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On the norm of the Poincaré series operator for a universal covering group

Dedicated to Professor Tatuo Fuji'i'e on his sixtieth birthday

By

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Introduction

Let Δ be the unit disk and Γ be a Fuchsian group acting on Δ which may be elementary or not. All Fuchsian groups considered in this paper are assumed to be torsionfree, that is, they are covering groups of some universal coverings. We denote by $A(\Delta, \Gamma)$ the Banach space of all holomorphic functions ϕ on Δ with

$$\gamma^*\phi = \phi \qquad \text{for all } \gamma \in \Gamma,$$

where $\gamma^*\phi = (\phi \circ \gamma)(\gamma')^2$, and norm

$$\|\phi\|_\Gamma = \iint_{\Delta/\Gamma} |\phi(z)|\, dx\, dy < \infty.$$

When Γ is the trivial group $\{1\}$, we abbreviate $A(\Delta, \{1\})$ and $\|\phi\|_{\{1\}}$ by $A(\Delta)$ and $\|\phi\|$, respectively.

For Γ and its subgroup Γ_1, the Poincaré series operator $\Theta_{\Gamma_1\backslash\Gamma}: A(\Delta, \Gamma_1) \to A(\Delta, \Gamma)$ is defined by

$$\Theta_{\Gamma_1\backslash\Gamma}\phi = \sum_{\gamma \in \Gamma_1\backslash\Gamma} \gamma^*\phi.$$

When $\Gamma_1 = \{1\}$, we simply denote Θ_Γ for $\Theta_{\{1\}\backslash\Gamma}$. It is known that $\Theta_{\Gamma_1\backslash\Gamma}$ is an open continuous surjection with norm at most one and satisfies $\Theta_\Gamma = \Theta_{\Gamma_1\backslash\Gamma} \circ \Theta_{\Gamma_1}$ (cf. Kra [3, p.91]).

We have shown in [9]

Theorem A. *Let Γ be a Fuchsian group acting on Δ and Γ_1 be a normal subgroup of Γ such that $\Gamma_1\backslash\Gamma$ is finitely generated and abelian. Then, for all nonzero $\Phi \in A(\Delta, \Gamma)$, we have*

$$\sup\left\{\frac{\|\Phi\|_\Gamma}{\|\phi\|_{\Gamma_1}} : \Phi = \Theta_{\Gamma_1\backslash\Gamma}\phi,\ \phi \in A(\Delta, \Gamma_1)\right\} = 1,$$

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in particular, $\|\Theta_{\Gamma_1\backslash\Gamma}\| = 1$.

Recently, McMullen [7] has extended this result.

Theorem B. *Let Γ be a Fuchsian group acting on Δ and Γ_1 be a subgroup of Γ.*

(a) *If the covering: $\Delta/\Gamma_1 \to \Delta/\Gamma$ is amenable, then the conclusion of Theorem A holds.*

(b) *If the Riemann surface Δ/Γ is of finite type, in other words, Γ is finitely generated and of the first kind, and if the covering: $\Delta/\Gamma_1 \to \Delta/\Gamma$ is nonamenable, then $\|\Theta_{\Gamma_1\backslash\Gamma}\| < 1$.*

(See McMullen [7] for the definition of amenability.)

In the present paper, we investigate when $\|\Theta_\Gamma\| = 1$ for an arbitrary universal covering group Γ, and show

Theorem 1. *For a Fuchsian group Γ acting on Δ, the norm of the Poincaré series operator Θ_Γ is one if and only if Γ satisfies one of the following conditions:*

(O_1) *For any positive ρ, there exists a hyperbolic disk D in Δ with radius ρ such that*

$$D \cap \gamma D = \emptyset \qquad \text{for all } \gamma \in \Gamma - \{1\}.$$

(O_2) *For any positive ε, there exists a hyperbolic element $\gamma \in \Gamma$ whose multiplier λ satisfies $1 < \lambda < 1 + \varepsilon$.*

The above theorem can be rewritten as follows:

Theorem 1′. *Let X be a Riemann surface whose universal covering surface is the unit disk, and Γ be the covering group. Then a necessary and sufficient condition that $\|\Theta_\Gamma\| = 1$ is that either (O_1') or (O_2') below holds.*

(O_1') *For any positive ρ, there is a point in X at which the injectivity radius (in the hyperbolic metric) is greater than ρ.*

(O_2') *For any positive ε, there is a geodesic in X whose length is less than ε.*

Thus the condition which determines whether $\|\Theta_\Gamma\| = 1$ or not for a universal covering group Γ is not combinatorial but geometric.

