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Non-existence of positive eigenvalues of the Schrödinger operator in a domain with unbounded boundary

By

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Introduction

In this paper we shall concern ourselves with the solution of the differential equation

$$(0.1)\qquad (-\Delta+q-\lambda)u=0$$

in a domain $D\subset R^n (n\geqq 2)$, where $\Delta=\sum_j(\partial/\partial x_j)^2$, $\lambda>0$, and q is a complex valued function. Define a domain D_α of R^n by

$$(0.2)\qquad D_\alpha=\{x\in R^n \mid x_1>|x|\cos(\alpha\pi/2)\},$$

where $1<\alpha<2$, $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$. We shall prove the following theorem.

Theorem 0.1. *Assume that D is larger than the half space $x_1>0$ in the sense that there exists a constant c with $1<c<2$ such that*

$$(0.3)\qquad D\supset D_c,$$

and assume that q can be written as $q=q_1+q_2$ such that the following conditions (0.4)~(0.6) *are satisfied.*

(0.4) *q_1 is real valued, of class $C^1(D)$, and*

$$q_1(x)=o(1)\qquad (|x|\to\infty \text{ in } D_c).$$

$$(0.5)\qquad |\nabla q_1(x)|+|q_2(x)|=o(|x|^{-1})\qquad (|x|\to\infty \text{ in } D_c).$$

(0.6) *There exist constants d and $\delta>0$ such that $1<d<c$, and*

$$|\nabla q_1(x)|+|q_2(x)|=O(|x|^{-(2/c)-\delta})\qquad (|x|\to\infty \text{ in } D_c-D_d).$$

In addition assume that q is such that the unique continuation property holds for equation (0.1), *i.e. if a solution u of* (0.1) *vanishes in an open set of D, u vanishes in all of D. Then if a solution u belongs to $L^2(D)$, u vanishes identically: $u\equiv 0$.*

Here it should be noted that in the hypotheses of the theorem we assume no conditions on the values of the solution u on the boundary ∂D of D.

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The above theorem is a consequence of a more general result, Theorem 0.2, which we shall state soon after the preparation of some notations. Define real valued functions $\xi_\alpha(x)$ and $\eta_\alpha(x)$ $(x\in D_\alpha)$ by

(0.7) $$(\xi_\alpha(x)+i\eta_\alpha(x))^\alpha=x_1+iy(x) \qquad (y(x)=(x_2^2+\cdots+x_n^2)^{1/2},\ i=\sqrt{-1}),$$

where we agree that we choose the argument of $\xi_\alpha(x)+i\eta_\alpha(x)$ such that

(0.8) $$0\leqq\arg(\xi_\alpha(x)+i\eta_\alpha(x))<\frac{\pi}{2} \qquad (x\in D_\alpha).$$

Define, in addition,

(0.9) $$\rho_\alpha(x)=(\xi_\alpha(x))^\alpha \qquad (x\in D_\alpha).$$

Notice that $\rho_\alpha(x)\leqq|x|$ and that $\rho_\alpha(x)=|x|$ when x is on the positive x_1-axis. We set

(0.10) $$D_\alpha(s)=\{x\in D_\alpha\,|\,\rho_\alpha(x)>s\}.$$

Further define

(0.11) $$\Theta_\alpha(x)=\xi_\alpha(x)^2/(\xi_\alpha(x)^2+\eta_\alpha(x)^2).$$

Theorem 0.2. *Assume that there exists a constant c such that $2>c>1$, and the condition* (0.3) *is satisfied. Moreover assume that q can be written as $q=q_1+q_2$ such that the condition* (0.4) *and the following* (0.12) *are satisfied.*

(0.12) *There exist a constant c' and a function $f(t)=o(1)$ $(t\to\infty)$ such that $c>c'>1$, and for any α with $c>\alpha>c'$*

$$\Theta_\alpha^{(1-\alpha)/2}|\nabla q_1|+|q_2|\leqq(\Theta_\alpha\rho_\alpha^{-1}+\rho_\alpha^{-3})\cdot f(\rho) \qquad (in\ D_\alpha(1)).$$

Then for a solution $u\in L^2(D)$ of (0.1) *there exists $T>0$ such that $u\equiv0$ in the domain $D_{c'}(T)$.*

The theorems will be proved in Section 2 after the preparatory Section 1, where some technical lemmas are proved. Section 3 is devoted to calculation of some quantities which are used in the discussions in the preceding sections. For some results previously known we refer to Konno [2], Mochizuki [3], Murata-Shibata [4] and Tayoshi [5].

Here we collect more notational conventions to be used in the following sections.

(0.13) $$\begin{cases} D_\alpha(s_1,s_2;s_3)=\{x\in D_\alpha\,|\,s_1<\rho_\alpha(x)<s_2,\ \eta_\alpha(x)<s_3\},\\ D_\alpha(s_1,s_2)=\{x\in D_\alpha\,|\,s_1<\rho_\alpha(x)<s_2\},\\ S_\alpha(s;t)=\{x\in D_\alpha\,|\,\rho_\alpha(x)=s,\ \eta_\alpha(x)<t\},\\ S_\alpha(s)=\{x\in D_\alpha\,|\,\rho_\alpha(x)=s\},\\ \Gamma_\alpha(s_1,s_2;s_3)=\{x\in D_\alpha\,|\,s_1<\rho_\alpha(x)<s_2,\ \eta_\alpha(x)=s_3\}.\end{cases}$$

Denoting the standard flat metric $\sum(\partial/\partial x_j)\otimes(\partial/\partial x_j)$ by G (we use the contravariant representation), we write $\nabla f=\sum(\partial f/\partial x_j)(\partial/\partial x_j)$ (f: a function), $\Delta f=\operatorname{div}\nabla f$, $G(df,dg)=\sum(\partial f/\partial x_j)\cdot(\partial g/\partial x_j)$ (df: the differential of f) $|df|^2=G(df,df)=|\nabla f|^2$ etc.. The following function appears in various places in the paper:

$$\mathrm{a}_\alpha(x)=|d\rho_\alpha|^2 \qquad (x\in D_\alpha). \tag{0.14}$$

The volume element $dx_1\wedge\cdots\wedge dx_n$ is denoted by Ω. If volume integrals over a domain B and surface integrals over its boundary ∂B appear in an expression, we choose the surface element Σ of ∂B such that $\Omega=n\wedge\Sigma$ where n is the outer normal covector to ∂B. The contraction of a vector field V with a 1-form w is written as $w(V)$. We set $\tilde{n}_\alpha=|d\rho_\alpha|^{-1}d\rho_\alpha$. We use the expression

$$\left[\int_{S_\alpha(s_2)}-\int_{S_\alpha(s_1)}\right]\cdots\tilde{n}_\alpha(V)\Sigma \tag{0.15}$$

for

$$\int_{S_\alpha(s_2)}\cdots\tilde{n}_\alpha(V)\Sigma-\int_{S_\alpha(s_1)}\cdots\tilde{n}_\alpha(V)\Sigma.$$

We let C, C_1, C_2, $\cdots$, T, T_1, $T_2\cdots$ etc. denote positive constants, and let $I(S)$, $I_1(S)$, $\cdots$ and $J(B)$, $J_1(B)$, $\cdots$ etc. denote some quantities given by certain surface integrals over an $(n-1)$-surface S and volume integrals over a certain n-region B. We shall specify the meaning of these notations wherever ambiguity may be caused, while the same letters with the same subscripts do not necessarily mean the same things if they appear in different contexts.

1. Some Lemmas

Throughout this section we let q_1 denote a real valued function of class $C^1(D)$, and put $q_2=q-q_1$, where q is as in (0.1). Let $B\subset D$ a domain. Take a function $\phi\in C^\infty(B)$ and set

$$v=\phi u \tag{1.1}$$

in equation (0.1). Then we have

$$-\Delta v+2\phi^{-1}G(d\phi,\, dv)+(Q+q_2-\lambda)v=0 \qquad (\text{in } B), \tag{1.2}$$

where we have put

$$Q=-2\phi^{-2}|d\phi|^2+\phi^{-1}\Delta\phi+q_1. \tag{1.3}$$

Lemma 1.1. *Assume that $D\supset D_\alpha(T)$ $(2>\exists\alpha>1,\ \exists T>0)$, and that q is bounded in $D_\alpha(T)$. Let ϕ, $\psi\in C^\infty(D_\alpha)$ be positive functions. Let B be a bounded subdomain of $D_\alpha(T)$ with piecewise smooth boundary ∂B. Let $\boldsymbol{Z}$ be a real smooth vector field in a neighborhood of the closure of B. Let u be a solution of* (0.1), *and further let v and Q be as in* (1.1) *and* (1.3). *Then the following identity holds:*

$$\int_{\partial B}\psi\left\{\mathrm{Re}[L_Z\bar{v}\cdot G(n,\, dv)]-\frac{1}{2}(|\nabla v|^2+(Q-\lambda)|v|^2)n(Z)\right\}\Sigma \tag{1.4}$$

$$-\int_B\mathrm{Re}\left[L_Z\bar{v}\cdot G\left(2\left(\frac{\psi}{\phi}\right)d\phi+d\psi,\, dv\right)\right]\Omega$$

$$+\int_B \frac{1}{2}\operatorname{div}(\psi Z)\{|\nabla v|^2+(Q-\lambda)|v|^2\}\Omega$$

$$+\int_B \frac{1}{2}\{\psi(L_Z G)(d\bar{v},\, dv)+\psi(L_Z Q)|v|^2\}\Omega$$

$$-\int_B \psi\,\mathrm{Re}[q_2 L_Z\bar{v}\cdot v]\Omega=0.$$

Here n is the outer normal covector to ∂B, and L_Z denotes the Lie differentiation with respect to the vector field Z.

