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On a multiplicative structure of BP-cohomology operation algebra

By

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§ 0. Introduction

The BP-theory is obtained as a factor of the p-localized complex cobordism theory, and has a close relation to the theory of p-typical formal group laws. For example, (BP_*, BP_*BP) has a particular algeraic structure, named Hopf-algebroid [10], and we can formulate its left unit, right unit, coproduct and canonical antipodal isomorphism in terms of the formal group law obtained from the complex orientation of the BP-theory. Since the E_2-term of the Adams-Novikov spectral sequence is a cohomology of the Hopf-algebroid (BP_*, BP_*BP), we can obtain many useful information from these formulae.

BP^*BP, which is a dual of BP_*BP, is a cohomology operation algera of the BP-theory, and can be regarded as a kind of (non-commutative) Hopf-algebra. D. Quillen [9] studied its Hopf-algebraic structure, and asserted that $BP^*BP \cong \mathrm{Hom}_{BP_*}(BP_*BP, BP_*) \cong \mathrm{Hom}_{\boldsymbol{Z}}(\boldsymbol{Z}[t_1, t_2, \cdots], BP_*) \cong BP^* \hat{\otimes} R$, where $R = \{r_E \colon E = (e_1, e_2, \cdots)\}$ is a dual basis of $\{t^E = t_1^{e_1} t_2^{e_2} \cdots\}$, the BP_*-free basis of BP_*BP (see also [1]). We call these r_E the Quillen elements. But its multiplicative structure has been expressed as a dual of the comultiplicative structure of BP_*BP, so that the complicatedness of this coproduct formula seems to prevent our intimate studying of the multiplicative structure of BP^*BP. Exceptionally, R. Kane [5], [6] demonstrated some interesting results about BP-operations and Steenrod operations from their behavior under the rationalization and the product formula modulo $(v_1, v_2, \cdots)$.

The purposes of this paper are to describe the complete formula for the product of BP-operations and to study the algebraic structure of BP^*BP by means of this formula.

§ 1. Product formula

BP^*BP is a stable cohomology operation algebra of the BP-theory. This is a dual algebra of $BP_*BP \cong BP_*[t_1, t_2, \cdots]$ ($\deg t_i = 2(p^i - 1)$) because BP_*BP is a free left module over the coefficient ring $BP_* \cong \boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$ ($\deg v_i = 2(p^i - 1)$).

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So we can describe BP^*BP as stated in the introduction. Notice that we understand $BP^* \cong BP_{-*}$, so that $r_E \in BP^nBP$ with $n = \sum 2e_i(p^i - 1)$ and $v_i \in BP^{2(1-p^i)}$. As also stated in the introduction, BP^*BP has not only a ring structure but a BP^*-Hopf-algebraic structure. Its multiplicative structure depends on the composition of operations, and its comultiplicative structure depends on the action on the product of two elements of BP^*X, where X is a CW-complex. In the sequel, η, ε, ψ and μ denote unit, counit (augmentation), coproduct and product, respectively.

$$\eta\colon BP^* \longrightarrow BP^*BP \qquad : \eta(a) = a \cdot 1.$$

$$\varepsilon\colon BP^*BP \longrightarrow BP^* \qquad : \varepsilon(r_E) = \begin{cases} 1: E = 0 \\ 0: \text{otherwise.} \end{cases}$$

$$\psi\colon BP^*BP \longrightarrow BP^*BP \mathop{\hat{\otimes}}_{BP^*} BP^*BP \qquad : \psi(r_E) = \sum_{F+G=E} r_E \otimes r_G.$$

The above infinite tensor product over BP^* is a tensor product of left BP^*-modules.

$$\mu\colon BP^*BP \otimes BP^*BP \longrightarrow BP^*BP\colon \text{Theorem 1.2.}$$

Notation 1.1. Throughout this paper, x^E denotes a monomial $x_1^{e_1}x_2^{e_2}\cdots$, where x_is are some specified elements, and we call $E = (e_1, e_2, \cdots)$ an exponential sequence.

