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Relations on pfaffians I: plethysm formulas

By

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1. Introduction

Let R be a commutative ring with unity, and fix an integer $n \geq 1$. Suppose x_{ij} be variables with $1 \leq i < j \leq n$. Denote by $S = R[X]$ the polynomial ring with $n(n-1)/2$ variables x_{ij}. Put $x_{ij} = -x_{ji}$ for $1 \leq i < j \leq n$ and $x_{ii} = 0$ for $i = 1, \ldots, n$. (x_{ij}) is the generic n by n antisymmetric matrix with entries in S. For a positive integer t such that $1 < 2t \leq n$ and a strictly increasing sequence $(1 \leq) p_1 < \cdots < p_{2t} (\leq n)$,

$$\frac{1}{2^t t!} \sum_{\sigma \in \mathbf{S}_{2t}} (\operatorname{sgn} \sigma) x_{p_{\sigma(1)} p_{\sigma(2)}} x_{p_{\sigma(3)} p_{\sigma(4)}} \cdots x_{p_{\sigma(2t-1)} p_{\sigma(2t)}}$$

is called a *$2t$-order pfaffian.* (This polynomial is defined over an arbitrary commutative ring R.) It is well-known that the square of this $2t$-order pfaffian coincides with the determinant of the $2t$ by $2t$ antisymmetric matrix $(x_{p_i p_j})_{i,j=1,\ldots,2t}$. We denote by Pf_{2t} the ideal generated by all $2t$-order pfaffians of (x_{ij}) and call it the *pfaffian ideal* of order $2t$.

It is well-known that, if R is Gorenstein, Pf_{2t} is a Gorenstein ideal with $\operatorname{grade}(Pf_{2t}) = hd_S(S/Pf_{2t}) = (n-2t+1)(n-2t+2)/2$ ([7] or [9]). Furthermore any Gorenstein subscheme of codimension 3 is known to be defined by certain pfaffians of a certain antisymmetric matrix ([3]).

The main purpose of this article is to investigate when the first syzygy modules of pfaffian ideals are generated by their relations of degree 1. When R contains the rationals $\mathbf{Q}$, any relation on pfaffians can be written by relations of degree 1 (in this case all the syzygies have been determined in [7] or [8]). In the case of arbitrary characteristic, when $t = 1$, $n = 2t$, $n = 2t + 1$ ([3]) or $n = 2t + 2$ ([14]), minimal free resolutions have been already constructed and the relation modules are generated by relations of degree 1. The main result of this article is

Theorem 5.3. 1. *The first syzygy of the pfaffian ideal Pf_{2t} is generated over $\mathrm{S}(\wedge^2 E)$ by relations of degree at most t, i.e.,*

$$\operatorname{Ker}(M_t) = \mathrm{S}(\wedge^2 E) \cdot \Big(\sum_{r=1}^{t} \operatorname{Ker}(M_{t,r})\Big).$$

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2. *Let R be a field and regard the rational number field as of characteristic infinity. Then the first syzygy of Pf_{2t} is generated over $\mathrm{S}(\wedge^2 E)$ by relations of degree* 1 *when $2p > n - 2t$. (p is the characteristic of R.)*

By using this theorem we will know that the first syzygies of pfaffian ideals are generated by their relations of degree 1 over an arbitrary commutative ring R when $n - 2t \leq 3$. The main ideas are the same as the case of determinantal ideals of generic matrices ([11]). But, in general, the relation modules of pfaffians are not generated only by their relations of degree 1 ([12]).

Section 2 is devoted to introducing the basic facts of characteristic free representation theory. All propositions of this section are proved in [1].

We will construct the plethysm formulas for $\mathrm{S}_r(\wedge^2 E)$ in Section 3. When R is a field of characteristic zero, any finite dimensional polynomial representation of $\mathrm{GL}(E)$ are completely reducible. In fact we have

$$\mathrm{S}_r(\wedge^2 E) = \oplus_{\lambda \in \Gamma_r,} L_\lambda E,$$

where Γ_r is a set of partitions defined in Definition 3.1. The plethysm formula is to give a natural filtration over an arbitrary commutative ring R whose associated graded object coincides with $\oplus_{\lambda \in \Gamma_r} L_\lambda E$.

By using this formula and the Knuth correspondence ([10]) we can show that the usual minimal generating set of the pfaffian ideal Pf_{2t} forms a Gröbner basis for any field R and for any integers $1 < 2t \leq n$. In Section 4 this is proved and some spectral sequences are constructed from the general theory of Gröbner bases in Remark 4.15.

From plethysm formulas and the theory of Gröbner bases, Theorem 5.3 is proved in Section 5.

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2. Preliminaries

In this section, we review the characteristic free representation theory of GL. For the proofs of propositions, we refer to Akin *et al.* [1]. Throughout this section, we denote by R a commutative ring with unity and all tensor products are defined over R.

Definition 2.1. A partition is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_q)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q$. The weight of λ is defined to be $\lambda_1 + \lambda_2 + \cdots + \lambda_q$, and denoted by $|\lambda|$. To each partition λ we associate its transposed partition $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{\lambda_1})$, where $\tilde{\lambda}_k$ is the number of λ_s such that $\lambda_s \geq k$. We introduce the lexicographic order to the set of partitions; i.e., for two partitions $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_r)$, we say that λ is higher than μ and write $\lambda > \mu$, if there exists i such that $\lambda_k = \mu_k$ for all $k < i$ and $\lambda_i > \mu_i$, regarding $\lambda_{q+1} = \lambda_{q+2} = \cdots = 0$ and $\mu_{r+1} = \mu_{r+2} = \cdots = 0$. $\lambda \geq \mu$ means $\lambda > \mu$ or $\lambda = \mu$.

Definition 2.2. Let E be a finitely generated free R-module and k a non-negative integer. We denote the kth exterior and symmetric modules of E by $\wedge^k E$ and $S_k E$, respectively. Furthermore, we define $\wedge E = \bigoplus_{k\geq 0} \wedge^k E$ and $SE = \bigoplus_{k\geq 0} S_k E$. Then they become Hopf algebras with the multiplications m and the comultiplications Δ. (m is defined as usual, and Δ is induced by the diagonalization. Note that $S_k E$ and $\wedge^k E$ are polynomial $GL(E)$-modules; moreover m and Δ are morphisms of $GL(E)$-modules.)

For a sequence of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_q)$, we denote by $\wedge_\alpha E$ and $S_\alpha E$ the tensor products $\wedge^{\alpha_1} E \otimes \cdots \otimes \wedge^{\alpha_q} E$ and $S_{\alpha_1} E \otimes \cdots \otimes S_{\alpha_q} E$, respectively. (Note that SE is a polynomial ring over R with $\operatorname{rank}(E)$ variables.)

For a partition $\lambda = (\lambda_1, \ldots, \lambda_q)$, let (a_{ij}) be a $(q \times \lambda_1)$-matrix over the ring of integers $\mathbf{Z}$ such that $a_{ij} = 1$ when $j \leq \lambda_i$ and $a_{ij} = 0$ when $j > \lambda_i$. Now consider the $GL(E)$-morphisms

$$\begin{array}{c}
\wedge_\lambda E = \wedge^{\lambda_1} E \otimes \cdots \otimes \wedge^{\lambda_q} E \\
\big\downarrow {\scriptstyle \Delta \otimes \cdots \otimes \Delta} \\
(\wedge^{a_{11}} E \otimes \cdots \otimes \wedge^{a_{1\lambda_1}} E) \otimes \cdots \otimes (\wedge^{a_{q1}} E \otimes \cdots \otimes \wedge^{a_{q\lambda_1}} E) \\
\big\downarrow \\
(S_{a_{11}} E \otimes \cdots \otimes S_{a_{1\lambda_1}} E) \otimes \cdots \otimes (S_{a_{q1}} E \otimes \cdots \otimes S_{a_{q\lambda_1}} E) \\
\big\downarrow \\
(S_{a_{11}} E \otimes \cdots \otimes S_{a_{q1}} E) \otimes \cdots \otimes (S_{a_{1\lambda_1}} E \otimes \cdots \otimes S_{a_{q\lambda_1}} E) \\
\big\downarrow {\scriptstyle m \otimes \cdots \otimes m} \\
S_{\tilde{\lambda}} E = S_{\tilde{\lambda}_1} E \otimes \cdots \otimes S_{\tilde{\lambda}_{\lambda_1}} E,
\end{array}$$

where the second map is induced by $\wedge^{a_{ij}} E = S_{a_{ij}} E$ (recall that $a_{ij} = 0$ or 1) and the third one is the permutation according to the index a_{ij}. Denote this composite map by $d_\lambda(E)$ or d_λ.

