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Canonical forms of 3×3 strongly hyperbolic systems with real constant coefficients

By

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1. Introduction

Consider an $m\times m$ system of differential equations

$$\frac{\partial u}{\partial t}=\sum_{i=1}^{n}A_i\frac{\partial u}{\partial x_i} \tag{1.1}$$

where u is an m-vector and A_i are real constant $m\times m$ matrix coefficients. For simplicity, we further assume any nontrivial linear combination of A_i is not equal to the zero matrix or the identity. Otherwise, the system (1.1) can be reduced to the one with a smaller n. (See the comments between Definition 2.6 and 2.7)

It was Yamaguti and Kasahara [3], [7] who gave the definition and a criterion for the system (1.1) to be strongly hyperbolic. Later, Strang [5] proved that (1.1) is strongly hyperbolic if and only if its initial value problem is L^2-wellposed. However, few attempts have been made to find out all the canonical forms of strongly hyperbolic systems (1.1). It is perhaps because the criterion of Yamaguti and Kasahara is stated in terms of the linear combinations of $A_1, A_2, \cdots, A_n$ and seems difficult to verify directly. The only exception is the case of $m=2$ (2×2 systems). In fact, Strang [5] proved that every strongly 2×2 system is simultaneously symmetrizable (see Definition 2.5). However, the case $m\geqq3$ is much more delicate. For instance, Lax [4] already gave an example of strongly hyperbolic 3×3 system which cannot be simulaneously symmetrized. But, as far as the author knows, no one has fully investigated the 3×3 systems.

In the present paper, we will give all the canonical forms of 3×3 ($m=3$) strongly hyperbolic systems (1.1). The result depends drastically on n. If $n=2$, the systems are either strictly hyperbolic (see Definition 2.6) or simultaneously symmetrizable. And there are eight canonical forms for the strictly hyperbolic systems. If $n=3$, the systems cannot be strictly hyperbolic and they are simultaneously symmetrizable or can be reduced to either

$$\frac{\partial u}{\partial t}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\frac{\partial u}{\partial x_1}+\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}\frac{\partial u}{\partial x_2}+\begin{bmatrix}0&\alpha&1\\-\alpha&0&0\\1&0&0\end{bmatrix}\frac{\partial u}{\partial x_3}$$

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where $0<\alpha<1$, or

$$\frac{\partial u}{\partial t}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\frac{\partial u}{\partial x_1}+\begin{bmatrix}0&\alpha&0\\\alpha&1&0\\0&0&-1\end{bmatrix}\frac{\partial u}{\partial x_2}+\begin{bmatrix}0&\beta&\gamma\\\beta'&0&0\\\gamma'&0&0\end{bmatrix}\frac{\partial u}{\partial x_3}$$

where

$$\alpha>0,\quad \gamma>\frac{1}{2}\left(\alpha+\frac{\beta^2}{\alpha}\right),\quad \gamma'>\frac{1}{2}\left(\alpha+\frac{\beta'^2}{\alpha}\right),\quad |\beta'-\beta|+|\gamma'-\gamma|>0.$$

Lastly, if $n\geqq 4$, all the strongly hyperbolic systems are simultaneously symmetrizable.

We will also consider a class of systems including Petrovsky's example

$$\frac{\partial u}{\partial t}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\frac{\partial u}{\partial x_1}+\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}\frac{\partial u}{\partial x_2}$$

which is non-uniformly real-diagonalizable and give their canonical forms.

All the results of this paper shall be summarized in the last section in terms of matrix families.

2. Definitions and preliminaries

Throughout this paper, we consider only real square (actually 3×3) matrices and their linear combinations with real coefficients.

Definition 2.1. The set of all linear combinations $A(\xi)=A(\xi_1, \xi_2, \cdots, \xi_n)=\sum_{j=1}^{n}\xi_j A_j$ $(\xi_1, \xi_2, \cdots, \xi_n\in \boldsymbol{R})$ of the $m\times m$ matrices $A_1, A_2, \cdots, A_n$ is said to be the matrix family spanned by $A_1, A_2, \cdots, A_n$ and is denoted by $\langle A_1, A_2, \cdots, A_n\rangle$.

Definition 2.2. A matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ is called real-diagonalizable if for every $A(\xi)\in\langle A_1, A_2, \cdots, A_n\rangle$, there exists a nonsingular matrix $S(\xi)$ (called a diagonalizer) such that

$$S(\xi)^{-1}A(\xi)S(\xi)$$

is a real diagonal matrix.

Definition 2.3. A matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ is called uniformly real-diagonalizable if it is real-diagonalizable and there is a diagonalizer $S(\xi)$ such that

$$\|S(\xi)\|,\ \|S(\xi)^{-1}\|\leqq\text{const}.$$

when ξ runs over $\boldsymbol{R}^n$. Similarly, a matrix family is called non-uniformly real-diagonalizable if there are no bounded diagonalizers.

We state here the most fundamental theorem concerning the equation (1.1).

Theorem 2.4 (Yamaguti-Kasahara [7]). *Equation* (1.1) *is strongly hyperbolic if and only if the matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ *is uniformly real-diagonalizable.*

Remark. As mentioned in Introduction, for constant-coefficient equations, strong hyperbolicity is equivalent to L^2-wellposedness. For the proof of Theorem 2.4, see Yamaguti-Kasahara [7], Kasahara-Yamaguti [3] (B^∞-theory), or Strang [5] (L^2-theory).

We now refer to two important subclasses of the uniformly real-diagonalizable matrix families.

Definition 2.5. A matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ is called simultaneously symmetrizable if there exists a nonsingular matrix T such that all $T^{-1}A_jT$ $(j=1, 2, \cdots, n)$ are simultaneously symmetric. In addition, Equation (1.1) with such A_j is called simultaneously symmetrizable.

Definition 2.6. A matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ is said to have real distinct eigenvalues if every $A(\xi)\in\langle A_1, A_2, \cdots, A_n\rangle$ with $\xi\neq 0$ has real distinct eigenvalues. In addition, Equation (1.1) with such A_j is called a strictly (or regularly) hyperbolic system.

Let us now consider what equivalence relation should be introduced for matrix families. It is easy to see the following three operations $\langle A_1, \cdots, A_n\rangle\rightarrow\langle B_1, \cdots, B_{n'}\rangle$ do not affect the real-diagonalizabilty (uniform or not) of matrix families.

a) *Change of basis.*

$$\begin{aligned} B_1 &= m_{11}A_1+m_{12}A_2+\cdots+m_{1n}A_n \\ B_2 &= m_{21}A_1+m_{22}A_2+\cdots+m_{2n}A_n \\ &\vdots \\ B_n &= m_{n1}A_1+m_{n2}A_2+\cdots+m_{nn}A_n \end{aligned}$$

where $M=(m_{ij})$ is a nonsingular real $n\times n$ matrix.

b) *Addition of scalar multiples of identity.*

$$\begin{aligned} B_1 &= A_1+\mu_1 I \\ B_2 &= A_2+\mu_2 I \\ &\vdots \\ B_n &= A_n+\mu_n I \end{aligned}$$

where I is the identity matrix and μ_i $(1\leqq i\leqq n)$ are reals.

c) *Similarity transformation.*

$$\begin{aligned} B_1 &= T^{-1}A_1T \\ B_2 &= T^{-1}A_2T \\ &\vdots \\ B_n &= T^{-1}A_nT \end{aligned}$$

where T is a nonsingular $m\times m$ real matrix arbitrarily fixed.

It is perhaps worth noting how the above three operations transform the original differential equation (1.1). First, a) corresponds to the change of space variables:

$$(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)^{\mathrm{T}}=M(x_1, x_2, \cdots, x_n)^{\mathrm{T}}.$$

Second, b) corresponds to the change of time-space variables of the type:

$$\tilde{x}_i = x_i - \mu_i t \qquad (1 \leq i \leq n).$$

Note that if some space variables disappear from (1.1) by these operations, they can be regarded as parameters for the solution of the *reduced* equation. Finally, c) corresponds to the change of unknowns:

$$(\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_m)^T = T^{-1}(u_1, u_2, \cdots, u_m)^T.$$

Combining the above operations a), b) and c), we are led to the following definition.

Definition 2.7. Matrix families $\langle A_1, A_2, \cdots, A_n\rangle$ and $\langle B_1, B_2, \cdots, B_{n'}\rangle$ are called equivalent if there exist a nonsingular matrix T and $\mu_j \in \boldsymbol{R}$ $(j=1, 1, \cdots, n)$ such that

$$\langle T^{-1}A_1T - \mu_1 I, T^{-1}A_2T - \mu_2 I, \cdots, T^{-1}A_nT - \mu_n I\rangle$$
$$= \langle B_1, B_2, \cdots, B_{n'}\rangle$$

And we denote the equivalence relation by

$$\langle A_1, A_2, \cdots, A_n\rangle \sim \langle B_1, B_2, \cdots, B_{n'}\rangle.$$

Remark. Note that if one of the equivalent matrix families is simultaneously symmetrizable then the others are also simultaneously symmetrizable.

By using the above operations a) and b), it is easy to see that any matrix family is equivalent to some $\langle B_1, \cdots, B_n\rangle$ where $B_1, B_2, \cdots, B_n$ are linearly independent and none of their nonzero linear combinations is equal to any scalar multiple of identity. Let us define a word indicating this property for later convenience.

Definition 2.8. A matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ is called nondegenerate if $I, A_1, A_2, \cdots, A_n$ are linearly independent over reals.

Note that the definitions in this section are valid for the square matrices of an arbitrary size, though we limit ourselves to study 3×3 matrix families which are uniformly or non-uniformly real-diagonalizable. And we shall treat the problem purely as the one in the matrix theory and shall not refer to the differential equation (1.1) any more.

3. Matrix families with multiple eigenvalues

We study first the real-diagonalizable matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ such that $A(\xi)$ has a multiple eigenvalue for some $\xi \neq 0$, and study later matrix families with real distinct eigenvalues. However, we had better cite the following result of Friedland, Robbin and Sylvester at this stage.

Theorem 3.1. *Let A, B, C be arbitrary real 3×3 matrices. Then it is impossible that $\langle A, B, C\rangle$ has real distinct eigenvalues.*

For the proof, see Friedland-Robbin-Sylvester [2] where these authors treat the problem for square matrices of arbitrary size.

Let us turn to the matrix families with multiple eigenvalues. For such a family $\langle A_1, A_2, \cdots, A_n\rangle$, changing the basis if necessary, we may assume A_1 has a multiple eigenvalue. If this multiple eigenvalue is triple, the 3×3 matrix A_1 must be a scalar multiple of identity and we may ignore it (see Definition 2.7). So we may assume A_1 has a double eigenvalue. By use of the similarity transformation diagonalizing A_1 and the addition of an appropriate scalar multiple of identity, we have

$$\langle A_1, A_2, \cdots, A_n\rangle$$
$$\sim\langle B_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, B_2, B_3, \cdots, B_n\rangle$$

with certain $B_2, B_3, \cdots, B_n$. We now reduce B_2 by the similarity transformation with

$$T=\begin{bmatrix}1&0&0\\0& & \\0& \multicolumn{2}{c}{U}\end{bmatrix}$$

where U is a nonsingular 2×2 matrix. Note that this type of similarity transformation leaves B_1 invariant and reduces the right-lower submatrix of B_2 to one of the following canonical forms.

$$\begin{bmatrix}\alpha&0\\0&\alpha\end{bmatrix}, \begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}, \begin{bmatrix}\alpha&1\\0&\alpha\end{bmatrix}, \begin{bmatrix}\alpha&\beta\\-\beta&\alpha\end{bmatrix}.$$

By use of this similarity transformation, $\langle A_1, A_2, \cdots, A_n\rangle$ and $\langle B_1, B_2, \cdots, B_n\rangle$ are equivalent to one of the following.

$$\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&b_1&b_2\\b_3&0&0\\b_4&0&0\end{bmatrix}, \cdots\rangle,$$

$$\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&b_1&b_2\\b_3&1&0\\b_4&0&-1\end{bmatrix}, \cdots\rangle,$$

$$\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&b_1&b_2\\b_3&0&1\\b_4&0&0\end{bmatrix}, \cdots\rangle,$$

$$\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&b_1&b_2\\b_3&0&1\\b_4&-1&0\end{bmatrix}, \cdots\rangle$$

where b_1, b_2, b_3, b_4 are certain real constants.

Let us consider each case separately, first for familes spanned by two matrices.