Theorem 1.2. *Let r_E, r_F be two Quillen elements. Then the product of them is described as follows*:

$$r_E \circ r_F = \sum_{\substack{n \geq 0 \\ E(X_0) = E \\ F(X_0) = F \\ E(X_1) = V(X_0) \\ \vdots \\ E(X_n) = V(X_{n-1}) \\ M(X_1) \neq 0, \dots, M(X_n) \neq 0}} (-1)^n \beta(X_0) \cdots \beta(X_n) m^{M(X_0) + \cdots + M(X_n)} r_{V(X_n)}.$$

In the above formula, X_0 is a three-dimensional tensor, and $X_1, \dots, X_n$ are two-dimensional matrices, i.e., $X_0 = (x_{ijk})$, where non-negative integers x_{ijk} are defined for $i, j, k \geq 0$ except $(j, k) = (0, 0)$, and $X_m = (X_{ij})$, where non-negative integers x_{ij} are defined for $i \geq 0$, $j \geq 1$.

$$X = (x_{ijk}) = \begin{array}{c} i \uparrow \qquad \vdots \\ x_{110}\ x_{120}\ x_{130} \\ x_{101}\ x_{111}\ x_{121}\ x_{131} \\ x_{102}\ x_{112}\ x_{122}\ x_{132} \qquad \cdots \\ x_{010}\ x_{020}\ x_{030} \longrightarrow j \\ x_{001}\ x_{011}\ x_{021}\ x_{031} \\ x_{002}\ x_{012}\ x_{022}\ x_{032} \\ \swarrow k \qquad \ddots \end{array} \qquad X = (X_{ij}) = \begin{array}{c} \longrightarrow i \\ *\ \ *\ \ *\ \ * \\ x_{01}\ x_{11}\ x_{21}\ x_{31} \\ x_{02}\ x_{12}\ x_{22}\ x_{32} \quad \cdots \\ x_{03}\ x_{13}\ x_{23}\ x_{33} \\ \downarrow j \qquad \vdots \end{array}$$

m_i's are the polynomial generators of $H_*BP \cong Z_{(p)}[m_1, m_2, \cdots]$ ($\deg m_i = 2(p^i - 1)$), *which contains $\pi_*BP = BP_* \cong Z_{(p)}[v_1, v_2, \cdots]$ as a subring. And the functions V, M, E, F and β are defined as follows:*

$$V(X) = (a_1, a_2, \cdots), \ a_m = \sum_{i+j+k=m} x_{ijk} \quad (= \sum_{i+j=m} x_{ij}),$$

$$M(X) = (b_1, b_2, \cdots), \ b_m = \sum_{j,k} x_{mjk} \quad (= \sum_j x_{mj}),$$

$$E(X) = (c_1, c_2, \cdots), \ c_m = \sum_{i,k} p^i x_{imk} \quad (= \sum_i p^i x_{im}),$$

$$F(X) = (d_1, d_2, \cdots), \ d_m = \sum_{i,j} p^{i+j} x_{ijm},$$

$$\beta(X) = \prod a_m!/\prod x_{ijk}! \quad (= \prod a_m!/\prod x_{ijk}!).$$

The right expression of the product formula is summed over all $n \geq 0$ and X_0, $X_1, \ldots, X_n$ satisfying the conditions written below $\sum$.

