-2\alpha(k-1)^{\alpha-1}\} \longrightarrow 1 \qquad \text{as } k \longrightarrow \infty. \end{aligned}$$

By (2.3), for $T_{k-1} \leq T \leq T_k$

$$1 \geq \frac{a_{T_{k-1}}}{a_{T_k}} \geq \frac{T_{k-1}}{T_k} \geq \exp\{-\alpha(k-1)^{\alpha-1}\}.$$

Thus it follows from the regularity of $\sigma(\cdot)$ at ∞ that

$$(4.6) \qquad 1 \geq \frac{\sigma(a_{T_{k-1}})}{\sigma(a_{T_k})} \geq \frac{\sigma(\exp\{-\alpha(k-1)^{\alpha-1}\}a_{T_k})}{\sigma(a_{T_k})} \longrightarrow 1 \qquad \text{as } k \longrightarrow \infty.$$

This proves the inequality (4.4).

Before proving Theorem 2.2 we begin with the lemma:

Lemma 4.6. *Let* a_T *be as in Theorem* 2.2. *Take* $0 < \alpha < 1$ *and set* $T_k = \exp(k^\alpha)$, $k \in \mathbb{N}$. *Then*

$$\liminf_{T\to\infty} \mathbf{B}(T, a_T) \geq \liminf_{k\to\infty} \mathbf{B}(T_k, a_{T_k}), \qquad \text{a.s.}$$

Proof. Let $T_k \leq T \leq T_{k+1}$. Denote

$$\mathscr{D}_k = \{(t, s);\, 0 \leq t \leq T_{k+1} - a_{T_k},\, a_{T_k} \leq s \leq a_{T_{k+1}}\}.$$

Since $T_k - a_{T_k} \leq T - a_T$, we have

$$\begin{aligned}
\mathbf{B}(T_k, a_{T_k}) &\leq \sup_{0\leq t\leq T_k - a_{T_k}} \frac{\xi(t+a_T) - \xi(t)}{\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}\sigma(a_{T_k})} \\
&\quad + \sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t+a_{T_k}) - \xi(t+s)|}{\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}\sigma(a_{T_k})} \\
&\leq \left\{\frac{\log(T_{k+1}/a_{T_{k+1}}) + \log\log T_{k+1}}{\log(T_k/a_{T_k}) + \log\log T_k}\right\}^{1/2} \frac{\sigma(a_{T_{k+1}})}{\sigma(a_{T_k})} \mathbf{B}(T, a_T) \\
&\quad + \sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t+a_{T_k}) - \xi(t+s)|}{\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}\sigma(a_{T_k})}.
\end{aligned}$$

As in (4.5) and (4.6), we have

$$\left\{\frac{\log(T_{k+1}/a_{T_{k+1}}) + \log\log T_{k+1}}{\log(T_k/a_{T_k}) + \log\log T_k}\right\}^{1/2} \frac{\sigma(a_{T_{k+1}})}{\sigma(a_{T_k})} \longrightarrow 1 \qquad \text{as } k \longrightarrow \infty.$$

Let us prove that

$$(4.7) \qquad \lim_{k\to\infty} \sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t+a_{T_k}) - \xi(t+s)|}{\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}\sigma(a_{T_k})} = 0, \qquad \text{a.s.}$$

If a_T is a constant in T, then the proof of (4.7) is not necessary. Set

$$X(t, s) = \frac{\xi(t+s) - \xi(t+a_{T_k})}{\sigma(a_{T_k})}, \quad (t, s) \in \mathscr{D}_k$$

and $\varphi(z) = 2\sigma(2z)/\sigma(a_{T_k})$, $z > 0$. Now, for $0 < \alpha < 1$

$$\frac{a_{T_{k+1}}}{a_{T_k}} \leq \frac{T_{k+1}}{T_k} \leq \exp(\alpha k^{\alpha-1})$$

and

(4.8)
$$\frac{a_{T_{k+1}} - a_{T_k}}{a_{T_k}} \le \exp(\alpha k^{\alpha-1}) - 1.$$

Since $\sigma(\cdot)$ is regularly varying at ∞, it follows from (4.8) that for any $\varepsilon'' > 0$

$$\sup_{(t,s)\in\mathscr{D}_k} EX(t, s)^2 = \frac{\sigma^2(a_{T_{k+1}} - a_{T_k})}{\sigma^2(a_{T_k})} = \Gamma^2 < (\varepsilon'')^2,$$

provided k is sufficiently large. It is easy to show that

$$E\{X(t_1, s_1) - X(t_2, s_2)\}^2 \le \varphi^2(\{(t_1 - t_2)^2 + (s_1 - s_2)^2\}^{1/2}).$$

To make use of Lemma 4.1, set $\lambda = ca_{T_k}$ where c is small enough. As in the proof of (4.2), we obtain that for any $\varepsilon' > 0$ we can choose a small $c > 0$ so that

$$(2\sqrt{2} + 2)A \int_0^\infty \varphi(\sqrt{2}ca_{T_k}2^{-y^2})dy < \varepsilon'$$

for all large k. For given $\varepsilon > 0$ and large k, we set

$$x = \varepsilon\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}/(\Gamma + \varepsilon').$$