Remark. Here and in the following we interpret the values of v and ∇v on a piecewise smooth $(n-1)$-surface S ($S=\partial B$ in the above lemma) in the sense of trace, which is meaningful because v is a solution of the elliptic equation (1.2) ([5], Section 6) and because we use traces of v and ∇v only when S is laid in a domain where q is bounded.

Proof of Lemma 1.1. Multiplying (1.2) by $\psi L_Z\bar{v}$, we have

(1.5) $$-\psi L_Z\bar{v}\cdot\Delta v+2\frac{\psi}{\phi}L_Z\bar{v}\cdot G(d\phi,\, dv)+\psi L_Z\bar{v}\cdot(Q+q_2-\lambda)v=0.$$

Let us write this as $V_1+V_2+V_3=0$. Integrating $-\mathrm{Re}[V_1]$ by parts over B, we have

(1.6) $$-\int_B \mathrm{Re}[V_1]\Omega=\int_{\partial B}\psi\,\mathrm{Re}[L_Z\bar{v}\cdot G(n,\, dv)]\Sigma-\int_B \mathrm{Re}[G(d(\psi L_Z\bar{v}),\, dv)]\Omega.$$

We write the right-hand side of (1.6) as $I_{11}(\partial B)+J_1(B)$. Using partial integration again, we have

$$J_1(B)=-\frac{1}{2}\int_B L_Z(\psi|\nabla v|^2)\Omega$$

$$+\int_B\left\{\frac{1}{2}(L_Z(\psi G))(d\bar{v},\, dv)-\mathrm{Re}[L_Z\bar{v}\cdot G(d\psi,\, dv)]\right\}\Omega$$

$$=-\frac{1}{2}\int_{\partial B}\psi|\nabla v|^2 n(Z)\Sigma$$

$$+\frac{1}{2}\int_B(\operatorname{div}(\psi Z))|\nabla v|^2\Omega+\frac{1}{2}\int_B\psi(L_Z G)(d\bar{v},\, dv)\Omega$$

$$-\int_B \mathrm{Re}[L_Z\bar{v}\cdot G(d\psi,\, dv)]\Omega.$$

Let us write this as

(1.7) $$J_1(B)=I_{12}(\partial B)+J_{11}(B)+J_{12}(B)+J_{13}(B).$$

We set

(1.8) $$J_2(B)=-\mathrm{Re}\int_B V_2\Omega=\int_B\left\{-2\frac{\psi}{\phi}\mathrm{Re}[L_Z\bar{v}\cdot G(d\phi,\, dv)]\right\}\Omega.$$

Integrating $-\mathrm{Re}[V_3]$, we have

$$(1.9)\qquad -\int_B \mathrm{Re}[V_3]\Omega = -\int_B \psi\,\mathrm{Re}[q_2 L_Z\bar{v}\cdot v]\Omega - \frac{1}{2}\int_B \psi(Q-\lambda)L_Z(|v|^2)\Omega$$

$$= -\frac{1}{2}\int_{\partial B}\psi(Q-\lambda)|v|^2 n(Z)\Sigma$$

$$+\frac{1}{2}\int_B (\mathrm{div}(\psi Z))(Q-\lambda)|v|^2\Omega$$

$$+\frac{1}{2}\int_B \psi L_Z Q\cdot|v|^2\Omega - \int_B \psi\,\mathrm{Re}[q_2 L_Z\bar{v}\cdot v]\Omega,$$

We write the last member of (1.9) as $I_3(\partial B)+J_{31}(B)+J_{32}(B)+J_{33}(B)$. Collecting (1.6)-(1.9), we have

$$(1.10)\qquad \{I_{11}(\partial B)+I_{12}(\partial B)+I_3(\partial B)\}+\{J_2(B)+J_{13}(B)\}$$

$$+\{J_{11}(B)+J_{31}(B)\}+\{J_{12}(B)+J_{32}(B)\}+J_{33}(B)=0.$$

This gives (1.4). Q. E. D.

Lemma 1.2. *Assume the hypotheses of Lemma* 1.1. *Let* ψ, ϕ, v, Q, Z *and* B *be as in Lemma* 1.1. *Further assume that there exist a constant* $\delta>0$ *and smooth positive functions* $\sigma(x)$, $\tau(x)$ *such that*

$$(1.11)\qquad 2\delta|d\sigma|^2(Z\otimes Z)-\sigma L_Z G \geqq 2\delta\tau G \quad \text{in } B.$$

Let E *be a real smooth function. Set*

$$(1.12)\qquad F=\mathrm{div}(\psi Z)-2E.$$

Define the vector field W *by*

$$(1.13)\qquad W=2\frac{F}{\phi}\nabla\phi+\nabla F=\phi^{-2}\nabla(\phi^2 F)).$$

Then we have the inequality

$$(1.14)\qquad \int_{\partial B}\Big\{\mathrm{Re}\Big[\Big(\psi L_Z\bar{v}+\frac{F}{2}\bar{v}\Big)G(n,\,dv)\Big]-\frac{\psi}{2}(|\nabla v|^2+(Q-\lambda)|v|^2)n(Z)-\frac{1}{4}|v|^2 n(W)\Big\}\Sigma$$

$$+\int_B\Big\{\psi\Big(\frac{\delta}{\sigma}|d\sigma|^2+\frac{1}{2}|q_2|\Big)|L_Z v|^2-\mathrm{Re}\Big[L_Z\bar{v}\cdot G\Big(\frac{\psi}{\phi}d\phi+d\psi,\,dv\Big)\Big]\Big\}\Omega$$

$$+\int_B\Big(-\frac{\delta\tau}{\sigma}\psi+E\Big)|\nabla v|^2\Omega$$

$$+\int_B\Big\{\frac{1}{2}\psi L_Z Q+E(Q-\lambda)+\frac{1}{2}\psi|q_2|-\frac{F}{2}\mathrm{Re}[q_2]+\frac{1}{4}\mathrm{div}\,W\Big\}|v|^2\Omega$$

$$\geqq 0.$$

Proof. Let us write the identity (1.4) of Lemma 1.1 as

$$(1.15)\qquad I(\partial B)+J_1(B)+J_2(B)+J_3(B)+J_4(B)=0.$$

Then $J_2(B)$ can be rewritten as follows:

$$J_2(B)=\int_B \frac{1}{2} F\left\{|\nabla v|^2+2\,\mathrm{Re}\left[\frac{1}{\phi}G(d\phi,\ d\bar{v})v\right]+(Q+\mathrm{Re}[q_2]-\lambda)|v|^2\right\}\Omega \tag{1.16}$$

$$-\int_B \frac{F}{\phi}\mathrm{Re}[G(d\phi,\ d\bar{v})v]\Omega$$

$$+\int_B E|\nabla v|^2\Omega+\int_B\left\{E(Q-\lambda)-\frac{F}{2}\mathrm{Re}[q_2]\right\}|v|^2\Omega.$$

We write the right-hand side of (1.16) as $J_{21}(B)+J_{22}(B)+J_{23}(B)+J_{24}(B)$. Taking equation (1.2) into consideration, by the Gauss-Green formula, we have

$$J_{21}(B)=\int_{\partial B}\frac{1}{2}\mathrm{Re}[F\bar{v}\cdot G(n,\ dv)]\Sigma-\int_B\frac{1}{2}\mathrm{Re}[\bar{v}\cdot G(dF,\ dv)]\Omega. \tag{1.17}$$

Let us write this as $J_{21}(B)=I_{21}(\partial B)+J_{211}(B)$. In view of the definition of W ((1.12)), we see that

$$\frac{F}{\phi}\mathrm{Re}[\bar{v}G(d\phi,\ dv]+\frac{1}{2}\mathrm{Re}[\bar{v}\cdot G(dF,\ dv)]=\frac{1}{2}\mathrm{Re}[\bar{v}\cdot L_W v].$$

Consequently, by partial integration, we obtain

$$J_{22}(B)+J_{211}(B)=-\int_B\frac{1}{2}\mathrm{Re}[\bar{v}\cdot L_W v]\Omega \tag{1.18}$$

$$=-\int_{\partial B}\frac{1}{4}|v|^2 n(W)\Sigma+\int_B\frac{1}{4}\mathrm{div}(W)|v|^2\Omega.$$