Definition 2.3 (Schur functors). Let $L_\lambda E$ be $\operatorname{Im}(d_\lambda(E))$. L_λ is called the Schur functor of the partition λ. (When R is a field of characteristic 0, $L_\lambda E$ is the irreducible polynomial $GL(E)$-module of degree $|\lambda|$ corresponding to the partition λ.)

Proposition 2.4 (Universal freeness of Schur functors). *For any R, E and λ, $L_\lambda E$ is a free R-module. If S is an R-algebra, then $(L_\lambda E) \otimes_R S = L_\lambda(E \otimes_R S)$.*

Definition 2.5. When s_1, s_2, and k are positive integers such that $k \leq s_2$, we have the $GL(E)$-morphisms

$$\wedge^{s_1+k} E \otimes \wedge^{s_2-k} E \xrightarrow{\Delta \otimes 1} \wedge^{s_1} E \otimes \wedge^k E \otimes \wedge^{s_2-k} E \xrightarrow{1 \otimes m} \wedge^{s_1} E \otimes \wedge^{s_2} E.$$

This composite map is denoted by $\square_k(E)$ or $\square_k$.

Similarly when $\alpha = (\alpha_1, \ldots, \alpha_q)$ is a sequence of positive integers, we define a $GL(E)$-morphism as

$$\sum_{t=1}^{q-1}\sum_{k=1}^{\alpha_{t+1}} \wedge^{\alpha_1} E \otimes \cdots \otimes \wedge^{\alpha_{t-1}} E \otimes \wedge^{\alpha_t + k} E \otimes \wedge^{\alpha_{t+1}-k} E \otimes \wedge^{\alpha_{t+2}} \otimes \cdots \otimes \wedge^{\alpha_q} E$$

$$\Big\downarrow \sum_{t=1}^{q-1}\sum_{k=1}^{\alpha_{t+1}} 1 \otimes \cdots \otimes 1 \otimes \square_k(E) \otimes 1 \otimes \cdots \otimes 1$$

$$\wedge_\alpha E = \wedge^{\alpha_1} E \otimes \cdots \otimes \wedge^{\alpha_q} E$$

and denote it by $\square_\alpha(E)$ or $\square_\alpha$.

Proposition 2.6 (Theorem II.2.16 in [1]). *For any partition λ, the sequence of* GL(E)*-morphisms*

$$0 \longrightarrow \mathrm{Im}(\square_\lambda(E)) \longrightarrow \wedge_\lambda E \xrightarrow{d_\lambda(E)} L_\lambda E \longrightarrow 0$$

is exact.

Definition 2.7. For a partition $\lambda = (\lambda_1, \dots, \lambda_q)$, we associate this partition with the set

$$B_\lambda = \{b_{11}, b_{12}, \dots, b_{1\lambda_1}, b_{21}, \dots, b_{2\lambda_2}, \dots, b_{q1}, \dots, b_{q\lambda_q}\},$$

which consists of $|\lambda|$ variables. Let n be a positive integer. $\mathrm{Tab}_\lambda\{1, \dots, n\}$ is defined to be the set of maps from B_λ to $\{1, \dots, n\}$, and each element of this set is called a *tableau.* Further, a tableau T is said to be standard if the following two conditions are satisfied:

(I) For any i and j such that $1 \le j < \lambda_i$, $T(b_{ij}) < T(b_{ij+1})$ holds.
(II) For any i and j such that $\lambda_{i+1} \ge j$, $T(b_{ij}) \le T(b_{i+1j})$ holds.

We denote by $\mathrm{St.Tab}_\lambda\{1, \dots, n\}$ the subset of $\mathrm{Tab}_\lambda\{1, \dots, n\}$ consists of all standard tableaux.

Moreover let $\{e_1, \dots, e_n\}$ be a free basis of E. For a tableau T contained in $\mathrm{Tab}_\lambda\{1, \dots, n\}$, e_T is defined to be an element of $\wedge_\lambda E$ as follows:

$$e_T = e_{T(b_{11})} \wedge \cdots \wedge e_{T(b_{1\lambda_1})} \otimes \cdots \otimes e_{T(b_{q1})} \wedge \cdots \wedge e_{T(b_{q\lambda_q})}$$

Proposition 2.8 (Theorem II.2.16 in [1]). *For any λ, E and R, the following set forms an R-free basis of $L_\lambda E$:*

$$\{d_\lambda(e_T) \mid T \in \mathrm{St.Tab}_\lambda\{1, \dots, n\}\}$$

Using these basic facts about the characteristic free representation theory of GL, we will introduce plethysm formulas in the next section.

3. Plethysm formulas

This section is devoted to introducing plethysm formulas for $S_r(\wedge^2 E)$, which play very important roles later. Essentially, these formulas are found in [4].

For a non-negative integer r, $S_r(\wedge^2 E)$ has a polynomial GL(E)-module structure. It is completely reducible when R is a field of characteristic zero. In fact by computing the character of $S_r(\wedge^2 E)$ as in [13], we know that it is decomposed as

$$S_r(\wedge^2 E) = \bigoplus_{\lambda \in \Gamma_r} L_\lambda E.$$

Here, Γ_r is a set of partitions which will be defined in Definition 3.1. Unfortunately, over an arbitrary commutative ring R, such decompositions do not exist in general. But $S_r(\wedge^2 E)$ has a natural filtration whose associated graded module coincides with $\bigoplus_{\lambda \in \Gamma_r} L_\lambda E$. We call such filtrations *plethysm formulas.*

Filtrations as above will be constructed in this section.

Throughout this section, R is a commutative ring with unity, and E a finitely generated free R-module of rank n.

Definition 3.1. For a non-negative integer r, Γ_r is defined as

$$\Gamma_r = \{\lambda\colon \text{partition } ||\lambda| = 2r; \text{ when } \lambda = (\lambda_1, \dots, \lambda_q), \text{ all } \lambda_i\text{'s are even}\}.$$

Definition 3.2. For a positive integer t, pf_{2t} is defined to be the map

$$pf_{2t}\colon \wedge^{2t} E \longrightarrow S_t(\wedge^2 E),$$

where, when $f_1, \dots, f_{2t}$ are elements of E, pf_{2t} sends $f_1 \wedge \cdots \wedge f_{2t}$ to the polynomial so-called *2t-order pfaffian*

$$\frac{1}{2^t t!} \sum_{\sigma \in \mathbf{S}_{2t}} (\operatorname{sgn} \sigma)(f_{\sigma(1)} \wedge f_{\sigma(2)})(f_{\sigma(3)} \wedge f_{\sigma(4)}) \cdots (f_{\sigma(2t-1)} \wedge f_{\sigma(2t)}),$$

where $\mathbf{S}_{2t}$ is the symmetric group on $\{1, \dots, 2t\}$ and the sum above runs over all permutations. (Note that each monomial appears just $2^t t!$ times with the same signature in the above sum.)

Moreover, for a partition $\lambda = (2r_1, \dots, 2r_q)$ contained in Γ_r, the following composite map is denoted by pf_λ;

$$\begin{array}{c} \wedge_\lambda E = \wedge^{2r_1} E \otimes \cdots \otimes \wedge^{2r_q} E \\ \Big\downarrow {\scriptstyle pf_{2r_1} \otimes \cdots \otimes pf_{2r_q}} \\ S_{r_1}(\wedge^2 E) \otimes \cdots \otimes S_{r_q}(\wedge^2 E) \\ \Big\downarrow {\scriptstyle m} \\ S_r(\wedge^2 E). \end{array}$$

(It is easily verified that pf_{2t} and pf_λ are well-defined and have the structures of $\mathrm{GL}(E)$-morphisms.)