Remark. 1. A universal covering is amenable if and only if the covering group is cyclic.

2. It is obvious that $\|\Theta_\Gamma\| = 1$ if and only if $\|\Theta_{\Gamma_1\backslash\Gamma}\| = 1$ for every subgroup Γ_1 of Γ.

3. Nakanishi and Yamamoto [8] have shown that the outradius of the Teichmüller space $T(\Gamma)$ equals six if and only if (O_1) or (O_2) holds.

By using Theorem 1, we obtain

Theorem 2. *Let Γ be a Fuchsian group acting on Δ. If neither (O_1) nor (O_2) holds, then the inclusion map of $T(\Gamma)$ into the universal Teichmüller space*

$T(1)$ is strictly distance decreasing in the Teichmüller metric.

The sufficiency of the condision in Theorem 1 is shown in Section 1. On the other hand, the proof of the necessity is divided into four steps and these correspond to Sections 2–5. Theorem 2 is proved in Section 6.

§1. Sufficiency

We denote by Möb the group of all Möbius transformations and set

$$\text{Möb}(\Delta) = \{\gamma \in \text{Möb}: \gamma(\Delta) = \Delta\}.$$

We sometimes write $m(E, f)$ instead of $\iint_E f\,dx\,dy$, for simplicity.

Let Γ be a Fuchsian group acting on the unit disk Δ. First, suppose that a hyperbolic disk D with center at $a \in \Delta$ and radius $\rho > 0$ satisfies

$$\gamma(D) \cap D = \emptyset \qquad \text{for all } \gamma \in \Gamma - \{1\},$$

that is, D is contained in a fundamental domain ω for Γ. Take a Möbius transformation $\eta_0 \in \text{Möb}(\Delta)$ with $\eta_0(a) = 0$, and set $\phi = (\eta_0')^2$. Then ϕ is an elemant in $A(\Delta)$ with norm π and $m(D, |\phi|) = \pi \tanh^2 \rho$. Therefore

$$\begin{aligned} \|\Theta_\Gamma \phi\|_\Gamma &\geq m(\omega, |\phi| - \sum_{\gamma \in \Gamma'} |\gamma^* \phi|) \\ &\geq m(D, |\phi|) - m(\Delta - D, |\phi|) \\ &= (2\tanh^2 \rho - 1)\|\phi\|. \end{aligned}$$

Letting $\rho \to \infty$, we obtain $\|\Theta_\Gamma\| \geq 1$, hence $\|\Theta_\Gamma\| = 1$.

Next, suppose that Γ contains a hyperbolic γ whose multiplier $\lambda(>1)$ is close to one. Take a Möbius transformation η_0 which maps Δ onto the upper half-plane U and the fixed points of γ to $\{0, \infty\}$, and set $\Phi = (\eta_0'/\eta_0)^2$, then $\Phi \in A(\Delta, \langle\gamma\rangle)$ and its norm is $\pi \log \lambda$. Let

$$S_\theta = \eta_0^{-1}\{z \in U: \theta < \arg z < \pi - \theta,\ 1 < |z| < \lambda\},$$

then, from the so-called collar lemma (cf. Abikoff [1, p.95]), it follows that there are a positive constant $\theta = \theta(\lambda)$ and a fundamental domain ω for Γ such that

$$\theta(\lambda) \longrightarrow 0 \qquad \text{as} \qquad \lambda \longrightarrow 1, \text{ and}$$
$$S_\theta \subset \omega \subset S_0.$$

We see by Theorem A that for any $t < 1$, there is $\phi \in A(\Delta)$ such that

$$\Theta_{\langle\gamma\rangle}\phi = \Phi \qquad \text{and} \qquad t\|\phi\| < \|\Phi\|_{\langle\gamma\rangle}.$$

Therefore we have

$$\begin{aligned}\|\Theta_\Gamma\phi\|_\Gamma &= \|\Theta_{\langle\gamma\rangle\backslash\Gamma}\Phi\|_\Gamma \\ &\geq m(\omega, |\Phi| - \sum_{\eta\in\langle\gamma\rangle\backslash(\Gamma-\langle\gamma\rangle)} |\eta^*\Phi|) \\ &\geq m(S_\theta, |\Phi|) - m(S_0 - S_\theta, |\Phi|) \\ &= (\pi - 4\theta)\log\lambda \\ &> \left(1 - \frac{4\theta}{\pi}\right) t\|\phi\|.\end{aligned}$$

Letting $t\to 1$ and $\lambda\to 1$, we obtain $\|\Theta_\Gamma\| = 1$. This completes the proof of the sufficiency part.