Choosing $\varepsilon'' = \varepsilon' = \varepsilon^2$, we have

$$x \ge \{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}/(2\varepsilon).$$

Now applying Lemma 4.1, we have for any $\varepsilon > 0$

(4.9)
$$\begin{aligned}
&P\left\{\sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t + a_{T_k}) - \xi(t + s)|}{\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}\sigma(a_{T_k})} > \varepsilon\right\} \\
&\le P\left\{\sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t + a_{T_k}) - \xi(t + s)|}{\sigma(a_{T_k})} > x\left\{\Gamma + (2\sqrt{2} + 2)A \int_0^\infty \varphi(\sqrt{2}\lambda 2^{-y^2})dy\right\}\right\} \\
&\le C_\varepsilon \frac{T_k}{a_{T_k}} \exp\left\{-\frac{1}{4\varepsilon^2}(\log(T_k/a_{T_k}) + \log\log T_k)\right\} \\
&= C_\varepsilon \left(\frac{T_k}{a_{T_k}}\right)^{-(\frac{1}{4\varepsilon^2} - 1)} (\log T_k)^{-\frac{1}{4\varepsilon^2}}.
\end{aligned}$$

For $\varepsilon < 1/2$, the last term of (4.9) is less than or equal to $C_\varepsilon k^{-\alpha/(4\varepsilon^2)}$. Taking $\varepsilon < \min(1/2, \sqrt{\alpha}/2)$, the series

$$\sum_k P\left\{\sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t + a_{T_k}) - \xi(t + s)|}{\{2(\log(T_k/a_{T_k}) + \log\log T_k)\}^{1/2}\sigma(a_{T_k})} > \varepsilon\right\}$$

is convergent and using the Borel-Cantelli lemma, we obtain (4.7).

Proof of Theorem 2.2. The proof will be shown in two steps.

Step 1. First consider the case in which the condition (2.5) be satisfied. For given T large, define a positive integer h_T by $h_T = [T/a_T]$ where $[y]$ denotes the greatest integer not exceeding y. It is obvious from (2.3) and (2.4) that h_T is nondecreasing and $h_T \to \infty$ as $T \to \infty$. For $i = 1, 2, \ldots, h_T$, let us define the incremental random variable

$$Y_T(i) = \xi(ia_T) - \xi((i-1)a_T).$$

Then, $Y_T(i)/\sigma(a_T)$ is a standard normal random variable. From the concavity of $\sigma^2(t)$ it follows that for $m := |i - j| \geq 1$

$$\begin{aligned}
&covariance(Y_T(i), Y_T(j)) \\
&= E\{\xi(ia_T)\xi(ja_T) - \xi(ia_T)\xi((j-1)a_T) - \xi((i-1)a_T)\xi(ja_T) \\
&\quad + \xi((i-1)a_T)\xi((j-1)a_T)\} \\
&= \tfrac{1}{2}(\{\sigma^2((m+1)a_T) - \sigma^2(ma_T)\} - \{\sigma^2(ma_T) - \sigma^2((m-1)a_T)\}) \\
&\leq 0.
\end{aligned}$$

Applying Lemma 4.2 for $\xi_j = Y_T(j)/\sigma(a_T)$, $j = 1, 2, \ldots, h_T$, we have for any $0 < \varepsilon < 1$

$$\begin{aligned}
&P\{\mathbf{B}(T, a_T) < \sqrt{1-\varepsilon}\} \\
&= P\left\{\sup_{0 \leq t \leq T - a_T} \frac{\xi(t + a_T) - \xi(t)}{\sigma(a_T)} < \{2(1-\varepsilon)(\log(T/a_T) + \log\log T)\}^{1/2}\right\} \\
&\leq P\left\{\sup_{1 \leq j \leq h_T} \frac{Y_T(j)}{\sigma(a_T)} < \{2(1-\varepsilon)(\log(T/a_T) + \log\log T)\}^{1/2}\right\} \\
&\leq \Phi(u_T)^{h_T}
\end{aligned}$$

where $u_T = \{2(1-\varepsilon)(\log(T/a_T) + \log\log T)\}^{1/2}$. Since, for large T,

$$\Phi(u_T) \leq \exp\left(-c\left\{\frac{T}{a_T}\log T\right\}^{-1+\varepsilon}\right),$$

we have

$$P\{\mathbf{B}(T, a_T) < \sqrt{1-\varepsilon}\} \leq \exp\left\{-c\left(\frac{T}{a_T}\right)^{\varepsilon}\left(\frac{1}{\log T}\right)^{1-\varepsilon}\right\}. \tag{4.10}$$

