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Stability of foliations of 4-manifolds by Klein bottles

Dedicated to Professor Masahisa Adachi on his 60th birthday

By

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§0. Introduction

This paper is a complement to my paper [8]. Let $\mathrm{Fol}_q(M)$ denote the set of condimension q C^∞-foliations of a closed manifold M. $\mathrm{Fol}_q(M)$ carries a natural weak C^r-topology ($0 \leqq r \leqq \infty$), which is described in [6], [9]. We denote this space by $\mathrm{Fol}_q^r(M)$. We say a foliation F is C^r-stable if there exists a neighborhood V of F in $\mathrm{Fol}_q^r(M)$ such that every foliation in V has a compact leaf. We say F is C^r-unstable if not. We say a foliation in a small neighborhood of F in $\mathrm{Fol}_q^r(M)$ is a small C^r-perturbation of F. It seems to be of interest to determine if F is C^r-stable or not.

In this paper we shall give a sufficient condition for a foliation of a closed 4-manifold by Klein bottles to be C^1-stable. More precisely we have the following.

Theorem. *Let F be a foliation of a closed 4-manifold M by Klein bottles. If $\chi_V(M/F)^2 + \chi(M/F)^2 \neq 0$, then F is C^1-stable.*

1. Foliations of 4-manifolds by Klein bottles

Let M be a closed manifold and F a compact foliation of M of codimension two. By the results of Epstein [4] and Edwards-Millett-Sullivan [3], we have a nice picture of the local behavior of F as follows.

Proposition 1 (Epstein [5]). *There is a generic leaf L_0 with property that there is an open dense saturated subset of M, where all leaves have trivial holonomy and are diffeomorphic to L_0. Given a leaf L, we can describe a neighborhood $U(L)$ of L, together with the foliation on the neighborhood as follows. There is a finite subgroup $G(L)$ of $O(2)$ such that $G(L)$ acts freely on L_0 on the right and $L_0/G(L) \cong L$. Let D^2 be the unit disk. We foliate $L_0 \times D^2$ with leaves of form $L_0 \times \{pt\}$. This foliation is preserved by the diagonal action of $G(L)$, defined by $g(x, y) = (x \cdot g^{-1}, g \cdot y)$ for $g \in G(L)$, $x \in L_0$ and $y \in D^2$, where $G(L)$ acts linearly on D^2. So we have a foliation induced on $U = L_0 \times D^2/G(L)$. The leaf corresponding to $y = 0$ is $L_0/G(L)$. Then there is a C^∞-imbedding $\varphi\colon U \to M$ with $\varphi(U) = U(L)$, which*

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preserves leaves and $\varphi(L_0/G(L)) = L$.

A finite subgroup of $O(2)$ is either a group of m rotations which is isomorphic to $\boldsymbol{Z}_m$ or a group of m rotations and m reflections which is isomrophic to $\boldsymbol{D}_m = \{u, v; u^m = v^2 = (uv)^2 = 1\}$. We give D^2 an orientation. Then $\boldsymbol{Z}_m$ has a natural generator u, the rotation by an angle of arc length $2\pi/m$. In the same way, $\boldsymbol{D}_m$ is regarded as a fixed group of rotations and reflections with fixed generators u, v, where u is the rotation by $2\pi/m$ and v is a reflection.

Definition 2. A leaf is *singular* if $G(L)$ is not trivial. We say such an L is a rotation leaf, a reflection leaf or a dihedral leaf if $G(L)$ is $\boldsymbol{Z}_m$, $\boldsymbol{D}_1$ or $\boldsymbol{D}_m (m > 1)$ respectively.

We consider foliations of closed 4-manifolds by Klein bottles. Then we have the following.

Proposition 3 (Proposition 4 of [8]). *If F is a foliation of a closed* 4*-manifold by Klein bottles, then F has no dihedral leaves.*

2. Stability of foliations of 4-manifolds by Klein bottles

Let F be a foliation of a closed 4-manifold M by Klein bottles. In the case F has no singular leaves, Bonatti ([1], [2]) has determined a sufficient and necessary condition for such an F to be C^1-stable. So we consider the stability of such a foliation F with singular leaf. By proposition 3, we know that any dihedral leaves do not appear in F. Furthermore in case that F has a rotation leaf, we have the following.

Proposition 4 (Theorem 13 of [8]). *Let F be a foliation of a closed* 4*-manifold by Klein bottles. If F has a rotation leaf, then F is C^1-stable.*

So we consider here the case that F has no rotation leaves. We denote by B the leaf space M/F, which is a compact 2-manifold with boundary and $\pi\colon M \to B$ its quotient map. Note that each point of ∂B corresponds to a reflection leaf of type $(1, v)$ (see Case B in §1 of [8]).