Lemma 3.3. *Let h_i's and f be elements of E. Then, for positive integers c and d, we have*

(1) $\displaystyle\binom{c+d}{c} pf_{2(c+d)}(h_1 \wedge \cdots \wedge h_{2(c+d)})$

$$= \sum_{\sigma \in \mathbf{S}_{2(c+d)}^{[1,2c;2c+1,2(c+d)]}} (\operatorname{sgn} \sigma)\, pf_{2c}(h_{\sigma(1)} \wedge \cdots \wedge h_{\sigma(2c)}) \cdot pf_{2d}(h_{\sigma(2c+1)} \wedge \cdots \wedge h_{\sigma(2(2+d))}),$$

(2) $$pf_{2(c+1)}(h_1 \wedge \cdots \wedge h_{2c+1} \wedge f) = \sum_{i=1}^{2c+1} (-1)^{i+1} pf_{2c}(h_1 \wedge \cdots \wedge \hat{h}_i \wedge \cdots \wedge h_{2c+1}) \cdot pf_2(h_i \wedge f),$$

where $h_1 \wedge \cdots \wedge \hat{h}_i \wedge \cdots \wedge h_{2c+1}$ *means* $h_1 \wedge \cdots \wedge h_{i-1} \wedge h_{i+1} \wedge \cdots \wedge h_{2c+1}$, *and* $\binom{c+d}{c}$ *is the binomial coefficient. For integers satisfying* $1 \leq i < j < k < l \leq m$, $\mathbf{S}_m^{[i,j;k,l]}$ *is the subset of the symmetric group* $\mathbf{S}_m$ *defined by*

$$\mathbf{S}_m^{[i,j;k,l]} = \{\sigma \in \mathbf{S}_m | \sigma(i) < \sigma(i+1) < \cdots < \sigma(j),\ \sigma(k) < \sigma(k+1) < \cdots < \sigma(l)\}.$$

Proof. It is easy to see that we may assume that R is the complex number field $\mathbf{C}$. First, consider the diagram

$$\begin{tikzcd} \wedge^{2(c+d)} E \arrow[d, "\Delta"] \arrow[ddr, "pf_{2(c+d)}"] & \\ \wedge^{2c} E \otimes \wedge^{2d} E \arrow[d, "pf_{2c} \otimes pf_{2d}"] & \\ S_c(\wedge^2 E) \otimes S_d(\wedge^2 E) \arrow[r, "m"] & S_{c+d}(\wedge^2 E). \end{tikzcd}$$

Since $\wedge^{2(c+d)} E$ is irreducible and $S_{c+d}(\wedge^2 E) = \bigoplus_{\lambda \in \Gamma_{c+d}} L_\lambda E$, $\operatorname{Hom}_{GL(E)}(\wedge^{2(c+d)} E, S_{c+d}(\wedge^2 E)) \simeq \mathbf{C}$ by Schur's lemma. So, there exists a complex number z which satisfies

$$m \circ (pf_{2c} \otimes pf_{2d}) \circ \Delta = z \cdot pf_{2(c+d)}.$$

It is easily verified that z coincides with the binomial coefficient $\binom{c+d}{c}$.

Second, consider the diagram

$$\begin{tikzcd} \wedge^{2c+1} E \otimes E \arrow[r, "\Delta \otimes 1"] \arrow[dd, "m"] & \wedge^{2c} E \otimes E \otimes E \arrow[d, "pf_{2c} \otimes \wedge^2"] \\ & S_c(\wedge^2 E) \otimes \wedge^2 E \arrow[d, "m"] \\ \wedge^{2(c+1)} E \arrow[r, "pf_{2(c+1)}"] & S_{c+1}(\wedge^2 E). \end{tikzcd}$$

Since $\wedge^{2c+1} E \otimes E \simeq L_{(2(c+1))} E \oplus L_{(2c+1,1)} E$, by the Pieri formula [13], we have $\operatorname{Hom}_{GL(E)}(\wedge^{2c+1} E \otimes E, S_{c+1}(\wedge^2 E)) \simeq \mathbf{C}$ by Schur's lemma. So, the second equation will be proved in the same way. Q.E.D.

Definition 3.4. For each partition λ in Γ_r, GL(E)-submodules $\mathcal{M}_\lambda$ and $\mathring{\mathcal{M}}_\lambda$ of $S_r(\wedge^2 E)$ are defined as

$$\mathcal{M}_\lambda = \sum_{\substack{\mu \in \Gamma_r \\ \mu \geq \lambda}} \operatorname{Im}(pf_\mu), \quad \mathring{\mathcal{M}}_\lambda = \sum_{\substack{\mu \in \Gamma_r \\ \mu > \lambda}} \operatorname{Im}(pf_\mu)$$

where the sums run over the partitions such that the above conditions are satisfied.

(It is easy to see that pf_ν is surjective when ν is $(2, 2, \dots, 2)$ which is the lowest partition in Γ_r under the lexicographic order, because $pf_2\colon \wedge^2 E \to S_1(\wedge^2 E)$ is the isomorphism. So, we have $\mathscr{M}_\nu = S_r(\wedge^2 E)$. Therefore $\{\mathscr{M}_\lambda\}_{\lambda \in \Gamma_r}$ gives a natural filtration on $S_r(\wedge^2 E)$.)

Proposition 3.5 (Plethysm formulas). *For an arbitrary commutative ring R and an arbitrary non-negative integer r, $\{\mathscr{M}_\lambda\}_{\lambda \in \Gamma_r}$ is a natural filtration of $S_r(\wedge^2 E)$ whose associated graded object coincides with $\bigoplus_{\lambda \in \Gamma_r} L_\lambda E$.*

Proof. We have only to prove that for any partition λ in Γ_r, $\mathscr{M}_\lambda/\mathring{\mathscr{M}}_\lambda$ is isomorphic to $L_\lambda E$ as a $GL(E)$-module. Consider the diagram

$$\begin{array}{ccc} \wedge_\lambda E & \xrightarrow{pf_\lambda} & \mathscr{M}_\lambda \subset S_r(\wedge^2 E) \\ \downarrow{\scriptstyle d_\lambda} & \searrow^{\phi_\lambda} & \downarrow{\scriptstyle \rho_\lambda} \\ L_\lambda E & & \mathscr{M}_\lambda/\mathring{\mathscr{M}}_\lambda, \end{array}$$

where ρ_λ is the projection and ϕ_λ is the composite map $\rho_\lambda \circ pf_\lambda$. In order to construct the isomorphism $L_\lambda E \xrightarrow{\sim} \mathscr{M}_\lambda/\mathring{\mathscr{M}}_\lambda$, it is sufficient to show that $\operatorname{Ker}(d_\lambda) = \operatorname{Ker}(\phi_\lambda)$ since both d_λ and ϕ_λ are surjective.

First we will prove $\operatorname{Ker}(d_\lambda) \subset \operatorname{Ker}(\phi_\lambda)$. By Proposition 2.6, this is equivalent to show $\operatorname{Im}(\square_\lambda) \subset \operatorname{Ker}(\phi_\lambda)$ for any partition λ in Γ_r. In short it suffices to show that $\phi_\lambda \circ \square_\lambda = 0$ for each partition λ in Γ_r. By definitions of $\mathscr{M}_\lambda$ and $\square_\lambda$, we may assume that partition λ consists of two integers.

Suppose $\lambda = (2a, 2b)$, where $a + b = r$ and $a \geq b$. It suffices to prove the next claim.

CLAIM. *The composite map*

$$\wedge^{2a+k} E \otimes \wedge^{2b-k} E \xrightarrow{\square_k} \wedge^{2a} E \otimes \wedge^{2b} E \xrightarrow{pf_{(2a,2b)}} \mathscr{M}_{(2a,2b)} \xrightarrow{\rho_{(2a,2b)}} \mathscr{M}_{(2a,2b)}/\mathring{\mathscr{M}}_{(2a,2b)}$$

is the 0*-morphism for* $k = 1, 2, \dots, 2b$.

Our proof of this claim will be proceeded by the induction on $a + b$.

When $a + b = 1$, it is trivial. (If $b = 0$, we can not choose k satisfying $1 \leq k \leq 2b$.)

So, assume $a + b \geq 2$ and $b \geq 1$.

First, suppose $k < 2b$. For any f_i's and g_j's in E, we have

$$\begin{aligned} & pf_{(2a,2b)} \circ \square_k(f_1 \wedge \cdots \wedge f_{2a+k} \otimes g_{k+1} \wedge \cdots \wedge g_{2b}) \\ &= \sum_{\sigma \in S_{2a+k}^{[1,2a;2a+1,2a+k]}} (\operatorname{sgn}\sigma)\, pf_{2a}(f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(2a)}) \\ &\quad \cdot pf_{2b}(f_{\sigma(2a+1)} \wedge \cdots \wedge \widehat{f_{\sigma(2a+k)}} \wedge g_{k+1} \wedge \cdots \wedge g_{2b}) \\ &= \sum_{\sigma \in S_{2a+k}^{[1,2a;2a+1,2a+k]}} \sum_{l=1}^{k} (\operatorname{sgn}\sigma)(-1)^{l+1}\, pf_{2a}(f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(2a)}) \end{aligned}$$

$$\cdot pf_{2(b-1)}(f_{\sigma(2a+1)} \wedge \cdots \wedge \widehat{f_{\sigma(2a+l)}} \wedge \cdots \wedge f_{\sigma(2a+k)} \wedge g_{k+1} \wedge \cdots \wedge g_{2b-1})$$

$$\cdot(f_{\sigma(2a+l)} \wedge g_{2b}) \tag{1}$$

$$+ \sum_{\sigma \in S_{2a+k}^{[1,2a;2a+1,2a+k]}} \sum_{t=k+1}^{2b-1} (\operatorname{sgn} \sigma)(-1)^{t+1} pf_{2a}(f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(2a)})$$

$$\cdot pf_{2(b-1)}(f_{\sigma(2a+1)} \wedge \cdots \wedge f_{\sigma(2a+k)} \wedge g_{k+1} \wedge \cdots \wedge \hat{g}_t \wedge \cdots \wedge g_{2b-1})$$

$$\cdot(g_t \wedge g_{2b}), \tag{2}$$

where $S_{2a+k}^{[1,2a;2a+1,2a+k]}$ is a set of permutations defined in Lemma 3.3.