Step 2. Next we consider the case in which the condition (2.6) is satisfied. Choose large integer N and real number T satisfying $N < (\log T)^{B_1}$ for some $B_1 > 0$. Define a positive integer k_T by $k_T = [T/(Na_T)]$. Then by (2.3) and (2.4), k_T is nondecreasing and $k_T \to \infty$ as $T \to \infty$. For $i = 1, \ldots, [T]$, we also define

$$X_T(i) = \xi(Nia_T) - \xi((Ni-1)a_T).$$

Then $X_T(i)/\sigma(a_T)$ is a standard normal random variable. It follows from (2.4) that for any $0 < \varepsilon' < \varepsilon < 1$ and large T

(4.11)

$$
\begin{aligned}
&P\{\mathbf{B}(T, a_T) < \sqrt{1-\varepsilon}\} \\
&= P\left\{\sup_{0 \le t \le T - a_T} \frac{\xi(t+a_T) - \xi(t)}{\sigma(a_T)} < \{2(1-\varepsilon)(\log(T/a_T) + \log\log T)\}^{1/2}\right\} \\
&\le P\left\{\sup_{0 \le t \le T - a_T} \frac{\xi(t+a_T) - \xi(t)}{\sigma(a_T)} < \{2(1-\varepsilon')\log k_T\}^{1/2}\right\} \\
&\le P\left\{\sup_{1 \le j \le k_T} \frac{X_T(j)}{\sigma(a_T)} < \{2(1-\varepsilon')\log k_T\}^{1/2}\right\}.
\end{aligned}
$$

Define

$$r_T(i, j) = correlation(X_T(i), X_T(j)),\ i \neq j$$

and let $m = |i - j|$; then by the assumption (2.6) we have

(4.12)

$$
\begin{aligned}
&|r_T(i, j)| \le c|\{\sigma^2((Nm+1)a_T) - \sigma^2(Nma_T)\} - \{\sigma^2(Nma_T) - \sigma^2((Nm-1)a_T)\}|/\sigma^2(a_T) \\
&\le c\left|\int_{Nma_T}^{(Nm+1)a_T} (\sigma^2(x))'\,dx - \int_{(Nm-1)a_T}^{Nma_T} (\sigma^2(x))'\,dx\right|/\sigma^2(a_T) \\
&\le c\int_{(Nm-1)a_T}^{Nma_T}\left(\int_x^{x+a_T} |(\sigma^2(y))''|\,dy\right)dx/\sigma^2(a_T) \\
&\le c\int_{(Nm-1)a_T}^{Nma_T}\left(\int_x^{x+a_T} \frac{\sigma^2(y)}{y^2}\,dy\right)dx/\sigma^2(a_T) \\
&\le \frac{c\sigma^2((Nm+1)a_T)a_T{}^2}{\sigma^2(a_T)(Nm-1)^2 a_T{}^2}.
\end{aligned}
$$

Since $\sigma(t)$ is regularly varying with index $\gamma(0 < \gamma < 1)$ at ∞, it follows that for any $\zeta > 0$ there exists $C_\zeta > 0$ such that

$$\frac{\sigma((Nm+1)a_T)}{\sigma(a_T)} \le C_\zeta(Nm+1)^{\gamma+\zeta}.$$

Taking $\zeta = (1-\gamma)/2$ for γ fixed, we get

$$\frac{\sigma^2((Nm+1)a_T)}{\sigma^2(a_T)} \le C_\gamma{}^2(Nm+1)^{1+\gamma}.$$

Thus from (4.12) we have

$$
\begin{aligned}
|r_T(i, j)| &\le C_\gamma{}^2\left(\frac{Nm+1}{Nm-1}\right)^2(Nm+1)^{-(1-\gamma)} \\
&\le 4C_\gamma{}^2(Nm)^{-(1-\gamma)}.
\end{aligned}
\tag{4.13}
$$

Choosing N so large that $N \ge (8C_\gamma^2)^{1/(1-\gamma)}$ for given γ, the last term of (4.13) is less than $m^{-\nu}(\nu = 1 - \gamma)$. Now let us apply Lemma 4.3 and Lemma 4.4 (ii) for

$$\xi_j = \frac{X_T(j)}{\sigma(a_T)}, \qquad j = 1, 2, \dots, [T];$$

$$\delta = \max_{i \neq j} |covariance(\xi_i, \xi_j)| < 1;$$

$$\xi_{\ell_j} = \frac{X_T(j)}{\sigma(a_T)}, \; j = 1, 2, \dots, k_T$$

and

$$u_T = \{(2-\eta)\log k_T\}^{1/2}, \; \eta = 2\varepsilon', \; \varepsilon' < v(1-\delta)/\{2(1+\delta+v)\}.$$

Then the last term of (4.11) is less than or equal to

$$\Phi(u_T)^{k_T} + ck_T{}^{-\delta_0}.$$

Thus we have

$$P\{\mathbf{B}(T, a_T) < \sqrt{1-\varepsilon}\} \leq \exp(-k_T{}^{\varepsilon'}) + ck_T{}^{-\delta_0}. \tag{4.14}$$