Proposition 3.2. *The matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0&b_1&b_2\\b_3&0&0\\b_4&0&0\end{bmatrix}\neq 0$$

is real-diagonalizable if and only if

$$b_1b_3+b_2b_4>0.$$

Proof. It suffices to find out under what condition

$$\xi A+B=\begin{bmatrix}\xi&b_1&b_2\\b_3&0&0\\b_4&0&0\end{bmatrix} \tag{3.1}$$

is similar to a real diagonal matrix for any $\xi\in\boldsymbol{R}$. The characteristic equation of $\xi A+B$ turns out to be

$$\begin{aligned}&\det(-\lambda I+\xi A+B)=0,\\&-\lambda(\lambda^2-\xi\lambda-b_1b_3-b_2b_4)=0.\end{aligned} \tag{3.2}$$

We split the case into three; $b_1b_3+b_2b_4>, =, <0$. When $b_1b_3+b_2b_4>0$, (3.2) has three real distinct roots (zero, positive, negative) for any $\xi\in\boldsymbol{R}$ and (3.1) is similar to a real diagonal matrix. When $b_1b_3+b_2b_4<0$, (3.2) has imaginary roots for $\xi=0$. When $b_1b_3+b_2b_4=0$, (3.2) with $\xi=0$ has 0 as a triple root but $0A+B\neq 0$ is not similar to the zero matrix. Thus we have completed the proof. □

Proposition 3.3. *Let the matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0&b_1&b_2\\b_3&0&0\\b_4&0&0\end{bmatrix}\neq 0$$

be real-diagonalizable. Then it is simultaneously symmetrized by some T *as follows.*

$$T^{-1}AT=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad T^{-1}BT=\alpha\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$$

where α *is some real constant.*

Proof. From Proposition 3.2, we have

$$b_1b_3+b_2b_4>0.$$

Setting

$$\alpha=\sqrt{b_1b_3+b_2b_4}$$

$$T=\begin{bmatrix}1 & 0 & 0\\ 0 & b_3/\alpha & -b_2/\alpha\\ 0 & b_4/\alpha & b_1/\alpha\end{bmatrix}$$

we obtain the conclusion. □

Proposition 3.4. *The matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ B=\begin{bmatrix}0 & b_1 & b_2\\ b_3 & 1 & 0\\ b_4 & 0 & -1\end{bmatrix}$$

is real-diagonalizable if and only if

$$\operatorname{sgn} b_1=\operatorname{sgn} b_3 \quad \textit{and} \quad \operatorname{sgn} b_2=\operatorname{sgn} b_4 .$$

Proof. It suffices to find out under what condition

$$\xi A+B=\begin{bmatrix}\xi & b_1 & b_2\\ b_3 & 1 & 0\\ b_4 & 0 & -1\end{bmatrix} \tag{3.3}$$

is similar to a real diagonal matrix for any $\xi\in\boldsymbol{R}$. Now the characteristic equation of $\xi A+B$ turns out to be

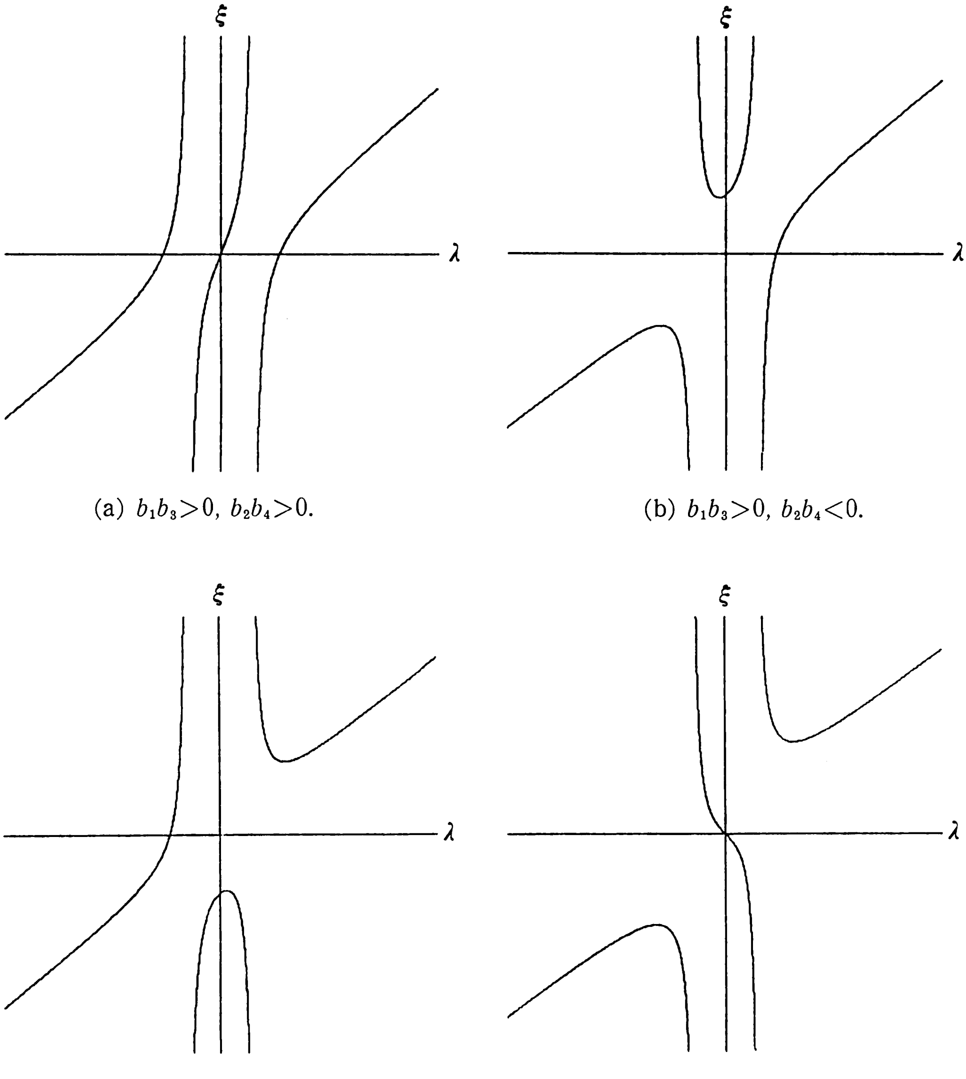
$$\begin{gathered}\det(-\lambda I+\xi A+B)=0,\\ (\xi-\lambda)(\lambda^2-1)+b_1b_3(\lambda+1)+b_2b_4(\lambda-1)=0.\end{gathered} \tag{3.4}$$

First we consider the case when $b_1b_3\neq0$ and $b_2b_4\neq0$. In this case, (3.4) has never ±1 as roots, its graph in λ, ξ-plane is the same as that of

$$\xi=\lambda-\frac{b_1b_3}{\lambda-1}-\frac{b_2b_4}{\lambda+1} \tag{3.5}$$

Let us plot the graph of (3.4) in the form of (3.5), noticing (3.4) has at most three real roots (counting their multiplicity) for each fixed ξ. From Fig. 1.(a), (b), (c), (d), we have the following. If $b_1b_3>0$ and $b_2b_4>0$ then (3.3) with any $\xi\in\boldsymbol{R}$ has three real distinct eigenvalues and is similar to a real diagonal matrix. If $b_1b_3<0$ or $b_2b_4<0$ then (3.3) with some $\xi\in\boldsymbol{R}$ has just one real eigenvalue and two imaginary ones. Thus the proposition is proved when $b_1b_3\neq0$ and $b_2b_4\neq0$ hold at once.

Each of remaining two cases is much simpler because (3.4) has always 1 or -1 as a root when $b_1b_3=0$ or $b_2b_4=0$ respectively. This root is repeated when $\xi=-b_2b_4/2+1$ (resp. $\xi=b_1b_3/2-1$) and corresponding eigenspace of (3.3) is 2-dimensional if $b_1=b_3=0$ (resp. $b_2=b_4=0$) and 1-dimensional if $b_1\neq0$ or $b_3\neq0$ (resp. $b_2\neq0$ or $b_4\neq0$). This and



(a) $b_1b_3>0,\ b_2b_4>0.$ (b) $b_1b_3>0,\ b_2b_4<0.$

(c) $b_1b_3<0,\ b_2b_4>0.$ (d) $b_1b_3<0,\ b_2b_4<0.$

Fig. 1.

the following facts complete the proof of the proposition. If one of b_1b_3 and b_2b_4 is zero and the other negative, then (3.4) has just one real root and two imaginary ones for some $\xi\in\boldsymbol{R}$. □

Proposition 3.5. *Let the matrix family $\langle A, B\rangle$ spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0&b_1&b_2\\b_3&1&0\\b_4&0&-1\end{bmatrix}$$

be real-diagonalizable. Then it is simultaneously symmetrized by some T as follows.

$$T^{-1}AT=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ T^{-1}BT=\begin{bmatrix}0&\alpha&\beta\\\alpha&1&0\\\beta&0&-1\end{bmatrix}$$

where α, β are some real constants.

Proof. From Proposition 3.4, we have

$$\operatorname{sgn} b_1=\operatorname{sgn} b_3 \quad\text{and}\quad \operatorname{sgn} b_2=\operatorname{sgn} b_4 .$$

Setting

$$\alpha=\sqrt{b_1b_3},\quad \beta=\sqrt{b_2b_4},$$

$$u\begin{cases}=\alpha/b_1 & (\text{if } b_1b_3>0),\\ =1 & (\text{if } b_1=b_3=0),\end{cases}$$

$$v\begin{cases}=\beta/b_2 & (\text{if } b_2b_4>0),\\ =1 & (\text{if } b_2=b_4=0),\end{cases}$$

$$T=\begin{bmatrix}1&0&0\\0&u&0\\0&0&v\end{bmatrix},$$

we have the desired result. □

Proposition 3.6. *The matrix family $\langle A, B\rangle$ spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&b_1&b_2\\b_3&0&1\\b_4&0&0\end{bmatrix}$$

is real-diagonalizable if and only if one of the following holds.

1) $b_1b_3>0$ *and* $b_4=0$.

2) $b_2b_4>0$ *and* $b_1=0$.

Proof. It suffices to find out under what condition

$$\xi A+B=\begin{bmatrix}\xi&b_1&b_2\\b_3&0&1\\b_4&0&0\end{bmatrix} \tag{3.6}$$

is similar to a real-diagonal matrix for any $\xi\in\boldsymbol{R}$. Now the characteristic equation of $\xi A+B$ turns out to be

$$\det(-\lambda I+\xi A+B)=0,$$

$$\lambda^2(\xi-\lambda)+(b_1b_3+b_2b_4)\lambda+b_1b_4=0. \tag{3.7}$$

We first consider the case $b_1b_4\neq 0$. In this case, the graph of (3.7) in λ, ξ-plane is the same as that of

$$\xi=\lambda-\frac{b_1b_3+b_2b_4}{\lambda}+\frac{b_1b_4}{\lambda^2}. \tag{3.8}$$

From Fig. 2.(a), (b), we obtain that for some ξ, (3.8), namely (3.7) has just one real root and two imaginary ones.

Let us go on to the case $b_1b_4=0$. In this case (3.7) turns to be

$$-\lambda(\lambda^2-\xi\lambda-b_1b_3-b_2b_4)=0. \tag{3.9}$$

So if $b_1b_4=0$ and $b_1b_3+b_2b_4>0$ then $\xi A+B$ for any $\xi\in\boldsymbol{R}$ has three real distinct eigenvalues (zero, positive, negative) and is similar to a real diagonal matrix. If $b_1b_4=0$ and $b_1b_3+b_2b_4<0$ then $0A+B$ (i.e., $\xi=0$) has imaginary eigenvalues. If $b_1b_4=0$ and $b_1b_3+b_2b_4=0$ then $B=0A+B$ has 0 as a triple eigenvalue but is not similar to the zero matrix.

We have proved $\langle A, B\rangle$ is real-diagonalizable if and only if

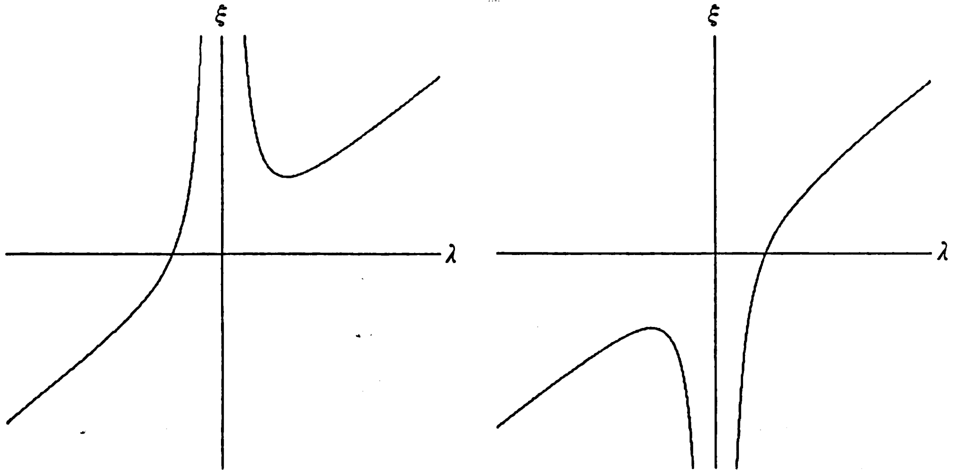
$$b_1b_4=0 \quad\text{and}\quad b_1b_3+b_2b_4>0.$$

From this, the conclusion follows. □

Proposition 3.7. *The following holds.*

1) *Let* $b_1b_3>0$. *Then*

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&b_1&b_2\\b_3&0&1\\0&0&0\end{bmatrix}\right\rangle\sim\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}\right\rangle$$



(a) $b_1b_4>0$. (b) $b_1b_4<0$.

Fig. 2.

2) *Let* $b_2b_4>0$. *Then*

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&b_2\\b_3&0&1\\b_4&0&0\end{bmatrix}\right\rangle\sim\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&1&0\end{bmatrix}\right\rangle$$

And both matrix families are non-uniformly real-diagonalizable.

