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Homology of the Kac-Moody groups III

By

Akira KONO and Kazumoto KOZIMA

§ 1. Introduction

Let G be a compact, connected, simply connected simple Lie group, $\mathfrak{g}$ its Lie algebra and $1=m(1)<m(2)\leqq \cdots \leqq m(l)$ its exponents where l is the rank of G. The homotopy type of the Kac-Moody group $\mathfrak{K}(\mathfrak{g}^{(1)})$ is $G\times\Omega G\langle 2\rangle$, where $\Omega G\langle 2\rangle$ is the 2-connected cover of ΩG, the based loop space on G. (See [9].) For a free graded module $M=\bigoplus M_j$ of finite type over a ring R,

$$\sum_{j=0}^{\infty}(\text{rank } M_j)t^j\in \boldsymbol{Z}[[t]]$$

is denoted by $P(M; R)$ and for a space X such that $H^*(X; R)$ is free of finite type $P(X; R)=P(H^*(X; R); R)$. V.G. Kac and D.H. Peterson defined a positive integer $d(G, p)$ for any prime p and showed that

$$P(\Omega G(2); \boldsymbol{F}_p)=(1+t^{2a(G,p)-1})(1-t^{2a(G,p)})^{-1}R_G(t) \tag{1.1}$$

where $a(G, p)=p^{d(G,p)}$

$$R_G(t)=\prod_{j=2}^{l}(1-t^{2n(j)})^{-1} \tag{1.2}$$

in terms of the Affine Weyl groups ([9], [11]). On the other hand there is a fibering

$$S^1 \longrightarrow \Omega G\langle 2\rangle \xrightarrow{\pi} \Omega G. \tag{1.3}$$

In [13], the first named author shows (1.1) by use of the cohomology Gysin sequence and determines $d(G, p)$.

Since $G\simeq_0\prod_{j=1}^{l}S^{2n(j)+1}$ by Serre ([22]), $\Omega G\langle 2\rangle\simeq_0\prod_{j=2}^{l}\Omega S^{2n(j)+1}$ and therefore the odd dimensional rational homology of $\Omega G\langle 2\rangle$ is zero. Since $H_{2m}(\Omega G; \boldsymbol{Z})$ is free and $H_{2m-1}(\Omega G; \boldsymbol{Z})=0$ for any m by Bott ([7]), the homology Gysin sequence of (1.2) with R-coefficient is

$$\begin{aligned} 0 &\longrightarrow H_{2m}(\Omega G\langle 2\rangle; R)\xrightarrow{\pi_*} H_{2m}(\Omega G; R)\xrightarrow{\chi_R} H_{2m-2}(\Omega G; R) \\ &\longrightarrow H_{2m-1}(\Omega G\langle 2\rangle; R)\longrightarrow 0. \end{aligned} \tag{1.4}$$

Using (1.4), we deduce that $H_{2m}(\Omega G\langle 2\rangle; \boldsymbol{Z})$ is free and $H_{2m-1}(\Omega G\langle 2\rangle; \boldsymbol{Z})$ is a finite group for any m. Therefore to determine $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z})$, it is sufficient to determine $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$.

Define a graded $\boldsymbol{Z}_{(p)}$-module $M(d, p)=\bigoplus_{j\geq 0}M(d, p)_j$ by

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$$M(d, p)_j=\begin{cases} \boldsymbol{Z}_{(p)}, & \text{if } j=0 \\ \boldsymbol{Z}/p^{r-d}, & \text{if } j+1=2p^r k,\ (k, p)=1 \text{ and } r\geqq d, \\ 0, & \text{otherwize.} \end{cases}$$

and denote by $L(G, p)$ a free graded $\boldsymbol{Z}_{(p)}$-module satisfying

$$P(L(G, p);\ \boldsymbol{Z}_{(p)})=R_G(t).$$

The purpose of this paper is to show

Theorem 1.1. *$H_*(\Omega G\langle 2\rangle;\ \boldsymbol{Z}_{(p)})$ is isomorphic to $M(d(G, p)-1, p)\otimes_{Z_{(p)}}L(G, p)$.*

Since $H_*(G;\ \boldsymbol{Z})$ is known, the integral homology of the Kac-Moody group is determined.