§2. Removing a neighborhood of punctures

Let $\gamma\in$ Möd(Δ) be parabolic and $0 < t \leq 1$. Choose an element $\eta\in$ Möb with $\eta(U) = \Delta$ for which $\eta^{-1}\gamma\eta(z) = z + 1$. We define a horocycle $H_t(\gamma)$ at the fixed point of γ in Δ by

$$H_t(\gamma) = \eta\left\{z\in U\colon \operatorname{Im} z > \frac{1}{t}\right\}.$$

The following lemma is well-known (cf. Matelski [6, p.831]).

Lemma 2.1. *Let Γ be a Fuchsian group acting on Δ. For a parabolic element $\gamma\in\Gamma$, $H_t(\gamma)$ is precisely invariant under $\langle\gamma\rangle$ in Γ. If two parabolic elements γ_1 and γ_2 have distinct fixed points, then*

$$H_1(\gamma_1)\cap H_1(\gamma_2) = \emptyset.$$

Let p be puncture of $X = \Delta/\Gamma$ and γ be a prime parabolic element in Γ corresponding to p. Set

$$N_t(p) = \Gamma(H_t(\gamma))/\Gamma.$$

Then $N_t(p)$ is a deleted neighborhood of the puncture p homeomorphic to a punctured disk. It follows from Lemma 2.1 that $N_1(p)\cap N_1(q) = \emptyset$ for distinct punctures p, q of X. We define a domain Δ^* invariant under Γ by

$$\Delta^* = \Delta - \operatorname{Cl}(\bigcup H_{1/2}(\gamma)),$$

where Cl means its closure and the union is over all parabolic $\gamma\in\Gamma$, and set

$$X^* = \Delta^*/\Gamma.$$

This is a subsurface of X obtained by deleting all parabolic cusps.

Lemma 2.2. *For every ϕ holomorphic and integrable on $\Delta - \{0\}$, we have*

$$\iint_{0<|z|<1/2} |\phi|\,dx\,dy \leq \iint_{1/2<|z|<1} |\phi|\,dx\,dy. \tag{2.1}$$

Proof. Since ϕ is holomorphic at 0 or has a simple pole there, $|z\phi(z)|$ is subharmonic on Δ. Hence the mean value over a circle

$$\mu(r) = \frac{1}{2\pi}\int_0^{2\pi} r|\phi(re^{i\theta})|\,d\theta$$

is a nondecreasing function of r, in particular, $\mu(r) \le \mu(r+1/2)$. Integrating this inequality from 0 to 1/2, we obtain (2.1).

The above two lemmas imply that

$$m(X, |\Phi|) \le 2m(X^*, |\Phi|) \qquad \text{for } \Phi \in A(\Delta, \Gamma).$$

Thus for $\phi \in A(\Delta)$ we have

$$\begin{aligned} m(X, |\Theta_\Gamma \phi|) &\le 2m(X^*, |\Theta_\Gamma \phi|) \\ &\le 2m(\Delta^*, |\phi|), \end{aligned}$$

hence either

$$m(\Delta^*, |\phi|) \ge \frac{1}{3}\|\phi\| \qquad \text{or} \qquad \|\Theta_\Gamma \phi\|_\Gamma = m(X, |\Theta_\Gamma \phi|) \le \frac{2}{3}\|\phi\|.$$

Therefore, to show $\|\Theta_\Gamma\| < 1$, it suffices to consider only such ϕ that

$$\|\phi\| = m(\Delta, |\phi|) = 1 \qquad \text{and} \qquad m(\Delta^*, |\phi|) \ge \frac{1}{3}.$$

We denote the set of such $\phi \in A(\Delta)$ by $A(\Delta; 1, 1/3)$.