Proof. We begin with 1). Setting

$$\alpha=\sqrt{b_1b_3},$$

$$T=\begin{bmatrix}1/b_3&0&0\\0&1/\alpha&-b_2/b_1\\0&0&1\end{bmatrix},$$

we obtain

$$T^{-1}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}T=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

$$T^{-1}\begin{bmatrix}0&b_1&b_2\\b_3&0&1\\0&0&0\end{bmatrix}T=\alpha\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}.$$

The case of 2) can be reduced to the transposition of 1) because

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}^{-1}\begin{bmatrix}0&0&b_2\\b_3&0&1\\b_4&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}0&b_2&0\\b_4&0&0\\b_3&1&0\end{bmatrix}.$$

To end the proof, we have only to show the real-diagonalizabilty of

$$\xi\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\eta\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}\qquad(\xi,\ \eta\in\boldsymbol{R})$$

is not uniform. For this purpose, it suffices to calculate its three eigenvectors and construct a diagonalizer. See Kasahara-Yamaguti [3] for detail. □

Proposition 3.8. *Given a matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B=\begin{bmatrix} 0 & b_1 & b_2 \\ b_3 & 0 & 1 \\ b_4 & -1 & 0 \end{bmatrix}$$

where b_1, b_2, b_3, b_4 *are arbitrary real constants. Then* $\langle A, B\rangle$ *is not real-diagonalizable.*

Proof. It is sufficient to prove, for ξ large enough,

$$\xi A+B=\begin{bmatrix} \xi & b_1 & b_2 \\ b_3 & 0 & 1 \\ b_4 & -1 & 0 \end{bmatrix} \tag{3.10}$$

has imaginary eigenvalues. Now the characteristic equation of $\xi A+B$ turns out to be

$$\begin{aligned} &\det(-\lambda I+\xi A+B)=0, \\ &(\lambda^2+1)(\xi-\lambda)+(b_1b_3+b_2b_4)\lambda+b_1b_4-b_2b_3=0. \end{aligned} \tag{3.11}$$

The graph of (3.11) in λ, ξ-plane is clearly the same as that of

$$\xi=\lambda-\frac{(b_1b_3+b_2b_4)+b_1b_4-b_2b_3}{\lambda^2+1}. \tag{3.12}$$

It is easy to see from Fig. 3, (3.11) has just one real root and two imaginary ones when ξ is large enough. □

Combining the results obtained in this section, we have the following theorem.

Theorem 3.9. *Let* $\langle A, B\rangle$ *be a nondegenerate* 3×3 *matrix family. Then the following holds.*

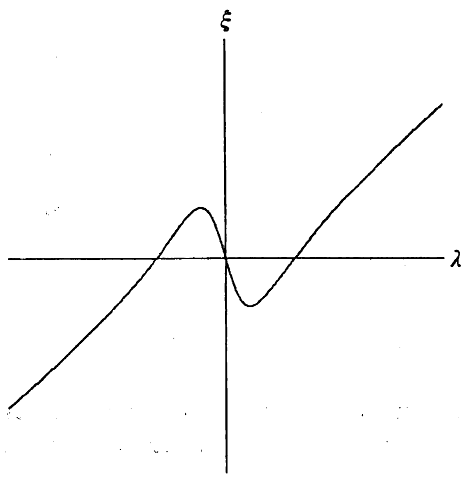


Fig. 3.

1) *Suppose that* $\langle A, B\rangle$ *has multiple eigenvalues and is uniformly real-diagonalizable. Then* $\langle A, B\rangle$ *is simultaneously symmetrizable.*

2) *Suppose that* $\langle A, B\rangle$ *is non-uniformly real-diagonalizable (consequently,* $\langle A, B\rangle$ *must have multiple eigenvalues). Then* $\langle A, B\rangle$ *is equivalent to either*

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle$$

or its transposition

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle.$$

4. Real-diagonalizable families spanned by three matrices

In this section, we study nondegenerate real-diagonalizable families spanned by three matrices. From Theorem 3.1, such a family, say $\langle A, B, C\rangle$, contains a certain member which has a double eigenvalue (recall that the nondegenerate $\langle A, B, C\rangle$ does not contain I). Therefore

$$\langle A, B, C\rangle \sim \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B', C' \right\rangle$$

where B', C' are appropriate matrices. From the propositions in Section 3, we may specify B' as one of the following.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_1 & b_2 \\ b_1 & 1 & 0 \\ b_2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We begin with the first two cases, when every member of the matrix family has a right-lower 2×2 submatrix similar to a real diagonal one. Changing the basis if necessary, such a matrix family must be equivalent to one of the following.

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & c_1 & c_2 \\ c_3 & 0 & 0 \\ c_4 & 0 & 0 \end{bmatrix} \right\rangle, \tag{4.1}$$

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_1 & b_2 \\ b_1 & 1 & 0 \\ b_2 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & c_1 & c_2 \\ c_3 & 0 & 0 \\ c_4 & 0 & 0 \end{bmatrix} \right\rangle, \tag{4.2}$$

$$\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&\alpha&\beta\\\alpha&1&0\\\beta&0&-1\end{bmatrix},\begin{bmatrix}0&a&b\\c&0&e\\d&f&0\end{bmatrix}\rangle \tag{4.3}$$

where $ef>0$. The property $ef>0$ of (4.3) is derived as follows. Because every member of (4.3) must have a right-lower 2×2 matrix similar to a real diagonal one, $e=f=0$ or $ef>0$. But (4.3) is reduced to (4.2) if $e=f=0$. Let us further reduce (4.3). By use of the similarity transformation with

$$T=\begin{bmatrix}1&0&0\\0&\pm1&0\\0&0&\sqrt{f/e}\end{bmatrix},$$

(4.3) is equivalent to

$$\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&b_1&b_2\\b_3&1&0\\b_4&0&-1\end{bmatrix},\begin{bmatrix}0&c_1&c_2\\c_3&0&1\\c_4&1&0\end{bmatrix}\rangle. \tag{4.3'}$$

Let us treat (4.1), (4.2), (4.3′) separately.

Proposition 4.1. *The nondegenerate matrix family $\langle A, B, C\rangle$ spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\ C=\begin{bmatrix}0&c_1&c_2\\c_3&0&0\\c_4&0&0\end{bmatrix}$$

is real-diagonalizable if and only if

$$\langle A, B, C\rangle\sim\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&\alpha&1\\-\alpha&0&0\\1&0&0\end{bmatrix}\rangle$$

where $0\leqq\alpha<1$ is satisfied. And in this case, $\langle A, B, C\rangle$ is uniformly real-diagonalizable.

Remark. The matrix family in the proposition

$$\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&\alpha&1\\-\alpha&0&0\\1&0&0\end{bmatrix}\rangle$$

is not simultaneously symmetrizable when $0<\alpha<1$ as will be proved in the following Lemma 4.2.

Proof. It is easy to see

$$\langle A, B, C\rangle=\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&c&c_2\\-c&0&0\\c_4&0&0\end{bmatrix}\rangle$$

for some $c \in \boldsymbol{R}$. Applying Proposition 3.2 to the real-diagonalizable subfamily

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & c & c_2 \\ -c & 0 & 0 \\ c_4 & 0 & 0 \end{bmatrix} \right\rangle$$

we have

$$c_2 c_4 > c^2 \geqq 0. \tag{4.4}$$

Using the similarity transformation with

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \sqrt{c_4/c_2} \end{bmatrix}$$

we obtain

$$\langle A, B, C \rangle \sim \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 1 \\ -\alpha & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \tag{4.5}$$

where $0 \leqq \alpha < 1$. The last inequality follows from (4.4) because α is determined by

$$\alpha = \frac{|c|}{\sqrt{c_2 c_4}}.$$

Let us now prove the right side family of (4.5) is uniformly real-diagonalizable. We set

$$\varphi = \varphi(\eta, \zeta) = [\eta^2 + (1-\alpha^2)\zeta^2]^{1/2}$$

$$S(\eta, \zeta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\eta - \alpha\zeta)/\varphi & -\zeta/\varphi \\ 0 & \zeta/\varphi & (\eta + \alpha\zeta)/\varphi \end{bmatrix} \quad \text{for} \quad (\eta, \zeta) \neq (0, 0),$$

$$S(0, 0) = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus we obtain the uniformity of $S(\eta, \zeta)$ and $S(\eta, \zeta)^{-1}$ as well as

$$S(\eta, \zeta)^{-1}\{\xi A + \eta B + \zeta C\} S(\eta, \zeta) = \begin{bmatrix} \xi & \varphi(\eta, \zeta) & 0 \\ \varphi(\eta, \zeta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $\xi A + \eta B + \zeta C$ is uniformly symmetrized. Therefore $\langle A, B, C \rangle$ is uniformly real-diagonalizable. □

Lemma 4.2. *The matrix family $\langle A, B, C \rangle$ spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\ C=\begin{bmatrix}0&\alpha&1\\-\alpha&0&0\\1&0&0\end{bmatrix}\qquad(\alpha\neq 0)$$

is not simultaneously symmetrizable.

Proof. In order to prove the lemma by contradiction, we assume there exists T such that

$$T^{-1}AT,\quad T^{-1}BT,\quad T^{-1}CT$$

are simultaneously symmetric. So we can diagonalize $T^{-1}AT$ by an orthogonal O as follows.

$$O^{-1}T^{-1}ATO=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}=A.$$

This means A and TO commute. Hence replacing TO by its appropriate scalar multiple if necessary, we may conclude it has the following form.

$$TO=\begin{bmatrix}1&0&0\\0&a&b\\0&c&d\end{bmatrix}\qquad(ad-bc\neq 0).$$

We set another orthogonal matrix O_1.

$$O_1=\begin{bmatrix}1&0&0\\0&\dfrac{a}{\sqrt{a^2+b^2}}&\dfrac{-b}{\sqrt{a^2+b^2}}\\0&\dfrac{b}{\sqrt{a^2+b^2}}&\dfrac{a}{\sqrt{a^2+b^2}}\end{bmatrix}.$$

Then $T_1=TOO_1$ has the following form.

$$T_1=\begin{bmatrix}1&0&0\\0&a'&0\\0&c'&d'\end{bmatrix}$$

where $a'>0$, c', d' are certain real constants. Because

$$T_1^{-1}BT_1=(OO_1)^{-1}T^{-1}BT(OO_1)$$

is symmetric, its (2, 1)- and (1, 2)-entries are equal and so are its (3, 1)- and (1, 3)-entries;

$$1/a'=a'>0$$
$$-c'/a'd'=0.$$

From this we have $a'=1$ and $c'=0$, that is, T_1 has the following form;

$$T_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & d'\end{bmatrix}.$$

On the other hand,

$$T_1^{-1}C\,T_1=(O\,O_1)^{-1}T^{-1}C\,T(O\,O_1)$$

must also be symmetric. However, its (2, 1)- and (1, 2)-entries are not equal because they are α and $-\alpha$ respectively (recall that $\alpha\neq 0$ by assumption). We are thus led to a contradiction. □

Proposition 4.3. *The nondegenerate matrix family $\langle A, B, C\rangle$ spanned by*

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ B=\begin{bmatrix}0 & b_1 & b_2\\ b_3 & 1 & 0\\ b_4 & 0 & -1\end{bmatrix},\ C=\begin{bmatrix}0 & c_1 & c_2\\ c_3 & 0 & 0\\ c_4 & 0 & 0\end{bmatrix}\neq 0$$

is real-diagonalizable if and only if $\langle A, B, C\rangle$ is simultaneously symmetrizable.

Proof. With any fixed $s\in\boldsymbol{R}$,

$$\langle A, B+sC\rangle=\left\langle\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\begin{bmatrix}0 & b_1+c_1s & b_2+c_2s\\ b_3+c_3s & 1 & 0\\ b_4+c_4s & 0 & -1\end{bmatrix}\right\rangle$$

is real-diagonalizable. So, from Proposition 3.4,

$$\begin{aligned}&\operatorname{sgn}(b_3+c_3s)=\operatorname{sgn}(b_1+c_1s),\\ &\operatorname{sgn}(b_4+c_4s)=\operatorname{sgn}(b_2+c_2s)\end{aligned}\tag{4.6}$$

for any $s\in\boldsymbol{R}$. Notice that two linear polynomials have always the same sign if and only if one of them is a positive multiple of the other. From this fact and (4.6), there exist positive constants $k_1>0$ and $k_2>0$ such that

$$b_3=k_1b_1,\quad c_3=k_1c_1,$$

$$b_4=k_2b_2,\quad c_4=k_2c_2.$$

This means $\langle A, B, C\rangle$ is simultaneously symmetrizable by the similarity transformation with

$$T=\begin{bmatrix}1 & 0 & 0\\ 0 & \sqrt{k_1} & 0\\ 0 & 0 & \sqrt{k_2}\end{bmatrix}.$$

Thus the proof is complete. □

Proposition 4.4. *The nondegenerate matrix family $\langle A, B, C\rangle$ spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&b_1&b_2\\b_3&1&0\\b_4&0&-1\end{bmatrix},\ C=\begin{bmatrix}0&c_1&c_2\\c_3&0&1\\c_4&1&0\end{bmatrix}$$

is real-diagonalizable if and only if $\langle A, B, C\rangle$ *is either simultaneously symmetrizable or*

$$\langle A, B, C\rangle\sim\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&\alpha&0\\\alpha&1&0\\0&0&-1\end{bmatrix},\begin{bmatrix}0&\beta&\gamma\\\beta'&0&1\\\gamma'&1&0\end{bmatrix}\right\rangle$$

where

$$\alpha>0,\quad \gamma>\frac{1}{2}\left(\alpha+\frac{\beta^2}{\alpha}\right),\quad \gamma'>\frac{1}{2}\left(\alpha+\frac{\beta'^2}{\alpha}\right).$$

Also in the second case, $\langle A, B, C\rangle$ *is uniformly real-diagonalizable.*

Remark. The matrix family in the proposition

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&\alpha&0\\\alpha&1&0\\0&0&-1\end{bmatrix},\begin{bmatrix}0&\beta&\gamma\\\beta'&0&1\\\gamma'&1&0\end{bmatrix}\right\rangle$$

with

$$\alpha>0,\quad \gamma>\frac{1}{2}\left(\alpha+\frac{\beta^2}{\alpha}\right),\quad \gamma'>\frac{1}{2}\left(\alpha+\frac{\beta'^2}{\alpha}\right).$$

is not simultaneously symmetrizable when $(\beta', \gamma')\neq(\beta, \gamma)$ as will be proved in Lemma 4.5.