By the inductive assumption on $a+b$,

$$\sum_{\sigma \in S_{2a+k}^{[1,2a;2a+1,2a+k]}} (\operatorname{sgn} \sigma) pf_{2a}(f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(2a)})$$

$$\cdot pf_{2(b-1)}(f_{\sigma(2a+1)} \wedge \cdots \wedge f_{\sigma(2a+k)} \wedge g_{k+1} \wedge \cdots \wedge \hat{g}_t \wedge \cdots \wedge g_{2b-1})$$

is contained in $\mathring{\mathcal{M}}_{(2a,2(b-1))}$ for each t. So, it is easy to see that the term (2) is contained in $\mathring{\mathcal{M}}_{(2a,2b)}$.

On the other hand, (1) is arranged as

$$\sum_{\sigma \in S_{2a+k}^{[1,2a;2a+1,2a+k-1]}} (\operatorname{sgn} \sigma)(-1)^{k+1} pf_{2a}(f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(2a)})$$

$$\cdot pf_{2(b-1)}(f_{\sigma(2a+1)} \wedge \cdots \wedge f_{\sigma(2a+k-1)} \wedge g_{k+1} \wedge \cdots \wedge g_{2b-1}) \cdot (f_{\sigma(2a+k)} \wedge g_{2b})$$

$$= \sum_{t=1}^{2a+k} (-1)^{t+1} \left(\sum_{\tau \in S_{2a+k-1}^{[1,2a;2a+1,2a+k-1]}} (\operatorname{sgn} \sigma) pf_{2a}(f^t_{\tau(1)} \wedge \cdots \wedge f^t_{\tau(2a)}) \right.$$

$$\left. \cdot pf_{2(b-1)}(f^t_{\tau(2a+1)} \wedge \cdots \wedge f^t_{\tau(2a+k-1)} \wedge g_{k+1} \wedge \cdots \wedge g_{2b-1}) \right) \cdot (f_t \wedge g_{2b}),$$

where we define $f^t_1 = f_1, \ldots, f^t_{t-1} = f_{t-1}$, $f^t_t = f_{t+1}, \ldots, f^t_{2a+k-1} = f_{2a+k}$ for each t. So, by induction, it is easily verified that the term (1) is also contained in $\mathring{\mathcal{M}}_{(2a,2b)}$. (When $k=1$, the term (1) is obviously contained in $\mathring{\mathcal{M}}_{(2a,2b)}$ from the second formula in Lemma 3.3.)

Next assume $k = 2b$. Then for any f_i's in E, we have

$$pf_{(2a,2b)} \circ \square_{2b}(f_1 \wedge \cdots \wedge f_{2(a+b)})$$

$$= \sum_{\sigma \in S_{2(a+b)}^{[1,2a;2a+1,2(a+b)]}} (\operatorname{sgn} \sigma) pf_{2a}(f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(2a)}) \cdot pf_{2b}(f_{\sigma(2a+1)} \wedge \cdots \wedge f_{\sigma(2(a+b))})$$

$$= \binom{a+b}{a} pf_{2(a+b)}(f_1 \wedge \cdots \wedge f_{2(a+b)}),$$

by the first formula in Lemma 3.3. We have completed the proof of the claim.

By the claim above, for each partition λ in Γ_r, there exists a natural surjective map $\iota_\lambda \colon L_\lambda E \to \mathcal{M}_\lambda / \mathring{\mathcal{M}}_\lambda$ which makes the following diagram commutative;

$$\begin{tikzcd} \wedge_\lambda E \arrow[r, "pf_\lambda"] \arrow[d, "d_\lambda"] \arrow[dr, "\phi_\lambda"] & \mathscr{M}_\lambda \subset S_r(\wedge^2 E) \arrow[d, "\rho_\lambda"] \\ L_\lambda E \arrow[r, "\iota_\lambda"] & \mathscr{M}_\lambda/\mathring{\mathscr{M}}_\lambda. \end{tikzcd}$$

Since $\bigoplus_{\lambda\in\Gamma_r} L_\lambda E$ and $S_r(\wedge^2 E)$ are free R-modules with the same rank over an arbitrary commutative ring R (see Propositions 2.4 and 2.8 and note the formula $S_r(\wedge^2 E) \simeq \bigoplus_{\lambda\in\Gamma_r} L_\lambda E$ over the complex number field $\mathbf{C}$), ι_λ's must be injective for all λ in Γ_r. Q.E.D.

4. Gröbner bases of pfaffian ideals

Throughout this section R is a field of arbitrary characteritic and E is an R-vector space of dimension n. Let $\{e_1, \dots, e_n\}$ be a basis of E and t an integer such that $1 \le 2t \le n$. We denote by Pf_{2t} the ideal of $S(\wedge^2 E)$ generated by $pf_{2t}(\wedge^{2t} E)$ and call it the *pfaffian ideal* of order $2t$.

It is easily verified that

$$\{pf_{2t}(e_{i_1} \wedge \cdots \wedge e_{i_{2t}}) | 1 \le i_1 < \cdots < i_{2t} \le n\}$$

is a homogeneous minimal generating system of the ideal Pf_{2t}. In this section, by using the Knuth correspondence [10], we will show that this homogeneous minimal generating set forms a Gröbner basis of Pf_{2t}. By the general theory of Gröbner bases, the first syzygy module of Pf_{2t} is proved to be generated by relations of degree at most t (see Theorem 5.3).

First, we review some basic facts about Gröbner bases.

Let $A = R[x_1, \dots, x_q]$ be the polynomial ring over a field R with variables $x_1, \dots, x_q$ and let $\mathbf{M}$ be the set of monomials of A, i.e.,

$$\mathbf{M} = \{x_1^{\alpha_1} \cdots x_q^{\alpha_q} | \alpha_i\text{'s are non-negative integers}\}.$$

Definition 4.1. $\mathbf{M}$ has a structure of a totally ordered set, i.e., for two monomials $M = x_1^{\alpha_1} \cdots x_q^{\alpha_q}$ and $N = x_1^{\beta_1} \cdots x_q^{\beta_q}$, we say that M is higher than N and write $M > N$ if

$$(\sum_{i=1}^{q} \alpha_i, \alpha_q, \dots, \alpha_1) \succ (\sum_{i=1}^{q} \beta_i, \beta_q, \dots, \beta_1),$$

where $\succ$ is the usual lexicographic order. $M \ge N$ means $M = N$ or $M > N$. This order is sometimes called the *reverse lexicographic order.*

Definition 4.2. For each non-zero polynomial f in A, a monomial M in $\mathbf{M}$ is called the *highest term* of f and denoted by $\mathrm{Hterm}(f)$ if M is the highest in the set of monomials which appear in f with non-zero coefficients. Moreover, the coefficient of M in f is denoted by $\mathrm{Hcoeff}(f)$.

Remark 4.3. For monomials L, M, N satisfying $M \ge N$, $ML \ge NL$

holds. So, for two non-zero polynomials f and g, we have $\mathrm{Hterm}(fg) = \mathrm{Hterm}(f)$ $\mathrm{Hterm}(g)$.

Definition 4.4. A set of non-zero polynomials $\{f_1,\dots,f_r\}$ is called a *Gröbner basis* of the polynomial ideal $(f_1,\dots,f_r)$ if, for any non-zero polynomial g in $(f_1,\dots,f_r)$, $\mathrm{Hterm}(g)$ can be divided by $\mathrm{Hterm}(f_i)$ for some i.

It is easily verified that any non-zero polynomial ideal have a Gröbner basis and any minimal Gröbner basis consists of a fixed number (determined by the given ideal) of elements.

By the general theory on Gröbner bases, we can find a generating set of the first syzygy module as follows.

Definition 4.5. For two non-zero polynomials f and g, we define

$$S(f, g) = \mathrm{Hcoeff}(g)\frac{\mathrm{l.c.m.}(\mathrm{Hterm}(f), \mathrm{Hterm}(g))}{\mathrm{Hterm}(f)}f - \mathrm{Hcoeff}(f)\frac{\mathrm{l.c.m.}(\mathrm{Hterm}(f), \mathrm{Hterm}(g))}{\mathrm{Hterm}(g)}g,$$

where l.c.m.(,) means the least common multiple.