Comparing (4.10) with (4.14) under the condition (2.4), we see that it is sufficient to estimate the inequality (4.14). The condition (2.4) also implies that there exists B_2 big enough such that for large T

$$\log(T/a_T) \geq B_2 \log\log T \tag{4.15}$$

and

$$B_2 > 1/(\alpha\delta_0) \tag{4.16}$$

where $1 > \alpha > 0$ is that given in Lemma 4.6 and δ_0 is that in (4.14). Hence from (4.15) we have $k_T \geq c(\log T)^{B_2}$ for some $c > 0$. Let $T_k = \exp(k^\alpha)$ as in Lemma 4.6. Then the inequality (4.14) yields

$$P\{\mathbf{B}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\} \leq \exp(-ck^{\alpha B_2 \varepsilon'}) + ck^{-\alpha B_2 \delta_0}$$

and hence from (4.16)

$$\textstyle\sum_k P\{\mathbf{B}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\} < \infty.$$

Using the Borel-Cantelli lemma, we obtain

$$\liminf_{k\to\infty} \mathbf{B}(T_k, a_{T_k}) \geq \sqrt{1-\varepsilon}, \qquad \text{a.s.}$$

Letting T be in $T_k \leq T \leq T_{k+1}$ for given T_k, the proof of Theorem 2.2 immediately follows from Lemma 4.6.

4.2. *Proofs of theorems in section* 3. The method of the proofs is similar to those in section 4.1. In Lemma 4.1, if we set

$$\varepsilon' = (2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{2}\lambda 2^{-y^2})\,dy$$

and $u = x(1+\varepsilon')$, $x \geq 1$, then by choosing sufficiently small λ, we can easily obtain the following

Lemma 4.7. *Let* $\mathscr{D} = \{(t, s); 0 \leq t \leq T - a_T, 0 \leq s \leq a_T\}$ *be as in Lemma* 4.2. *Set*

$$X(t, s) = \frac{\xi(t+s) - \xi(t)}{\sigma(a_T)}, \quad (t, s) \in \mathscr{D}$$

and

$$\varphi(z) = \frac{2\sigma(2z)}{\sigma(a_T)}, \quad z > 0$$

where $\varphi(z)$ *is the function defined in* (4.1). *Then for any* $\varepsilon' > 0$ *there exists a constant* $K_{\varepsilon'}$ *depending only on* ε' *such that for all* $u > 0$

$$P\{\sup_{(t,s)\in\mathscr{D}} X(t, s) \geq u\} \leq K_{\varepsilon'} T a_T \exp\left(-\frac{u^2}{2+\varepsilon'}\right).$$

Proof of Theorem 3.1. Applying Lemma 4.7, we have for any $\varepsilon > 0$

$$\begin{aligned} &P\{\mathbf{D}^*(T, a_T) > \sqrt{1+\varepsilon}\} \\ &= P\left\{\sup_{(t,s)\in\mathscr{D}} \frac{\xi(t+s) - \xi(t)}{\sigma(a_T)} > \{2(1+\varepsilon)\log(Ta_T)\}^{1/2}\right\} \\ &\leq K_{\varepsilon'} T a_T \exp\left\{-\frac{1}{2+\varepsilon'}(2(1+\varepsilon)\log(Ta_T))\right\} \qquad (4.17) \\ &\leq K_{\varepsilon'} (Ta_T)^{1 - \frac{2+2\varepsilon}{2+\varepsilon'}}. \end{aligned}$$

Taking $\varepsilon' < 2\varepsilon$ for given ε and setting $\delta_2 = (2\varepsilon - \varepsilon')/(2+\varepsilon')$, the last term of (4.17) is less than or equal to $K_{\varepsilon'} T^{-\delta_2}$. For given $k \in \mathbb{N}$, let $T_k = \exp(k^\alpha)$ $(0 < \alpha < 1)$. Then we have

$$P\{\mathbf{D}^*(T_k, a_{T_k}) > \sqrt{1+\varepsilon}\} \leq K_{\varepsilon'} \exp(-\delta_2 k^\alpha).$$

Since ε is arbitrary, we obtain

$$\limsup_{k\to\infty} \mathbf{D}^*(T_k, a_{T_k}) \leq 1, \qquad \text{a.s.}$$

Now let us prove

$$\limsup_{T\to\infty} \mathbf{D}^*(T, a_T) \leq \limsup_{k\to\infty} \mathbf{D}^*(T_k, a_{T_k}). \tag{4.18}$$

Since $T_k - a_{T_k} \geq T - a_T$ for $T_{k-1} \leq T \leq T_k$, we have

$$\begin{aligned} \mathbf{D}^*(T_k, a_{T_k}) &\geq \sup_{(t,s)\in\mathscr{D}} \frac{\xi(t+s) - \xi(t)}{\{2\log(T_k a_{T_k})\}^{1/2} \sigma(a_{T_k})} \\ &\geq \left(\frac{\log(T_{k-1} a_{T_{k-1}})}{\log(T_k a_{T_k})}\right)^{1/2} \frac{\sigma(a_{T_{k-1}})}{\sigma(a_{T_k})} \mathbf{D}^*(T, a_T). \end{aligned}$$

