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On the composition of functions of bounded mean oscillation with meromorphic functions

Dedicated to Professor Tatsuo Fuji'i'e on his sixtieth birthday

By

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Introduction

A quasi-conformal mapping preserves *BMO*, that is, if $g: \Omega_1 \to \Omega_2$ is a quasiconformal mapping between plane domains, then for every $f \in BMO(\Omega_2)$, $f \circ g$ belongs to $BMO(\Omega_1)$. In our former paper we partially extended this result by characterizing the analytic functions which preserve *BMO*. In this paper we treat more generally meromorphic functions. We shall characterize the Blaschke type meromorphic functions preserving *BMO* (Theorem 1).

§1. Main Theorem

Let Ω be a domain on complex plane **C**. $BMO(\Omega)$ is the space of all locally integrable functions f on Ω such that

$$\|f\|_{*,\Omega} = \sup m(B)^{-1} \int_B |f - f_B|\, dm < \infty$$

where dm is the 2-dimensional Lebesgue measure, f_B is the integral mean of f on B and the supremum is taken for every disk B in Ω. $BMO(\mathbf{C})$ coinsides with the *BMO* space on the complex sphere $\hat{\mathbf{C}}$ with respect to its surface measure (cf. [10]), and $BMO(\mathbf{C})$ is obviously invariant under dilations and translations, especially it is invariant under Möbius transformations of $\hat{\mathbf{C}}$. More generally, Reimann and Jones proved the following result;

Proposition 1 ([7], [9]). *Let Ω_1 and Ω_2 be plane domains and $g: \Omega_1 \to \Omega_2$ a quasi-conformal mapping then for every $f \in BMO(\Omega_2)$, $f \circ g$ belongs to $BMO(\Omega_1)$ and it holds that $\|f \circ g\|_{*,\Omega_1} \leq C \|f\|_{*,\Omega_2}$ where $C > 0$ is a constant depending only on the maximal dilatation of g. Conversely if g is an absolutely continuous homeomorphism which preserves BMO then g is a quasi-conformal mapping.*

In our former papers, we characterized the analytic function which preserves *BMO*, *BMOH*, and *BMOA* as follows, where *BMOH* (resp. *BMOA*) is the space of all harmonic (analytic) *BMO* functions;

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Proposition 2 ([4], [5]). *Let Ω_1, Ω_2 be plane domains and $g: \Omega_1 \to \Omega_2$ an analytic function then*

(1) *g preserves BMO if and only there exists a integer $p > 0$ such that for every disk B in Ω_1 satisfying $rad(B) < d(B, \Omega_1)$, g is p-valent on B.*

(2) *g preserves BMOH if and only if there exists a constant $C > 0$ such that*

$$|dg(z)|/d(g(z), \partial\Omega_2) \leq C|dz|/d(z, \partial\Omega_1), \qquad z \in \Omega_1$$

(3) *g always preserves BMOA,*

where $rad(B)$ is the Euclidean radius of B and $d(\ , \ )$ denotes the Euclidean distance.

Corollary 1 ([5]). *An entire function $g: \mathbf{C} \to \mathbf{C}$ preserves $BMO(\mathbf{C})$ if and only if g is a polynomial.*

Especially let an analytic function $g: \Omega_1 \to \Omega_2$ form an unbranched and unbounded covering, then the conditions (1) and (2) of Proposition 2 are equivalent, hence in this case g preserves $BMOH$ if and only if g preserves BMO.

We extend the usual definition of BMO for plane domains to the subdomains Ω of the complex sphere $\hat{\mathbf{C}}$ by $BMO(\Omega) = BMO(\Omega \backslash \{\infty\})$. This extension is a natural one because if $g: \Omega_1 \to \Omega_2$ is a conformal mapping between subdomains of $\hat{\mathbf{C}}$, then we can identify the space $BMO(\Omega_1)$ with $BMO(\Omega_2)$, since one point is removable for BMO, to be precise, let Ω be a plane domain and $p \in \Omega$ then $BMO(\Omega \backslash \{p\}) = BMO(\Omega)$ and it holds that $\|f\|_{*, \Omega} \leq A\|f\|_{*, \Omega\backslash\{p\}}$, where $A > 0$ is a universal constant (cf. [10]).