Let $\{f_1,\dots,f_r\}$ be a Gröbner basis of $I = (f_1,\dots,f_r)$. For i and j such that $1 \leq i < j \leq n$, we can describe $S(f_i, f_j)$ in the form

$$S(f_i, f_j) = \sum_{k=1}^{r} g_k f_k.$$

Since $\{f_1,\dots,f_r\}$ is a Gröbner basis, we can choose each g_k which satisfies

$$\mathrm{Hterm}(g_k f_k) = \mathrm{Hterm}(g_k)\mathrm{Hterm}(f_k) \leq \mathrm{Hterm}(S(f_i, f_j)).$$

Let L be a finitely generated free A-module of rank r, and let $\{l_1,\dots,l_r\}$ be a basis of L. We have a exact sequence

$$L \xrightarrow{\delta} A \longrightarrow A/I \longrightarrow 0,$$

where $\delta(l_i) = f_i$ for $i = 1,\dots,r$. Then $\mathrm{Ker}(\delta)$ is the first syzygy module of I.

Definition 4.6. For i and j such that $1 \leq i < j \leq r$, we define

$$R(i, j) = \mathrm{Hcoeff}(f_j)\frac{\mathrm{l.c.m.}(\mathrm{Hterm}(f_i), \mathrm{Hterm}(f_j))}{\mathrm{Hterm}(f_i)}l_i - \mathrm{Hcoeff}(f_i)\frac{\mathrm{l.c.m.}(\mathrm{Hterm}(f_i), \mathrm{Hterm}(f_j))}{\mathrm{Hterm}(f_j)}l_j - \sum_{k=1}^{r} g_k l_k.$$

By definition, $R(i, j)$ is contained in $\mathrm{Ker}(\delta)$.

Proposition 4.7 ([2]). $\mathrm{Ker}(\delta)$ *is generated by*

$$\{R(i, j) \mid 1 \leq i < j \leq r\}.$$

Now we start ot prove that a homogeneous minimal generating set of the pfaffian ideal Pf_{2t} is a Gröbner basis.

First, we introduce an order to the set of monomials of $S(\wedge^2 E)$.

When $\{e_1, \ldots, e_n\}$ is a fixed basis of E, let $x_1 = e_1 \wedge e_2$, $x_2 = e_2 \wedge e_3, \ldots, x_{n-1} = e_{n-1} \wedge e_n$, $x_n = e_1 \wedge e_3$, $x_{n+1} = e_2 \wedge e_4, \ldots, x_{n(n-1)/2} = e_1 \wedge e_n$. Identify $S(\wedge^2 E)$ with $A = R[x_1, \ldots, x_{n(n-1)/2}]$, and introduce the reverse lexicographic order to the set of monomials $\mathbf{M}$ of A as in Definition 4.1.

Definition 4.8. Define the monomial ideals Σ and Σ' as

$$\Sigma = (\mathrm{Hterm}(g) \mid 0 \neq g \in Pf_{2t}),$$
$$\Sigma' = (\mathrm{Hterm}(pf_{2t}(e_{i_1} \wedge \cdots \wedge e_{i_{2t}})) \mid 1 \leq i_1 < \cdots < i_{2t} \leq n).$$

It is easy to see that for any monomial M contained in Σ, there exists a homogeneous element g in Pf_{2t} such that $\mathrm{Hterm}(g) = M$.

For a homogeneous ideal J, we denote by $(J)_r$ the rth homogeneous component of J.

We have to note that every x_k's and every $e_i \wedge e_j$'s are of degree 1.

Lemma 4.9. *For each positive integer r,*

$$\dim_R(\Sigma')_r \leq \dim_R(\Sigma)_r = \dim_R(Pf_{2t})_r.$$

Moreover, $\{pf_{2t}(e_{i_1} \wedge \cdots \wedge e_{i_{2t}}) \mid 1 \leq i_1 < \cdots < i_{2t} \leq n\}$ is a Gröbner basis of the pfaffian ideal Pf_{2t} if and only if $\dim_R(\Sigma')_r = \dim_R(\Sigma)_r$ holds for any r.

Proof. We can find monomials $M_1, \ldots, M_q$ such that $\{M_1, \ldots, M_q\}$ is an R-basis of $(\Sigma)_r$. Then there exist $g_1, \ldots, g_q$ in $(Pf_{2t})_r$ satisfying $\mathrm{Hterm}(g_i) = M_i$ for $i = 1, \ldots, q$. It is easy to see that $\{g_1, \ldots, g_q\}$ is an R-basis of $(Pf_{2t})_q$.

The second assertion is obvious. Q.E.D.

We will prove that $\dim_R(\Sigma')_r = \dim_R(Pf_{2t})_r$ for any r. We may assume $r \geq t$, because $(\Sigma)_r = (Pf_{2t})_r = 0$ if $r < t$. By the plethysm formula (Proposition 3.5), we have $(Pf_{2t})_r = \mathcal{M}_{\lambda'}$, where $\lambda' = (2t, 2, \ldots, 2)$ in Γ_r. Since

$$\mathcal{M}_{\lambda'} = \bigoplus_{\substack{\mu \in \Gamma_r \\ \mu \geq \lambda'}} L_\mu E$$

up to filtration, we have

$$\dim_R(Pf_{2t})_r = \sum_{\substack{\mu \in \Gamma_r \\ \mu \geq \lambda'}} {}^{\#}\{\mathrm{St.Tab}_\mu\{1, \ldots, n\}\}$$

by Proposition 2.8. (${}^{\#}\{\ \ \}$ means the cardinary of the given set.)

So, we have only to show that

$$\dim_R(\Sigma')_r = \sum_{\substack{\mu\in\Gamma_r \\ \mu\geq\lambda'}} {}^{\#}\{\mathrm{St.Tab}_\mu\{1,\dots,n\}\}$$

for $r \geq t$.

Definition 4.10. For each positive integer s, define subsets of n by n matrices

$$\mathrm{Mat}(s) = \{(a_{ij}) \mid a_{ij}\text{'s are non-negative integers such that } \Sigma_{i,j}a_{ij} = s\},$$
$$\mathrm{S.Mat}(s) = \{(a_{ij})\in \mathrm{Mat}(s) \mid a_{ij} = a_{ji} \text{ for } i \neq j, \text{ and } a_{ii} = 0 \text{ for } i = 1,\dots,n\}.$$

Definition 4.11. A 2 by u matrix

$$\begin{pmatrix} b_1 \cdots b_u \\ c_1 \cdots c_u \end{pmatrix}$$

is said a *two-line array* of weight u if the following conditions are satisfied;

- b_i's and c_j's are positive integers less than or equal to n,
- $1 \leq b_1 \leq \cdots \leq b_u \leq n$,
- if $b_i = b_{i+1}$, then $c_i \leq c_{i+1}$.

The set of two-line arrays of weight u is denoted by $\mathrm{TLA}(u)$.

Let E_{ij} be the elementary n by n matrix whose (i,j) entry is 1, and others are 0. For a two-line array $T = \begin{pmatrix} b_1 \cdots b_v \\ c_1 \cdots c_v \end{pmatrix}$ of weight v, $\psi_v(T) = \sum_{i=1}^{v} E_{b_ic_i}$ gives a bijection $\psi_v\colon \mathrm{TLA}(v) \to \mathrm{Mat}(v)$.

For a monomial $M = (e_{i_1} \wedge e_{j_1})\cdots(e_{i_u} \wedge e_{j_u})$ in $\mathbf{M}$, $\xi_u(M) = \sum_{k=1}^{u}(E_{i_kj_k} + E_{j_ki_k})$ gives a injection $\xi_u\colon \mathbf{M}_u \to \mathrm{Mat}(2u)$, where $\mathbf{M}_u$ is the set of monomials of degree u. It is easy to see that $\xi_u(\mathbf{M}_u) = \mathrm{S.Mat}(2u)$.

Definition 4.12. For a two-line array $T = \begin{pmatrix} b_1 \cdots b_u \\ c_1 \cdots c_u \end{pmatrix}$, the length of the longest strictly decreasing subsequences of $c_1 \cdots c_u$ is denoted by $l(T)$.