Proof. Let us begin with the necessity. For any fixed $s\in\boldsymbol{R}$, the subfamily

$$\langle A, (s^2-1)B+2sC\rangle$$

$$=\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&b_1(s^2-1)+2c_1s&b_2(s^2-1)+2c_2s\\b_3(s^2-1)+2c_3s&s^2-1&2s\\b_4(s^2-1)+2c_4s&2s&-(s^2-1)\end{bmatrix}\right\rangle \tag{4.7}$$

is also real-diagonalizable. Using the similarity transformation with

$$T=\begin{bmatrix}1&0&0\\0&s&-1\\0&1&s\end{bmatrix},$$

we know that (4.7) is equivalent to

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & f_1(s) & f_2(s) \\ \dfrac{f_3(s)}{s^2+1} & s^2+1 & 0 \\ \dfrac{f_4(s)}{s^2+1} & 0 & -(s^2+1) \end{bmatrix} \right\rangle$$

where

$$\begin{aligned} f_1(s)&=(s^2-1)(b_1s+b_2)+2s(c_1s+c_2), \\ f_2(s)&=(s^2-1)(b_2s-b_1)+2s(c_2s-c_1), \\ f_3(s)&=(s^2-1)(b_3s+b_4)+2s(c_3s+c_4), \\ f_4(s)&=(s^2-1)(b_4s-b_3)+2s(c_4s-c_3). \end{aligned} \tag{4.8}$$

From Proposition 3.4,

$$\operatorname{sgn} f_1(s)=\operatorname{sgn} f_3(s) \quad \text{and} \quad \operatorname{sgn} f_2(s)=\operatorname{sgn} f_4(s). \tag{4.9}$$

These equalities hold for any $s\in\boldsymbol{R}$ (recall that we have chosen s arbitrarily). Especially, the cubic equations $f_2(s)=0$ and $f_4(s)=0$ have the same real roots one of which we denote by s_0. So, using the similarity transformation with

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & s_0 & -1 \\ 0 & 1 & s_0 \end{bmatrix},$$

we have

$$\begin{aligned} &\langle A,\ B,\ C\rangle \\ &=\langle A,\ (s_0^2-1)B+2s_0C,\ -2s_0B+(s_0^2-1)C\rangle \\ &\sim\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ * & s_0^2+1 & 0 \\ 0 & 0 & -(s_0^2+1) \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ * & 0 & s_0^2+1 \\ * & s_0^2+1 & 0 \end{bmatrix} \right\rangle \\ &=\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ * & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{bmatrix} \right\rangle \end{aligned}$$

where each $*$ stands for a certain real constant.

From the result just obtained, we may assume $b_2=b_4=0$ in B from the beginning. And (4.8) is reduced to

$$\begin{aligned} f_1(s)&=s(b_1s^2+2c_1s-b_1+2c_2), \\ f_2(s)&=(-b_1+2c_2)s^2-2c_1s+b_1, \\ f_3(s)&=s(b_3s^2+2c_3s-b_3+2c_4), \\ f_4(s)&=(-b_3+2c_4)s^2-2c_3s+b_3. \end{aligned}$$

From (4.9), we have

$$\operatorname{sgn}\{(-b_1+2c_2)s^2-2c_1s+b_1\} = \operatorname{sgn}\{(-b_3+2c_4)s^2-2c_3s+b_3\} \tag{4.10}$$

for any $s\in \boldsymbol{R}$. There are only two cases. In the first case, there exists a positive constant $k>0$ such that

$$(-b_3+2c_4)s^2-2c_3s+b_3\equiv k\{(-b_1+2c_2)s^2-2c_1s+b_1\}$$

which means

$$b_3=kb_1,\quad c_3=kc_1,\quad c_4=kc_2.$$

Therefore $\langle A, B, C\rangle$ can be symmetrized by the similarity transformation with

$$T=\begin{bmatrix} 1/\sqrt{k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us go on to the other case where (4.10) holds. In this case,

$$(-b_1+2c_2)s^2-2c_1s+b_1,$$
$$(-b_3+2c_4)s^2-2c_3s+b_3$$

have the same constant sign and they do not vanish for any $s\in \boldsymbol{R}$. Thus we have

$$\begin{aligned} &b_1b_3>0,\\ &c_1^2<b_1(-b_1+2c_2),\\ &c_3^2<b_3(-b_3+2c_4). \end{aligned} \tag{4.11}$$

We set

$$\alpha=\sqrt{b_1b_3}$$

$$T=\begin{bmatrix} \pm\sqrt{b_1/b_3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the sign $\pm$ is taken appropriately. With this T and (4.11),

$$T^{-1}AT=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ T^{-1}BT=\begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\ T^{-1}CT=\begin{bmatrix} 0 & \beta & \gamma \\ \beta' & 0 & 1 \\ \gamma' & 1 & 0 \end{bmatrix}$$

where the constants α, β, β', γ, γ' satisfy

$$\alpha>0,\quad \gamma>\frac{1}{2}\left(\alpha+\frac{\beta^2}{\alpha}\right),\quad \gamma'>\frac{1}{2}\left(\alpha+\frac{\beta'^2}{\alpha}\right). \tag{4.12}$$

In order to complete the proof, we have only to prove the uniform real-diagonalizability of

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \beta & \gamma \\ \beta' & 0 & 1 \\ \gamma' & 1 & 0 \end{bmatrix} \right\rangle$$

with (4.12). Let us symmetrize

$$M(\xi, \eta, \zeta)=\xi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +(\eta^2-\zeta^2) \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} +2\eta\zeta \begin{bmatrix} 0 & \beta & \gamma \\ \beta' & 0 & 1 \\ \gamma' & 1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} \xi & \alpha(\eta^2-\zeta^2)+2\beta\eta\zeta & 2\gamma\eta\zeta \\ \alpha(\eta^2-\zeta^2)+2\beta'\eta\zeta & \eta^2-\zeta^2 & 2\eta\zeta \\ 2\gamma'\eta\zeta & 2\eta\zeta & -\eta^2+\zeta^2 \end{bmatrix}.$$

Notice that $(\eta, \zeta)\to(\eta^2-\zeta^2, 2\eta\zeta)$ maps $\boldsymbol{R}^2$ onto $\boldsymbol{R}^2$ because

$$(\eta+\zeta\sqrt{-1})^2=\eta^2-\zeta^2+2\eta\zeta\sqrt{-1}.$$

We set

$$V(\eta, \zeta)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\eta}{\sqrt{\eta^2+\zeta^2}} & \dfrac{-\zeta}{\sqrt{\eta^2+\zeta^2}} \\ 0 & \dfrac{\zeta}{\sqrt{\eta^2+\zeta^2}} & \dfrac{\eta}{\sqrt{\eta^2+\zeta^2}} \end{bmatrix},$$

$$W(\eta, \zeta)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \left[\dfrac{\alpha\eta^2+2\beta'\eta\zeta+(2\gamma'-\alpha)\zeta^2}{\alpha\eta^2+2\beta\eta\zeta+(2\gamma-\alpha)\zeta^2}\right]^{1/2} & 0 \\ 0 & 0 & \left[\dfrac{(2\gamma'-\alpha)\eta^2-2\beta'\eta\zeta+\alpha\zeta^2}{(2\gamma-\alpha)\eta^2-2\beta\eta\zeta+\alpha\zeta^2}\right]^{1/2} \end{bmatrix}$$

for $(\eta, \zeta)\neq(0, 0)$ and

$$V(0, 0)=W(0, 0)=I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We set also

$$S(\eta, \zeta)=V(\eta, \zeta)W(\eta, \zeta).$$

Thus we obtain the symmetricity of

$$S(\eta, \zeta)^{-1}M(\xi, \eta, \zeta)S(\eta, \zeta)$$

as well as the uniformity of $S(\eta, \zeta)$ and $S(\eta, \zeta)^{-1}$. Because $M(\xi, \eta, \zeta)$ is uniformly symmetrized, it can be uniformly real-diagonalizable. The proof is complete. □

Lemma 4.5. *Let the matrix family $\langle A, B, C\rangle$ be spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&\alpha&0\\\alpha&1&0\\0&0&-1\end{bmatrix},\ C=\begin{bmatrix}0&\beta&\gamma\\\beta'&0&1\\\gamma'&1&0\end{bmatrix}$$

where $\alpha>0$ and $(\beta', \gamma')\neq(\beta, \gamma)$. Then $\langle A, B, C\rangle$ is not simultaneously symmetrizable.

Proof. In order to prove the lemma by contradiction, we assume there exists T_1 such that

$$T_1^{-1}AT_1,\quad T_1^{-1}BT_1,\quad T_1^{-1}CT_1$$

are simultaneously symmetric. The same procedure as in the proof of Lemma 4.2 shows that we may assume T_1 has the following form.

$$T_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&d'\end{bmatrix}.$$

Because the (3, 2)- and (2, 3)-entries of $T_1^{-1}CT_1$ are equal, we have

$$1/d'=d'.$$

From this we obtain $d'=\pm1$, that is, T_1 must actually have the following form.

$$T_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\pm1\end{bmatrix}.$$

With this T_1, however, $T_1^{-1}CT_1$ is not symmetric because of $(\beta', \gamma')\neq(\beta, \gamma)$. We are thus led to a contadiction. □

Let us now work on the last type of real-diagonalizable families mentioned at the beginning of this section, namely,

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}, C\right\rangle \tag{4.13}$$

with some matrix C or its transposed family. We shall consider only (4.13) without loss of generality. Changing the basis if necessary, we may assume (4.13) is expressed as follows.

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&c_1&c_2\\c_3&c_4&0\\c_5&c_6&-c_4\end{bmatrix}\right\rangle \tag{4.14}$$

where c_1, c_2, c_3, c_4, c_5, c_6 are certain real constants.

Proposition 4.6. *The nondegenerate matrix family* $\langle A, B, C\rangle$ *spanned by*

$$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix},\ C=\begin{bmatrix}0&c_1&c_2\\c_3&c_4&0\\c_5&c_6&-c_4\end{bmatrix}\neq 0$$

is (non-uniformly) real-diagonalizable if and only if C *is a scalar multiple of*

$$\begin{bmatrix}0&\beta&-\gamma\\\beta(1-\alpha)&1&0\\-2\alpha&0&-1\end{bmatrix}$$

where $0<\alpha<1$, $\gamma>\beta^2/8$ *are satisfied.*

Proof. Taking an arbitrary $s\in\boldsymbol{R}$, the subfamily

$$\langle A, sB+C\rangle=\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&s+c_1&c_2\\s+c_3&c_4&s\\c_5&c_6&-c_4\end{bmatrix}\right\rangle$$

is also real-diagonalizable. From Propositions 3.2, $\cdots$, 3.8, the 2×2 submatrix

$$\begin{bmatrix}c_4&s\\c_6&-c_4\end{bmatrix}$$

must have only real eigenvalues for any fixed $s\in\boldsymbol{R}$. This means

$$c_6=0.$$

Let us now prove $c_4\neq 0$ by contradiction. We assume $c_4=0$. Because the subfamily

$$\langle A, B+sC\rangle=\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&c_1s+1&c_2s\\c_3s+1&0&1\\c_5s&0&0\end{bmatrix}\right\rangle$$

with any fixed $s\in\boldsymbol{R}$ is real-diagonalizable, Proposition 3.6 is applicable. So we have

$$c_5s(c_1s+1)=0$$

$$(c_1s+1)(c_3s+1)+c_2c_5s^2>0$$

for any $s\in\boldsymbol{R}$. This implies

$$c_5=0,\quad c_1=c_3=0$$

and C has actually the following form.

$$C=\begin{bmatrix}0&0&c_2\\0&0&0\\0&0&0\end{bmatrix}\neq 0.$$

This contradicts the (real-)diagonalizability of C.