Let $g: \Omega_1 \to \Omega_2$ be a meromorphic function. We now consider the problem that under what condition does $f \circ g$ belong to $BMO(\Omega_1)$ for every $f \in BMO(\Omega_2)$. If Ω_2 is a proper subdomain of $\hat{\mathbf{C}}$, we can reduce this problem to the case of analytic function. Therefore we can restrict our attention to the case $\Omega_2 = \hat{\mathbf{C}}$. In the beginning we give one example.

Let $g: \mathbf{C} \to \hat{\mathbf{C}}$ be an elliptic function, f a $BMO(\hat{\mathbf{C}})$ function and B a disk on $\mathbf{C}$. If $rad(B) < 1$ then by Proposition 2, we have $m(B)^{-1}\int_B |f \circ g - (f \circ g)_B| dm \leq C_1$, where C_1 (and C_2 below) > 0 is constant independent on B. Further if $rad(B) \geq 1$ then by the periodicity of g we have $m(B)^{-1}\int_B |f \circ g - (f \circ g)_B| dm \leq 2m(B)^{-1}\int_B |f \circ g| dm \leq C_2$. Hence $f \circ g$ belongs to $BMO(\mathbf{C})$. Remark that the boundedness of the norm of g as a linear operator is the consequence of the category theory. Thus there exists an infinite valence meromorphic function f on $\mathbf{C}$ which preserves BMO. Similarly the meromorphic function $g_1(z) = g(1/z)$ on $\mathbf{C} \backslash \{0\}$ gives the example of a meromorphic function preserving BMO which does not satisfy the condition of (1) of Proposition 2. From these examples, it seems to be much more difficult to characterize such meromorphic functions than the analytic case.

In this paper we treat Blaschke type meromorphic functions. In the following D always denotes the unit disk in $\mathbf{C}$ and $D(z, r)$ denotes the disk in $\mathbf{C}$ having z and r as its center and radius. Let B be a finite Blaschke product on D. Its zeros $\{z_n\}$, which is to be counted with their multiplicity, induce a measure $d\mu(z) = \sum_n (1 - |z_n|^2) d\delta_{z_n}$, where δ_{z_n} is the Dirac measure at z_n. We denote its

Carleson constant $Car(\mu)$ by $Car(B)$, that is,

$$Car(B) = \sup\{\mu(S_{\theta,h})/h : 0 < h \leq 1, 0 \leq \theta < 2\pi\}$$

where $S_{\theta,h} = \{re^{i\varphi} : 1 - h \leq r < 1, \theta - h \leq \varphi \leq \theta + h\}$. Further we set

$$Car^*(B) = \sup\{Car(B_\zeta) : B_\zeta(z) = (B(z) - \zeta)/(1 - \bar{\zeta}B(z)),\ \zeta \in D\}.$$

We now state our main result.

Theorem 1. *Let $B : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a finite Blaschke product on D and $\|B\|$ its norm as a linear operator between $BMO(\hat{\mathbf{C}})$, then*

(1) $\|B\| \leq C_1(Car^*(B))$,

(2) $Car^*(B) \leq C_2(\|B\|)$,

where $C_1(Car^*(B)) > 0$ is a constant depending only on $Car^*(B)$ and $C_2(\|B\|) > 0$ is a constant depending only on $\|B\|$.

Lemma 1 ([3]). *Let μ be a positive measure on D and G^μ its Green potential. We extend G^μ as 0 to $\hat{\mathbf{C}}\backslash D$, which we denote by $\tilde{G}^\mu$, then $\tilde{G}^\mu$ belongs to $BMO(\hat{\mathbf{C}})$ if and only if $(1 - |z|^2)d\mu$ is a Carleson measure and it holds that*

(1) $Car((1 - |z|^2)d\mu) \leq C_1(\|\tilde{G}^\mu\|_{*,\hat{\mathbf{C}}})$,

(2) $\|\tilde{G}^\mu\|_{*,\hat{\mathbf{C}}} \leq C_2(Car((1 - |z|^2 d\mu))$,

where $C_1(\|\tilde{G}^\mu\|_{*,\hat{\mathbf{C}}}) > 0$ is a constant depending only on $\|\tilde{G}^\mu\|_{*,\hat{\mathbf{C}}}$ and $C_2(Car((1 - |z|^2)d\mu))) > 0$ is a constant depending only on $Car((1 - |z|^2)d\mu)$.