Proposition 4.13 (Knuth correspondence, Theorem 4 in [10], [16]). *For each positive integer r, there exist bijections κ_{2r} and κ'_{2r} which make the following diagram commutative;*

$$\begin{tikzcd}
\mathrm{TLA}(2r) \arrow[r, "\kappa_{2r}"] & \coprod\limits_{\substack{\lambda:\text{partition}\\ |\lambda|=2r}} \mathrm{St.Tab}_\lambda\{1,\dots,n\} \times \mathrm{St.Tab}_\lambda\{1,\dots,n\} \\
\mathrm{Mat}(2r) \arrow[u, "\psi_{2r}^{-1}"'] & \\
\mathrm{S.Mat}(2r) \arrow[u, "\iota"'] & \coprod\limits_{\substack{\lambda:\text{partition}\\ |\lambda|=2r}} \mathrm{St.Tab}_\lambda\{1,\dots,n\} \arrow[uu, "d"'] \\
\mathbf{M}_r \arrow[u, "\xi_{2r}"'] \arrow[r, "\kappa'_{2r}"] & \coprod\limits_{\lambda\in\Gamma_r} \mathrm{St.Tab}_\lambda\{1,\dots,n\}, \arrow[u, "\varsigma"']
\end{tikzcd}$$

where ι and ς are inclusion, and d is the diagonal map.

Furthermore, for each two-line array T in TLA$(2r)$, *if $\kappa_{2r}(T)$ is contained in* $\mathrm{St.Tab}_\lambda\{1,\ldots,n\} \times \mathrm{St.Tab}_\lambda\{1,\ldots,n\}$ *for some partition $\lambda = (\lambda_1, \lambda_2,\ldots)$, then $l(T)$ coincides with λ_1.*

By using this result, we can prove

Theorem 4.14. *For any positive integer t such that $1 < 2t \leq n$,*

$$\{pf_{2t}(e_{i_1} \wedge \cdots \wedge e_{i_{2t}}) | 1 \leq i_1 < \cdots < i_{2t} \leq n\}$$

forms a Gröbner basis of the pfaffian ideal Pf_{2t}.

Proof. It suffices to show that

$$\dim_R(\Sigma')_r = \sum_{\substack{\mu \in \Gamma_r \\ \mu \geq \lambda'}} {}^{\#}\{\mathrm{St.Tab}_\mu\{1,\ldots,n\}\}$$

for any positive integer r not less that t.

For a $2t$-order pfaffian $pf_{2t}(e_{i_1} \wedge \cdots \wedge e_{i_{2t}})(1 \leq i_1 < \cdots < i_{2t} \leq n)$, it is easy to see that

$$\mathrm{Hterm}\,(pf_{2t}(e_{i_1} \wedge \cdots \wedge e_{i_{2t}})) = (e_{i_1} \wedge e_{i_{2t}}) \cdot (e_{i_2} \wedge e_{i_{2t-1}}) \cdots (e_{i_t} \wedge e_{i_{t+1}}).$$

So, we have

$$\Sigma' = ((e_{i_1} \wedge e_{i_{2t}}) \cdot (e_{i_2} \wedge e_{i_{2t-1}}) \cdots (e_{i_t} \wedge e_{i_{t+1}}) | 1 \leq i_1 < \cdots < i_{2t} \leq n).$$

Let M be a monomial contained in $\mathbf{M}_r$. It is easy to check that $l(\psi_{2r}^{-1} \circ \iota \circ \xi_{2r}(M)) \geq 2t$ if and only if M is contained in $(\Sigma')_r$. Hence, the assertion is clear by the Knuth correspondence. Q.E.D.

Remark 4.15. Let $A = R[x_1,\ldots,x_q]$ be a polynomial ring over a field R and $\mathbf{M}$ (resp. $\mathbf{M}_r$) the set of monomials (resp. monomials of degree r). Suppose that $\{f_1,\ldots,f_l\}$ is a Gröbner basis of a polynomial ideal I, and $J = (\mathrm{Hterm}(f_i) | i = 1,\ldots,l)$. By the general theory of Gröbner basis ([2]) we can construct a finite free resolution of A/I which is isomorphic to Taylor's resolution ([17]) of A/J up to some filtration. So there exist spectral sequences

$$\{E_M^1 = \mathrm{Tor}_i^A(A/J, A/(x_1,\ldots,x_q))_M\}_{M \in \mathbf{M}} \Longrightarrow \mathrm{Tor}_i^A(A/I, A/(x_1,\ldots,x_q))$$

for every i. (Note that A/J and $A/(x_1,\ldots,x_q)$ are $\mathbf{M}$-graded, so is $\mathrm{Tor}_i^A(A/J, A/(x_1,\ldots,x_q))$.)

Furthermore, when I is homogeneous, there exist

$$\{E_M^1 = \mathrm{Tor}_i^A(A/J, A/(x_1,\ldots,x_q))_M\}_{M \in \mathbf{M}_r} \Longrightarrow \mathrm{Tor}_i^A(A/I, A/(x_1,\ldots,x_q))_r$$

for every i and r.

We can describe E_M^1 in terms of cohomology groups of some simplicial complex using the same method as in the proof of Hochster's formula ([6]).

So the system of above spectral sequences seems to be a natural extension of Hochster's formula.

5. Main theorem

In this section R is a prime field and E is an R-vector space with basis $e_1, \dots, e_n$.

Suppose that t is a positive integer such that $2t \le n$.

Definition 5.1. Denote by M_t the composite map

$$\wedge^{2t} E \otimes S(\wedge^2 E) \xrightarrow{pf_{2t} \otimes 1} S_t(\wedge^2 E) \otimes S(\wedge^2 E) \xrightarrow{\quad m \quad} S(\wedge^2 E),$$

and for a non-negative integer r, define $M_{t,r}$ to be the composite map

$$\wedge^{2t} E \otimes S_r(\wedge^2 E) \xrightarrow{pf_{2t} \otimes 1} S_t(\wedge^2 E) \otimes S_r(\wedge^2 E) \xrightarrow{\quad m \quad} S_{t+r}(\wedge E).$$

$\mathrm{Ker}(M_t)$ is usually called the first syzygy, or the relation module of $2t$-order pfaffians.

By definitions, $\mathrm{Ker}(M_t) = \bigoplus_{r \ge 1} \mathrm{Ker}(M_{t,r})$. (It is easy to see that $\mathrm{Ker}(M_{t,0}) = 0$ because of the linear independence of $2t$-order pfaffians.) Each element of $\mathrm{Ker}(M_{t,r})$ is called a *relation of degree* r on $2t$-order pfaffians.

Definition 5.2. For a positive integer r more than or equal to t, $\Gamma_{r,t}$ is defined as

$$\Gamma_{r,t} = \{\lambda \in \Gamma_r \mid \lambda \ge (2t)\},$$

where $\ge$ means the lexicographic order.

Our main purpose is to prove the following result.

Theorem 5.3. 1. *The first syzygy of the pfaffian ideal Pf_{2t} is generated over* $S(\wedge^2 E)$ *by relations of degree at most t, i.e.,*

$$\mathrm{Ker}(M_t) = S(\wedge^2 E) \cdot \left(\sum_{r=1}^{t} \mathrm{Ker}(M_{t,r})\right).$$

2. *Regard the rational number field as of characteristic infinity. Then the first syzygy of Pf_{2t} is generated over* $S(\wedge^2 E)$ *by relations of degree* 1 *when* $2p > n - 2t$. *(p is the characteristic of R.)*

Before proving this, we define some more notation.

Definition 5.4. For non-negative integers $r \ge t$, and for a partition λ in $\Gamma_{r,t}$, $\delta_{r,t}$ and $\delta_{\lambda,t}$ are defined as follows:

1. $\delta_{r,t}$: $\wedge^{2r} E \to \wedge^{2t} E \otimes S_{r-t}(\wedge^2 E)$ is defined inductively.

   Let $\delta_{t,t} = \mathrm{id}$ and $\delta_{r,t}$ is determined by

$$\delta_{r,t}(e_{i_1}\wedge\cdots\wedge e_{i_{2r}})=\sum_{l=1}^{2r-1}(-1)^{l+1}\delta_{r-1,t}(e_{i_1}\wedge\cdots\wedge\widehat{e_{i_l}}\wedge\cdots\wedge e_{i_{2r-1}})\cdot(e_{i_l}\wedge e_{i_{2r}}),$$

where $1\le i_1<\cdots<i_{2r}\le n$.