Theorem 5. *Let F be a foliation of a closed* 4*-manifold M by Klein bottles and B its leaf space. Suppose that* 1) *F has only reflection leaves as singular leaves and* 2) $\chi(B) \neq 0$, where $\chi(B)$ *denotes the euler characteristic. Then F is C^1-stable, that is, any C^1-perturbation of F has a compact leaf.*

Proof. We denote by $R(F)$ the union of all reflection leaves of F. Since $\pi\colon M - R(F) \to B - \partial B$ is a fibration with generic leaf L_0 as a fibre, $\pi_1(B - \partial B)$ $(\cong \pi_1(B))$ acts on $H_1(L_0; \boldsymbol{R}) \cong \boldsymbol{R}$. It is sufficient to prove the case that $\pi_1(B)$ acts on $H_1(L_0; \boldsymbol{R})$ trivially. For, if necessary, take an appropriate double cover $\tilde{M}$ of M. Let $\tilde{B}$ be the leaf space of the foliation $\tilde{F}$ induced on $\tilde{M}$. Then we may consider that $\pi_1(\tilde{B} - \partial\tilde{B})$ acts on $H_1(\tilde{L}_0; R)$ trivially, where $\tilde{L}_0$ is a generic leaf of

$\tilde{F}$. Furthermore we have $\chi(\tilde{B}) = 2\chi(B) \neq 0$. Hence the stability of $\tilde{F}$ implies the stability of F.

We fix a Riemannian metric on M. Let $p: N \to M$ be the normal bundle associated with F, $\sigma: M \to N$ the zero section, $\mathcal{N}$ a neighborhood of $\sigma(M)$ in N. Let $q: \mathcal{N} \to M$ be the submersion induced from the exponential map. Let $\mathcal{N}_0$ be a relative compact neighborhood of $\sigma(M)$ in $\mathcal{N}$ such that $p^{-1}(x) \cap \mathcal{N}_0$ is a disk $\hat{D}_x$ centered at $\sigma(x)$ for $x \in M$. Let $D_x = q(\hat{D}_x)$. Let F' be a sufficiently small C^1-perturbation of F. Then we construct a local first return map for F' following Bonatti [1]. See [1] for more details. Let U_0 be a small open set of M and x, y any points in U_0. Then we have a canonical isomorphism $j_*: \pi_1(L_x, x) \to \pi_1(L_y, y)$ if L_x and L_y are generic leaves, where L_x is the leaf of F through x. In the case L_y is a reflection leaf, we have that j_* is injective. Let U'_0 be a relative compact open set in U_0. We take and fix $x_0 \in U'_0$ such that L_{x_0} is a generic leaf and $a \in \pi_1(L_{x_0}, x_0) = \{a, b; aba^{-1}b = 1\}$. Then we can construct the return map $H(F', a): U'_0 \to M$ (see [1], [7]) which is a local diffeormophism and C^1-close to $1_{U'_0}$. We denote by $\hat{F}$ and $\hat{F}'$ the foliations on defined by $\hat{F} = q^*F$ and $\hat{F}' = q^*F'$. Let $\hat{U}_0 = q^{-1}(U_0) \cap \mathcal{N}_0$ and $\hat{U}'_0$ be a relative compact open neighborhood of $\sigma(U'_0)$ in U_0. We have a canonical injective $\pi_1(L_{x_0}, x_0) \to \pi_1(\hat{L}_z, z)$, where $\hat{L}_z$ is the leaf of $\hat{F}$ through $z (\in \hat{U}'_0)$. In the same way as above (see also [1]), we can construct a local first return map for $\hat{F}'$, $H(\hat{F}', a)$: $\hat{U}'_0 \to \mathcal{N}_0$ which is a local diffeomorphism. Note that $H(F', a)(x) = q(H(\hat{F}', a))(\sigma(x))$ for any $x \in U'_0$.

Take families of open sets $\{U_i\}$, $\{U'_i\}$ and $\{U''_i\}$ of M such that 1) $U_i \supset U'_i \supset U''_i$ and 2) $\{U''_i\}$ is a finite open covering of M. We denote by $\{\hat{U}_i\}$, $\{\hat{U}'_i\}$ and $\{\hat{U}''_i\}$ the families of open sets of $\mathcal{N}_0$ associated with $\{U_i\}$, $\{U'_i\}$ and $\{U''_i\}$. We take $x_i \in U_i$ with L_{x_i} generic leaf and $a_i \in \pi_1(L_{x_i}, x_i)$ for each i, where $\pi_1(L_{x_i}, x_i) = \{a_i, b_i; a_i b_i a_i^{-1} b_i = 1\}$. Let $\{\phi_i\}$ be a partition of unity associated with $\{U''_i\}$. For a sufficiently small C^1-perturbation F' of F and each i, $H(F', a_i)$ and $H(\hat{F}', a_i)$ are defined on U'_i and $\hat{U}'_i$ respectively. We define the map $H(F')$: $M \to M$ by

$$H(F')(x) = q\Big(\sum_{\{i \mid x \in U''_i\}} \phi_i(x)\, H(\hat{F}', a_i)(\sigma(x))\Big)(\in D_x).$$

$H(F')$ is a diffeomrophism and C^1-close to 1_M. We have x and $H(F')(x)$ in the geodesic disk D_x for $x \in M$. So we define $X(x)$ to be the vector tangent to the geodesic in D_x from x to $H(F')(x)$. By Corollary 3(ii) of [1], we see that $X(x_0) = 0$ and $x_0 \in M - R(F)$ implies that the leaf of F' through x_0 is compact.