We write the last member of (1.18) as $I_{221}(\partial B)+J_{221}(B)$. Then we have

$$J_2(B)=I_{21}(\partial B)+J_{211}(B)+J_{22}(B)+J_{23}(B)+J_{24}(B) \tag{1.19}$$

$$=I_{21}(\partial B)+I_{221}(\partial B)+J_{221}(B)+J_{23}(B)+J_{24}(B).$$

By assumption (1.11),

$$\int_B \phi\frac{\delta}{\sigma}|d\sigma|^2|L_Z v|^2\Omega-\int_B\phi\frac{\delta\tau}{\sigma}|\nabla v|^2\Omega+\int_B\frac{1}{2}\phi(L_Z Q)|v|^2\Omega\geqq J_3(B), \tag{1.20}$$

which we write as

$$J_{31}(B)+J_{32}(B)+J_{33}(B)\geqq J_3(B). \tag{1.21}$$

Finally let us write the obvious inequality

$$\int_B\frac{1}{2}\phi|q_2||L_Z v|^2\Omega+\int_B\frac{1}{2}\phi|q_2||v|^2\Omega\geqq J_4(B)$$

as

$$J_{41}(B)+J_{42}(B)\geqq J_4(B). \tag{1.22}$$

Collecting (1.19)-(1.22), and comparing them with (1.15) we see that

$$\{I(\partial B)+I_{21}(\partial B)+I_{221}(\partial B)\}+\{J_1(B)+J_{31}(B)+J_{41}(B)\} \tag{1.23}$$

$$+\{J_{23}(B)+J_{32}(B)\}+\{J_{221}(B)+J_{24}(B)+J_{33}(B)+J_{42}(B)\}\geqq 0.$$

This proves (1.14). Q. E. D.

Lemma 1.3. *Let* $2>\alpha>\beta>1$. *Then we have*

(1.24) $$\sup_{x\in D_\beta}\frac{|x|}{\rho_\alpha(x)}=(\cos(\beta\pi/2\alpha))^{-\alpha},$$

(1.25) $$\inf_{x\in D_\beta}\Theta_\alpha(x)=(\cos(\beta\pi/2\alpha))^2.$$

Proof. From (0.2), (0.7) and (0.8) we see that $\arg(\xi_\alpha(x)+i\eta(x)_\alpha)<\beta\pi/2\alpha$ if and only if $x\in D_\beta$. Combining this fact with the definition of ρ_α and Θ_α ((0.9), (0.11)), we obtain (1.24) and (1.25) through straightforward calculation. Q. E. D.

Lemma 1.4. *Let* $2>\alpha>1$, *and* $h_\alpha=(\alpha-1)/\alpha$. *Define*

(1.26) $$X_\alpha=\mathrm{a}_\alpha^{-1}\nabla\rho_\alpha.$$

Then

(1.27) $$2h_\alpha\mathrm{a}_\alpha(X_\alpha\otimes X_\alpha)-\rho_\alpha L_{X_\alpha}G\geqq 2h_\alpha\Theta_\alpha G \qquad (in\ D_\alpha).$$

If the space dimension $n=2$ *we have the equality in* (1.27).

The proof of Lemma 1.4 will be given in Section 3.

To proceed further we introduce more notations. Let m, γ and γ_0 be real numbers. Let us put

(1.28) $$\boldsymbol{E}_\alpha=\gamma_0\Theta_\alpha\rho_\alpha^{\gamma-1},$$

Moreover in the definition of Q, F and W ((1.3), (1.12) and (1.13)) let us set

(1.29) $$\phi=\rho_\alpha^m,\quad \psi=\rho^\gamma_\alpha,\quad E=\boldsymbol{E}_\alpha,\quad Z=X_\alpha.$$

Let the functions and vector field thus obtained be denoted by $\boldsymbol{Q}_\alpha$, $\boldsymbol{F}_\alpha$ and $\boldsymbol{W}_\alpha$. Strictly speaking we should label these quantities not only by α but by m, γ and γ_0 also. However, such omission will cause little fear of confusion. Using the fact that $d\rho_\alpha^m=m\rho_\alpha^{m-1}d\rho$ and that $\Delta(\rho_\alpha^m)=m(m-1)\rho_\alpha^{m-2}G(d\rho_\alpha, d\rho_\alpha)+m\rho_\alpha^{m-1}\Delta\rho$, we have from (1.3)

(1.30) $$\boldsymbol{Q}_\alpha=-(m^2+m)\rho_\alpha^{-2}\mathrm{a}_\alpha+m\rho_\alpha^{-1}\Delta\rho_\alpha+q_1.$$

We shall show in Section 3 that

$$\operatorname{div}X_\alpha=2h_\alpha(\Theta_\alpha-1)\rho_\alpha^{-1}+\mathrm{a}_\alpha^{-1}\Delta\rho_\alpha \qquad ((3.22)\ \text{in Section 3}).$$

Using this we have from (1.12)

(1.31) $$\boldsymbol{F}_\alpha=\{\gamma-2h_\alpha+2\Theta(h_\alpha-\gamma_0)\}\rho_\alpha^{\gamma-1}-\rho^\gamma_\alpha\mathrm{a}_\alpha^{-1}\Delta\rho_\alpha.$$

Furthermore, from (1.13) and (1.31), after rather elementary but somewhat cumbersome calculation we have

(1.32) $$\begin{aligned}\boldsymbol{W}_\alpha&=\rho_\alpha^{-2m}\nabla(\rho_\alpha^{2m}\boldsymbol{F}_\alpha)\\&=(2m+\gamma-1)\{\gamma-2h_\alpha+2\Theta_\alpha(h_\alpha-\gamma_0)\}\rho_\alpha^{\gamma-2}\nabla\rho_\alpha+2(h_\alpha-\gamma_0)\rho_\alpha^{\gamma-1}\nabla\Theta_\alpha\\&\quad+\frac{1}{\mathrm{a}_\alpha}(2m+\gamma)(\Delta\rho_\alpha)\rho_\alpha^{\gamma-1}\nabla\rho_\alpha+\rho^\gamma_\alpha\nabla\Big(\frac{1}{\mathrm{a}_\alpha}\Delta\rho_\alpha\Big).\end{aligned}$$

Lemma 1.5. *In relation to* ρ_α *we have the following* (1.33)～(1.35).

$$|\nabla\rho_\alpha|^2(=|d\rho_\alpha|^2=\mathrm{a}_\alpha)=\Theta_\alpha^{\alpha-1}\leqq 1. \tag{1.33}$$

$$\frac{1}{\mathrm{a}_\alpha}\Delta\rho_\alpha=O(\rho_\alpha^{-1}) \qquad (\rho_\alpha\to\infty \text{ in } D_\alpha). \tag{1.34}$$

$$L_{X_\alpha}(\Delta\rho_\alpha)=O(\rho_\alpha^{-2}) \qquad (\rho_\alpha\to\infty \text{ in } D_\alpha). \tag{1.35}$$

$\boldsymbol{F}_\alpha$, $|\boldsymbol{W}_\alpha|$ *and* div $\boldsymbol{W}_\alpha$ *are bounded in* $D_\alpha(s)$ *for each* $s>0$. *Moreover, if we choose* $m\geqq 0$, *there exist positive constants* C_1, C_2 *and* C_3 *which depend on* γ_0 *but not on* $m\geqq 0$ *and* γ *such that in* $D_\alpha(1)$ *the following* (1.36)～(1.38) *hold good:*

$$|\boldsymbol{F}_\alpha|\leqq(|\gamma|+1)C_1\rho_\alpha^{\gamma-1}. \tag{1.36}$$

$$|\boldsymbol{W}_\alpha|\leqq(m+|\gamma|+1)(|\gamma|+1)C_2\rho_\alpha^{\gamma-2}. \tag{1.37}$$

$$|\mathrm{div}\,\boldsymbol{W}_\alpha|\leqq\{2\mathrm{a}_\alpha m\gamma^2+C_3(m|\gamma|+1)\}\rho_\alpha^{\gamma-3}. \tag{1.38}$$

The proof of Lemma 1.5 will be given in Section 3.

Lemma 1.6. *Assume that* $D\supset D_\alpha(T)(2>\exists\alpha>1, \exists T>0)$, *and that* q *in* (0.1) *is bounded in* $D_\alpha(T)$. *Let* $u\in L^2(D)$ *be a solution of equation* (0.1), *and* m *be any real number. Set* $v=\rho_\alpha^m u$. *Then we have* $v\in L^2(D_\alpha(s_1, s_2))$,

$$|\nabla v|\in L^2(D_\alpha(s_1, s_2)) \qquad (T<s_1<s_2<\infty) \tag{1.39}$$

and

$$v\in L^2(S_\alpha(t)), \quad |\nabla v|\in L^2(S_\alpha(t)) \qquad (T<t<\infty). \tag{1.40}$$

If we assume $v\in L^2(D_\alpha(s_1))$ $(s_1>T)$ *for a fixed* m, *we have*

$$|\nabla v|\in L^2(D_\alpha(s_1)) \tag{1.41}$$

for the same m.