2. Suppose $\lambda=(2r_1,\dots,2r_q)$ is a partition such that $r_1\ge t$, $r_1+\cdots+r_q=r$. Denote by $\delta_{\lambda,t}$ the composite map

$$\begin{array}{c}\wedge_\lambda E=\wedge^{2r_1}E\otimes\cdots\otimes\wedge^{2r_q}E\\ \Big\downarrow{\scriptstyle \delta_{r_1,t}\otimes pf_{2r_2}\otimes\cdots\otimes pf_{2r_q}}\\ \wedge^{2t}E\otimes S_{r_1-t}(\wedge^2E)\otimes S_{r_2}(\wedge^2E)\otimes\cdots\otimes S_{r_q}(\wedge^2E)\\ \Big\downarrow{\scriptstyle 1\otimes m}\\ \wedge^{2t}E\ \otimes S_{r-t}(\wedge^2E).\end{array}$$

($\delta_{r,t}$ and $\delta_{\lambda,t}$ depend on a choice of basis elements of E. So, they are not $GL(E)$-morphisms.)

It is easy to check that $M_{t,r-t}\circ\delta_{r,t}=pf_{2r}$ and $M_{t,r-t}\circ\delta_{\lambda,t}=pf_\lambda$ (see the second formula in Lemma 3.3).

Lemma 5.5. *Let a and k be integers such that $a\ge t$ and $1\le k\le 2(a+1)$. Then for any increasing sequence $1\le i_1<\cdots<i_{2(a+1)}\le n$,*

$$\begin{aligned}&\delta_{a+1,t}(e_{i_1}\wedge\cdots\wedge e_{i_{2(a+1)}})\\&-\sum_{j=1}^{k-1}(-1)^{k+j+1}\delta_{a,t}(e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge e_{i_{2(a+1)}})\cdot(e_{i_j}\wedge e_{i_k})\\&-\sum_{j=k+1}^{2(a+1)}(-1)^{k+j}\delta_{a,t}(e_{i_1}\wedge\cdots\wedge\widehat{e_{i_k}}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_{2(a+1)}})\cdot(e_{i_j}\wedge e_{i_k})\end{aligned}$$

is contained in $S_{a-t}(\wedge^2E)\cdot\mathrm{Ker}(M_{t,1})$.

Lemma 5.6. *Let a be an integer more than or equal to t. Then for any increasing sequence $1\le i_1<\cdots<i_{2a+1}\le n$ and any integer k such that $1\le k\le 2a+1$,*

$$\sum_{l=1}^{2a+1}(-1)^{l+1}\delta_{a,t}(e_{i_1}\wedge\cdots\wedge\widehat{e_{i_l}}\wedge\cdots\wedge e_{i_{2a+1}})\cdot(e_{i_l}\wedge e_{i_k})$$

is contained in $S_{a-t}(\wedge^2E)\cdot\mathrm{Ker}(M_{t,1})$.

Lemma 5.7. *Let a be an integer more than or equal to t. Then for any increasing sequence $1\le i_1<\cdots<i_{2(a+1)}\le n$,*

$$\begin{aligned}&(a+1)\cdot\delta_{a+1,t}(e_{i_1}\wedge\cdots\wedge e_{i_{2(a+1)}})\\&-\sum_{\sigma\in S_{2(a+1)}^{[1,2a;2a+1,2(a+1)]}}(\operatorname{sgn}\sigma)\,\delta_{a,t}(e_{i_{\sigma(1)}}\wedge\cdots\wedge e_{i_{\sigma(2a)}})\cdot(e_{i_{\sigma(2a+1)}}\wedge e_{i_{\sigma(2(a+1))}})\end{aligned}$$

is contained in $S_{a-t}(\wedge^2E)\cdot\mathrm{Ker}(M_{t,1})$.

Direct computations and induction on a give these lemmas. So we omit the proofs.

Proof of Theorem 5.3. From Theorem 4.14,

$$\{pf_{2t}(e_{i_1} \wedge \cdots \wedge e_{i_{2t}}) | 1 \le i_1 < \cdots < i_{2t} \le n\}$$

is a Gröbner basis of the pfaffian ideal Pf_{2t}. So, by Proposition 4.7, we have a system of generators of $\mathrm{Ker}(M_t)$. Since all $2t$-order pfaffians are homogeneous of degree t, all $R(i, j)$'s are relations of degree at most t. Therefore, the first assertion is clear.

Now we prove the second assertion. Let $\mathbf{X}$ be a given homogeneous relation of degree s. We shall prove that $\mathbf{X}$ is contained in $\mathrm{S}_{s-1}(\wedge^2 E)\mathrm{Ker}(M_{t,1})$. By the first assertion, we may assume $s \le t$.

Let $\lambda' = (2t, 2, \ldots, 2)$ in $\Gamma_{s+t,t}$. Obviously, $\delta_{\lambda',t}$ is surjective. So, there exists C in $\wedge_{\lambda'} E$ which satisfies

$$\delta_{\lambda',t}(C) = \mathbf{X}.$$

By adding an appropriate element of $\mathrm{S}_{s-1}(\wedge^2 E)\cdot\mathrm{Ker}(M_{t,1})$ to $\mathbf{X}$, we wish to reduce $\mathbf{X}$ to higher partitions, i.e., whenever we have a equation

$$\sum_{\alpha \in \Gamma_{s+t,t}} \delta_{\alpha,t}(T_\alpha) \equiv \mathbf{X} \text{ modulo } \mathrm{S}_{s-1}(\wedge^2 E)\cdot\mathrm{Ker}(M_{t,1}),$$

we shall show that we can always find a new equation

$$\sum_{\beta \in \Gamma_{s+t,t}} \delta_{\beta,t}(T'_\beta) \equiv \mathbf{X} \text{ modulo } \mathrm{S}_{s-1}(\wedge^2 E)\cdot\mathrm{Ker}(M_{t,1})$$

such that the lowest partition β for which $T'_\beta \ne 0$ is strictly higher than the lowest partition α for which $T_\alpha \ne 0$ under the lexicographic order. (Note that $\delta_{\lambda',t}(C) = \mathbf{X}$ itself is one of such equations.)

CASE I. Assume $\delta_{\nu,t}(T_\nu) \equiv \mathbf{X}$ modulo $\mathrm{S}_{s-1}(\wedge^2 E)\cdot\mathrm{Ker}(M_{t,1})$, where $\nu = (2(s+t))$ is the highest partition in $\Gamma_{s+t,t}$. Since $\mathbf{X}$ and all elements of $\mathrm{S}_{s-1}(\wedge^2 E)\cdot\mathrm{Ker}(M_{t,1})$ are contained in $\mathrm{Ker}(M_{t,s})$, we obtain

$$M_{t,s} \circ \delta_{\nu,t}(T_\nu) = pf_\nu(T_\nu) = 0.$$

Because $pf_\nu = pf_{2(s+t)}\colon \wedge^{2(s+t)} E \to \mathrm{S}_{s+t}(\wedge^2 E)$ is injective (see Proposition 3.5), T_ν must be equal to 0. So, $\mathbf{X}$ is contained in $\mathrm{S}_{s-1}(\wedge^2 E)\cdot\mathrm{Ker}(M_{t,1})$.

CASE II. Assume that there exist partitions $\lambda_1 > \cdots > \lambda_r$ in $\Gamma_{s+t,t}$ and T_{λ_k} in $\wedge_{\lambda_k} E$ such that

$$\sum_{k=1}^{r} \delta_{\lambda_k,t}(T_{\lambda_k}) \equiv \mathbf{X} \text{ modulo } \mathrm{S}_{s-1}(\wedge^2 E)\cdot\mathrm{Ker}(M_{t,1})$$

with $T_{\lambda_r} \ne 0$, where $(2(s+t)) > \lambda_r$. Since $M_{t,s}(\mathbf{X}) = 0$, we have

$$M_{t,s}(\sum_{k=1}^{r} \delta_{\lambda_k,t}(T_{\lambda_r})) = \sum_{k=1}^{r} pf_{\lambda_r}(T_{\lambda_r}) = 0.$$

By the definitions of $\mathscr{M}_{\lambda_k}$ and $\mathring{\mathscr{M}}_{\lambda_k}$ (see Definition 3.4), $pf_{\lambda_r}(T_{\lambda_r})$ is contained in $\mathring{\mathscr{M}}_{\lambda_r}$. From the commutative diagram

$$\begin{array}{ccc} \wedge_\lambda E & \xrightarrow{pf_\lambda} & \mathscr{M}_\lambda \subset S_r(\wedge^2 E) \\ \downarrow{\scriptstyle d_\lambda} & & \downarrow{\scriptstyle \rho_\lambda} \\ L_\lambda E & \xrightarrow{\iota_\lambda} & \mathscr{M}_\lambda/\mathring{\mathscr{M}}_\lambda, \end{array}$$

T_{λ_r} is contained in $\mathrm{Ker}(d_{\lambda_r}) = \mathrm{Im}(\square_{\lambda_r})$.