Proof. The coproduct map of BP_*BP is defined recursively by the following equations (see [1], [2], [9]):

$$\sum_{i+j=n} m_i(\psi t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}, \text{ for } n = 1, 2, \cdots.$$

We now follow the same way as J. Milnor [8] obtained the product formula for the Steenrod algebra. We apply the formula $(y_1 + \cdots + y_n)^e = \sum_{i_1+\ldots+i_n=e} (e!/i_1! \cdots i_n!) y_1^{i_1} \cdots y_n^{i_n}$ to the left and right expressions of the above equations. Then we obtain:

$$\begin{aligned}
&\Big(\sum_{i+j=n, j\geq 1} m_i(\psi t_j)^{p^i}\Big)^{e_n} \\
&= \sum_{x_{n-1,1}+\cdots+x_{0,n}=e_n} (e_n!/x_{n-i,1}! \cdots x_{0,n}!)(m_{n-1}\psi t_1^{p^{n-1}})^{x_{n-1,1}} \cdots (\psi t_n)^{x_{0,n}} \\
&= \sum_{x_{n-1,1}+\cdots+x_{0,n}=e_n} (e_n!/x_{n-1,1}! \cdots x_{0,n}!) m_1^{x_{1,n-1}} \cdots m_{n-1}^{x_{n-1,1}} \psi t_1^{p^{n-1}x_{n-1,1}} \cdots \psi t_n^{x_{0,n}}
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\sum_{i+j+k=n, (j,k)\neq(0,0)} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}\Big)^{e_n} \\
&= \sum_{\substack{\Sigma x_{ijk}=e_n \\ i+j+k=n, (j,k)\neq(0,0)}} (e_n!/\prod x_{ijk}!) m_1^{\Sigma x_{ijk}} \cdots m_{n-1}^{\Sigma x_{n-1,jk}} t_1^{\Sigma p^i x_{i1k}} \cdots t_n^{x_{0n0}} \otimes t_1^{\Sigma p^{i+j} x_{ij1}} \cdots t_n^{x_{00n}}.
\end{aligned}$$

Multiplying the corresponding expressions for $n = 1, 2, \ldots,$ we obtain:

$$\sum_{V(X)=V} \beta(X) m M^{(X)} \psi t^{E(X)} = \sum_{V(X)=V} \beta(X) m^{M(X)} t^{E(X)} \otimes t^{F(X)}, \tag{1.3}$$

where $V = (e_1, e_2, \cdots)$ is a fixed exponential sequence. The left expression is summed over all matrices $X=(x_{ij}: i\geq 0, i\geq 1)$ satisfying $V(X)=V$, and the right expression is summed over all tesors $X=(x_{ijk}: i, j, k\geq 0, (j, k)\neq(0, 0))$ satisfying $V(X)=V$.

Here we regard the collection of all equations (1.3) corresponding to every exponential sequence as simultaneous equations of ψt^{E}. That is, if we define a sequence-indexed matrix $A=(A_{E}^{V})$ by $A_{E}^{V}=\sum_{\substack{V(X)=V\\E(X)=E}}\beta(X)m^{M(X)}$, then the above equation is described as follows:

$$A(\psi t^{E})=(u_{V}),\ \text{where}\ u_{V}=\sum_{V(X)=V}\beta(X)m^{M(X)}t^{E(X)}\otimes t^{F(X)}.$$

Obviously, ψt^{V} has the highest degree in the left expression of (1.3). This fact suggests that A is indeed a lower triangular matrix whose diagonal entries are 1's. Hence $A^{-1}=\sum_{n\geq 0}(-1)^{n}(A-I)^{n}$. Therefore we obtain the formula of ψt^{V} as follows:

$$\psi t^{V}=\sum_{\substack{n\geq 0\\V(X_{n})=V\\V(X_{n-1})=E(X_{n})\\ \vdots \\V(X_{0})=E(X_{1})\\M(X_{1})\neq 0,\dots,M(X_{n})\neq 0}}(-1)^{n}\beta(X_{0})\cdots\beta(X_{n})m^{M(X_{0})+\cdots+M(X_{n})}t^{E(X_{0})}\otimes t^{F(X_{0})}. \tag{1.4}$$

Theorem follows from (1.4) and $\langle r_{E}\circ r_{F}, x\rangle=\sum\langle r_{E}, x_{1}\eta_{R}\langle r_{F}, x_{2}\rangle\rangle$, where $x\in BP_{*}BP$, $\psi x=\sum x_{1}\otimes x_{2}$. Q.E.D.