By (3.5), we have

$$\frac{a_{T_k}}{a_{T_{k-1}}} \le \frac{T_k}{T_{k-1}} \le \exp\{\alpha(k-1)^{\alpha-1}\}$$

and

$$1 \ge \frac{\log(T_{k-1}a_{T_{k-1}})}{\log(T_k a_{T_k})} \ge \frac{T_{k-1}a_{T_{k-1}}}{T_k a_{T_k}} \ge \frac{1}{\exp\{2\alpha(k-1)^{\alpha-1}\}} \longrightarrow 1 \qquad \text{as } k \longrightarrow \infty.$$

Since $\sigma(t) \to \sqrt{2}$ as $t \to \infty$,

$$1 \ge \frac{\sigma(a_{T_{k-1}})}{\sigma(a_{T_k})} \longrightarrow 1 \qquad \text{as } k \longrightarrow \infty.$$

This proves (4.18) and thus the proof of Theorem 3.1 is complete.

For proving Theorems 3.2–3.4, we shall need the following

Lemma 4.8. *Let* a_T *be as in Theorem* 3.2. *Take* $0 < \theta < 1$ *and set* $T_k = k^{1/\theta}/\log k$, $k \in \mathbb{N}$. *Then we have*

$$\liminf_{T\to\infty} \mathbf{D}(T, a_T) \ge \liminf_{k\to\infty} \mathbf{D}(T_k, a_{T_k}), \qquad \text{a.s.}$$

Proof. Let $T_k \le T \le T_{k+1}$. Denote

$$\mathscr{D}_k = \{(t, s);\, 0 \le t \le T_{k+1} - a_{T_k},\ a_{T_k} \le s \le a_{T_{k+1}}\}.$$

Since $T_k - a_{T_k} \le T - a_T$, we have

$$\begin{aligned}
\mathbf{D}(T_k, a_{T_k}) &\le \sup_{0\le t\le T-a_T} \frac{\xi(t+a_{T_k}) - \xi(t)}{\{2\log(T_k/a_{T_k})\}^{1/2}\sigma(a_{T_k})} \\
&\le \sup_{0\le t\le T-a_T} \frac{\xi(t+a_T) - \xi(t)}{\{2\log(T_k/a_{T_k})\}^{1/2}\sigma(a_{T_k})} \\
&\quad + \sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t+a_{T_k}) - \xi(t+s)|}{\{2\log(T_k/a_{T_k})\}^{1/2}\sigma(a_{T_k})} \\
&\le \left(\frac{\log(T_{k+1}/a_{T_{k+1}})}{\log(T_k/a_{T_k})}\right)^{1/2} \frac{\sigma(a_{T_{k+1}})}{\sigma(a_{T_k})} \mathbf{D}(T, a_T) \\
&\quad + \sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t+a_{T_k}) - \xi(t+s)|}{\{2\log(T_k/a_{T_k})\}^{1/2}\sigma(a_{T_k})}.
\end{aligned}$$

By (3.5),

$$\frac{a_{T_{k+1}}}{a_{T_k}} \le \frac{T_{k+1}}{T_k} \le \left(1 + \frac{1}{k}\right)^{1/\theta} \tag{4.19}$$

and thus

$$1 \le \frac{\log(T_{k+1}/a_{T_{k+1}})}{\log(T_k/a_{T_k})} \le \frac{T_{k+1}/a_{T_{k+1}}}{T_k/a_{T_k}} \le \left(1 + \frac{1}{k}\right)^{1/\theta} \longrightarrow 1 \qquad \text{as } k \longrightarrow \infty.$$

Since $\sigma(t) \to \sqrt{2}$ as $t \to \infty$,

$$1 \le \frac{\sigma(a_{T_{k+1}})}{\sigma(a_{T_k})} \longrightarrow 1 \qquad \text{as } k \longrightarrow \infty.$$

Now let us prove

$$\text{(4.20)} \qquad \lim_{k\to\infty} \sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t + a_{T_k}) - \xi(t + s)|}{\{2\log(T_k/a_{T_k})\}^{1/2}\sigma(a_{T_k})} = 0, \qquad \text{a.s.}$$

Set

$$X(t, s) = \frac{\xi(t + s) - \xi(t + a_{T_k})}{\sigma(a_{T_k})}, \ (t, s) \in \mathscr{D}_k$$

and $\varphi(z) = 2\sigma(2z)/\sigma(a_{T_k})$, $z > 0$. For some $c > 0$, we have from (3.4) and (4.19)

$$a_{T_{k+1}} - a_{T_k} \le \left\{\left(1 + \frac{1}{k}\right)^{1/\theta} - 1\right\} a_{T_k}$$

$$\le c\frac{a_{T_k}}{k} \le c(\log k)^{-\theta} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty.$$