To prove Theorem 1.1, we show the following:

Theorem 1.2. *There are elements a, b and a subalgebra $A(G, p)$ of $H_*(\Omega G\langle 2\rangle;\ \boldsymbol{F}_p)$ satisfying*

(1) *$|a|=2a(G, p)$, $|b|=|a|-1$, $b\in\mathrm{Im}\,\rho$ and $A(G, p)\subset\mathrm{Im}\,\rho$ where ρ is the mod p reduction,*

(2) *$H_*(\Omega G\langle 2\rangle;\ \boldsymbol{F}_p)\cong\boldsymbol{F}_p[a]\otimes\Lambda(b)\otimes A(G, p)$ as an algebra.*

Using the fact that $H_{2a(G,p)-1}(\Omega G\langle 2\rangle;\ \boldsymbol{Z}_{(p)})\cong\boldsymbol{Z}/p$ (Theorem 3.1 of [16]), Lemma 2.1 of [16] and Theorem 1.2, we can compute the Bockstein spectral sequence constructed in [16] and get Therem 1.1. If $(G, p)\neq(B_n, 2)$ or $(D_n, 2)$, Theorem 1.1 is proved in [15] and [16]. But the proof of this paper is applicable for any (G, p) and is an improvement of [15] and [16].

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§2. Some properties of ρ and $\chi_{Z_{(p)}}$

Denote by $P_{2m}(R)$ the submodules of the primitive elements in $H_{2m}(\Omega G;\ R)$ and by $Q_{2m}(R)$ the indecomposable quotient $Q^{2m}(H_*(\Omega G;\ R))$.

Consider the commutative diagram of the Gysin exact sequence (1.4):

$$\begin{array}{ccccccccc}
0\to H_{2m}(\Omega G\langle 2\rangle;\ \boldsymbol{Z}_{(p)}) & \xrightarrow{\pi_*} & H_{2m}(\Omega G;\ \boldsymbol{Z}_{(p)}) & \xrightarrow{\chi} & H_{2m-1}(\Omega G;\ \boldsymbol{Z}_{(p)}) & \to & H_{2m-1}(\Omega G\langle 2\rangle;\ \boldsymbol{Z}_{(p)}) & \to & 0 \\
\downarrow\rho & & \downarrow\rho & & \downarrow\rho & & \downarrow\rho & & \\
0\to H_{2m}(\Omega G\langle 2\rangle;\ \boldsymbol{F}_p) & \xrightarrow{\pi_*} & H_{2m}(\Omega G;\ \boldsymbol{F}_p) & \xrightarrow{\bar{\chi}} & H_{2m-2}(\Omega G;\ \boldsymbol{F}_p) & \to & H_{2m-1}(\Omega G\langle 2\rangle;\ \boldsymbol{F}_p) & \to & 0
\end{array}$$

where ρ is the mod p reduction, $\chi=\chi_{Z_{(p)}}$ and $\bar{\chi}=\chi_{F_p}$. Note that χ_R is a derivation and given by the formula

$$\chi_R(\alpha)=t\diagdown\Delta_*\alpha \tag{2.1}$$

where $\diagdown$ is the slant product, Δ is the diagonal map and $t\in H^2(\Omega G;R)\cong R$ is a generator. Using (2.1), we have

Lemma 2.1. *$P_{2m}(R)\subset \operatorname{Ker}\chi_R$ for $m>1$.*

On the other hand by the Bockstein exact sequence, we have

Lemma 2.2. *$\rho: H_m(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})\to H_m(\Omega G\langle 2\rangle \boldsymbol{F}_p)$ is epic for $m\leqq 2a(G, p)-1$.*

Denote by $P(p)$ the subset $\{1, p, p^2, \cdots, p^j, \cdots\}$ of integers. Consider the map

$$\xi_{2m}: H_{2m}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \longrightarrow H_{2m}(\Omega G\langle 2\rangle; \boldsymbol{F}_p) \xrightarrow{\pi_*} H_{2m}(\Omega G; \boldsymbol{F}_p) \xrightarrow{\lambda_{2m}} Q_{2m}(\boldsymbol{F}_p).$$

where λ_{2m} is the projection. Then we have

Proposition 2.3. *Suppose $m<a(G, p)$, then*
(1) *ξ_{2m} is epic if $m\notin P(p)$,*
(2) *$\dim \operatorname{Coker}\xi_{2m}=1$ if $m\in P(p)$.*