Proof. The assertion that $v\in L^2(D_\alpha(s_1, s_2))$ is obvious because ρ_α is bounded in $D_\alpha(s_1, s_2)$. We may assume $s_1<t<s_2$. Let us choose s_0 such that $T<s_0<s_1$. Note that v is a solution of equation (1.2) with $\phi=\rho_\alpha^m$, i. e.

$$-\Delta v+\frac{2m}{\rho_\alpha}G(d\rho_\alpha, dv)+(Q_\alpha+q_2-\lambda)v=0 \qquad (\text{in } D_\alpha(T)), \tag{1.42}$$

which is uniformly elliptic. Hence we can apply the theorem on a priori estimates of solutions of elliptic equations to obtain

$$\int_{D_\alpha(s_1, s_2; s_3)}\{|\Delta v|^2+|\nabla v|^2\}\Omega\leqq C_1\int_{D_\alpha(s_0, s_2+1; s_3+1)}|v|^2\Omega, \tag{1.43}$$

where $s_3>1$ (see e. g. [1], Theorem 6.3.). In addition, from (1.43) and the theorem concerning the traces of functions in the Sobolev spaces we have

$$\int_{S_\alpha(t; s_3-1)}\{|\nabla v|^2+|v|^2\}\Sigma\leqq C_2\int_{D_\alpha(s_0, s_2+1; s_3+1)}|v|^2\Omega. \tag{1.44}$$

([1], Theorem 3.10.) Notice that, in view of lemma 1.5, the coefficients of equation (1.42) are bounded functions, and that

$$\inf\{|x-x'|\,|x\in D_\alpha(s_1, s_2; s_3),\ x'\in R^n-D_\alpha(s_0, s_2+1; s_3+1)\}>0.$$

Noting that $S_\alpha(t)$ is close to a cone near infinity, it turns out possible to choose C_1 and C_2 in (1.43) and (1.44) such that they do not depend on s_2 and s_3. Passing to the limit for $s_3\to\infty$ we have (1.39) and (1.40). If $v\in L^2(D_\alpha(s_1))$ is assumed, in (1.43) we can pass to the limit for $s_2\to\infty$ after the limiting procedure for $s_3\to\infty$ to obtain the final assertion (1.41) of the lemma. Q. E. D.

Lemma 1.7. *Assume that* $D\supset D_\alpha(T)\,(2>\exists\alpha>1,\ \exists T>0)$. *Assume that* q, q_2 *and* $L_{X_\alpha}q_1$ *are bounded in* $D_\alpha(T)$. *Choose* γ_0 *such that* $h_\alpha>\gamma_0>0$. *Let* $m\geqq 0$, *and let* γ *be real. Further let* $u\in L^2(D)$ *be a solution of* (0.1). *Then for* $v=\rho_\alpha^m u$ *we have the following inequality:*

$$\begin{aligned}(1.45)\qquad &\left[\int_{S_\alpha(s_2)}-\int_{S_\alpha(s_1)}\right]\left[\rho_\alpha^\gamma\left\{\mathrm{a}_\alpha|L_{X_\alpha}v|^2-\frac{1}{2}(|\nabla v|^2+(Q_\alpha-\lambda)|v^2|)\right\}\tilde{n}_\alpha(X_\alpha)\right.\\
&\qquad\left.+\frac{F_\alpha}{2}\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]\tilde{n}_\alpha(X)-\frac{1}{4}|v|^2\tilde{n}_\alpha(W_\alpha)\right]\Sigma\\
&+\int_{D_\alpha(s_1,s_2)}\rho_\alpha^{\gamma-1}\left\{-2m+h_\alpha-\gamma+\frac{\rho_\alpha}{2\mathrm{a}_\alpha}|q_2|\right\}\mathrm{a}_\alpha|L_{X_\alpha}v|^2\Omega\\
&+\int_{D_\alpha(s_1,s_2)}\rho_\alpha^{\gamma-1}\Theta_\alpha\{-h_\alpha+\gamma_0\}|\nabla v|^2\Omega\\
&+\int_{D_\alpha(s_1,s_2)}\left[\rho_\alpha^{\gamma-3}\mathrm{a}_\alpha\{(1-h_\alpha+(h_\alpha-\gamma_0)\Theta_\alpha)m^2+Cm\}\right.\\
&\qquad+\rho_\alpha^{\gamma-1}\left\{-\lambda\gamma_0\Theta_\alpha+\frac{\rho_\alpha}{2}|L_{X_\alpha}q_1|+\gamma_0\Theta_\alpha q_1+\frac{\rho_\alpha}{2}|q_2|\right\}\\
&\qquad\left.-\frac{F_\alpha}{2}\mathrm{Re}[q_2]+\frac{1}{4}\mathrm{div}(W_\alpha)\right]|v|^2\Omega\\
&\geqq 0\qquad(s_2>s_1\geqq T).\end{aligned}$$

Here C *is a positive constant not depending on* γ *and* m.

Proof. As a consequence of Lemma 1.4, the condition (1.11) in Lemma 1.2 is satisfied if we assign ρ_α, h_α, Θ_α and X_α to σ, δ, τ and Z. In the inequality (1.14) of Lemma 1.2 let us set ϕ, ψ, E and Z as in (1.28) and (1.29), and put

$$(1.46)\qquad B=D_\alpha(s_1, s_2; s_3)\qquad(T\leqq s_1<s_2<\infty,\ 0<s_3<\infty),$$

and write the inequality thus obtained from (1.14) as

$$\begin{aligned}(1.47)\qquad &I_1(S_\alpha(s_1; s_3)\cup S_\alpha(s_2; s_3)\cup\Gamma_\alpha(s_1, s_2; s_3))\\
&\quad+J_1(D_\alpha(s_1, s_2; s_3))+J_2(D_\alpha(s_1, s_2; s_3))+J_3(D_\alpha(s_1, s_2; s_3))\\
&\geqq 0.\end{aligned}$$

On the set $S_\alpha(s_2; s_3)$ we have $n=\tilde{n}_\alpha$ and $G(n, dv)=\mathrm{a}_\alpha L_{X_\alpha}v$. In addition we have $n=-\tilde{n}_\alpha$ on $S_\alpha(s_1; s_3)$, and $n(X_\alpha)=0$ on $\Gamma_\alpha(s_1, s_2; s_3)$. Consequently we have

$$(1.48)\qquad I_1(S_\alpha(s_1; s_3)\cup S_\alpha(s_2; s_3)\cup\Gamma_\alpha(s_1, s_2; s_3))$$

$$=\left[\int_{S_\alpha(s_2,s_3)}-\int_{S_\alpha(s_1,s_3)}\right]\left[\rho_\alpha^\gamma\left\{\mathrm{a}_\alpha|L_{X_\alpha}v|^2-\frac{1}{2}(|\nabla v|^2+(\boldsymbol{Q}_\alpha-\lambda)|v|^2)\right\}\tilde{n}_\alpha(X_\alpha)\right.$$

$$\left.+\frac{\boldsymbol{F}_\alpha}{2}\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]\tilde{n}_\alpha(X)-\frac{1}{4}|v|^2\tilde{n}_\alpha(\boldsymbol{W}_\alpha)\right]\Sigma$$

$$+\int_{\Gamma_\alpha(s_1,s_2;s_3)}\left\{\mathrm{Re}\left[\left(\rho^\gamma L_{X_\alpha}\bar{v}+\frac{\boldsymbol{F}_\alpha}{2}\bar{v}\right)G(n, dv)\right]-\frac{1}{4}|v|^2n(\boldsymbol{W}_\alpha)\right\}\Sigma\,.$$

Let us write the right-hand side of (1.48) as $I_{11}(S_\alpha(s_2, s_3))+I_{12}(S_\alpha(s_1, s_3))+I_{13}(\Gamma_\alpha(s_1, s_2; s_3))$. Here I_{13} is estimated as follows:

$$(1.49)\qquad I_{13}(\Gamma_\alpha(s_1, s_2; s_3))\leqq\int_{\Gamma_\alpha(s_1,s_2;s_3)}\left\{\left(\rho_\alpha^\gamma|L_{X_\alpha}v|+\frac{1}{2}|\boldsymbol{F}_\alpha v|\right)|\nabla v|+\frac{1}{4}|v|^2n(\boldsymbol{W}_\alpha)\right\}\Sigma.$$

Let us write the right-hand side as $I_{131}(\Gamma_\alpha(s_1, s_2; s_3))$. Because of the setting (1.29), we have $d\phi=m\rho_\alpha^{m-1}d\rho_\alpha$ and $d\phi=\gamma\rho_\alpha^{\gamma-1}d\rho_\alpha$, which implies

$$(1.50)\qquad J_1(D_\alpha(s_1, s_2; s_3))=\int_{D_\alpha(s_1,s_2;s_3)}\rho_\alpha^{\gamma-1}\left\{-2m+h_\alpha-\gamma+\frac{\rho_\alpha}{2\mathrm{a}_\alpha}|q_2|\right\}\mathrm{a}_\alpha|L_{X_\alpha}v|^2\Omega\,.$$