Combining the above results, we find that

$$C=\begin{bmatrix} 0 & c_1 & c_2 \\ c_3 & c_4 & 0 \\ c_5 & 0 & -c_4 \end{bmatrix}$$

with $c_4 \neq 0$. So, multiplying C by a scalar if necessary, we may assume

$$C=\begin{bmatrix} 0 & c_1 & c_2 \\ c_3 & 1 & 0 \\ c_5 & 0 & -1 \end{bmatrix}.$$

Let us consider again the real-diagonalizability of

$$\langle A,\ sB+C\rangle=\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & s+c_1 & c_2 \\ s+c_3 & 1 & s \\ c_5 & 0 & -1 \end{bmatrix}\right\rangle.$$

By the similarity transformation with

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -s/2 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$\langle A,\ sB+C\rangle\sim\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & s+c_1 & -\frac{1}{2}s^2-\frac{c_1}{2}s+c_2 \\ \left(1+\frac{c_5}{2}\right)s+c_3 & 1 & 0 \\ c_5 & 0 & -1 \end{bmatrix}\right\rangle.$$

Thus Proposition 3.4 is applicable and we have

$$\operatorname{sgn}(s+c_1)=\operatorname{sgn}\left\{\left(1+\frac{c_5}{2}\right)s+c_3\right\}$$

$$\operatorname{sgn} c_5=\operatorname{sgn}\left\{-\frac{s^2}{2}-\frac{c_1}{2}s+c_2\right\}$$

with any $s\in \boldsymbol{R}$. This means

$$c_5>-2,\quad -c_1\left(1+\frac{c_5}{2}\right)+c_3=0,$$

$$c_5<0,\quad c_1^2+8c_2<0.$$

Putting

$$\alpha=-\frac{c_5}{2},\quad \beta=c_1,\quad \gamma=-c_2,$$

we have the desired result. $\square$

Summing up the results of this section, we have the following theorem.

Teeorem 4.7. *Let $\langle A, B, C\rangle$ be a nondegenerate* 3×3 *matrix family. Then the following statements hold.*

1) *$\langle A, B, C\rangle$ is uniformly real-diagonalizable and is not simultaneously symmetrizable if and only if it is equivalent to either*

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 1 \\ -\alpha & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle \quad (0<\alpha<1)$$

or

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \beta & \gamma \\ \beta' & 0 & 1 \\ \gamma' & 1 & 0 \end{bmatrix} \right\rangle$$

with real constants

$$\alpha>0, \quad \gamma>\frac{1}{2}\left(\alpha+\frac{\beta^2}{\alpha}\right), \quad \gamma'>\frac{1}{2}\left(\alpha+\frac{\beta'^2}{\alpha}\right), \quad (\beta, \gamma)\neq(\beta', \gamma').$$

2) *$\langle A, B, C\rangle$ is non-uniformly real-diagonalizable if and only if it is equivalent to either*

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \beta & -\gamma \\ \beta(1-\alpha) & 1 & 0 \\ -2\alpha & 0 & -1 \end{bmatrix} \right\rangle$$

with real constants $0<\alpha<1$, β, $\gamma>\beta^2/8$ or its transposition.

5. Real-diagonalizable families spanned by four or more matrices

The goal of this section is the following theorem.

Theorem 5.1. *Suppose that a nondegenerate* 3×3 *matrix family $\langle A_1, A_2, \cdots, A_n\rangle$ $(n\geqq4)$ is real-diagonalizable. Then $\langle A_1, A_2, \cdots, A_n\rangle$ is simultaneously symmetrizable.*

We shall prove this theorem by several lemmas. First, by the same argument as at the beginning of section 4, we may assume

$$A_1=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Lemma 5.2. *Suppose that the same hypotheses as in Theorem* 5.1 *are satisfied. Suppose also that*

$$A_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

Then every member $A(\xi)$ *of* $\langle A_1, A_2, \cdots, A_n\rangle$ *has a real-diagonalizable right lower* 2×2 *submatrix.*

Proof. In order to prove the lemma by contradiction, we assume a certain member has a right-lower 2×2 submatrix which is not similar to any real 2×2 diagonal matrix. By the same procedure as at the beginning of section 4, we may further assume

$$A_2=\begin{bmatrix}0&1&0\\1&0&1\\0&0&0\end{bmatrix}.$$

Recalling the comment just before Proposition 4.6 and applying the proposition itself to $\langle A_1, A_2, A_3\rangle$ and $\langle A_1, A_2, A_4\rangle$, we may assume

$$A_3=\begin{bmatrix}0&*&*\\ *&1&0\\ *&0&-1\end{bmatrix},\quad A_4=\begin{bmatrix}0&*&*\\ *&1&0\\ *&0&-1\end{bmatrix}\tag{5.1}$$

where each $*$ stands for a certain real. On the other hand $\langle A_1, A_2, A_3-A_4\rangle$ must be real-diagonalizable. From (5.1), A_3-A_4 must have the form

$$A_3-A_4=\begin{bmatrix}0&*&*\\ *&0&0\\ *&0&0\end{bmatrix}\neq0$$

because A_3 and A_4 are linearly independent. This contradicts Proposition 4.6. □

With this Lemma 5.2 in mind, we repeat the same argument as the one just before Proposition 4.1. Thus, we know $\langle A_1, A_2, \cdots, A_n\rangle$ is equivalent to one of the following.

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\ *&0&0\\ *&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\ *&0&0\\ *&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\ *&0&0\\ *&0&0\end{bmatrix},\cdots\right\rangle$$

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\ *&1&0\\ *&0&-1\end{bmatrix},\begin{bmatrix}0&*&*\\ *&0&0\\ *&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\ *&0&0\\ *&0&0\end{bmatrix},\cdots\right\rangle$$

$$\left\langle\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\ *&1&0\\ *&0&-1\end{bmatrix},\begin{bmatrix}0&*&*\\ *&0&1\\ *&1&0\end{bmatrix},\begin{bmatrix}0&*&*\\ *&0&0\\ *&0&0\end{bmatrix},\cdots\right\rangle.$$

where each $*$ stands for a real number. We treat each of the cases separtely.

Lemma 5.3. *Suppose that the matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ $(n\geqq 4)$ *spanned by*

$$A_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ A_2=\begin{bmatrix}0&b_1&b_2\\b_3&0&0\\b_4&0&0\end{bmatrix},\ A_3=\begin{bmatrix}0&c_1&c_2\\c_3&0&0\\c_4&0&0\end{bmatrix},\ A_4=\begin{bmatrix}0&d_1&d_2\\d_3&0&0\\d_4&0&0\end{bmatrix}$$

is real-diagnalizable. Then $A_1, A_2, \cdots, A_n$ *are linearly dependent.*

Proof. Assume the contrary. Applying Proposition 4.1 to $\langle A_1, A_2, A_3\rangle$, we may assume

$$A_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ A_2=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\ A_3=\begin{bmatrix}0&\alpha&1\\-\alpha&0&0\\1&0&0\end{bmatrix}\qquad(0\leqq\alpha<1).$$

On the other hand

$$\langle A_1,\ A_4+(-d_1+\alpha d_2)A_2-d_2A_3\rangle$$

must be real-diagonalizable. However, since

$$A_4+(-d_1+\alpha d_2)A_2-d_2A_3\neq 0$$

has vanishing (1, 2)- and (1, 3)-entries, we get a contradiction from Proposition 3.2. □

Lemma 5.4. *Suppose that the nondegenerate matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ $(n\geqq 4)$ *spanned by*

$$A_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ A_2=\begin{bmatrix}0&b_1&b_2\\b_3&1&0\\b_4&0&-1\end{bmatrix},\ A_3=\begin{bmatrix}0&c_1&c_2\\c_3&0&0\\c_4&0&0\end{bmatrix},\ A_4=\begin{bmatrix}0&d_1&d_2\\d_3&0&0\\d_4&0&0\end{bmatrix},\ \cdots$$

is real-diagonalizable. Then $\langle A_1, A_2, \cdots, A_n\rangle$ *is simultaneously symmetrizable.*

Proof. For any fixed $s, t, u, \cdots\in\boldsymbol{R}$,

$$\langle A_1,\ A_2+sA_3+tA_4+uA_5+\cdots\rangle$$

is real-diagonalizable. We assume $n=4$ for simplicity because the argument remain the same for the other case. From Proposition 3.4,

$$\operatorname{sgn}(b_1+c_1s+d_1t)=\operatorname{sgn}(b_3+c_3s+d_3t)$$

$$\operatorname{sgn}(b_2+c_2s+d_2t)=\operatorname{sgn}(b_4+c_4s+d_4t)$$

for any $s, t\in\boldsymbol{R}$. Regarding these linear polynomials as ones in t with $s\in\boldsymbol{R}$ arbitrarily fixed, we have

$$\operatorname{sgn} d_1=\operatorname{sgn} d_3,\qquad \frac{c_1s+b_1}{d_1}=\frac{c_3s+b_3}{d_3},$$

$$\operatorname{sgn} d_2 = \operatorname{sgn} d_4, \qquad \frac{c_2 s + b_2}{d_2} = \frac{c_4 s + b_4}{d_4}.$$

Because the last equalities hold for any fixed $s \in \boldsymbol{R}$, we obtain

$$b_3 = k_1 b_1, \qquad c_3 = k_1 c_1, \qquad d_3 = k_1 d_1,$$

$$b_4 = k_2 b_2, \qquad c_4 = k_2 c_2, \qquad d_4 = k_2 d_2$$

with some $k_1 > 0$ and $k_2 > 0$. Therefore $\langle A_1, A_2, A_3, A_4 \rangle$ is simultaneously symmetrized by

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{k_1} & 0 \\ 0 & 0 & \sqrt{k_2} \end{bmatrix}.$$

Thus the proof is complete. □

Lemma 5.5. *Suppose that the nondegenerate matrix family* $\langle A_1, A_2, \cdots, A_n \rangle$ $(n \geqq 4)$ *spanned by*

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & b_1 & b_2 \\ b_3 & 1 & 0 \\ b_4 & 0 & -1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & c_1 & c_2 \\ c_3 & 0 & 1 \\ c_4 & 1 & 0 \end{bmatrix}, \ A_4 = \begin{bmatrix} 0 & d_1 & d_2 \\ d_3 & 0 & 0 \\ d_4 & 0 & 0 \end{bmatrix}, \cdots$$

is real-diagonalizable. Then $\langle A_1, A_2, \cdots, A_n \rangle$ *is simultaneously symmetrizable.*

Proof. For any fixed $s, t, u, \cdots \in \boldsymbol{R}$,

$$\langle A_1, (s^2-1)A_2 + 2sA_3 + tA_4 + uA_5 + \cdots \rangle \tag{5.2}$$

is real-diagonalizable. We assume $n=4$ for simplicity because the argument remains the same for the other cases. Using the similarity transformation with

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & -1 \\ 0 & 1 & s \end{bmatrix},$$

we know that (5.2) is equivalent to

$$\left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & f_1(s) + t g_1(s) & f_2(s) + t g_2(s) \\ \dfrac{f_3(s) + t g_3(s)}{s^2+1} & s^2+1 & 0 \\ \dfrac{f_4(s) + t g_4(s)}{s^2+1} & 0 & -(s^2+1) \end{bmatrix} \right\rangle$$

where

$$\begin{aligned} f_1(s) &= (s^2-1)(b_1 s + b_2) + 2s(c_1 s + c_2), \\ f_2(s) &= (s^2-1)(b_2 s - b_1) + 2s(c_2 s - c_1), \\ f_3(s) &= (s^2-1)(b_3 s + b_4) + 2s(c_3 s + c_4), \end{aligned} \tag{5.3}$$

$$f_4(s)=(s^2-1)(b_4 s-b_3)+2s(c_4 s-c_3),$$

(5.4)
$$g_1(s)=d_1 s+d_2,$$
$$g_2(s)=d_2 s-d_1,$$
$$g_3(s)=d_3 s+d_4,$$
$$g_4(s)=d_4 s-d_3.$$

From Proposition 3.4,

$$\operatorname{sgn}\{f_1(s)+tg_1(s)\}=\operatorname{sgn}\{f_3(s)+tg_3(s)\}$$
$$\operatorname{sgn}\{f_2(s)+tg_2(s)\}=\operatorname{sgn}\{f_4(s)+tg_4(s)\}$$

for any $s, t\in \boldsymbol{R}$. Regarding

$$f_1(s)+tg_1(s), \qquad f_3(s)+tg_3(s)$$

as linear polynomials in t for an arbitrarily fixed s, we have

$$\operatorname{sgn} g_1(s)=\operatorname{sgn} g_3(s),$$
$$\frac{f_1(s)}{g_1(s)}=\frac{f_3(s)}{g_3(s)}$$

for any fixed $s\in \boldsymbol{R}$. Hence there exists a positive constant $k>0$ such that

$$b_3=kb_1, \qquad b_4=kb_2,$$
$$c_3=kc_1, \qquad c_4=kc_2,$$
$$d_3=kd_1, \qquad d_4=kd_2.$$

Therefore $\langle A_1, A_2, A_3, A_4\rangle$ is simultaneously symmetrized by the similarity transformation with

$$T=\begin{bmatrix} 1/\sqrt{k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the proof is complete. □

Proof of Theorem 5.1. First, use Lemma 5.2. Next, use one of Lemmas 5.3, 5.4, 5.5. Then the claim follows.