Proof. Since $\rho: H_{2m}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})\to H_{2m}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$ is epic for $m<a(G, p)$ by Lemma 2.2, we can replace ξ_{2m} by ξ'_{2m} where ξ'_{2m} is the composition

$$H_{2m}(\Omega G\langle 2\rangle; \boldsymbol{F}_p) \xrightarrow{\pi_*} H_{2m}(\Omega G; \boldsymbol{F}_p) \xrightarrow{\lambda_{2m}} Q_{2m}(\boldsymbol{F}_p).$$

Choose a generator t of $H^2(\Omega G; \boldsymbol{Z})\cong \boldsymbol{Z}$. There is a fibering

(2.2) $$\Omega G\langle 2\rangle \xrightarrow{\pi} \Omega G \xrightarrow{t} CP^\infty = K(\boldsymbol{Z}, 2)$$

which is the loop of the fibering

(2.3) $$G\langle 3\rangle \xrightarrow{\pi} G \xrightarrow{x} K(\boldsymbol{Z}, 3)$$

where x is a generator of $H^3(G; \boldsymbol{Z})\cong \boldsymbol{Z}$ satisfying $\sigma(x)=t$ (σ denotes the cohomology suspension). Therefore the mod p homology Serre spectral sequence for (2.2) is multiplicative. By the fact that $H_{2m-1}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)=0$ for $m<a(G, p)$, this spectral sequence is trivial for up to degree less than $2a(G, p)$. Therefore we have as an algebra

$$H_*(CP^\infty; \boldsymbol{F}_p)\cong H_*(\Omega G; \boldsymbol{F}_p)/(\operatorname{Im}\pi_*^+)$$

for $*<2a(G, p)$. Proposition 2.3 follows from the fact that

$$H_*(CP^\infty; \boldsymbol{F}_p)\cong \boldsymbol{F}_p[\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_j, \cdots]/(\bar{e}_1{}^p, \bar{e}_2{}^p, \cdots, \bar{e}_j{}^p, \cdots)$$

where $|\bar{e}_j|=2p^{j-1}$.

Lemma 2.4. *If $x\in H_{2m}(\Omega G; \boldsymbol{F}_p)$ such that $pm<a(G, p)$, then $x^p\in \operatorname{Im}\pi_*\circ\rho$.*

Proof. Since $\bar{\chi}$ is derivative, $\bar{\chi}x^p=0$. Therefore $x^p\in \operatorname{Im}\pi_*$ and Lemma 2.4 follows from Lemma 2.2.

§ 3. Proof of Theorem 1.2

First we prove the following:

Lemma 3.1. *There are an element a' and a subalgebra $B(G, p)$ of $H_*(\Omega G; \boldsymbol{F}_p)$ such that*

(1) $|a'|=2a(G, p)$, $a'\in \operatorname{Im}\pi_*$ *and* $B(G, p)\subset \operatorname{Im}\pi_*\circ\rho$,

(2) $\operatorname{Im}\pi_*$ *is a polynomial algebra generated by a' over $B(G, p)$.*

Proof. First we assume that $H_*(G; \boldsymbol{Z}_{(p)})$ is torsion free and $n(l)<a(G, p)$. Then the above lemma follows easily from Proposition 2.3 and Lemma 2.4. Next we assume that $H_*(G; \boldsymbol{Z}_{(p)})$ is torsion free and $n(l)\geqq a(G, p)$. Then $(G, p)=(C_n, 2)$ or $(G_2, 3)$ by [13] and $d(G, p)=1$. Since $H_*(\Omega C_n; \boldsymbol{Z}_{(2)})$ is primitively generated by [14] and $H_*(G_2; \boldsymbol{Z}_{(3)})$ is primitively generated by the dimensional reasons, the above lemma follows from Lemma 2.1 and Lemma 2.4. Note that $H_*(G; \boldsymbol{Z}_{(p)})$ is not torsion free if and only if (G, p) is one of the following:

$$(B_n, 2), (D_{n+1}, 2) \quad (n\geqq 3), \tag{3.1}$$

$$(G_2, 2), (F_4, 2), (E_l, 2) \quad (l=6, 7, 8), \tag{3.2}$$

$$(F_4, 3), (E_l, 3) \quad (l=6, 7, 8), (E_8, 5). \tag{3.3}$$

First we consider the case (3.3). Using the structure of $H^*(G\langle 3\rangle; \boldsymbol{F}_p)$ in [12], [18] and [20], we can easily show as an algebra

$$H_*(\Omega G\langle 2\rangle; \boldsymbol{F}_p)\cong \boldsymbol{F}_p[\tilde{t}_{2n(j)}\mid j=2, 3, \cdots, l]\otimes \boldsymbol{F}_p[a]\otimes \Lambda(b)$$

where $l=\operatorname{rank} G$, $|\tilde{t}_{2n(j)}|=2n(j)$, $|a|=2a(G, p)$ and $|b|=|a|-1$. We may assume $\tilde{t}_{2n(j)}$, $b\in \operatorname{Im}\rho$ by the dimensional reasons. In fact, $\beta_*a=b$ and for any $j\geqq 2$, $H_{2n(j)-1}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)=0$ or

$$\beta_*(H_{2n(k)}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)H_{2a(G,p)}(\Omega G\langle 2\rangle; \boldsymbol{F}_p))=H_{2n(j)-1}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$$

for some $k<j$ where β_* is the Bockstein operation. Therefore $\xi_{2n(j)}$ is epic for any $j=2, 3, \cdots, l$.

For the case (3.2), $H_*(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$ is known in [17] and the proof is similar. Now we consider the case (3.1). We only give a proof for $(B_{2n}, 2)$ since the other cases are quite similar. Denote the mod 2 reduction of σ_j $(1\leqq j\leqq 2n-1)$, $2\sigma_j$ $(2\leqq j\leqq 4n-1)$ and $2p_j$ $(2n\leqq j\leqq 4n-1)$ of Bott ([7], section 9) by x_j, y_j, γ_j respectively. Put $d=d(B_{2n}, 2)$. Then by [13], $2^{d-1}<4n\leqq 2^d$. By [7], as an algebra

$$H_*(\Omega B_{2n}; \boldsymbol{F}_2)\cong \boldsymbol{F}_2[x_1, x_2, \cdots, x_{n-1}]/(x_1^2, x_2^2, \cdots, x_{n-1}^2)$$
$$\otimes \boldsymbol{F}_2[x_n, x_{n+1}, \cdots, x_{2n-1}]\otimes \boldsymbol{F}_2[y_{2n+1}, y_{2n+3}, \cdots y_{4n-1}]$$

(Note that $|x_j|=2j$ and $|y_{2j+1}|=4j+2$). Since $y_{2j+1}\equiv\gamma_{2j+1}$ mod decomposables, we may replace y_{2j+1}'s by γ_{2j+1}'s. By Proposition 2.3, if $j\notin P(2)$ and $3\leqq j\leqq 2n-1$, there is an element $u_j\in H_{2j}(\Omega B_{2n}; \boldsymbol{Z}_{(2)})$ such that

$$\pi_*\rho(u_j)\equiv x_j \quad \text{mod decomposables.}$$

Since p_j is primitive by [7], we get $\gamma_{2j+1} \in \mathrm{Im}\, \pi_* \circ \rho$. Put $B(B_{2n}, 2)$ the subalgebra generated by $\{\pi_* \circ \rho u_j | 3 \leqq j \leqq 2n-1,\ j \notin P(2)\} \cup \{\gamma_{2n+1}, \gamma_{2n+3}, \cdots, \gamma_{4n-1}\}$. Put $a' = x_{2d-2}^2$, then $\bar{\chi} a' = 0$ and so $a' \in \mathrm{Im}\, \pi_*$. a' generates a polynomial subalgebra over $B(B_{2n}, 2)$ and $B(B_{2n}, 2)[a'] \subset \mathrm{Im}\, \pi_*$. But

$$P(B(B_{2n}, 2)[a']; \boldsymbol{F}_2) = (1-t^{2^d})^{-1} R_{B_{2n}}(t) = P(\mathrm{Im}\, \pi_*; \boldsymbol{F}_2)$$

and so $B(B_{2n}, 2)[a'] = \mathrm{Im}\, \pi_*$.