Proof of Theorem 1 (2). We extend the Green function $g_\zeta(z) = \log(|1 - \bar{\zeta}z| / |z - \zeta|)$ on D as 0 to $\hat{\mathbf{C}}\backslash D$, which we denote by $\tilde{g}_\zeta$. Since $\tilde{g}_0$ belongs to $BMO(\hat{\mathbf{C}})$ and $\tilde{g}_\zeta = \tilde{g}_0 \circ A_\zeta$ where $A_\zeta(z) = (z - \zeta)/(1 - \bar{\zeta}z)$, we have $\sup\{\|g_\zeta\|_{*,\hat{\mathbf{C}}} : \zeta \in D\}$ $(=: M) < \infty$ by Proposition 1. Let $\mu_\zeta = \sum_n \{\delta_{z_n} : B(z_n) = \zeta\}$, then $\tilde{g}_\zeta \circ B = \tilde{G}^{\mu_\zeta}$, hence by Lemma 1,

$$Car(B_\zeta) = Car((1 - |z|^2)d\mu_\zeta) \leq C_1(\|\tilde{G}^{\mu_\zeta}\|_{*,\hat{\mathbf{C}}}) \leq C_1(\|B\|M).$$

and so $Car^*(B) \leq C_1(\|B\|M)$.

To prove Theorem 1 (1), we need several lemmas. We say a sequence $\{z_n\}$ on D is an interpolating sequence if

$$I(\{z_n\}) = \inf_{k \in N} \prod_{l \neq k} \left| \frac{z_k - z_l}{1 - \bar{z}_k z_l} \right| > 0,$$

Let B be a Blaschke product having $\{z_n\}$ as its zeros, then

$$I(\{z_n\}) = \inf_{n \in N} (1 - |z_n|^2)|B'(z_n)|.$$

Lemma 2 (cf. [8]). *Let $\{z_n\}$ be a sequence on D and assume its corresponding* *measure* $d\mu(z)=\sum_n(1-|z_n|^2)d\delta_{z_n}$ *form a Carleson measure. Then we can partition* $\{z_n\}$ *into a finite number of interpolating sequences* $\{z_n^{(k)}\}$, $k=1,2,\dots,s$, *such that*

(1) $I(\{z_n^{(k)}\})\geq C_1(Car(\mu))$, $k=1,2,\dots,s$,

(2) $s\leq C_2(Car(\mu))$,

where $C_1(Car(\mu))$, $C_2(Car(\mu))>0$ *are constants depending only on* $Car(\mu)$.

Lemma 3 (cf. [6]). *Let* $\{z_n\}$ *be an interpolating sequence and* B *the Blaschke product having* $\{z_n\}$ *as its zeros, then there exists a constant* $\varepsilon>0$ *which depends only on* $I(\{z_n\})$ *such that*

(1) $B^{-1}(D(0,\varepsilon))=\bigcup_n U_n$, $z_n\in U_n$, *(disjoint union)*,

(2) B *is conformal on each* U_n.

Lemma 4 (cf. [4]). *Let* g *be a p-valent locally univalent analytic function on* D *then there exists a constant* $r>0$ *which depends only on* p *such that* g *is conformal on a disk* $D(z,r)\subset D$.

Lemma 5. *Let* B *be a finite Blaschke product on* D, *then there exist constants* $\alpha,\beta,\gamma,\delta>0$ *which depend only on* $Car^*(B)$ *such that for every disk* $D_1=D(z_1,d(z_1,\partial D)/2)\subset D$ *there exists a disk* $D_2=D(z_2,r_2)\subset D_1$ *such that*

(1) $r_2>\alpha d(z_1,\partial D)$,

(2) B *is conformal on* D_2,

(3) $\gamma^{-1}d(B(z_2),\partial D)\leq\max\{|B(z)-B(z_2)|: z\in\partial D_2\}$
$\leq\gamma\min\{|B(z)-B(z_2)|: z\in\partial D_2\}$,

(4) $\delta^{-1}|B'(z_2)|\leq|B'(z)|\leq\delta|B'(z_2)|$, $z\in D_2$.