The formula of this theorem seems to be quite complicated, but has a remarkbable resemblance to the product formula of the Steenrod operations under some conditions.

Corollary 1.5. *If* $E=(e_{1},\dots,e_{n-1})$ *and* $F=(f_{1}, f_{2},\cdots)$, $f_{i}=0$ *unless* $n|i$, *then*

$$r_{E}\circ r_{F}=\sum_{\substack{M(X)=E\\E(X)=F}}r_{V(X)}.$$

where $X=(x_{ij}: i, j\geq 0$ *and* $(i, j)\neq(0, 0))$.

Proof. These conditions imply:

$$\sum_{\substack{V(X)=V\\E(X)=E\\F(X)=F}}\beta(X)m^{M(X)}=\sum_{\substack{M(Y)=0\\E(Y)=E\\F(Y)=F}}\ \sum_{\substack{V(Z)=V\\E(Z)=V(Y)}}\beta(Z)m^{M(Z)},$$

where X, Y are tensors and Z is a matrix stated in Theoem 1.2. Then the corollary easily follows. Q.E.D.

Remark. This corollary holds when the conditions for E and F are inverted.

Corollary 1.6. $r_{E}\circ r_{F}\equiv\sum_{\substack{M(X)=E\\E(X)=F}}\beta(X)r_{V(X)}$ mod $(v_{1}, v_{2},\cdots)$, *where* $X=(x_{ij})$, *i*,

$j \geq 0$ *and* $(i, j) \neq (0.0)$.

Proof. The ideal $(v_1, v_2, \cdots)$ equals to $(m_1, m_2, \cdots) \cap BP_*$. The terms of $n \geq 1$ or $M(X_0) \neq 0$ vanish modulo $(m_1, m_2, \cdots)$. Q.E.D.

Remark. This corollary is first stated by R. Kane [6].

We must consider the action of r_E on the coefficient ring for the purpose of the complete description of the product of BP-operations since $(x \cdot r_E) \circ (y \cdot r_F) = \sum_{E_1+E_2=E} x \cdot r_{E_1}(y) \cdot r_{E_2} \circ r_F$, where x and y are some elements of the coefficient ring.

Lemma 1.7. $r_E(m^V) = \sum_{\substack{V(X)=V \\ E(X)=E}} \beta(X) m^{M(X)}$, *where* $X = (x_{ij}: j \geq 0,\ (i, j) \neq (0, 0))$.

Proof. $r_E(m_n) = \begin{cases} m_{n-i}: E = (0, \dots, 0, e_i),\ e_i = p^{n-i} \\ 0 \quad : \text{otherwise} \end{cases}$ (See [9]).

Then the coproduct formula of r_E easily certifies this lemma Q.E.D.

We close this section by the mention of two little lemmas about converting v_i's and m_i's.

Lemma 1.8. *Let* v_i *be Hazewinkel's generator* [3] *and* $I = (i_1, i_2, \dots, i_m)$ *be a finite (possibly empty) sequence of positive integers. Let* $|I| = m$ *and* $\|I\| = \sum i_k$. *We define* v_I *recursively by* $v_\phi = 1$, and $v_I = v_{i_1}(v_{I'})^a$, where $a = p^{i_1}$ *and* $I' = (i_2, i_3, \cdots)$. *Then the following equation holds*:

$$m_n = \sum_{\|I\|=n} \frac{v_I}{p^{|I|}}.$$

This lemma is stated by D. C. Ravenel [10] rather for the case of Araki's generator [2].