Thus for large k and any $\varepsilon'' > 0$

$$\sup_{(t,s)\in\mathscr{D}_k} EX(t, s)^2 = \frac{\sigma^2(a_{T_{k+1}} - a_{T_k})}{\sigma^2(a_{T_k})} = \Gamma^2 < (\varepsilon'')^2.$$

Take $\lambda > 0$ so small that for large k and any $\varepsilon' > 0$

$$(2\sqrt{2} + 2)A \int_0^\infty \varphi(\sqrt{2}\lambda 2^{-y^2})\,dy = (2\sqrt{2} + 2)A \int_0^\infty \frac{2\sigma(2\sqrt{2}\lambda 2^{-y^2})}{\sigma(a_{T_k})}\,dy < \varepsilon'.$$

For given $\varepsilon > 0$ and large k we set

$$x = \varepsilon\{2\log(T_k/a_{T_k})\}^{1/2}/(\Gamma + \varepsilon').$$

Choosing $\varepsilon'' = \varepsilon' = \varepsilon^2$, it follows from (3.4) that

$$x \ge \{2(1 - \theta)\log T_k\}^{1/2}/(2\varepsilon).$$

Now applying Lemma 4.1, we have for any $\varepsilon > 0$

$$P\left\{\sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t + a_{T_k}) - \xi(t + s)|}{\{2\log(T_k/a_{T_k})\}^{1/2}\sigma(a_{T_k})} > \varepsilon\right\}$$

$$\le P\left\{\sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t + a_{T_k}) - \xi(t + s)|}{\sigma(a_{T_k})} > x\{\Gamma + (2\sqrt{2} + 2)A \int_0^\infty \varphi(\sqrt{2}\lambda 2^{-y^2})\,dy\}\right\}$$

$$\le 2K_\varepsilon T_k a_{T_k} \exp\left(-\frac{(1 - \theta)\log T_k}{4\varepsilon^2}\right)$$

$$\le K_\varepsilon T_k^{-\{\frac{1-\theta}{4\varepsilon^2} - 2\}} \le K_\varepsilon k^{-\{\frac{1-\theta}{4\varepsilon^2} - 2\}/\theta}.$$

Taking $\varepsilon < ((1-\theta)/12)^{1/2}$, the series

$$\sum_k P\left\{\sup_{(t,s)\in\mathscr{D}_k} \frac{|\xi(t+a_{T_k})-\xi(t+s)|}{\{2\log(T_k/a_{T_k})\}^{1/2}\sigma(a_{T_k})} > \varepsilon\right\}$$

is convergent and hence we obtain (4.20). This proves Lemma 4.8.

Proof of Theorem 3.2. Define

$$Y_T(i) = \xi(ia_T) - \xi((i-1)a_T), \; i = 1, 2, \ldots, h_T$$

where $h_T = [T/a_T]$, as in Step 1 of the proof of Theorem 2.2. Then it follows from the convexity of $r(t)$ that for $m = |i-j| \geq 1$

$$\begin{aligned} &covariance(Y_T(i), Y_T(j)) \\ &= \{r(ma_T) - r((m-1)a_T)\} - \{r((m+1)a_T) - r(ma_T)\} \\ &\leq 0. \end{aligned}$$

From Lemma 4.2, we have for any $0 < \varepsilon < 1$

$$\begin{aligned} &P\{\mathbf{D}(T, a_T) < \sqrt{1-\varepsilon}\} \\ &\leq P\left\{\sup_{1\leq j\leq h_T} \frac{Y_T(j)}{\sigma(a_T)} < \{2(1-\varepsilon)\log(T/a_T)\}^{1/2}\right\} \\ &\leq \Phi(\{2(1-\varepsilon)\log(T/a_T)\}^{1/2})^{h_T} \qquad (4.21) \\ &\leq \exp(-c(T/a_T)^{\varepsilon}) \leq \exp(-cT^{(1-\theta)\varepsilon}). \end{aligned}$$

As in Lemma 4.8, let $T_k = k^{1/\theta}/\log k$. Then (4.21) yields

$$P\{\mathbf{D}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\} \leq \exp(-ck^{(1-\theta)\varepsilon/\theta}).$$

Thus

$$\sum_k P\{\mathbf{D}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\} < \infty$$

and

$$\liminf_{k\to\infty} \mathbf{D}(T_k, a_{T_k}) \geq \sqrt{1-\varepsilon}, \qquad \text{a.s.}$$

The proof of Theorem 3.2 immediately follows from Lemma 4.8.