6. Matrix families with real distinct eigenvalues

In this section, we study matrix families with real distinct eigenvalues. From Theorem 3.1, we know such a family is spanned by two matrices, say A and B. Thus the cubic equation in λ

(6.1)
$$\det(-\lambda I+\xi A+\eta B)=0$$

has three real distinct roots for any choice of $(\xi, \eta)\neq(0, 0)$.

Let us consider how the equivalence relation between matrix families is expressed in terms of the characteristic polynomial:

$$\det(-\lambda I+\xi A+\eta B). \tag{6.2}$$

The general form of matrix family equivalent to $\langle A, B\rangle$ is

$$\langle T^{-1}(c_{11}A+c_{21}B-c_{01}I)T,\ T^{-1}(c_{12}A+c_{22}B-c_{02}I)T\rangle$$

where $c_{11}c_{22}-c_{12}c_{21}\neq 0$. For this family, (6.2) becomes

$$\det\{-(\lambda+c_{01}\xi+c_{02}\eta)I+(c_{11}\xi+c_{12}\eta)A+(c_{21}\xi+c_{22}\eta)B\}$$

where $c_{11}c_{22}-c_{12}c_{21}\neq 0$. The last polynomial can also be obtained from (6.2) in another way, that is, by use of the new variables λ', ξ', η' determined by

$$\begin{aligned}\lambda&=\lambda'+c_{01}\xi'+c_{02}\eta'\\ \xi&=\qquad c_{11}\xi'+c_{12}\eta'\\ \eta&=\qquad c_{21}\xi'+c_{22}\eta'.\end{aligned} \tag{6.3}$$

So we shall use this type of change of variables to reduce the cubic polynomial (6.2).

Concerning the property of (6.2), there are two cases. In the first case, (6.2) is reducible and can be fatorized as a polynomial in λ, ξ, η. In the second case, (6.2) is an irreducible cubic polynomial in λ, ξ, η. We begin with the first case.

Lemma 6.1. *Suppose that a matrix family $\langle A, B\rangle$ has real distinct eigenvalues. Suppose also that the cubic polynomial* $\det(-\lambda I+\xi A+\eta B)$ *in λ, ξ, η can be factorized. Then there exists a matrix family $\langle A', B'\rangle$ equivalent to $\langle A, B\rangle$ such that*

$$\det(-\lambda I+\xi A'+\eta B')=-\lambda\{\lambda^2+k_1\xi\lambda-\xi^2-k_2\eta^2\}$$

where k_1 and $k_2>0$ are real constants.

Proof. First we show (6.2) $\det(-\lambda I+\xi A+\eta B)$ is the product of a linear factor and an irreducible quadratic one. This is the case because otherwise it would have three linear factors and $\det(-\lambda I+\xi A+\eta B)=0$ as an equation in λ would have a repeated root for certain $(\xi, \eta)\neq(0, 0)$.

The linear factor must have the form

$$\lambda+c_{01}\xi+c_{02}\eta.$$

So using the new variables λ', ξ', η' determined by

$$\begin{aligned}\lambda'&=\lambda+c_{01}\xi+c_{02}\eta,\\ \xi'&=\quad\ \xi\qquad\qquad,\\ \eta'&=\qquad\qquad\quad\eta,\end{aligned}$$

we have (6.2) is the product of λ' and an irreducible quadratic polynomial in λ', ξ', η'. Because (λ', ξ', η') corresponds to a certain matrix family $\langle A', B'\rangle$ equivalent to $\langle A, B\rangle$, we now drop primes from λ', ξ', η', A', B' for simplicity. Therefore we may assume

(6.2) has the following form from the beginning.

(6.4) $$\det(-\lambda I+\xi A+\eta B)=-\lambda\{\lambda^2+(q_{11}\xi+q_{12}\eta)\lambda+(q_{20}\xi^2+q_{21}\xi\eta+q_{22}\eta^2)\}$$

where q_{ij} are certain real constants.

We now show $q_{20}\xi^2+q_{21}\xi\eta+q_{22}\eta^2$ on the right side of (6.4) is a negative definite quadratic form in ξ, η. First we notice that it does not vanish for any $(\xi, \eta)\neq(0, 0)$ because otherwise $\det(-\lambda I+\xi A+\eta B)=0$ would have 0 as a double root for some pair of parameters $(\xi, \eta)\neq(0, 0)$. So $q_{20}\xi^2+q_{21}\xi\eta+q_{22}\eta^2$ is positive or negative definite. Actually it must be negative definite because otherwise $\det(-\lambda I+\xi A+\eta B)=0$ would have imaginary roots for the pair of parameters $(\xi, \eta)\neq(0, 0)$ satisfying $q_{11}\xi+q_{12}\eta=0$. Thus we have shown the quadratic form is negative definite.

From the results just obtained, we can introduce again new variables λ', ξ', η' determined by

$$\lambda'=\lambda$$

$$\xi'=c_{11}\xi+c_{12}\eta$$

$$\eta'=c_{21}\xi+c_{22}\eta$$

such that

$$\det(-\lambda' I+\xi' A+\eta' B)=-\lambda'\{\lambda'^2+k_1\xi'\lambda'-\xi'^2-k_2\eta'^2\}$$

where k_1 and $k_2>0$ are real constants. And this (λ', ξ', η') corresponds to another matrix family $\langle A', B'\rangle$ equivalent to $\langle A, B\rangle$. The proof is now complete. □

By virtue of Lemma 6.1, we may limit ourselves to study matrix familes $\langle A, B\rangle$ satisfying

(6.5) $$\det(-\lambda I+\xi A+\eta B)=-\lambda(\lambda^2+k_1\xi\lambda-\xi^2-k_2\eta^2)$$

where k_1 and $k_2>0$ are real constants. Substituting $(\xi, \eta)=(1, 0)$ in this equation, we know A has a zero, a positive, a negative eigenvalues. Using a similarity transformation which diagonalizes A, we may assume

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix}$$

for some $\alpha>0$.

Lemma 6.2. *Suppose*

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix},\ B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where $\alpha>0$ and b_{ij} are real constants. Then

(6.5) $$\det(-\lambda I+\xi A+\eta B)\equiv-\lambda^3-k_1\xi\lambda^2+(\xi^2+k_2\eta^2)\lambda$$

holds for some real constants k_1 and $k_2>0$ if and only if the following 1), 2), 3), 4) *hold at once.*

1) $b_{11}=b_{22}=b_{33}=0$,

2) $b_{12}b_{21}=\alpha^2 b_{13}b_{31}$,

3) $b_{12}b_{23}b_{31}+b_{13}b_{21}b_{32}=0$,

4) $b_{12}b_{21}+b_{23}b_{32}+b_{13}b_{31}>0$.

Proof. Comparing the coefficients of $\xi^2\eta$, $\lambda^2\eta$ and $\lambda\xi\eta$ of the both sides of (6.5), we have

$$b_{11}=0,$$

$$b_{11}+b_{22}+b_{33}=0,$$

$$\left(\alpha-\frac{1}{\alpha}\right)b_{11}-\frac{1}{\alpha}b_{22}+\alpha b_{33}=0$$

which implies

$$b_{11}=b_{22}=b_{33}=0.$$

Substituting this in (6.5) and comparing the coefficients of $\xi\eta^2$, η^3 and $\lambda\eta^2$ on the both sides, we have

$$\frac{b_{12}b_{21}}{\alpha}-\alpha b_{13}b_{31}=0$$

$$b_{12}b_{23}b_{31}+b_{13}b_{21}b_{32}=0$$

$$b_{12}b_{21}+b_{23}b_{32}+b_{13}b_{31}(=k_2)>0.$$

The converse is clear. Thus the proof is complete. □

Using Lemmas 6.1 and 6.2, we have the following proposition.

Proposition 6.3. *A matrix family $\langle A, B\rangle$ has real distinct eigenvalues and the cubic polynomial* $\det(-\lambda I+\xi A+\eta B)$ *in λ, ξ, η is factorizable if and only if $\langle A, B\rangle$ is equivalent to one of the following.*

1) $$\left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 1 \\ \alpha & 0 & \beta \\ 1 & -\beta & 0 \end{bmatrix} \right\rangle$$

where the real constants satisfy $\alpha>0$ and $\beta^2<\alpha^2+1$.

2) $$\left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix}, \begin{bmatrix} 0 & \beta & \gamma \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

where β and γ are arbitrary real constants.

3) *The transposition of* 2).

Proof. From Lemma 6.1, we may assume (6.5) is valid. Hence Lemma 6.2 is applicable and we have $b_{11}=b_{22}=b_{33}=0$. We split the situation according to

$$b_{12}b_{21}\neq 0 \quad \text{or} \quad b_{12}b_{21}=0.$$

We begin with the case $b_{12}b_{21}\neq 0$. From 2) of Lemma 6.2, we have

$$\operatorname{sgn} b_{12}b_{31}=\operatorname{sgn} b_{13}b_{21}\neq 0.$$

From this and 3) of Lemma 6.2, $\operatorname{sgn} b_{23}=-\operatorname{sgn} b_{32}$ which means

$$b_{23}b_{32}\leqq 0.$$

The last inequality and 2) and 4) of Lemma 6.2 show

$$(1+\alpha^2)b_{13}b_{31}=b_{12}b_{21}+b_{13}b_{31}>-b_{23}b_{32}\geqq 0.$$

Thus we have proved $b_{12}b_{21}>0$ and $b_{13}b_{31}>0$.

Using the similarity transformation with

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\sqrt{b_{12}b_{21}}}{b_{12}} & 0 \\ 0 & 0 & \dfrac{\sqrt{b_{13}b_{31}}}{b_{13}} \end{bmatrix},$$

if necessary, we can reduce the situation as $b_{12}=b_{21}>0$ and $b_{13}=b_{31}>0$. Multiplying a positive scalar if necessary, we may further assume

$$b_{13}=b_{31}=1.$$

From 2) of Lemma 6.2 and $b_{13}=b_{31}>0$ and $\alpha>0$,

$$b_{12}=b_{21}=\alpha.$$

From 3) of Lemma 6.2, there exists some β such that

$$b_{23}=\beta, \qquad b_{32}=-\beta.$$

From 4) of Lemma 6.2,

$$\alpha^2-\beta^2+1>0.$$

Thus we have 1) of the present proposition.

Let us go on to the case where

$$b_{12}b_{21}=\alpha^2 b_{13}b_{31}=0.$$

From this and 4) of Lemma 6.2 with

$$b_{23}b_{32}>0.$$

Using the similarity transformation with

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{b_{23}b_{32}}}{b_{23}} \end{bmatrix}$$

and an appropriate scalar multiplication, we may assume

$$b_{23}=b_{32}=1.$$

From this and $b_{12}b_{21}=b_{13}b_{31}=0$ and 3) of Lemma 6.2, we have

$$b_{12}=b_{13}=0 \quad \text{or} \quad b_{21}=b_{31}=0.$$

Thus we have 2) or 3) of the proposition. □

We now go on to the case where (6.2) is an irreducible cubic polynomial in λ, ξ, η. In this case, (6.1) defines a nonsingular cubic curve in $\boldsymbol{RP}^2$ because it has three real distinct roots for any fixed $(\xi, \eta)\neq(0, 0)$.

Lemma 6.4. *Suppose that a matrix family $\langle A, B\rangle$ has real distinct eigenvalues. Suppose also that the cubic polynomial* $\det(-\lambda I+\xi A+\eta B)$ *in λ, ξ, η is irreducible. Then there exists a matrix family $\langle A', B'\rangle$ equivalent to $\langle A, B\rangle$ such that*

$$\det(-\lambda I+\xi A'+\eta B')=-\lambda\{\lambda^2+(k_1\xi+k_2\eta)\lambda-\xi^2-k_3\eta^2\}+k_4\eta^3$$

where k_1, k_2, k_3 and $k_4\neq 0$ are real constants.

Proof. First we notice that every nonsingular cubic curve in $\boldsymbol{RP}^2$ has just three real inflection points and six imaginary ones (see p 92 of van der Waerden [6] or Prop. 14 of Brieskorn-Knörrer [1]). We denote by $(\lambda_0, \xi_0, \eta_0)$ one of the real inflection points. It is easy to see

$$(\xi_0, \eta_0)\neq(0, 0)$$

because there are no points of $\det(-\lambda I+\xi A+\eta B)=0$ in $\boldsymbol{RP}^2$ satisfying $\xi=\eta=0$. And its tangent line at the inflection point $(\lambda_0, \xi_0, \eta_0)$ must have the form

$$\lambda+c_1\xi+c_2\eta=0 \tag{6.6}$$

where c_1 and c_2 are real constants. Let us prove this fact by contradiction. We assume the contrary, namely, that the tangent line at $(\lambda_0, \xi_0, \eta_0)$ is

$$\eta_0\xi-\xi_0\eta=0.$$

The definition of tangent lines and inflection points shows

$$\det(-\lambda I+\xi A+\eta B)=(\eta_0\xi-\xi_0\eta)Q(\lambda, \xi, \eta)-(\lambda+a\xi+b\eta)^3$$

where $Q(\lambda, \xi, \eta)$ is a quadratic polynomial and a, b are real constants. From this, $\det(-\lambda I+\xi A+\eta B)=0$ would have a triple root when $(\xi, \eta)=(\xi_0, \eta_0)\neq(0, 0)$. This contradiction shows the tangent line at $(\lambda_0, \xi_0, \eta_0)$ has the form (6.6).