Lemma 1.9.

$$v^V = (pm)^V + \sum_{\substack{n \geq 1 \\ V(X_1)=V \\ \vdots \\ V(X_n)=E(X_{n-1}) \\ M(X_1)\neq 0, \dots, M(X_n) \neq 0}} (-1)^n \beta(X_0) \cdots \beta(X_m) m^{M(X_0)+\cdots+M(X_n)} (pm)^{E(X_n)}.$$

Proof. v_n's are recursively defined by $pm_n = \sum_{0 \leq i < n} m_i v_{n-i}{}^{p^i}$. Then we obtain the following equation:

$$(pm)^V = \sum_{V(X)=V} \beta(X) m^{M(X)} v^{E(X)}.$$

Hence the lemma follows in the same way as Theorem 1.2. Q.E.D.

§ 2. BP^*-algebraic generators of FS^*

$FS^* = BP^* \otimes R$ is a dense BP^*-subcoalgebra of BP^*BP. But FS^* also has a BP^*-subalgebraic structure of BP^*BP, so that the (Hopf-) algebraic structure of BP^*BP is a completion of that of FS^*. BP^*BP is a quite large object whose cardinality is $\aleph$. On the other hand, FS^* is so small that BP_*BP can be reconstructed as $BP_*BP \cong \mathrm{Hom}_{BP^*}(FS^*, BP^*)$. This means that FS^* is accessible as well as essential.

Lemma 2.1. *FS^* is a BP^*-subHopf-algebra of BP^*BP.*

Proof. The formula of Theorem 1.2 is a finite summation. Q.E.D.

Remark. The target of the coproduct map ψ of FS^* is a finite tensor product $FS^* \otimes FS^*$.

FS^* has a close resemblance to the Steenrod algebra. First we observe the following commutative diagram of spectra constitutes a ring homomorphism ($\mathscr{P}(E)$ means $Sq(2E)$ for $p = 2$).

$$\begin{array}{ccc} BP & \xrightarrow{r_E} & BP \\ \downarrow & & \downarrow \\ HZ/p & \xrightarrow{\chi(\mathscr{P}(E))} & HZ/p \end{array}$$

Lemma 2.2. *Define $\rho: BP^*BP \to HZ/p^*HZ/p = \mathscr{A}$: the Steenrod algebra by the correspondence of the above diagram. Then ρ is a ring homomorphism.*

Proof. This is obvious from Corollary 1.6 and the fact that $(p, v_1, v_2, \cdots)$ is invariant under the action of the BP-operations (see [4], [7]). Q.E.D.

It is well-known that $\mathscr{P}^{p^n}$ for $n \geq 0$ generate $\mathscr{A}/(\beta)$ (in case of $p = 2$, Sq^{2^n} for $n \geq 1$ generate $\mathscr{A}/(Sq^1)$). FS^* also resembles the Steenrod algebra in that the following theorem holds.

Theorem 2.3. *$r(p^n)$ for $n \geq 0$ generate FS^* as a BP^*-algebra.*

Remark Since the cardinality of BP^*BP is $\aleph$, the cardinality of its algebraic generators is also $\aleph$.
2) Any proper subset of $\{r(p^n)\}$ cannot generate $r(p^n)$ which is not contained in this subset.

Proof. The necessity is immediate. If we can write down $r(p^n)$ from other elements which are not projected to $\chi(\mathscr{P}^{p^n})$ under the map ρ mentioned in lemma 2.2, then this equation remains valid after applying ρ. But this contradicts the fact that $\chi(\mathscr{P}^{p^n})$ is indecomposable.

On the other hand, an inductive argument is required to prove its sufficiency. To begin with, we give the following definitions.

Definition 2.4. 1) $F(n) = BP^* \otimes \{r(e_1, e_2, \cdots): e_i < p^{n+1-i}\}$.
2) $R_k^j = r(0, \dots, p^j)$, where p^j occurs at the k-th entry.
3) For an exponential sequence E (or for r_E), we define its excess by $ex(E) = \sum e_i$.

Suppose all elements of $F(n-1)$ are generated from $r(p^n)$'s. We will now argue by an inner induction on excess that any element in $F(n) - F(n-1)$ can be decomposed in terms of $r(p^n)$'s. The element of minimal excess is R_n^0. For this element we have the following lemma.