Proof of Theorem 3.3. For given T large, define a positive integer k_T by $k_T = [T/(2a_T)]$. For $i = 1, 2, \ldots, [T]$, we also define

$$X_T(i) = \xi(2ia_T) - \xi((2i-1)a_T).$$

Then $X_T(i)/\sigma(a_T)$ is a standard normal variable. Define

$$r_T(i,j) = correlation(X_T(i), X_T(j)), \; i \neq j$$

and let $m = |i-j|$; then by (3.9) we have for large T

$$
\begin{aligned}
|r_T(i, j)| &\le c|E\{X_T(i)X_T(j)\}| \\
&\le c\{|r(2ma_T) - r((2m-1)a_T)| + |r((2m+1)a_T) - r(2ma_T)|\} \\
&\le c\left\{\int_{(2m-1)a_T}^{2ma_T} |r'(t)|\,dt + \int_{2ma_T}^{(2m+1)a_T} |r'(t)|\,dt\right\} \\
&\le c\left\{\int_{(2m-1)a_T}^{2ma_T} t^{-1-\nu}dt + \int_{2ma_T}^{(2m+1)a_T} t^{-1-\nu}dt\right\} \\
&\le c\{((2m-1)a_T)^{-1-\nu}a_T + (2ma_T)^{-1-\nu}a_T\} \\
&\le \mu m^{-1-\nu}, \quad m = 1, 2, \dots, k_T - 1
\end{aligned}
$$

where μ is small enough. Now let us apply Lemmas 4.3 and 4.4 (i) for

$$\xi_j = \frac{X_T(j)}{\sigma(a_T)}, \; j = 1, 2, \dots, [T];$$

$$\delta := \max_{i \ne j}|covariance(\xi_i, \xi_j)| < 1;$$

$$\xi_{\ell_j} = \frac{X_T(j)}{\sigma(a_T)}, \; j = 1, 2, \dots, k_T$$

and

$$u_T = \{2(1-\varepsilon)\log(T/a_T)\}^{1/2}, \; \eta = 2\varepsilon < \nu, \; \varepsilon < (1-\mu)/2.$$

Then we have

$$
\begin{aligned}
&P\{\mathbf{D}(T, a_T) < \sqrt{1-\varepsilon}\} \\
&\le P\left\{\sup_{1 \le j \le k_T} \frac{X_T(j)}{\sigma(a_T)} < \{2(1-\varepsilon)\log(T/a_T)\}^{1/2}\right\} \qquad (4.22) \\
&\le \Phi(u_T)^{k_T} + c(k_T)^{-1+\delta_0} \\
&\le \exp\{-cT^{(1-\theta)\varepsilon}\} + cT^{(1-\theta)(-1+\delta_0)}
\end{aligned}
$$

where $\delta_0 = (\eta + 2\mu)/(1+\mu)$. For given $k \in \mathbb{N}$, set $T_k = k^{1/\theta}/\log k$. Since $(1-\theta)(-1+\delta_0)/\theta < -1$, it follows from (4.22) that

$$\sum_k P\{\mathbf{D}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\} < \infty$$

and

$$\liminf_{k \to \infty} \mathbf{D}(T_k, a_{T_k}) \ge \sqrt{1-\varepsilon}, \qquad \text{a.s.}$$

Thus the proof of Theorem 3.3 immediately follows from Lemma 4.8.

Proof of Theorem 3.4. In the case when $r(t)$ is convex, the proof immediately follows from Corollaries 1 and 2. Under the condition (3.13), it is sufficient to prove

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{\xi(t+a_T)-\xi(t)}{(2\log T)^{1/2}\sigma(a_T)} \ge 1, \qquad \text{a.s.} \tag{4.23}$$

Let us define $X_T(i)$, δ and $r_T(i,j)$ as in the proof of Theorem 3.3. Then by (3.13) we have for large T

$$\begin{aligned} |r_T(i,j)| &\le c|E\{X_T(i)X_T(j)\}| \\ &= c|2r(2ma_T) - r((2m-1)a_T) - r((2m+1)a_T)| \\ &\le c|r((2m-1)a_T)| \\ &\le c\{(2m-1)a_T\}^{-v} \\ &< m^{-v}, \quad m = 1, 2, \ldots, k_T - 1. \end{aligned}$$

Apply Lemmas 4.3 and 4.4 (ii) with $u_T = \{2(1-\varepsilon)\log(T/a_T)\}^{1/2}$, $0<\varepsilon<1$, $\eta = 2\varepsilon < (1-\delta)v/(1+v+\delta)$. Then we have

$$\begin{aligned} &P\{\mathbf{D}(T, a_T) < \sqrt{1-\varepsilon}\} \\ &\le P\left\{\sup_{1\le j\le k_T} \frac{X_T(j)}{\sigma(a_T)} < \{2(1-\varepsilon)\log(T/a_T)\}^{1/2}\right\} \\ &\le \Phi(u_T)^{k_T} + ck_T^{-\delta_0} \\ &\le \exp\{-cT^{(1-\theta)\varepsilon}\} + cT^{-(1-\theta)\delta_0}. \end{aligned}$$

Defining T_k as in Lemma 4.8, we have

$$P\{\mathbf{D}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\} \le \exp\{-ck^{(1-\theta)\varepsilon/\theta}\} + ck^{-(1-\theta)\delta_0/\theta}.$$

Since θ is arbitrary, choosing θ so that $\theta/(1-\theta) < \delta_0$, the series

$$\sum_k P\{\mathbf{D}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\}$$

is convergent and hence

$$\liminf_{k\to\infty} \mathbf{D}(T_k, a_{T_k}) \ge \sqrt{1-\varepsilon}, \qquad \text{a.s.}$$

Using Lemma 4.8 and (3.10), we obtain (4.23).