We write the right-hand side of (1.50) as $J_{11}(D_\alpha(s_1, s_2; s_3))$. Recalling that we have set $E=\boldsymbol{E}_\alpha=\gamma_0\rho^{\gamma-1}\Theta_\alpha$, we obtain

$$(1.51)\qquad J_2(D_\alpha(s_1, s_2; s_3))=\int_{D_\alpha(s_1,s_2;s_3)}\rho_\alpha^{\gamma-1}\Theta_\alpha\{-h_\alpha+\gamma_0\}|\nabla v|^2\Omega\,,$$

which we write as $J_{21}(D_\alpha(s_1, s_2; s_3))$. To estimate J_3 we have to calculate $(1/2)\phi L_ZQ+E(Q-\lambda)$ with $Z=X_\alpha$ and $Q=\boldsymbol{Q}_\alpha$. Using (1.30) and the equality $L_{X_\alpha}\mathrm{a}_\alpha=2h_\alpha\mathrm{a}_\alpha(1-\Theta_\alpha)\rho_\alpha^{-1}$ (see (3.8)), we have

$$(1.52)\qquad \frac{1}{2}\rho_\alpha^\gamma L_{X_\alpha}\boldsymbol{Q}_\alpha+\boldsymbol{E}_\alpha(\boldsymbol{Q}_\alpha-\lambda)=(m^2+m)(1-h_\alpha(1-\Theta_\alpha)-\Theta_\alpha\gamma_0)\mathrm{a}_\alpha\rho^{\gamma-3}$$

$$+m\left(-\frac{1}{2}+\Theta_\alpha\gamma_0\right)\rho_\alpha^{\gamma-2}\Delta\rho_\alpha+\frac{m}{2}\rho_\alpha^{\gamma-1}L_{X_\alpha}(\Delta\rho_\alpha)$$

$$+(-\lambda+q_1)\gamma_0\rho_\alpha^{\gamma-1}\Theta_\alpha+\frac{1}{2}\rho_\alpha^\gamma L_{X_\alpha}q_1\,.$$

Combining this with (1.34) and (1.35) of Lemma 1.5 together, we have

$$(1.53)\qquad J_3(D_\alpha(s_1, s_2; s_3))\leqq\int_{D_\alpha(s_1,s_2;s_3)}\left[\rho_\alpha^{\gamma-3}\mathrm{a}_\alpha\{(1-h_\alpha+(h_\alpha-\gamma_0)\Theta_\alpha)m^2+Cm\}\right.$$

$$+\rho_\alpha^{\gamma-1}\left\{(-\lambda+q_1)\gamma_0\Theta_\alpha+\frac{\rho_\alpha}{2}|L_{X_\alpha}q_1|+\frac{\rho_\alpha}{2}|q_2|\right\}$$

$$\left.-\frac{\boldsymbol{F}_\alpha}{2}\mathrm{Re}[q_2]+\frac{1}{4}\mathrm{div}(\boldsymbol{W}_\alpha)\right]|v|^2\Omega\,.$$

Let us write the right-hand side of (1.53) as $J_{31}(D_\alpha(s_1, s_2; s_3))$. Form (1.47)~(1.51) and (1.53) we see that

(1.54) $$\boldsymbol{I}(s_1, s_2; s_3)+\boldsymbol{J}(s_1, s_2, s_3)+I_{131}(\Gamma_\alpha(s_1, s_2; s_3))\geqq 0,$$

where

(1.55) $$\boldsymbol{I}(s_1, s_2; s_3)=I_{11}(S_\alpha(s_2; s_3))+I_{12}(S_\alpha(s_1; s_3)),$$

(1.56) $$\boldsymbol{J}(s_1, s_2; s^8)=J_{11}(D_\alpha(s_1, s_2; s_3)+J_{21}(D_\alpha(s_1, s_2; s_3)+J_{31}(D_\alpha(s_1, s_2; s_3).$$

Using the facts $|X_\alpha|=|d\rho_\alpha|^{-1}=\Theta_\alpha^{(1-\alpha)/2}$ and $|d\eta_\alpha|=\alpha^{-1}(\xi_\alpha^2+\eta_\alpha^2)^{(1-\alpha)/2}$ (see (1.26) and (3.7)), we have from Lemma 1.6

$$|L_{X_\alpha}v||d\eta_\alpha|\leqq|X_\alpha||\nabla v||d\eta_\alpha|=\alpha^{-1}\xi_\alpha^{1-\alpha}|\nabla v|\in L^2(D_\alpha(s_1, s_2)).$$

Accordingly we have

(1.57) $$\int_0^\infty\left[\int_{\Gamma_\alpha(s_1, s_2; \eta)}\rho^\gamma|L_{X_\alpha}v||\nabla v|\Sigma\right]d\eta$$
$$=\int_{D_\alpha(s_1; s_2)}\rho^\gamma|L_{X_\alpha}v||\nabla v||d\eta|\Omega<\infty.$$

By Lemma 1.5, $|\boldsymbol{F}_\alpha|$ and $|\boldsymbol{W}_\alpha|$ are bounded in $D_\alpha(s_1, s_2)$. Hence (1.57) shows that $I_{131}(\Gamma_\alpha(s_1, s_2; s_3))$ tends to 0 as $s_3\to\infty$ along a suitable sequence. As a consequence of Lemma 1.6 and the assumptions we have imposed on q, q_2 and $L_{X_\alpha}q_1$, we see that $\boldsymbol{I}(s_1, s_2, s_3)$ and $\boldsymbol{J}(s_1, s_2, s_3)$ remain finite when $s_3\to\infty$. Thus in (1.54) we can pass to the limit for $s_3\to\infty$ along a suitable sequence, which proves the inequality (1.45).

Q. E. D.

Lemma 1.8. *Assume* $D\supset D_\alpha(T)$ $(2>\exists\alpha>1, \exists T>0)$, *and assume*

(1.58) $$q_1=o(1)\qquad (|x|\to\infty \text{ in } D_\alpha).$$

In addition assume that there exists a function $f(s)=o(1)$ $(s\to\infty)$ *such that*

(1.59) $$|L_{X_\alpha}q_1|+|q_2|\leqq(\Theta_\alpha\rho_\alpha^{-1}+\rho_\alpha^{-3})\cdot f(\rho)\qquad (\text{in } D_\alpha(1)).$$

Let $u\in L^2(D)$ *be a solution of* (0.1) *such that*

(1.60) $$\int_{D_\alpha(T)}\rho_\alpha^{2m}|u|^2\Omega<\infty\qquad (\forall m\geqq 0).$$

Then there exists $T_1>T$ *such that* u *vanishes identically in* $D_\alpha(T_1)$.

Proof. Set $v=\rho_\alpha^m u$. Let us notice that (1.60) and Lemma 1.6 imply $\rho_\alpha^l(|v|+|\nabla v|)\in L^2(D_\alpha(1))$ $(\forall l\geqq 0)$. Integrating the inequality $\mathrm{a}_\alpha|L_{X_\alpha}(\rho_\alpha^{\sqrt{m}/2}v)|^2\geqq 0$ over $D_\alpha(T)$ we have

$$\int_{D_\alpha(t)}\{\rho_\alpha^{\sqrt{m}}\mathrm{a}_\alpha|L_{X_\alpha}v|^2+\sqrt{m}\,\rho_\alpha^{-1+\sqrt{m}}\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]$$
$$+\frac{1}{4}m\rho_\alpha^{-2+\sqrt{m}}\mathrm{a}_\alpha|v|^2\}\Omega\geqq 0.$$

Then partial integration gives

$$\int_{D_\alpha(t)} \rho_\alpha^{\sqrt{m}} a_\alpha |L_{X_\alpha} v|^2 \Omega \tag{1.61}$$

$$+\int_{D_\alpha(t)} \rho_\alpha^{-2+\sqrt{m}} \left\{-\frac{1}{4} a_\alpha m + \frac{1}{2}\sqrt{m}\left(a_\alpha - \rho_\alpha \operatorname{div}(a_\alpha X_\alpha)\right)\right\} |v|^2 \Omega$$

$$-\frac{1}{2}\int_{S_\alpha(t)} \rho_\alpha^{-1+\sqrt{m}} \sqrt{m}\, a_\alpha |v|^2 \tilde{n}_\alpha(X_\alpha) \Sigma \geqq 0.$$