Proof. Remark that B is $p=p(Car^*(B))$-valent on D_1, hence by Lemma 4 there exists a disk $D_0=D(z_0,r_0)\subset D_1$ such that $r_0>C_1d(z_1,\partial D)$ and B is conformal on D_0, where C_1 (and C_2, $C_3,\cdots$ below) is a positive constant depending only on $Car^*(B)$. Therefore if we set $z_2=z_0$, $r_2=r_0/2$, then it is easy to show that $D(z_2,r_2)$ satisfys every condition of this lemma except for the inequality '$\gamma^{-1}d(B(z_2),\partial D)\leq\max\{|B(z)-B(z_2)|: z\in\partial D_2\}$'. To prove this remaining inequality, it suffices to prove

$$C_2/(1-|z_2|^2)\leq|B'(z_2)|/(1-|B(z_2)|^2).$$

By considering $(B-B(z_2))/(1-\overline{B(z_2)}B)$, we can assume $B(z_2)=0$ from the beginning, then above inequality reduces to $(1-|z_2|^2)|B'(z_2)|>C_2$. Let $\{\zeta_n\}$ be the zeros of B and we partition this sequence into finite number of interpolating sequences $\{\zeta_n^{(k)}\}$, $k=1,2,\dots,s$, following Lemma 2, where $\zeta_1^{(1)}=z_2$, and let $B^{(k)}$, $k=1,2,\dots,s$, the Blaschke products having $\{\zeta_n^{(k)}\}$ as its zeros. Then

$$(1-|z_2|^2)|B^{(1)'}(z_2)|\geq I(\{\zeta_n^{(1)}\})\geq C_3.$$

Further by Lemma 3, $|B^{(k)}(z_2)| \geq C_4$, $k = 2, 3, \ldots, s$. Summerizing above we have

$$(1 - |z_2|^2)|B'(z_2)| = (1 - |z_2|^2)|B^{(1)\prime}(z_2)| \prod_{k=2}^{s} |B^{(k)}(z_2)| \geq C_3 C_4^{s-1}.$$

Q.E.D.

Lemma 6 (cf. [2], [7]). *Let L be a circle or a line on $\mathbf{C}$, Ω_1 and Ω_2 the connected components of $\mathbf{C}\backslash L$, p the reflection with respect to L and f a $BMO(\Omega_1)$ function. If we extend f to $\mathbf{C}\backslash\Omega_1$ as $f \circ p$, which we denotes by $\tilde{f}$, then $\tilde{f}$ belongs to $BMO(\mathbf{C})$ and $\|\tilde{f}\|_{*,\mathbf{C}} \leq A\|f\|_{*,\Omega_1}$, here $A > 0$ is a universal constant.*

Lemma 7 (cf. [2], [7]). *Let $\Omega = D$ or $\mathbf{C}$, f a $BMO(\Omega)$ function, and $D_1 = D(z_1, r_1)$, $D_2 = D(z_2, r_2)$ disks in Ω, then*

$$|f_{D_1} - f_{D_2}| \leq A\left\{1 + \log\left(\frac{|z_1 - z_2|}{r_1} + 1\right)\left(\frac{|z_1 - z_2|}{r_2} + 1\right)\right\}\|f\|_{*,\Omega}$$

where $A > 0$ is a universal constant.