Lemma 2.5. $R_{n+1}^0 = [R_n^0, R_1^n]$, *for all* $n \geq 1$.

Proof. For $n = 1$,

$$R_1^0 \cdot R_1^1 = R_2^0 + (p+1)r(p+1) - v_1 r(1, 1),$$

$$R_1^1 \cdot R_1^0 = (p+1)r(p+1) - v_1 r(1, 1).$$

For $n > 1$, we obtain from Corollary 1.5:

$$R_n^0 \cdot R_1^n = R_{n+1}^0 + r(p^n, 0, \dots, 1),$$

$$R_1^n \cdot R_n^0 = r(p^n, 0, \dots, 1).$$

Hence the lemma follows. Q.E.D.

So consider $r_E \in F(n) - F(n-1)$ assuming that elements of lower excess can be generated from $r(p^n)$'s. If $E = (e_1, \dots, e_k)$ has one or more non-empty entries in addition to e_k, then Corollary 1.5 implies $r(e_1, \dots, e_{k-1})r(0, \dots, e_k) = r(e_1, \dots, e_k)$ + lower excess terms in $F(n)$. In case of $E = (0, \dots, 0, e_k)$ but $r_E \neq R_k^{n-k}$, e_k has a p-adic expansion of the form $e_k = \sum a_i p^i$, where $a_{n-k} \neq 0$. This implies $\binom{e_k}{p^{n-k}} \not\equiv 0$ mod p, i.e. $\binom{e_k}{p^{n-k}}$ is invertible in $\boldsymbol{Z}_{(p)}$. We have:

$$r(0, \dots, 0, e_k - p^{n-k})R_k^{n-k} = \binom{e_k}{p^{n-k}} r(0, \dots, 0, e_k) + \text{lower excess terms.} \tag{2.6}$$

To see that the lower excess terms in this expression are actually contained in $F(n)$, we need the following lemma.

Lemma 2.7. *Define the weight of an exponential sequence* E (*or of* r_E) *by* $w(E) = \sum e_i p^i$, *then the weight of* r_W *which appears in the right hand of the formula of Theorem* 1.2 *dose not exceed* $w(E) + w(F)$.

Proof. We also define $w(X) = \sum x_{ijk} p^{i+j+k}$ for X: tensor ($\sum x_{ij} p^{i+j}$ for matrices). Then $w(V(X)) = w(X) \leq w(E(X)) + w(F(X))$ for X: tensor, and $w(V(X)) = w(X) = w(E(X))$ for X: matrix. Hence the lemma follows. Q.E.D.

Therefore if $r(c_1, \dots, c_m)$ appears in the right expression of the formula (2.6), then $c_i p^i \leq w(r(c_1, \ldots, c_m)) \leq e_k p^k < p^{n+1}$. Hence $c_i < p^{n+1-i}$. This implies $r(c_1, \ldots, c_m) \in F(n)$.

R_k^{n-k} now remains to be considered. In case of $k > 2$, Corollary 1.5 implies:

$$\begin{aligned}[R_{k-1}^{n-k}, R_1^{n-1}] = R_k^{n-k} &+ \sum_{s=1}^{p^{n-k}-1} r(p^{n-1} - Sp^{k-1}, 0, \ldots, p^{n-k} - S, S) \\ &- \sum_{t=1}^{p^{n-k-1}} r(p^{n-1} - t, 0, \ldots, p^{n-k} - pt, t).\end{aligned}$$

where s and t occur at the k-th entry. Every term except R_k^{n-k} in the right expression is contained in $F(n-1)$. Hence we obtain the decomposition of R_k^{n-k} for $k > 2$.

R_2^{n-2} needs more deliberation. Here we consider $[R_1^{n-2}, R_1^{n-1}]$ as the case of $k > 2$, but Corollary 1.5 is not available any longer. First we observe the product formula of n Quillen operations is described in the same way as Theorem 1.2 by means of $(n+1)$-dimensional tensor.