In (1.45) of Lemma 1.7, let us choose

$$\gamma = 1+\sqrt{m}. \tag{1.62}$$

In addition, let us choose ε and γ_0 such that

$$\varepsilon < \frac{1}{2},\ \gamma_0 < h_\alpha,\quad 1-h_\alpha-\varepsilon+(h_\alpha-\gamma_0)\Theta_\alpha < -\frac{1}{4}, \tag{1.63}$$

which is possible if we take ε and γ_0 such that $1/2-\varepsilon$ and $h_\alpha-\gamma_0$ are sufficiently small. Because of (1.58) and (1.59) the assumptions on q in Lemma 1.7 are satisfied. Further (1.60) allows us to pass to the limit for $s_2 \to \infty$ in the inequality (1.45), where we put $s_1 = t$. Adding the inequality thus obtained to (1.61) multiplied by $4m\varepsilon$, we have

$$-\int_{S_\alpha(t)} \Big[\rho_\alpha^{1+\sqrt{m}} \Big\{a_\alpha |L_{X_\alpha} v|^2 - \frac{1}{2}(|\nabla v|^2 + (Q_\alpha - \lambda)|v|^2)\Big\} \tilde{n}_\alpha(X_\alpha) \tag{1.64}$$

$$+2\rho_\alpha^{-1+\sqrt{m}} \varepsilon m \sqrt{m} |v|^2 \tilde{n}_\alpha(X_\alpha) + \frac{\boldsymbol{F}_\alpha}{2} a_\alpha \operatorname{Re}[\bar{v}\cdot L_{X_\alpha} v] \tilde{n}_\alpha(X_\alpha)$$

$$-\frac{1}{4}|v|^2 \tilde{n}_\alpha(\boldsymbol{W}_\alpha)\Big]\Sigma$$

$$+\int_{D_\alpha(t)} \rho_\alpha^{\sqrt{m}} \Big\{-(2-4\varepsilon)m + h_\alpha - 1 - \sqrt{m} + \frac{\rho_\alpha}{2a_\alpha}|q_2| a_\alpha |L_{X_\alpha} v|^2\Big\} \Omega$$

$$+\int_{D_\alpha(t)} \rho_\alpha^{\sqrt{m}} \Theta_\alpha \{-h_\alpha + \gamma_0\} |\nabla v|^2 \Omega$$

$$+\int_{D_\alpha(t)} \Big[\rho_\alpha^{-2+\sqrt{m}} a_\alpha \Big\{(1-h_\alpha-\varepsilon+(h_\alpha-\gamma_0)\Theta_\alpha)m^2 + Cm$$

$$+2\varepsilon m\sqrt{m}\Big(1-\frac{\rho_\alpha}{a_\alpha}\operatorname{div}(a_\alpha X_\alpha)\Big)\Big\}$$

$$+\rho_\alpha^{\sqrt{m}} \Big\{(-\lambda+q_1)\gamma_0\Theta + \frac{\rho_\alpha}{2}|L_{X_\alpha} q_1| + \frac{\rho_\alpha}{2}|q_2|\Big\} - \frac{\boldsymbol{F}_\alpha}{2}\operatorname{Re}[q_2]$$

$$+\frac{1}{4}\operatorname{div}(\boldsymbol{W}_\alpha)\Big]|v|^2 \Omega$$

$$\geqq 0 \qquad (t \geqq T).$$

We write this as

$$I(S_\alpha(t)) + J_1(D_\alpha(t)) + J_2(D_\alpha(t)) + J_3(D_\alpha(t)) \geqq 0 \qquad (t > T).$$

From the assumption (1.59) and (1.33) of Lemma 1.5, we see that $\rho_\alpha |q_2| / a_\alpha = o(1)$.

Using this and $\varepsilon<1/2$ ((1.63)), we have

(1.65) $$J_1(D_\alpha(t))\leqq 0 \qquad (m\geqq\exists M_2\geqq 0,\ t\geqq\exists T_2>T).$$

Obviously

(1.66) $$J_2(D_\alpha(t))\leqq 0 \qquad (t>T).$$

From (1.36), (1.38) of Lemma 1.5 and (1.62) we have

(1.67) $$\frac{1}{2}|\boldsymbol{F}_\alpha \operatorname{Re}[q_2]|\leqq\rho_\alpha^{\sqrt{m}}\{C_1+\sqrt{m}\}|q_2|\leqq\rho_\alpha^{\sqrt{m}}\left\{\frac{m}{\rho_\alpha}+C_2\rho_\alpha\right\}|q_2|,$$

(1.68) $$\frac{1}{4}|\operatorname{div}\boldsymbol{W}_\alpha|\leqq \mathrm{a}_\alpha\rho_\alpha^{-2+\sqrt{m}}\left\{\frac{1}{2}m^2+C_3m\sqrt{m}+1\right\}.$$

Here C_1, C_2 and C_3 do not depend on m. Using inequalities (1.67) and (1.68), we see that the integrand of J_3 is bounded above by the function

(1.69) $$\Big[\rho_\alpha^{-2+\sqrt{m}}\mathrm{a}_\alpha\Big\{\Big(1-h_\alpha-\varepsilon+(h_\alpha-\gamma_0)\Theta_\alpha+\frac{1}{2}\Big)m^2+(C+\rho_\alpha|q_2|)m +2\varepsilon m\sqrt{m}\Big(\frac{C_3}{\varepsilon}+1-\frac{\rho_\alpha}{\mathrm{a}_\alpha}\Delta\rho_\alpha\Big)\Big\} +\rho_\alpha^{\sqrt{m}}\Big\{(-\lambda+q_1)\gamma_0\Theta_\alpha+\frac{\rho_\alpha}{2}|L_{X_\alpha}q_1|+\Big(\frac{1}{2}+C_2\Big)\rho_\alpha|q_2|\Big]|v|^2.$$

(Here we have used $\operatorname{div}(\mathrm{a}_\alpha X_\alpha)=\Delta\rho_\alpha$). In view of (1.34) of Lemma 1.5, the assumption (1.58), (1.59) and the choice (1.63), we see that (1.69) is non-positive for sufficiently large m and t. Thus we have

(1.70) $$J_3(D_\alpha(t))\leqq 0 \qquad (m\geqq\exists M_3>M_2,\ t\geqq\exists T_3>T_2).$$

Now let us prove the lemma by reductio ad absurdum. If the assertion of the lemma is false, for any solution u of (0.1) and for any $T_1>0$, there would exist $t>T_1$ such that the function obtained by tracing u onto $S_\alpha(t)$, which function we write u again, does not vanish as a function in $L^2(S_\alpha(t))$. Let us estimate $I(S_\alpha(t))$ in (1.64) for such t. We may assume $t>T_1>T_3$. Let us recall $v=\rho_\alpha^m u$. Then, after some calculation, we see that

(1.71) $$I(S_\alpha(t))=t^{-1+2m+\sqrt{m}}\left\{-m^2\int_{S_\alpha(t)}\mathrm{a}_\alpha|u|^2\tilde{n}(X_\alpha)\Sigma+m\sqrt{m}(\cdots)+m(\cdots)+(\cdots)\right\},$$

where $(\cdots)$'s are functions of t not depending on m. This shows that $I(S_\alpha(t))$ is negative for sufficiently large m, which, in view of (1.65), (1.66) and (1.70), leads us to an absurdity because the left-hand side of (1.64) should be nonnegative. Q. E. D.

2. Proof of the theorems

To prove Theorem 0.2 we prepare one more lemma.

Lemma 2.1. *Let us assume the hypotheses of Theorem* 0.2. *Let c and c' be as in* *the hypotheses of Theorem* 0.2. *Take* α *and* β *such that* $c\geqq\alpha>\beta>c'$. *Assume that* u *is a solution of* (0.1) *satisfying*

$$(1+\rho_\alpha)^m u\in L^2(D_\alpha) \qquad (\exists m\geqq 0). \tag{2.1}$$

Then

$$(1+\rho_\beta)^{m+(1/2)}u\in L^2(D_\beta), \qquad (1+\rho_\beta)^{m+(1/2)}|\nabla u|\in L^2(D_\beta). \tag{2.2}$$

Proof. From (0.11) and $|X_\alpha|=|d\rho_\alpha|^{-1}=\Theta_\alpha^{(1-\alpha)/2}$ (see (1.26) and (3.7)) we have

$$|L_{X_\alpha}q_1|+|q_2|\leqq|X_\alpha|\,|\nabla q_1|+|q_2|\leqq(\Theta_\alpha\rho_\alpha^{-1}+\rho_\alpha^{-3})\cdot f(\rho_\alpha) \qquad (\text{in } D_\alpha). \tag{2.3}$$

From Lemma 1.6 and (2.1) we see that

$$|\nabla((1+\rho_\alpha)^m u|\in L^2(D_\alpha). \tag{2.4}$$

Now let us set v as in Lemma 1.7. Because of (0.3) we may take $T=1$ in (1.45) of Lemma 1.7, where we further set $\gamma=1$ and fix $s_1=t>1$. Then (2.1) and (2.4) allow us to pass to the limit for $s_2\to\infty$ along a suitable sequence. Thus inequality (1.45) is rewritten as

$$-I(S_\alpha(t))+J_1(D_\alpha(t))+J_2(D_\alpha(t))+J_3(D_\alpha(t))\geqq 0. \tag{2.5}$$