Proof of Theorem 3.5. Again let $X_T(i)$, δ and $r_T(i,j)$ be defined as in the proof of Theorem 3.3. Then it follows from (3.16) that for large T

$$\begin{aligned} |r_T(i,j)| &\le c|E\{X_T(i)X_T(j)\}| \\ &\le c|r((2m-1)a_T)| \\ &\le c\{\log(2m-1) + \log a_T\}^{-1-v} \\ &\le c(\log m)^{-1-v}, \quad m = 2, 3, \ldots, k_T - 1, \\ &< \delta, \qquad\qquad m = 1. \end{aligned}$$

Applying Lemmas 4.3 and 4.4(iii) with

$$u_T = \{2\log(T/a_T) - (1+\rho)\log\log(T/a_T)\}^{1/2}, \qquad 0 < \rho < \nu,$$

we obtain for $0 < \varepsilon < 1$ and large T

$$\begin{aligned} &P\{\mathbf{D}(T, a_T) < \sqrt{1-\varepsilon}\} \\ &\le P\left\{\sup_{1 \le j \le k_T} \frac{X_T(j)}{\sigma(a_T)} < \{2(1-\varepsilon)\log(T/a_T)\}^{1/2}\right\} \\ &\le P\left\{\sup_{1 \le j \le k_T} \frac{X_T(j)}{\sigma(a_T)} < \{2\log(T/a_T) - (1+\rho)\log\log(T/a_T)\}^{1/2}\right\} \\ &\le \Phi(u_T)^{k_T} + c(\log(T/a_T))^{-\nu+\rho}. \end{aligned}$$

Since

$$\Phi(u_T) \le \exp\{-(T/a_T)^{-1}(\log(T/a_T))^{\rho/2}\},$$

we have

$$P\{\mathbf{D}(T, a_T) < \sqrt{1-\varepsilon}\} \le \exp\{-c(\log T)^{\rho/2}\} + c(\log T)^{-(\nu-\rho)}.$$

For given $k \in \mathbb{N}$, set $T_k = \exp(k^{1/\tau})$, where $0 < \omega < \tau < \nu - \rho$ for ρ small enough. Then we have

$$P\{\mathbf{D}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\} \le \exp(-ck^{\rho/(2\tau)}) + ck^{-(\nu-\rho)/\tau}.$$

Thus the series

$$\sum_k P\{\mathbf{D}(T_k, a_{T_k}) < \sqrt{1-\varepsilon}\}$$

is convergent and

$$\liminf_{k\to\infty} \mathbf{D}(T_k, a_{T_k}) \ge \sqrt{1-\varepsilon}, \qquad \text{a.s.}$$

Let T be in $T_k \le T \le T_{k+1}$. By (3.18)

$$1 \le \frac{a_{T_{k+1}}}{a_{T_k}} \le \left(\frac{\log T_{k+1}}{\log T_k}\right)^d = \left\{1 + \frac{1}{k}\right\}^{d/\tau}.$$

Since, by (3.17)

$$\frac{\log a_{T_{k+1}}}{\log T_{k+1}} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty,$$

we have

$$1 \le \frac{\log(T_{k+1}/a_{T_{k+1}})}{\log(T_k/a_{T_k})} = \left(\frac{k+1}{k}\right)^{1/\tau}\left\{\frac{1 - (\log a_{T_{k+1}}/\log T_{k+1})}{1 - (\log a_{T_k}/\log T_k)}\right\} \longrightarrow 1$$

as $k \to \infty$. We note that

$$a_{T_{k+1}} - a_{T_k} \le \left\{\left(1 + \frac{1}{k}\right)^{d/\tau} - 1\right\} a_{T_k} \le c a_{T_k}/k$$

$$\leq c(\log T_k)^{\omega}/k = ck^{-(1-\omega/\tau)} \longrightarrow 0 \qquad \text{as } k \longrightarrow \infty.$$

Proceeding to the same lines as in the proof of Lemma 4.8, we can easily deduce

$$\liminf_{T\to\infty} \mathbf{D}(T, a_T) \geq \liminf_{k\to\infty} \mathbf{D}(T_k, a_{T_k}), \qquad \text{a.s.}$$

This completes the proof of Theorem 3.5.

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DEPARTMENT OF MATHEMATICS
COLLEGE OF NATURAL SCIENCE
GYEONGSANG NATIONAL UNIVERSITY
JINJU, KOREA

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