(2.8) $$r_{E_1} \circ \cdots \circ r_{E_n} = \sum_{\substack{k \geq 0 \\ E_1(X_0) = E_1 \\ \vdots \\ E_n(X_0) = E_n \\ E(X_1) = V(X_0), \ldots, E(X_k) = V(X_{k-1}) \\ M(X_1) \neq 0, \ldots, M(X_n) \neq 0}} (-1)^k \beta(X_0) \cdots \beta(X_k) m^{M(X_0) + \cdots + M(X_k)} r_{V(X_k)}.$$

X_0 is an $(n+1)$-dimensional tensor, i.e., $X_0 = (x_{i_0 \ldots i_n})$, where non-negative integers $x_{i_0 \ldots i_n}$ are defined for $i_0, \ldots, i_n \geq 0$ except $(i_1, \ldots, i_n) = (0, \ldots, 0)$. $X_1, \ldots, X_k$ are two-dimensional matrices as stated in Theorem 1.2. The vector-valued functions V, M, $E_1, \ldots, E_n$ and the integer-valued function β are similarly defined as follows:

$$V(X) = (a_1, a_2, \cdots), \; a_m = \sum_{i_0 + \cdots + i_n = m} x_{i_0 \cdots i_n},$$

$$M(X) = (b_1, b_2, \cdots), \; b_m = \sum_{i_1, \ldots, i_n} x_{m, i_1 \cdots i_n},$$

$$E_k(X) = (c_1, c_2, \cdots), \; c_m = \sum_{i_0, \ldots (i_k) \ldots, i_n} p^{i_0 + \cdots + i_{k-1}} x_{i_0 \cdots m \cdots i_n},$$

where the k-th index of x is replaced with m, and the sum extends over all non-negative integers $i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n$.

$$\beta(X) = \prod a_m! / \prod x_{i_0 \ldots i_n}!.$$

Definition 2.9. Let X, Y be $(n+1)$-dimensional tensors.

1) X will be called simple if and only if $M(X) = 0$.
2) X will be called trivial if and only if $x_{i_0 \ldots i_n} = 0$ unless the positive-valued index i_k is unique.
3) X will be called a reduction of Y (Y will be called an expansion of X) if and only if X is simple and

$$X_{0,1_1\dots i_n} = \sum_{i_0} p^{i_0} y_{i_0\dots i_n}, \qquad \text{for all } i_1, \dots, i_n.$$

4) We also define the weight and the excess of X by

$$w(X) = \sum p^{i_0+\cdots+i_n} x_{i_0\dots i_n}, \quad ex(X) = \sum x_{i_0\dots i_n}.$$

The formula (2.8) is too troublesome to write repeatedly, so we will use the following notation.

Notation 2.10. For X: simple tensor of any dimension, we use the notation $P(X)$ for the following expression:

$$\sum_{\substack{k\geq 0 \\ X_0:\text{expansion of } X \\ E(X_1)=V(X_0) \\ \vdots \\ E(X_k)=V(X_{k-1}) \\ M(X_1)\neq 0,\dots,M(X_k)\neq 0}} (-1)^k \beta(X_0)\cdots\beta(X_k) m^{M(X_0)+\cdots+M(X_k)} r_{V(X_k)}.$$

Using this notation, we can concisely describe the product formula (2.8) as follows:

$$r_{E_1} \circ \cdots \circ r_{E_n} = \sum P(X),$$

where the sum extends over all $(n+1)$-dimensional simple tensors X satisfying $E_k(X) = E_k$ for $k = 1, 2, \dots, n$. Notice that the trivial tensor has the highest excess among these simple tensors.