Therefore we have only to show

CLAIM. *Let $\mu \neq (2(s+t))$ be a partition contained in $\Gamma_{s+t,t}$. If $C \in \wedge_\mu E$ is contained in $\mathrm{Im}(\square_\mu)$, then*

$$\delta_{\mu,t}(C) \in \sum_{\substack{\xi \in \Gamma_{s+t,t} \\ \xi > \mu}} \mathrm{Im}(\delta_{\xi,t}) + S_{s-1}(\wedge^2 E)\cdot \mathrm{Ker}(M_{t,1}).$$

This claim is proved by induction on s.

First suppose $s = 1$. Then μ must be $(2t, 2)$. Let μ' be $(2(t+1))$. Since C is in $\mathrm{Im}(\square_\mu)$, $pf_\mu(C)$ is contained in $\mathring{\mathscr{M}}_\mu$. So there exists $D \in \wedge_{\mu'} E$ such that

$$pf_\mu(C) = pf_{\mu'}(D).$$

In this case, $\delta_{\mu,t}(C) - \delta_{\mu',t}(D)$ is contained in $\mathrm{Ker}(M_{t,1})$.

Next, assume $s > 1$. Let $\mu = (2m_1, \dots, 2m_q)$. By the definition,

$$\square_\mu = \sum_{l=1}^{q-1} \sum_{k=1}^{2m_{l+1}} 1^{\otimes(l-1)} \otimes \square_k \otimes 1^{\otimes(q-l-1)}.$$

Suppose that C is contained in $\mathrm{Im}(1^{\otimes(l-1)} \otimes \square_k \otimes 1^{\otimes(q-l-1)})$.

When $l \geq 2$, we can show the claim immediately by the plethysm formula (as in the proof of Proposition 3.5, we can reduce C to higher partitions than μ from the assumption $s \leq t$).

Therefore, assume $l = 1$. If $q \geq 3$, the assertion is clear by induction on s. Hence we may assume that $\mu = (2a, 2b)$ such that $a \geq b > 0$, $a + b = s + t$, $a \geq t$, and that there exists k such that $1 \leq k \leq 2b$ and $C \in \mathrm{Im}(\square_k)$. Let C be $\square_k(e_{i_1} \wedge \cdots \wedge e_{i_{2a+k}} \otimes e_{j_{k+1}} \wedge \cdots \wedge e_{j_{2b}})$, where $1 \leq i_1 < \cdots < i_{2a+k} \leq n$ and $1 \leq j_{k+1} < \cdots < j_{2b} \leq n$.

CASE 1. Suppose $k \leq 2b - 1$.

Then, we have

$$\delta_{\mu,t}(C)$$

$$= \sum_{\sigma \in \mathbf{S}_{2a+k}^{[1,2a;2a+1,2a+k]}} (\mathrm{sgn}\,\sigma)\,\delta_{a,t}(e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(2a)}})$$

$$\cdot pf_{2b}(e_{i_{\sigma(2a+1)}} \wedge \cdots \wedge e_{i_{\sigma(2a+k)}} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_{2b}})$$

$$= \sum_{\sigma \in \mathbf{S}_{2a+k}^{[1,2a;2a+1,2a+k]}} (\operatorname{sgn}\sigma) \sum_{l=1}^{k} (-1)^{l+1} \delta_{a,t}(e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(2a)}}) \cdot (e_{i_{\sigma(2a+l)}} \wedge e_{j_{2b}})$$

$$\cdot pf_{2(b-1)}(e_{i_{\sigma(2a+1)}} \wedge \cdots \wedge \widehat{e_{i_{\sigma(2a+l)}}} \wedge \cdots \wedge e_{i_{\sigma(2a+k)}} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_{2b-1}}) \tag{3}$$

$$+ \sum_{\sigma \in \mathbf{S}_{2a+k}^{[1,2a;2a+1,2a+k]}} (\operatorname{sgn}\sigma) \sum_{l=k+1}^{2b-1} (-1)^{l+1} \delta_{a,t}(e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(2a)}})$$

$$\cdot pf_{2(b-1)}(e_{i_{\sigma(2a+1)}} \wedge \cdots \wedge e_{i_{\sigma(2a+k)}} \wedge e_{j_{k+1}} \wedge \cdots \wedge \widehat{e_{j_l}} \wedge \cdots \wedge e_{j_{2b-1}}) \cdot (e_{j_l} \wedge e_{j_{2b}}) \tag{4}$$

By the inductive assumption on s,

$$\sum_{\sigma \in \mathbf{S}_{2a+k}^{[1,2a;2a+1,2a+k]}} (\operatorname{sgn}\sigma) \delta_{a,t}(e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(2a)}})$$

$$\cdot pf_{2(b-1)}(e_{i_{\sigma(2a+1)}} \wedge \cdots \wedge e_{i_{\sigma(2a+k)}} \wedge e_{j_{k+1}} \wedge \cdots \wedge \widehat{e_{j_l}} \wedge \cdots \wedge e_{j_{2b-1}})$$

can be reduced to higher partitions than $(2a, 2(b-1))$. So, the second term (4) is reduced to higher partitions than $\mu = (2a, 2b)$. (The second term (4) does not appear when $k = 2b - 1$.)

On the other hand, the first term (3) is rewritten

$$\sum_{\sigma \in \mathbf{S}_{2a+k.}^{[1,2a;2(a+1),2a+k]}} (\operatorname{sgn}\sigma) \delta_{a,t}(e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(2a)}})$$

$$\cdot (e_{i_{\sigma(2a+1)}} \wedge e_{j_{2b}}) \cdot pf_{2(b-1)}(e_{i_{\sigma(2(a+1))}} \wedge \cdots \wedge e_{i_{\sigma(2a+k)}} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_{2b-1}}).$$

It is easy to see that we can reduce this to higher partitions by the induction on s. (When $k = b = 1$, use Lemmas 5.5 and 5.6.)

CASE 2. Suppose $k = 2b$.

We may assume $2(a+b) \leq n$.

If $k = 2$, the assertion is clear by Lemma 5.7.

When $k = 2b \geq 4$, we have

$$\delta_{\mu,t} \circ \square_{2b}(e_{i_1} \wedge \cdots \wedge e_{i_{2(a+b)}})$$

$$= \sum_{\sigma \in \mathbf{S}_{2(a+b)}^{[1,2a;2a+1,2(a+b)]}} (\operatorname{sgn}\sigma) \delta_{a,t}(e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(2a)}}) \cdot pf_{2b}(e_{i_{\sigma(2a+1)}} \wedge \cdots \wedge e_{i_{\sigma(2(a+b))}})$$

$$= \frac{1}{b}\Bigg(\sum_{\sigma \in \mathbf{S}_{2(a+b)}^{[1,2a;2a+1,2(a+1);2a+3,2(a+b)]}} (\operatorname{sgn}\sigma) \delta_{a,t}(e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(2a)}})$$

$$\cdot (e_{i_{\sigma(2a+1)}} \wedge e_{i_{\sigma(2(a+1))}}) \cdot pf_{2(b-1)}(e_{i_{\sigma(2a+3)}} \wedge \cdots \wedge e_{i_{\sigma(2(a+b))}}) \Bigg)$$

where $\mathbf{S}_{2(a+b)}^{[1,2a;2a+1,2(a+1);2a+3,2(a+b)]}$ is the subset of $\mathbf{S}_{2(a+b)}$ defined as

$$\{\sigma \in \mathbf{S} \mid \sigma(1) < \cdots < \sigma(2a),\ \sigma(2a+1) < \sigma(2(a+1)),\ \sigma(2a+3) < \cdots < \sigma(2(a+b))\}.$$

(Since $a \geq t$, $a+b=s+t$, $2p > n-2t$ and $2(a+b) \leq n$, we have $b < p$. So, b is a unit in R.)

By the inductive assumption on s, we can reduce $\delta_{\mu,t} \circ \square_{2b}(e_{i_1} \wedge \cdots \wedge e_{i_{2(a+b)}})$ to partitions higher than μ.

We have completed the proof of the claim. Q.E.D.

When $n \leq 2t+3$, $\mathrm{Ker}(M_t)$ is generated over $\mathrm{S}(\wedge^2 E)$ by relations of degree 1 over an arbitrary prime field R by (2) of Theorem 5.3. So, by Proposition 2 of Section 4 in [15] or Proposition II.3.4 in [5], we obtain

Corollary 5.8. *When $n \leq 2t+3$, $\mathrm{Ker}(M_t)$ is generated as an $\mathrm{S}(\wedge^2 E)$-module by relations of degree 1 over an arbitrary commutative ring R.*

In general, $\mathrm{Ker}(M_t)$ is not generated over $\mathrm{S}(\wedge^2 E)$ only by relations of degree 1 ([12]).

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