Lemma 8. *Let f be a $BMO(D)$ function satisfying $\|f\|_{*,D} \leq 1$ and*

$$\sup\{|f|_{D_1}: D_1 = D(z, d(z, \partial D)/2),\ z \in D\}\,(=: M) < \infty,$$

then if we extend f as 0 to $\hat{\mathbf{C}}\backslash D$, which we denote by $\tilde{f}$, $\tilde{f}$ belongs to *$BMO(\hat{\mathbf{C}})$ and $\|\tilde{f}\|_{*,\hat{\mathbf{C}}} \leq C(M)$, where $C(M) > 0$ is a constant depending only on M.*

Proof. Let $S_{\theta,h} = \{re^{i\varphi}: 1 - h \leq r < 1, \theta - h \leq \varphi \leq \theta + h\}$. Note that $|f| \in BMO(D)$ and $\||f|\|_{*,D} \leq 2\|f\|_{*,D} \leq 2$, indeed

$$m(B)^{-1}\int_B ||f| - |f|_B|\,dm \leq m(B)^{-2}\int_B\int_B |f(z) - f(\zeta)|\,dm(z)\,dm(\zeta) \leq 2\|f\|_{*,D},$$

for every disk $B \subset D$. And the similar argument, using the dyadic decomposition of $S_{\theta,h}$, as the proof of Hilfsatz 2 (p4 [10]) shows

$$\sup\{|f|_{S_{\theta,h}}: 0 < h \leq 1, 0 \leq \theta < 2\pi\} \leq C,$$

where $C > 0$ is a constant depending only M. Let $D_1 = D(z_1, r_1)$ be a disk such that $D_1 \not\subset D$. If $r_1 > 1/4$ then $|\tilde{f}|_{D_1} \leq \{m(D)/m(D_1)\}|f|_D \leq 64C$. On the other hand, if $r_1 \leq 1/4$ then we can choose $S_{\theta,h}$ such that $D \cap D_1 \subset S_{\theta,h}$ and $m(S_{\theta,h}) \leq Am(D_1)$, where $A > 1$ is a universal constant, hence $|f|_{D_1} \leq \{m(S_{\theta,h})/m(D_1)\}|f|_{S_{\theta,h}} \leq AC$. Thus, we have

$$m(D_1)^{-1}\int_{D_1} |\tilde{f} - \tilde{f}_{D_1}|\,dm \leq 2|\tilde{f}|_{D_1} \leq 2C\max\{64, A\}.$$

Q.E.D.

Proof of Theorem 1 (1). Since both $B|D$ and $B|(\mathbf{C}\backslash\bar{D})$ satisfy the condition

(1) of Proposition 2, we have $\|B|D\|$, $\|B|(C\backslash\bar{D})\| \leq C_1$, where C_1 (and C_2, C_3,... below) > 0 is a constant depending only on $Car^*(B)$. Let p be the reflection with respect to ∂D, and f a $BMO(\hat{\mathbf{C}})$ function. Set

$$f_1 = \begin{cases} f\circ p & \text{on } D \\ f & \text{on } \hat{\mathbf{C}}\backslash D \end{cases}, \qquad f_2 = \begin{cases} f - f\circ p & \text{on } D \\ 0 & \text{on } \hat{\mathbf{C}}\backslash D \end{cases}$$

then $f = f_1 + f_2$ and by Lemma 6, $\|f_1\|_{*,\hat{\mathbf{C}}} \leq A\|f\|_{*,D}$, where $A > 0$ is a universal constant, hence $\|f_2\|_{*,\hat{\mathbf{C}}} \leq \|f\|_{*,D} + \|f_1\|_{*,D} \leq (A+1)\|f\|_{*,\hat{\mathbf{C}}}$. Similarly, since $(f_1\circ B)\circ p = f_1\circ p\circ B = f_1\circ B$ we get the following estimate by using Lemma 6 again;

$$\|f_1\circ B\|_{*,\hat{\mathbf{C}}} \leq A\|f_1\circ B\|_{*,\hat{\mathbf{C}}\backslash D} \leq C_2\|f_1\|_{*,\hat{\mathbf{C}}\backslash D} \leq C_2\|f\|_{*,\hat{\mathbf{C}}}.$$