Proof of Theorem 1.2. Fix a generator b of $H_{2a(G,p)-1}(\Omega G; \boldsymbol{F}_p)$. By Lemma 2.2, $b \in \mathrm{Im}\, \rho$. Then using the fact that $\pi_*: H_{2m}(\Omega G\langle 2\rangle; R) \to H_{2m}(\Omega G; R)$ is monic for any m, we get Theorem 1.2 by Lemma 3.1.

DEPARTMENT OF MATHEMATICS,
KYOTO UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES,
UNIVERSITY OF ABERDEEN
DEPARTMENT OF MATHEMATICS,
KYOTO UNIVERSITY OF EDUCATION

References

[1] S. Araki, Differential Hopf algebra and the cohomology mod 3 of the compact exceptional groups E_7 and E_8, Ann. Math., **73** (1961), 404-436.

[2] S. Araki, Cohomology modulo 2 of the compact exceptional groups E_6 and E_7, J. Math. Osaka City Univ., **12** (1981), 43-65.

[3] S. Araki and Y. Shikata, Cohomology mod 2 of the compact exceptional group E_8, Proc. Japan Acad., **37** (1961), 619-622.

[4] A. Borel, Sur l'homologie et la cohomologie des groupes de Lie compacts connexes, Amer. J. Math., **76** (1954), 273-342.

[5] A. Borel and J.P. Serre, Groupes de Lie et puissances reduites de Steenrod, Amer. J. Math., **73** (1953), 409-448.

[6] R. Bott, An application of the Morse theory to the topology of Lie groups, Bull. Soc. Math. France, **84** (1956), 251-281.

[7] R. Bott, The space of loops on a Lie group, Michigan Math. J., **5** (1958), 35-61.

[8] K. Ishitoya, A. Kono and H. Toda, Hopf algebra structure of mod 2 cohomology of simple Lie group, Publ. R.I.M.S. Kyoto Univ., **12** (1976), 141-167.

[9] V.G. Kac, Torsion in cohomology of compact Lie group and Chow rings of reductive alg. groups, Inv. Math., **80** (1985), 69-79.

[10] V.G. Kac, Constructing groups associated to infinite dimensional Lie algebra, Infinite dimensional groups with applications, MSRI Publ., 4, 167-216.

[11] V.G. Kac and D. Peterson, Infinite flag varieties and conjugacy theorems, Proc. Nat. Acad. Sci., **80** (1983), 1778-1782.

[12] A. Kono, Hopf algebra structure of simple Lie groups, J. Math. Kyoto Univ., **17** (1977), 259-298.

[13] A. Kono, On the cohomology of the 2-connected cover of the loop space of simple Lie groups, Publ. R.I.M.S. Kyoto Univ., **22** (1986), 537-541.

[14] A. Kono and K. Kozima, The space of loops on a symplectic group, Japan. J. Math. **4** (1978), 461-486.

[15] A. Kono and K. Kozima, Homology of the Kac-Moody Lie groups, I, J. Math. Kyoto Univ., **29** (1989), 449-453.

[16] A. Kono and K. Kozima, Homology of the Kac-Moody Lie groups, II, to appear in J. Math. Kyoto Univ.

[17] A. Kono and K. Kozima, The mod 2 homology of the space of loops on the exceptional Lie group, Proc. Roy. Soc. Edinbrugh, **112A** (1989), 187-202.

[18] A. Kono and M. Mimura, Cohomology operations anf Hopf algebra structure of the compact, exceptional Lie group E_7 and E_8, Proc. London Math. Soc., III **35** (1977), 345-358.

[19] J. Milnor and J. Moore, On the structure of Hopf algebra, Ann. Math., **81** (1965), 211-264.

[20] M. Mimura, Homotopy groups of Lie groups of low rank, J. Math. Kyoto Univ., **6** (1967), 131-176.

[21] M. Mimura and H. Toda, Cohomology operations and homotopy of compact Lie groups I, Topology, **9** (1970), 317-336.

[22] J.P. Serre, Groupes d'homotopie et classes de groupes abeliens, Ann. Math., **58** (1953), 258-294.