Using the above λ_0, ξ_0, η_0, c_1, c_2, we introduce new coordinates λ', ξ', η' as

$$\lambda' = \lambda + c_1\xi + c_2\eta$$
$$\xi' = \quad \xi_0\xi + \eta_0\eta$$
$$\eta' = \quad \eta_0\xi - \xi_0\eta .$$

With these coordinates, the tangent line is expressed as $\lambda'=0$ and the inflection point is expressed as $(0, 1, 0)=(0, \xi_0^2+\eta_0^2, 0)\in \boldsymbol{RP}^2$. Therefore the cubic polynomial is expressed as

$$-\lambda'\{\lambda'^2+(q_{11}\xi'+q_{12}\eta')\lambda'+q_{20}\xi'^2+q_{21}\xi'\eta'+q_{22}\eta'^2\}+q_{33}\eta'^3$$

where q_{ij} are real constants. Notice that $q_{33}\neq 0$ because otherwise the cubic polynimial could be factorized. By virtue of the next Lemma 6.5, we also have $q_{20}<0$. So introducing new coordinates of the form

$$\lambda'' = \lambda'$$
$$\xi'' = \ c_{11}\xi' + c_{12}\eta'$$
$$\eta'' = \qquad \eta' ,$$

we can make $q_{20}=-1$ and $q_{21}=0$. Because $\langle \lambda'', \xi'', \eta'' \rangle$ corresponds to a certain family $\langle A'', B'' \rangle$ equivalent to $\langle A, B \rangle$, we have the desired result. □

Let us prove Lemma 6.5 used in the proof of Lemma 6.4.

Lemma 6.5. *Given a cubic polynomial* $Q(\lambda, \xi, \eta)$ *in* λ, ξ, η *as*

$$Q(\lambda, \xi, \eta) = -\lambda\{\lambda^2+(q_{11}\xi+q_{12}\eta)\lambda+q_{20}\xi^2+q_{21}\xi\eta+q_{22}\eta^2\}+q_{33}\eta^3$$

where $q_{33}\neq 0$. *Suppose that* $Q(\lambda, \xi, \eta)$, *as an equation in* λ, *has three real distinct roots for any* $(\xi, \eta)\neq(0, 0)$. *Then*

$$q_{20}<0 .$$

Proof. Let r_0 denote the middle one of the three real roots of $Q(\lambda, 1, 0)=0$. Let also $r(\xi)$ denote that of $Q(\lambda, \xi, 1)=0$.

We first show $r_0=0$ by contradiction. Assume $r_0\neq 0$. It is easy to see

$$\lim_{\xi\to\pm\infty}\frac{r(\xi)}{\xi}=r_0\neq 0 .$$

So $r(\xi)$ has the opposite sign according to $\xi\to\pm\infty$. Consequently $r(\xi)=0$ for some $\xi\in \boldsymbol{R}$. It means $Q(\lambda, \xi, 1)=0$ has 0 as one of its roots for such ξ. This contradicts

$$Q(0, \xi, 1)=q_{33}\neq 0$$

for all $\xi\in \boldsymbol{R}$.

Because $r_0=0$ is the middle root of

$$Q(\lambda, 1, 0)\equiv-\lambda(\lambda^2+q_{11}\lambda+q_{20})=0 ,$$

one of the remaining roots is positive and the other negative. This means the desired inequality

$$q_{20}<0.$$

Thus the proof is complete. □

Notice that we may assume also in the present case

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix} \quad (\alpha>0)$$

repeating the same argument just before Lemma 6.2.

Lemma 6.6. *A matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix} \quad (\alpha>0),\ B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

has real distinct eigenvalues and satisfies

$$\det(-\lambda I+\xi A+\eta B)=-\lambda\{\lambda^2+(k_1\xi+k_2\eta)\lambda-\xi^2-k_3\eta^2\}+k_4\eta^3$$

for some real constants k_1, k_2, k_3 *and* $k_4\neq 0$ *if and only if the following four conditions hold at once.*

1) $b_{11}=0$,

2) $b_{12}b_{21}=\alpha^2 b_{13}b_{31}$,

3) $b_{22}=\alpha^2 b_{33}$,

4) $\det\left\{-\lambda I+\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda A+B\right\}=0$ *has three nonzero real distinct roots including a positive one and a negative one.*

Proof. Substituting $(\xi, \eta)=(1, 0)$ in

(6.7) $$\det(-\lambda I+\xi A+\eta B)\equiv-\lambda\{\lambda^2+(k_1\xi+k_2\eta)\lambda-\xi^2-k_3\eta^2\}+k_4\eta^3$$

we get the characteristic polynomial of A. So we have

(6.8) $$k_1=\alpha-\frac{1}{\alpha}.$$

Comparing the coefficients of $\xi^2\eta$, $\xi\eta^2$ and $\lambda\xi\eta$ on the both sides of (6.7), we obtain 1), 2) and 3) of the lemma.

In order to obtain 4), we plot the graph of $\det(-\lambda I+\xi A+\eta B)=0$ in the λ, ξ-plane. Substituting $\eta=1$ in (6.7) and using (6.8), we obtain

$$\lambda\left\{\xi-\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda\right\}^2=\frac{1}{4}\left(\alpha+\frac{1}{\alpha}\right)^2\lambda^3+k_2\lambda^2-k_3\lambda-k_4$$

$$\equiv -\det\left\{-\lambda I+\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda A+B\right\}$$

Solving this with respect to ξ, we have

$$\xi=\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda\pm\left[-\frac{1}{\lambda}\det\left\{-\lambda+\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda A+B\right\}\right]^{1/2}$$

$$=\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda\pm\left[\frac{1}{2}\left(\alpha+\frac{1}{\alpha}\right)^2\lambda^2-\frac{k_4}{\lambda}+k_2\lambda-k_3\right]^{1/2}.$$

We plot $\det(-\lambda I+\xi A+B)=0$ in the last form, taking account of the curve:

$$\xi=F(\lambda)\equiv-\det\left\{-\lambda I+\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda A+B\right\}.$$

Observing Fig. 4(a), (b), (c), we obtain 4) of the lemma. □

Let the conditions in Lemma 6.6 be satisfied. Then we obtain

(6.9) $$b_{12}b_{31}\neq 0 \quad \text{or} \quad b_{13}b_{21}\neq 0$$

as follows. First we have

(6.10) $$(b_{12}, b_{13})\neq(0, 0) \quad \text{and} \quad (b_{21}, b_{31})\neq(0, 0)$$

because otherwise $k_4=0$ would hold and contradict Lemma 6.6. Hence (6.10) and 2) of Lemma 6.6 show (6.9). In addition, the second case in (6.9) can be reduced to the first case $b_{12}b_{31}\neq 0$. This is done by the similarity transformation with

$$T=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

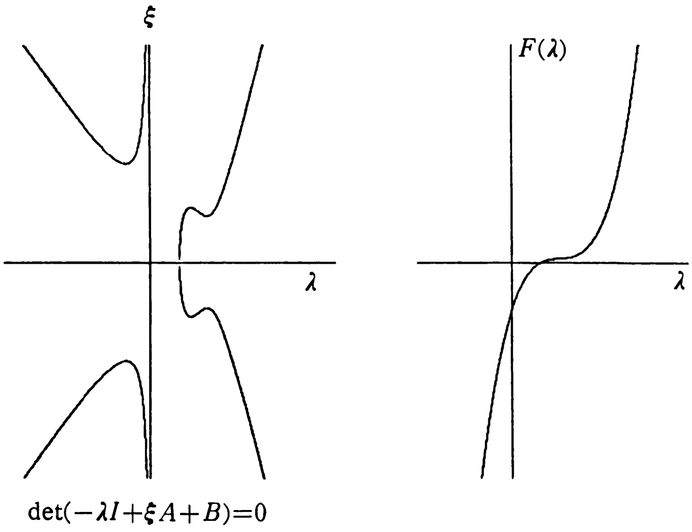


Fig. 4(a)

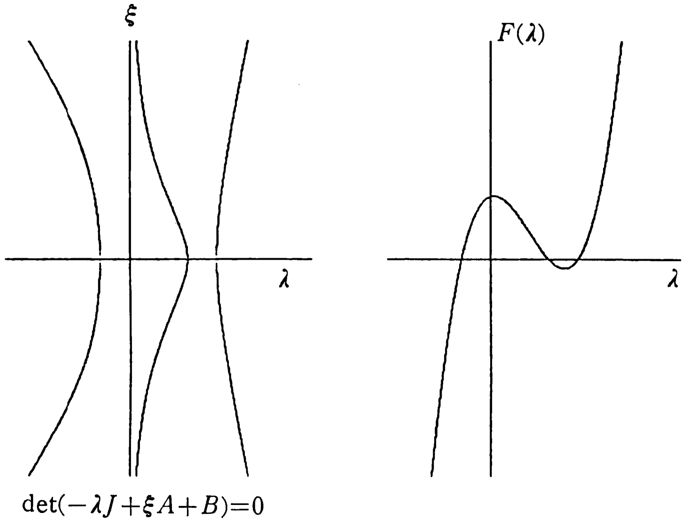


Fig. 4(b)

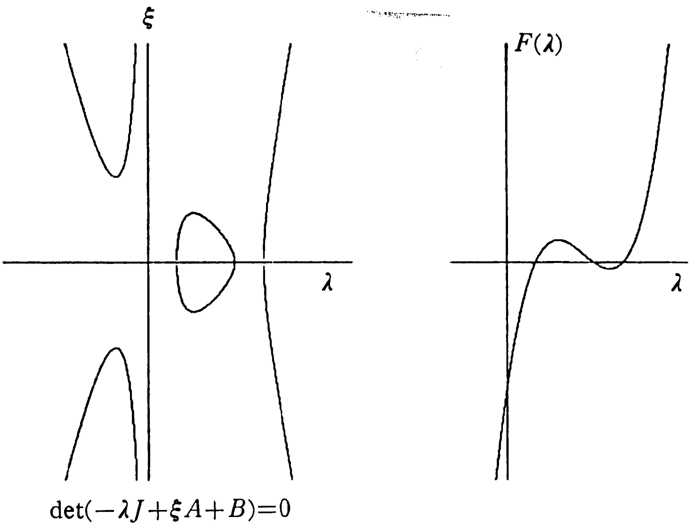


Fig. 4(c)

and the replacement of $\alpha(>0)$ by $1/\alpha$ in

$$\langle -T^{-1}AT,\ T^{-1}BT\rangle .$$

From the last argument, we may assume

(6.11) $$b_{12}b_{31}\neq 0 .$$

Let us now reduce B by a scalar multiplication and a similarity transformation with a diagonal T. Note that $T^{-1}AT=A$. Using 1), 2), 3) of Lemma 6.6, B is reduced to one of the following.

$$\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ a & (\alpha^2+1)/2 & (\alpha^2+1)b/2\alpha^2 \\ a & (\alpha^2+1)c/2 & (\alpha^2+1)/2\alpha^2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & (\alpha^2+1)/2 & 0 \\ 0 & (\alpha^2+1)/2 & (\alpha^2+1)b/2\alpha^2 \\ 1 & (\alpha^2+1)c/2 & (\alpha^2+1)/2\alpha^2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ 1 & 0 & (\alpha^2+1)b/2\alpha^2 \\ 1 & (\alpha^2+1)c/2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ -1 & 0 & (\alpha^2+1)b/2\alpha^2 \\ -1 & (\alpha^2+1)c/2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & (\alpha^2+1)/2 & 0 \\ 0 & 0 & (\alpha^2+1)b/2\alpha^2 \\ 1 & (\alpha^2+1)/2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & (\alpha^2+1)/2 & 0 \\ 0 & 0 & (\alpha^2+1)b/2\alpha^2 \\ 1 & 0 & 0 \end{bmatrix}$$

where $a\neq 0$, b, c are real constants. Here $\det B\neq 0$ because of the assumptions of Lemma 6.6, especially $k_4\neq 0$. We consider each of the above cases separately, sometimes putting $b=\delta+\varepsilon$, $c=\delta-\varepsilon$ for convenience of calculation. But we write a detailed argument only for the first one because the others can be discussed almost in the same way.

Lemma 6.7. *The matrix family $\langle A, B\rangle$ spanned by*

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix},\ B=\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ \gamma & (\alpha^2+1)/2 & (\alpha^2+1)(\delta+\varepsilon)/2\alpha^2 \\ \gamma & (\alpha^2+1)(\delta-\varepsilon)/2 & (\alpha^2+1)/2\alpha^2 \end{bmatrix}$$

which satisfy $\alpha>0$ and $\det B\neq 0$ has real distinct eigenvalues if and only if there exists $\beta\neq 0$ such that

$$\frac{\gamma(\delta-1)}{\beta}<\min(\beta-\beta^2, 0),$$

$$\left(\delta+2\frac{\gamma}{\beta}-\beta+1\right)(\delta+\beta-1)\geqq 0,$$

$$\varepsilon^2=\left(\delta+2\frac{\gamma}{\beta}-\beta+1\right)(\delta+\beta-1).$$

Remark. In order to determine adequate parameters, we have only to do in the following way. First determine $\alpha>0$, $\beta\neq 0$ and $\gamma\neq 0$ arbitrarily. Next, determine δ satisfying the above two inequalities. Finally, determine ε by the last equality.