Here

$$-I(S_\alpha(t))=-\int_{S_\alpha(s_1)}\Big[\rho_\alpha\Big\{\mathrm{a}_\alpha|L_{X_\alpha}v|^2-\frac{1}{2}(|\nabla v|^2+(\boldsymbol{Q}_\alpha-\lambda)|v|^2)\Big\}\tilde{n}_\alpha(X_\alpha) + \frac{\boldsymbol{F}_\alpha}{2}\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]\tilde{n}_\alpha(X)-\frac{1}{4}|v|^2\tilde{n}_\alpha(W)\Big]\Sigma. \tag{2.6}$$

$$J_1(D_\alpha(t))=\int_{D_\alpha(t)}\Big\{-2m+h_\alpha-1+\frac{\rho_\alpha}{2\mathrm{a}_\alpha}|q_2|\Big\}\mathrm{a}_\alpha|L_{X_\alpha}v|^2\Omega, \tag{2.7}$$

$$J_2(D_\alpha(t))=\int_{D_\alpha(t)}\Theta_\alpha\{-h_\alpha+\gamma_0\}|\nabla v|^2\Omega, \tag{2.8}$$

$$J_3(D_\alpha(t))=\int_{D_\alpha(t)}\Big[\rho_\alpha^{-2}\mathrm{a}_\alpha\{(1-h_\alpha+(h_\alpha-\gamma_0)\Theta_\alpha)m^2+Cm\Big\} +\Big\{-\lambda\gamma_0\Theta_\alpha+\frac{\rho_\alpha}{2}|L_{X_\alpha}q_1|+\gamma_0\Theta_\alpha q_1+\frac{\rho_\alpha}{2}|q_2|\Big\} -\frac{\boldsymbol{F}_\alpha}{2}\mathrm{Re}[q_2]+\frac{1}{4}\mathrm{div}(\boldsymbol{W}_\alpha)\Big]|v|^2\Omega. \tag{2.9}$$

We have chosen $\gamma=1$. Hence by Lemma 1.5 we have

$$\begin{cases} |\boldsymbol{F}_\alpha|\leqq M_1, \\ |\boldsymbol{W}_\alpha|\leqq M_1\rho_\alpha^{-1}, \\ |\mathrm{div}\,\boldsymbol{W}_\alpha|\leqq M_1\rho_\alpha^{-2}, \end{cases} \tag{2.10}$$

where M_1 is a constant depending on m. Furthermore, taking (2.3) into consideration, we can choose $\gamma_0>0$, $\mu>0$ and $T_1>1$ such that the following inequalities hold good:

$$(2.11)\quad \begin{cases} \gamma_0<h_\alpha, \quad -h_\alpha+\gamma_0<-\dfrac{\mu}{2}, \\ -2m+h_\alpha-1+\dfrac{\rho_\alpha}{2a_\alpha}|q_2|<0 \qquad (\text{in } D_\alpha(T_1)), \\ -\lambda\gamma_0\Theta_\alpha+\dfrac{\rho_\alpha}{2}|L_{X_\alpha}q_1|+\gamma_0\Theta_\alpha q_1+\dfrac{\rho_\alpha}{2}|q_2|-\dfrac{\boldsymbol{F}_\alpha}{2}\mathrm{Re}[q_2]<-\dfrac{\mu}{2}\Theta_\alpha+\rho_\alpha^{-2} \ (\text{in } D_\alpha(T_1)). \end{cases}$$

(Here we have used $\mathrm{a}_\alpha=\Theta_\alpha^{\alpha-1}\geqq\Theta_\alpha$.) From (2.10), taking sufficiently large $M_2>0$, we have

$$(2.12)\qquad -I(S_\alpha(t))\leqq\int_{S_\alpha(t)}\frac{1}{2}\rho_\alpha\{|\nabla v|^2+(\boldsymbol{Q}_\alpha+\mathrm{Re}[q_2]-\lambda)|v|^2\}\tilde{n}(X_\alpha)\Sigma$$
$$+\int_{S_\alpha(t)}\left\{M_2(|\nabla v|^2+|v|^2)-\frac{1}{2}\rho_\alpha\,\mathrm{Re}[q_2]|v|^2\right\}\tilde{n}(X_\alpha)\Sigma\,.$$

In addition, by (2.10) and (2.11), we obtain

$$(2.13)\qquad J_1(D_\alpha(t))\leqq 0 \qquad (t>T_1),$$

$$(2.14)\qquad J_2(D_\alpha(t))+J_3(D_\alpha(t))\leqq M_3\int_{D_\alpha(t)}\rho_\alpha^{-2}|v|^2\Omega$$
$$-\frac{\mu}{2}\int_{D_\alpha(t)}\Theta_\alpha\{|\nabla v|^2+|v|^2\}\Omega \qquad (t>T_1).$$

Using (2.12)～(2.14), we obtain

$$(2.15)\qquad \int_{S_\alpha(t)}\rho_\alpha\{|\nabla v|^2+(\boldsymbol{Q}_\alpha+\mathrm{Re}[q_2]-\lambda)|v|^2\}\tilde{n}(X_\alpha)\Sigma$$
$$+\int_{S_\alpha(t)}\{M(|\nabla v|^2+|v|^2)-\rho_\alpha\mathrm{Re}[q_2]|v|^2\}\tilde{n}(X_\alpha)\Sigma$$
$$+M\int_{D_\alpha(t)}\rho_\alpha^{-2}|v|^2\Omega$$
$$\geqq\mu\int_{D_\alpha(t)}\Theta_\alpha(|\nabla v|^2+|v|^2)\Omega \qquad (t>T_1).$$

Here M depends on m, but not on $t>T_1$. Integrating the first surface integral of the left-hand side with respect to t over the interval $[t_1, t_2]$ ($t_1>T_1$), we have a volume integral over $D(t_1, t_2)$, which is calculated as follows. By use of equation (1.42) and the Gauss-Green formula we have

$$\int_{D_\alpha(t_1,t_2)}\rho_\alpha\left\{|\nabla v|^2+\frac{2m}{\rho_\alpha}\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]+(Q_\alpha+\mathrm{Re}[q_2]-\lambda)|v|^2\right\}\Omega$$
$$-2m\int_{D_\alpha(t_1,t_2)}\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]\Omega$$
$$=\left[\int_{S_\alpha(t_2)}-\int_{S_\alpha(t_1)}\right]\rho_\alpha\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]\tilde{n}(X_\alpha)\Sigma$$
$$-\int_{D_\alpha(t_1,t_2)}(2m+1)\mathrm{a}_\alpha\,\mathrm{Re}[\bar{v}\cdot L_{X_\alpha}v]\Omega\,.$$

Therefore integrating (2.15) with respect to t over the interval $[t_1, t_2]$ $(t_1>T_1)$, we have

$$\left[\int_{S_\alpha(t_2)}-\int_{S_\alpha(t_1)}\right]\rho_\alpha \mathrm{a}_\alpha \operatorname{Re}[\bar{v}\cdot L_{X_\alpha}v]\tilde{n}(X_\alpha)\Sigma \tag{2.16}$$

$$+\int_{D_\alpha(t_1,t_2)}\{-(2m+1)\mathrm{a}_\alpha \operatorname{Re}[\bar{v}\cdot L_{X_\alpha}v]+M(|\nabla v|^2+|v|^2)-\rho_\alpha \operatorname{Re}[q_2]|v|^2\}\Omega$$

$$+M\int_{D_\alpha(t_1,t_2)}(\rho_\alpha-t_1)\rho_\alpha^{-2}|v|^2\Omega+(t_2-t_1)M\int_{D_\alpha(t_2)}\rho_\alpha^{-2}|v|^2\Omega$$

$$\geqq\mu\int_{D_\alpha(t_1,t_2)}(\rho_\alpha-t)\Theta_\alpha(|\nabla v|^2+|v|^2)\Omega$$

$$+\mu(t_2-t_1)\int_{D_\alpha(t_2)}\Theta_\alpha(|\nabla v|^2+|v|^2)\Omega.$$

Because of (2.1) and (2.4), the left-hand side of (2.16) remains finite when we let $t_2\to\infty$ along a suitable sequence. Thus we see that

$$\Theta_\alpha\rho_\alpha^{m+(1/2)}u\in L^2(D_\alpha), \qquad \Theta_\alpha\rho_\alpha^{m+(1/2)}|\nabla u|\in L^2(D_\alpha). \tag{2.17}$$

On the other hand, by Lemma 1.3, we have

$$\Theta_\alpha(x)>c_1, \qquad \rho_\beta(x)\leqq c_2|x|\leqq c_3\rho_\alpha(x) \qquad (x\in D_\beta) \tag{2.18}$$

$(\exists c_1, c_2, c_3>0)$. From (2.17) and (2.18) we have the assertion (2.2) of the lemma.

Q. E. D.

Proof of Theorem 0.2. Choose a sequence $\{\alpha_j\}$ $(j=0, 1, 2, \cdots)$ such that $c>\alpha_j>\alpha_{j+1}>c'$. We have $u\in L^2(D_{\alpha_0})$ by the assumption of the theorem. Applying Lemma 2.1 successively, we see that $\rho_{\alpha_j}^{j/2}u\in L^2(D_{\alpha_j})$ $(j=1, 2, \cdots)$. Thus we have $\rho_{c'}^m u\in L^2(D_{c'})$ $(\forall m\geqq 0)$. This together with Lemma 1.8 gives $u=0$ in $D_{c'}(T)$ $(\exists T>0)$. Q. E. D.