Thus we can assume $f = 0$ on $\hat{\mathbf{C}}\backslash D$ from the first. Let $D_1 = D(z, d(z_1, \partial D)/2)$, then by Lemma 8, it suffices to show $(|f|\circ B)_{D_1} \leq C_3\|f\|_{*,\hat{\mathbf{C}}}$. Let D_2 be the disk in D_1 satisfying the condition of Lemma 5. Then by Lemma 7,

$$(|f|\circ B)_{D_1} \leq (|f|\circ B)_{D_2} + C_4\||f|\circ B\|_{*,D} = I_1 + I_2,$$

here $I_2 \leq C_5\||f|\|_{*,D} \leq 2C_5\|f\|_{*,D}$. Next let $D_3 = D(z_3, r_3)$, where $z_3 = B(z_2)$ and $r_3 = \max\{|B(z) - B(z_2)|: |z - z_2| = r_2\}$, then $I_2 \leq C_6|f|_{B(D)} \leq C_7|f|_{D_3}$. Since we can take a disk $D_3' = D(z_3', r_3') \subset \mathbf{C}\backslash D$ so that $r_3' = r_3$ and $d(D_3', D_3) \leq C_8 r_3$, hence by applying Lemma 7 we obtain

$$|f|_{D_3} \leq |f|_{D_3'} + C_9\||f|\|_{*,\hat{\mathbf{C}}} \leq 2C_9\|f\|_{*,\hat{\mathbf{C}}}.$$

from these estimate we have $(|f|\circ B)_{D_1} \leq C_3\|f\|_{*,\hat{\mathbf{C}}}$.

Q.E.D.

§2. Some consequences

The assumption that B is finite Blaschke does not play an essential role to prove Theorem 1. Indeed, it is not difficult to verify that the same argumant holds for every indestructive Blaschke product, here we say a Blaschke product B is indestructive if for every Möbius transformation T of the unit disk D, $B\circ T$ is again a Blaschke product. Then B may have singularities on ∂D. But we can ignore it when we regard B as a linear operator between BMO since ∂D is a nul set.

We give some example of infinite Blaschke products which preserve $BMO(\hat{\mathbf{C}})$. Let R be a Riemann surface having $\pi: D \to R$ as its universal covering and we assume R has the Green function. We define a function c_R on $R \times R$ by

$$c_R(\pi(z), \pi(\zeta)) = \sum_{A\in\Gamma} c_D(z, A(\zeta)), \qquad z, \zeta \in D$$

where $c_D(z, \zeta) = (1 - |z|^2)(1 - |\zeta|^2)/|1 - \bar{\zeta}z|^2$ and Γ is the covering transformation group for π.

Proposition 3 ([3]). *Let R, c_R be as above, μ_R a positive measure on R, and μ_D its lift on D, then $(1-|z|^2)\,d\mu(z)$ is a Carleson measure if and only if*

$$\sup_{p\in R}\int_R c_R(p, q)\,d\mu(q) < \infty.$$

Proposition 4 ([3]). *Let R and c_R be as above and g_R the Green function on R. Then c_R is bounded above if and only if R satisfies the following condition;*

(*) *there exists a constant $M > 0$ such that for every $q \in R$ the domain $\{p \in R: g_R(p, q) > M\}$ is simply connected.*

With these propositions and Theorem 1 we have

Corollary 2. *Let R and π be as above and $\mu_p = \Sigma\{\delta_z: \pi(z) = p, z \in D\}$ then $\sup\{Car((1-|z|^2)\,d\mu_p): p \in R\} < \infty$ if and only if R satisfy the condition (*) in Proposition 4.*

Theorem 2. *Let R be a Riemann surface having $\pi: D \to R$ as its universal covering satisfying the conditon (*) in Proposition 4 and $h: R \to D$ a unbounded and branched covering with finite valence. Then $h \circ \pi$ is a indestructive Blaschke product and its natural extension to $\hat{\mathbf{C}}$ preserves $BMO(\hat{\mathbf{C}})$.*

For instance let

$$g(z) = \prod_{n=1}^{\infty} \frac{2^n z - i}{2^n z + i} \prod_{n=0}^{\infty} \frac{2^n i - z}{2^n i + z},$$

and H the upper half plane, then $g|H: H \to D$ is a 2-valenced unbounded branched covering map on the compact bordered surface $R = H/\Gamma$ where $\Gamma = \langle T \rangle$, $T(z) = 4z$, hence g preserves $BMO(\hat{\mathbf{C}})$.

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