Let D be a disk in B such that $D \cap \partial B = \phi$ and π is trivial over D. Note that $\pi^{-1}(D) = T \cong D \times K$, where K is the Klein bottle. Then we have thw following.

Proposition 6 (Fukui [7]). *There exists a compact connected 2-manifold B^* transverse to F over $B - \operatorname{int} D$ such that*

1) $\pi: B^* \to B - \operatorname{int} D$ *is a double covering expected for ∂B and*
2) B^* *meets ∂T in at most two simple closed curves ∂B^*, that is, we denote by r the number of the connected components of ∂B^*, then $1 \leqq r \leqq 2$.*

Let $h: S^1 \to T$ be a continuous map and $h(S^1) = C$. Then we define $I(F', C)$ to be the total number of times the vector $\pi_* X(h(\theta))$ rotates about the center of D as θ goes once around S^1.

We now assume that F' has no compact leaves. Then we have the following.

Proposition 7. *Let α be a simple closed curve in $L_0 = \pi^{-1}(0)$ with the homology class $[\alpha] \neq 0$ in $H_1(L_0; \boldsymbol{R}) = \boldsymbol{R}$. Then $I(F', \alpha) = 0$.*

Proof. From Corollary 2 of [1], it follows that if F' has no compact leaves, then the local first return map $H(F', \alpha)$ has no fixed points on T. We denote by G and G' the restrictions of F and F' to $D \times \alpha$, respectively, which are again foliations. We consider the restricted map $H(F', \alpha): D \to D$. Then we have $H(F', \alpha) = H(G', \alpha)$. Thus $H(G', \alpha)$ has no fixed points. Hence we have $I(F', \alpha) = I(G', \alpha) = 0$ from Seifert [10]. This completes the proof.

The proof of Theorem 5 continued. We denote by C a connected component of ∂B^*. Then $\{C\} = a^i b^j$ for some integers i and j, where $\{\ \}$ denotes homotopy class in $\pi_1(T)$ and $\pi_1(L_0, *) = \{a, b; aba^{-1}b = 1\}$. Since the homology classes $[a] \neq 0$ and $[ab^j] \neq 0$ in $H_1(L_0; \boldsymbol{R})$ by Proposition 7, we have $I(F', a) = I(F', ab^j) = I(F', a^{-1}) = 0$, hence $I(F', C) = 0$.

The vector field X projects naturally to a vector field X^* tangent to B^* since X and B^* are transverse to F. Now we construct a closed 2-manifold $\bar{B}^*$ pasting one or two disks $D_l\,(1 \leqq l \leqq r)$ to B^* along ∂B^*. The vector field $\pi_*(X|_C)$ on ∂D is homotopic to a constant vector field because $I(F', C) = 0$. Hence deforming X^* along ∂B^* in the homotopy class, we may assume that $\pi_*(X^*|_{\partial B^*})$ is a constant vector field on ∂D. Since $\pi: C \to \partial D$ is one or two fold covering, we easily see that the vector field X^* is extended to a vector field $\bar{X}^*$ on $\bar{B}^*$ with exactly one or two singular points of index $-2/r + 1$ in $D_l\,(1 \leqq l \leqq r)$. Here a singular point of index 0 means a nonsingular point. $\bar{X}^*$ may have singular points on $\pi^{-1}(\partial B)(\subset B^*)$. However we may assume that $\bar{X}^*$ has no singular points on B^* by slightly deforming $\bar{X}^*$ on a neighborhood of $\pi^{-1}(\partial B)$ in B^* because $\chi(\pi^{-1}(\partial B)) = 0$. Then the euler characteristic of $\bar{B}^*$ is equal to $\chi(B^*) = -2 + r$. On the other hand, we have $\chi(\bar{B}^*) = \chi(B^* \cup \{\bigcup_{l=1}^{r} D_l\}) = 2(\chi(B) - 1) + r$. Hence $\chi(B) = 0$. This contradicts the assumption 2). Therefore F' has a compact leaf. This completes the proof.

Combining Proposition 4, Theorem 5 and the result of Bonatti ([1], [2]), we have the following. See Theorem 6 and Remark 7 of [7] for the notation χ_V.

Theorem 8. *Let F be a foliation of a closed 4-manifold M by Klein bottles. If $\chi_V(M/F)^2 + \chi(M/F)^2 \neq 0$, then F is C^1-stable.*

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