Lemma 2.11. *If Y is an $(n+1)$-dimensinal simple tensor, then there exists a trivial tensor Z of some dimension and satisfies $P(Y) = P(Z)$.*

Proof. Suitably order the positive-valued entries of Y. If $y_{(k)} = y_{0,i_1\dots i_n}$ is the k-th term in this order, let $E_k = (0, \dots, y_{(k)})$, where $y_{(k)}$ occurs at the l-th entry with $l = \sum_{j=1}^{n} i_j$. Then can define Z so as to satisfy $E_k(Z) = E_k$ and to be trivial. It is evident that Z is what we need. Q.E.D.

Remark. $\sum w(E_k) = w(Y)$, and $w(E_k) \leq \max_j w(E_j(Y))$.

Here we consider the simple tensors which appear in the right expression of the equation $[R_1^{n-2}, R_1^{n-1}] = \sum \pm P(X)$. Observe that $P(X)$ corresponding to the trivial tensor is already canceled. By Lemma 2.11, we can cancel these $P(X)$'s by adding or subtracting $r_{E_1} \circ \cdots \circ r_{E_k} \circ \cdots$, so that the remainder is a summation of $\pm P(Y)$'s, where Y's are of lower excess than that of X's. This process can be continued until the remainder amounts to zero because the maximal excess of the tensor which appears in the remainder certainly decreases as the canceling proceeds. The weight of r_{E_k} which appears in the resulting equation is equal to or lower than p^n. This means $r_{E_k} \in F(n-1)$ or $r_{E_k} = R_m^{n-m}$. But R_m^{n-m} for $m > 2$ cannot appear because its degree is higher than that of R_2^{n-2}. And R_2^{n-2} appears in this equation once and only once because of the degree reason of the same kind. Therefore R_2^{n-2} can be decomposed in terms of $r(p^n)$'s. Q.E.D.

Theorem 2.3 asserts the resemblance between FS^* and the Steenrod algebra. But there should be many differences between them. The next proposition is one of such differences.

Proposition 2.12. *There is no zero-divisor in* FS^*.

Proof. Given $x, y \in FS^*$, $x \neq 0$, $y \neq 0$, we have only to show $xy(m^W) \neq 0$ for some W. To begin with, consider the case of $x = r_E$ and $y = r_F$. From lemma 1.7, we have $r_E \circ r_F(m^{W+E+F}) = [W, E][W+E, F]m^W$ + higher excess terms, where $[A, B] = \prod(a_i, b_i)$ is a product of binomial coefficients, and the excess means the excess of W. This equation implies $r_E \circ r_F \neq 0$.

Next consider the general case: $x = \sum c_i m^{M_i} r_{E_i}$, $y = \sum d_j m^{N_j} r_{F_j}$, where c_i, $d_j \in Z_{(p)}$. If W has sufficiently large entries, we have:

$$\begin{aligned}&(c_i m^{M_i} r_{E_i}) \circ (d_j m^{N_j} r_{F_j})(m^W)\\ &\qquad = c_i d_j [W + N_j - F_j - E_i, E_i][W - F_j, F_j] m^{W + M_i - E_i + N_j - F_j}\\ &\qquad\qquad + \text{higher excess terms.}\end{aligned}$$

Then the lowest excess term of $xy(m^W)$ is of the form:

$$\{\sum_{s,t} c_{i_s} d_{j_t} [W + N_{j_t} - F_{j_t} - E_{i_s}, E_{i_s}][W - F_{j_t}, F_{j_t}]\} m^{W + T(x) + T(y)},$$

where $T(x)$ (resp. $T(y)$) is the exponential sequence (with possibly negative entries) of the lowest excess of $M_i - E_i$ (resp. $N_j - F_j$), and i_s (resp. j_t) are such indices that $M_{i_s} - E_{i_s} = T(x)$ (resp. $N_{j_t} - F_{j_t} = T(y)$). Notice that the coefficient $b(W) = b(w_1, w_2, \cdots)$ of the above term is indeed a non-trivial polynomial expression for finitely many variables $w_1, w_2, \cdots$. Hence there exist non-negative integers w_i satisfying $b(W) \neq 0$. This fact implies $xy \neq 0$. Q.E.D.

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