Proof of Theorem 0.1. Let c, d and δ be as in the statement of Theorem 0.1. Let us choose δ' and c' such that

$$\delta>\delta'>0, \quad c>c'>\max\left(\frac{2c}{2+c\delta'}, d\right). \tag{2.19}$$

Let us show that the conditions (0.5) and (0.6) imply (0.12). To this end it suffices to show that there exists $C>0$ such that

$$\Theta_\alpha\rho_\alpha^{-1}\geqq C|x|^{-1} \qquad (\text{in } D_d)\ (c>\forall\alpha>c'), \tag{2.20}$$

and

$$\Theta_\alpha\rho_\alpha^{-1}\geqq C|x|^{-2(2/c)-\delta'} \qquad (\text{in } D_\alpha(1))\ (c>\forall\alpha>c'). \tag{2.21}$$

By (2.18) (in which we replace β with d), (2.20) is obvious. Furthermore we see that

$$\Theta_\alpha(x)\rho_\alpha^{-1}=|x|^{-2/\alpha}\rho_\alpha^{(2/\alpha)-1}\geqq|x|^{-2/\alpha} \qquad (\text{in } D_\alpha(1)), \tag{2.22}$$

which together with (2.19) shows (2.21). Therefore we have from Theorem 0.2 that any solution $u\in L^2(D)$ of (0.1) vanishes in an open set $D_{c'}(T)$ $(\exists T>0)$, which with the assumption of the unique continuation property for (0.1) implies that u vanishes identically in D. Q. E. D.

3. Calculation of quantities related to G

Let G, α, D_α, y, ξ_α, η_α etc. be as in the Introduction. In what follows we shall drop the subscript α if it does not raise any fear of confusion. From $x_1+iy=(\xi+i\eta)^\alpha$ we have

$$\text{(3.1)}\qquad \begin{cases} dx_1=\alpha(\xi^2+\eta^2)^{-1}\{(x_1\xi+y\eta)d\xi+(x_1\eta-y\xi)d\eta\}, \\ dy=\alpha(\xi^2+\eta^2)^{-1}\{(y\xi-x_1\eta)d\xi+(x_1\xi+y\eta)d\eta\}. \end{cases}$$

and

$$\text{(3.2)}\qquad G(d\xi, d\xi)=G(d\eta, d\eta)=\alpha^{-2}(\xi^2+\eta^2)^{1-\alpha}.$$

Let us introduce the coordinate systems $(w^1, \cdots, w^n)$ defined as follows. When $n=2$ we set the coordinate neighborhood of (w^1, w^2) to be D_α, and define

$$\text{(3.3)}_\text{a}\qquad w^1=\xi, \quad w^2=\eta \ (x_2\geqq 0), \quad w^2=-\eta \ (x_2\leqq 0).$$

When $n\geqq 3$, let us take integers k such that $2\leqq k\leqq n$, and consider the domains $D_{k,\pm}=\{x\in D_\alpha \mid \pm x_k>0\}$. In each $D_{k,+}$ or $D_{k,-}$ let us define,

$$\text{(3.3)}_\text{b}\qquad w^1=\xi, \quad w^2=\eta, \quad w^j=x_{j-1}/y \ (3\leqq j\leqq k), \quad w^j=x_j/y \ (k<j\leqq n).$$

Although the domains $D_{k,\pm}$ do not cover the x_1-axis, the results computed in the following can be extended to the positive x_1-axis by continuity. Writing $g^{jk}=G(dw^j, dw^k)$, we see that

$$\text{(3.4)}\qquad g^{11}=g^{22}=\alpha^{-2}(\xi^2+\eta^2)^{1-\alpha},$$

$$\text{(3.5)}\qquad g^{jk}=g^{kj}=0 \quad (j=1, 2;\ k\geqq 2;\ j\neq k),$$

$$\text{(3.6)}\qquad g^{jk}=g^{kj}=\frac{1}{y^2}(\delta^{jk}-w^jw^k) \quad (j, k\geqq 3).$$

For $\Theta(x)$ and $\mathrm{a}(x)$ defined in (0.10) and (0.13), we have from (3.4)

$$\text{(3.7)}\qquad \mathrm{a}=G(d\rho, d\rho)=\xi^{2\alpha-2}(\xi^2+\eta^2)^{1-\alpha}=\Theta^{\alpha-1}.$$

Recalling that the vector field $X=X_\alpha$ is defined by (1.26), we have

$$\text{(3.8)}\qquad L_X\mathrm{a}=\mathrm{a}^{-1}G(d\rho, d\Theta)=2h\mathrm{a}(1-\Theta)\rho^{-1} \quad (h=(\alpha-1)/\alpha).$$

Proof of Lemma 1.4. In the system (3.3) the vector field X has the components

$$\text{(3.9)}\qquad X^1=\alpha^{-1}\xi^{1-\alpha}, \quad X^j=0 \ (j\geqq 2).$$

The components of the tensor L_XG are given by

$$(L_XG)^{jk}=\sum_s(X^s\partial g^{jk}/\partial w^s-g^{sk}\partial X^j/\partial w^s-g^{js}\partial X^k/\partial w^s).$$

Through direct calculation we have

$$(3.10)\qquad (L_X G)^{11}=2\alpha^{-3}\xi^{-\alpha}(\xi^2+\eta^2)^{-\alpha}$$
$$=2\alpha^{-1}(\alpha-1)\rho^{-1}(1-\Theta)g^{11},$$

$$(3.11)\qquad (L_X G)^{22}=-2\alpha^{-1}(\alpha-1)\rho^{-1}\Theta g^{22},$$

$$(3.12)\qquad (L_X G)^{1j}=0\ (j\geqq 2),\qquad (L_X G)^{2j}=0\ (j\geqq 3),$$

$$(3.13)\qquad (L_X G)^{jk}=-2y^{-3}(\delta^{jk}-w^j w^k)\xi^{1-\alpha}(\xi y-\eta x_1)/(\xi^2+\eta^2)$$
$$=-2\rho^{-1}[1-(\eta/\xi)(x_1/y)]\Theta g^{jk}\qquad (j,\ k\geqq 3).$$

From (3.9)~(3.13) we have

$$(3.14)\qquad 2h\mathrm{a}(X\otimes X)-\rho L_X G=2h\Theta G+2(\alpha^{-1}-K)\Theta\sum_{j,k\geqq 3} g^{jk}(\partial/\partial w^j)\otimes(\partial/\partial w^k).$$

Here

$$(3.15)\qquad K=(\eta/\xi)\cdot(x_1/y).$$

Set

$$(3.16)\qquad b(x)=\arg(\xi+i\eta)=\frac{1}{\alpha}\arg(x_1+iy).$$

Then

$$(3.17)\qquad K=(\tan b)\ (\cos\alpha b)/\sin\alpha b.$$

In D_α we have $0<b(x)<\pi/2$, hence we have

$$(3.18)\qquad K\leqq 1/\alpha\qquad \text{in } D_\alpha.$$

Combining (3.14) and (3.18) we have Lemma 1.4. Q. E. D.

Additional calculation is necessary before we proceed to the proof of Lemma 1.5. Let us set $g=(\det(g^{jk}))^{-1}$. Then we see that

$$(3.19)\qquad g=(\xi^2+\eta^2)^{2\alpha-2}y^{2n-4}[\cdots].$$

Here $[\cdots]$ is a factor not depending on the first and second coordinates w^1 and w^2. Direct calculation shows

$$(3.20)\qquad \Delta\xi=\frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\,g^{11})}{\partial w^1}$$
$$=\alpha^{-1}(n-2)\{\xi-\eta(x_1/y)\}(\xi^2+\eta^2)^{-\alpha},$$

which gives

$$(3.21)\qquad \Delta\rho=\alpha(\alpha-1)\xi^{\alpha-2}G(d\xi,\ d\xi)+\alpha\xi^{\alpha-1}\Delta\xi$$
$$=\mathrm{a}\rho^{-1}\{h+(n-2)\Theta(1-K)\}.$$

Here K is as in (3.15) and (3.17). From (3.20) we have

$$(3.22)\qquad \mathrm{div}X=\mathrm{div}(\mathrm{a}^{-1}G(d\rho))=-\mathrm{a}^{-2}(\alpha-1)\Theta^{\alpha-2}G(d\Theta,\ d\rho)+\Theta^{1-\alpha}\Delta\rho$$
$$=2h(\Theta-1)\rho^{-1}+\mathrm{a}^{-1}\Delta\rho$$

$$=\rho^{-1}\{-h+2\Theta h+(n-2)(1-K)\Theta\},$$

which was used in Section 1 in the computation of $\boldsymbol{F}_{\alpha}$.

Proof of Lemma 1.5. From (3.7) we have (1.33). Using (3.16), (3.17), (3.20) and $\Theta=(\cos b)^2$ we see that

$$\frac{1}{a}\Delta\rho=\frac{1}{\rho}\left\{h+(n-2)\left(\cos^2 b-\frac{1}{2}\frac{\sin 2b\cos\alpha b}{\sin\alpha b}\right)\right\}, \tag{3.23}$$

which shows (1.34) and (1.36). The assertions (1.35), (1.37) and (1.38) are obtained through elementary but lengthy calculation from the facts that the first and second derivatives of $\sin b/\sin\alpha b$ with respect to b are bounded on the interval $0<b<\pi/2$ and that $|db|=\alpha^{-1}|x|^{-1}$. Q. E. D.

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