Proof. By virtue of Lemma 6.6, we have only to find the condition such that

$$\det\left\{-\lambda I+\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda A+B\right\}$$

$$=\det\begin{bmatrix}-\lambda & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ \gamma & (\alpha^2+1)(1-\lambda)/2 & (\alpha^2+1)(\delta+\varepsilon)/2\alpha^2 \\ \gamma & (\alpha^2+1)(\delta-\varepsilon)/2 & (\alpha^2+1)(1-\lambda)/2\alpha^2\end{bmatrix}$$

$$=\frac{(\alpha^2+1)^2}{4\alpha^2}\det\begin{bmatrix}-\lambda & 1 & 1 \\ \gamma & 1-\lambda & \delta+\varepsilon \\ \gamma & \delta-\varepsilon & 1-\lambda\end{bmatrix}=0$$

has three nonzero roots including a positive one and a negative one. The equation in question is simplified as following.

$$\det\begin{bmatrix}-\lambda & 1 & 1 \\ \gamma & 1-\lambda & \delta+\varepsilon \\ \gamma & \delta-\varepsilon & 1-\lambda\end{bmatrix}=0, \tag{6.12}$$

$$-\lambda^3+2\lambda^2+(2\gamma+\delta^2-\varepsilon^2-1)\lambda+2\gamma(\delta-1)=0.$$

We begin with the necessity. Let $\beta\neq 0$ be the middle one of the three real roots of (6.12). Substituting $\lambda=\beta$ and solving (6.12) with respect to ε^2, we obtain

$$\varepsilon^2=\left(\delta+2\frac{\gamma}{\beta}-\beta+1\right)(\delta+\beta-1). \tag{6.13}$$

Note that the right side of (6.13) must be nonnegative. And in this case where (6.13) holds, (6.12) is factorized as

$$-(\lambda-\beta)\left\{\lambda^2+(\beta-2)\lambda+\frac{2\gamma(\delta-1)}{\beta}\right\}=0. \tag{6.14}$$

Because the two roots of (6.14) other than β have the opposite signs,

$$\frac{2\gamma(\delta-1)}{\beta}<0. \tag{6.15}$$

Because β is the middle root of (6.14),

$$\beta^2+(\beta-2)\beta+\frac{2\gamma(\delta-1)}{\beta}<0. \tag{6.16}$$

Conversely if (6.13), (6.15) and (6.16) are satisfied for some $\beta\neq0$ and $\gamma\neq0$, (6.12) has three nonzero roots including a positive one and a negative one. Simplifying (6.13), (6.15), (6.16), we obtain the conclusion. □

Also, in the other cases, the same argument holds, introducing again a new parameter β as the middle root of

$$\det\left\{-\lambda I+\frac{1}{2}\left(\alpha-\frac{1}{\alpha}\right)\lambda A+B\right\}=0.$$

So let us write down only the results omitting their proofs.

Lemma 6.8. *The matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix}0&0&0\\0&-\alpha&0\\0&0&1/\alpha\end{bmatrix},\ B=\begin{bmatrix}0&(\alpha^2+1)/2&0\\0&(\alpha^2+1)/2&(\alpha^2+1)\delta/2\alpha^2\\1&(\alpha^2+1)\varepsilon/2&(\alpha^2+1)/2\alpha^2\end{bmatrix}$$

which satisfy $\alpha>0$ *and* $\det B\neq0$ *has real distinct eigenvalues if and only if there exists* $\beta\neq0$ *such that*

$$\frac{\delta}{\beta}<\min(2\beta-2\beta^2,\ 0),$$

$$\varepsilon=\frac{\beta^3-2\beta^2+\beta-\delta}{\beta\delta}.$$

Lemma 6.9. *The matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix}0&0&0\\0&-\alpha&0\\0&0&1/\alpha\end{bmatrix},\ B=\begin{bmatrix}0&(\alpha^2+1)/2&(\alpha^2+1)/2\alpha^2\\1&0&(\alpha^2+1)(\delta+\varepsilon)/2\alpha^2\\1&(\alpha^2+1)(\delta-\varepsilon)/2&0\end{bmatrix}$$

which satisfy $\alpha>0$ *and* $\det B\neq0$ *has real distinct eigenvalues if and only if there exists* $\beta\neq0$ *such that*

$$\frac{\delta}{\beta}<-\beta^2,$$

$$(\delta+\beta)\left(\delta-\beta+\frac{2}{\beta}\right)\geqq0,$$

$$\varepsilon^2=(\delta+\beta)\left(\delta-\beta+\frac{2}{\beta}\right).$$

Lemma 6.10. *The matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix},\ B=\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ -1 & 0 & (\alpha^2+1)(\delta+\varepsilon)/2\alpha^2 \\ -1 & (\alpha^2+1)(\delta-\varepsilon)/2 & 0 \end{bmatrix}$$

which satisfy $\alpha>0$ *and* $\det B\neq 0$ *has real distinct eigenvalues if and only if there exists* $\beta\neq 0$ *such that*

$$\frac{\delta}{\beta}>\beta^2,$$

$$(\delta+\beta)\left(\delta-\beta-\frac{2}{\beta}\right)\geqq 0,$$

$$\varepsilon^2=(\delta+\beta)\left(\delta-\beta-\frac{2}{\beta}\right).$$

Lemma 6.11. *The matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix},\ B=\begin{bmatrix} 0 & (\alpha^2+1)/2 & 0 \\ 0 & 0 & (\alpha^2+1)\delta/2\alpha^2 \\ 1 & (\alpha^2+1)/2 & 0 \end{bmatrix}$$

which satisfy $\alpha>0$ *and* $\det B\neq 0$ *has real distinct eigenvalues if and only if* $\delta>\dfrac{27}{4}$ *holds.*

Remark. Proceeding in the same way as in the proof of Lemma 6.7, we have

$$-\frac{3}{2}<\beta<-1, \qquad \delta=\frac{\beta^3}{\beta+1}.$$

From this, the statement of the lemma immediately follows.

Lemma 6.12. *The matrix family* $\langle A, B\rangle$ *spanned by*

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix},\ B=\begin{bmatrix} 0 & (\alpha^2+1)/2 & 0 \\ 0 & 0 & (\alpha^2+1)\delta/2\alpha^2 \\ 1 & 0 & 0 \end{bmatrix}$$

is not real-diagonalizable for any $\alpha>0$ *and* δ.

Proof. Clear from the fact that B is not similar to a real diagonal matrix. □

We shall summarize Lemma 6.7, ⋯, 6.12 not in this section, but in the next section as a part of the summary of all the present paper.

7. Summary

In this section, we summarize all the results obtained in the present paper. For the sake of the completeness of this section, let us first reproduce Theorem 5.1 as Theorem 7.1.

Theorem 7.1. *Suppose a nondegenerate* 3×3 *matrix family* $\langle A_1, A_2, \cdots, A_n\rangle$ $(n\geqq 4)$ *is real-diagonalizable. Then it is simultaneously symmetrizable.*

From Theorem 3.9, Theorem 4.7, Theorem 5.1, we have the following two theorems.

Theorem 7.2. *A uniformly real-diagonalizable* 3×3 *matrix family is neither simultaneously symmetrizable nor equivalent to any family with real distinct eigenvalues if and only if it is equivalent to one of the following* 1), 2).

$$1)\ \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 1 \\ -\alpha & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\rangle$$

where $0<\alpha<1$ *is satisfied.*

$$2)\ \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \beta & \gamma \\ \beta' & 0 & 1 \\ \gamma' & 1 & 0 \end{bmatrix} \right\rangle$$

where

$$\alpha>0,\quad \gamma>\frac{1}{2}\Big(\alpha+\frac{\beta^2}{\alpha}\Big),\quad \gamma'>\frac{1}{2}\Big(\alpha+\frac{\beta'^2}{\alpha}\Big),\quad |\beta'-\beta|+|\gamma'-\gamma|>0$$

are satisfied.

Therem 7.3. *A* 3×3 *matrix family is non-uniformly real-diagonalizable if and only if it is equivalent to one of the following* 1), 1′), 2), 2′).

$$1)\ \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle.$$

1′) *The transposition of* 1).

$$2)\ \left\langle \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \beta & -\gamma \\ \beta(1-\alpha) & 1 & 0 \\ -2\alpha & 0 & -1 \end{bmatrix} \right\rangle.$$

where $0<\alpha<1$ *and* $\gamma>\beta^2/8$ *are satisfied.*

2′) *The transposition of* 2).

We now summarize the results concerning 3×3 matrix families with real distinct eigenvalues. Recall that each of such families is spanned by two matrices (see Theorem 3.1). First we reproduce Proposition 6.3 as Theorem 7.4 for the sake of completeness of this section.

Theorem 7.4. *A* 3×3 *matrix family* $\langle A, B\rangle$ *has real distinct eigenvalues and the cubic polynomial* $\det(-\lambda I+\xi A+\eta B)$ *in* λ, ξ, η *is factorizable if and only if* $\langle A, B\rangle$ *is equivalent to one of the following* 1), 2), 2′).

$$1)\quad \left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix}, \begin{bmatrix} 0 & \alpha & 1 \\ \alpha & 0 & \beta \\ 1 & -\beta & 0 \end{bmatrix} \right\rangle$$

where the constants satisfy $\alpha>0$ *and* $\beta^2<\alpha^2+1$.

$$2)\quad \left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix}, \begin{bmatrix} 0 & \beta & \gamma \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle$$

where β *and* γ *are arbitrary constants.*

2′) *The transposition of* 2).

Now we summarize the results of Lemma 6.7 to 6.12.

Theorem 7.5. *A* 3×3 *matrix family* $\langle A, B\rangle$ *has real distinct eigenvalues and the cubic polynomial* $\det(-\lambda I+\xi A+\eta B)$ *in* λ, ξ, η *is irreducible if and only if* $\langle A, B\rangle$ *is equivalent to* $\langle A_0, B_0\rangle$ *where*

$$A_0=\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 1/\alpha \end{bmatrix} \qquad (\alpha>0)$$

and B_0 *is one of the following* 1), 2), 3), 4), 5).

$$1)\quad B_0=\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ \gamma & (\alpha^2+1)/2 & (\alpha^2+1)(\delta+\varepsilon)/2\alpha^2 \\ \gamma & (\alpha^2+1)(\delta-\varepsilon)/2 & (\alpha^2+1)/2\alpha^2 \end{bmatrix}$$

where there exist $\beta\neq0$ *and* $\gamma\neq0$ *such that*

$$\frac{\gamma(\delta-1)}{\beta}<\min(\beta-\beta^2, 0),$$

$$\left(\delta+2\frac{\gamma}{\beta}-\beta+1\right)(\delta+\beta-1)\geqq0,$$

$$\varepsilon^2=\left(\delta+2\frac{\gamma}{\beta}-\beta+1\right)(\delta+\beta-1)$$

are satisfied.

2) $$B_0=\begin{bmatrix} 0 & (\alpha^2+1)/2 & 0 \\ 0 & (\alpha^2+1)/2 & (\alpha^2+1)\delta/2\alpha^2 \\ 1 & (\alpha^2+1)\varepsilon/2 & (\alpha^2+1)/2\alpha^2 \end{bmatrix}$$

where there exists $\beta\neq 0$ *such that*

$$\frac{\delta}{\beta}<\min(2\beta-2\beta^2,\ 0),$$

$$\varepsilon=\frac{\beta^3-2\beta^2+\beta-\delta}{\beta\delta}$$

are satisfied.

3) $$B_0=\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ 1 & 0 & (\alpha^2+1)(\delta+\varepsilon)/2\alpha^2 \\ 1 & (\alpha^2+1)(\delta-\varepsilon)/2 & 0 \end{bmatrix}$$

where there exists $\beta\neq 0$ *such that*

$$\frac{\delta}{\beta}<-\beta^2,$$

$$(\delta+\beta)\left(\delta-\beta+\frac{2}{\beta}\right)\geqq 0,$$

$$\varepsilon^2=(\delta+\beta)\left(\delta-\beta+\frac{2}{\beta}\right)$$

are satisfied.

4) $$B_0=\begin{bmatrix} 0 & (\alpha^2+1)/2 & (\alpha^2+1)/2\alpha^2 \\ -1 & 0 & (\alpha^2+1)(\delta+\varepsilon)/2\alpha^2 \\ -1 & (\alpha^2+1)(\delta-\varepsilon)/2 & 0 \end{bmatrix}$$

where there exists $\beta\neq 0$ *such that*

$$\frac{\delta}{\beta}>\beta^2,$$

$$(\delta+\beta)\left(\delta-\beta-\frac{2}{\beta}\right)\geqq 0,$$

$$\varepsilon^2=(\delta+\beta)\left(\delta-\beta-\frac{2}{\beta}\right)$$

are satisfied.

5) $$B_0=\begin{bmatrix} 0 & (\alpha^2+1)/2 & 0 \\ 0 & 0 & (\alpha^2+1)\delta/2\alpha^2 \\ 1 & (\alpha^2+1)/2 & 0 \end{bmatrix}$$

where

$$\delta > \frac{27}{4}$$

is satisfied.

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