§3. Finding fitting pants

We denote by $d_\Delta(\cdot, \cdot)$, $d_X(\cdot, \cdot)$ etc. the hyperbolic distance funtions on Δ, X etc., respectively.

Our aim is to show that $\|\Theta_\Gamma\| < 1$ for a Fuchsian group Γ which acts on Δ and satisfies

Assumption. There exist positive constants m and M such that

(A_1) for every $z \in \Delta$

$$\inf\{d_\Delta(z, \gamma(z)) : \gamma \in \Gamma - \{1\}\} \le M, \quad \text{and}$$

(A_2) the multiplier of each hyperbolic element in Γ is not less than e^{2m}.

Let $\Omega(x; X)$ be the set of all loops in X whose initial and terminal point is x. Then the above (A_1) and (A_2) can be restated to the following:

(A_1') for every $x \in X$ there is a (homotopically) nontrivial loop in $\Omega(x; X)$ whose length is not more than M,

(A_2') every loop on X which is nontrivial and freely homotopic to no punctures of X has the length not less than m.

From now on to the end of Section 5, we assume that Γ is a Fuchsian group acting on Δ and satisfies the above assumption. We note that such Γ is necessarily of the first kind by (A_1). If Γ is finitely generated (and of the first kind)–of course, such a group satisfies our assumption–then by Theorem B we obtain what we desire. Therefore, we furthermore assume that

(A_3) Γ is infinitely generated.

The assumption (A_2) means that all nontrivial short loops are freely homotopic to punctures, thus we have

Lemma 3.1. *There is a positive constant m_1 depending only on m for which the following three equivalent conditions hold.*

(B_1) *For all $z \in \Delta^*$ and all $\gamma \in \Gamma - \{1\}$,*

$$D_\Delta(z; m_1) \cap \gamma D_\Delta(z; m_1) = \emptyset,$$

where $D_\Delta(z; m_1)$ is the (hyperbolic) disk in Δ with center at z and radius m_1.

(B_2) *The injectivity radius at every point in X^* is not less than m_1.*

(B_3) *All nontrivial loops C in X with $C \cap X^* \neq \emptyset$ has length not less than $2m_1$.*

For an arc C, we denote by $l(C)$ the length of C, and when C is a nontrivial loop, we define $g[C]$ as the geodesic or the puncture freely homotopic to C.

Take and fix an arbitrary point $x \in X^*$. Let C_1 be one of the shortest nontrivial loops in $\Omega(x; X)$, then C_1 is simple by the shortestness, and

$$l(C_1) \leq M$$

by (A_1'). Furthermore, this and (A_2) imply

Lemma 3.2. *There is a positive constant M_1 depending only on m and M such that*

$$d_X(x, g[C_1]) \leq M_1 \qquad \text{if } g[C_1] \text{ is a geodesic,}$$

$$d_X(x, \partial N_{1/2}(p)) \leq M_1 \qquad \text{if } g[C_1] \text{ is a puncture } p.$$

Proof. Consider a universal covering: $U \to X$, and fix a point $c \in U$ over x. Let c' be the terminal point of the lift of C_1 with initial point c, and let η be the covering transformation which maps c to c'. We can normalize η so that $\eta(z) = \lambda z$, $\lambda \geq e^{2m}$, or $\eta(z) = z + 1$ according as $g[C_1]$ is a geodesic or a puncture. In the first case, we have

$$M \geq l(C_1) \geq d_U(c, \lambda c) \geq d_U(c, e^{2m} c),$$

$$d_U(c, \{\operatorname{Re} z = 0\}) \geq d_X(x, g[C_1]), \text{ and}$$

$$d_U(c, e^{2m} c) \longrightarrow \infty \qquad \text{as} \qquad d_U(c, \{\operatorname{Re} z = 0\}) \longrightarrow \infty.$$

Hence we obtain what we desire.

The proof of the second case is the same. Q.E.D.

Next, let C_1' be one of the shortest loops in the class $\{C \in \Omega(x; X)$: C is not homotopic to any C_1^n, $n \in \mathbf{Z}$, (by a homotopy keeping x fixed)$\}$. Then we have

Lemma 3.3. *The loop C_1' is simple and does not intersect with C_1 except at x. Moreover, there is a positive constant M_2 depending only on m and M such that*

$$l(C_1') \le M_2. \tag{3.1}$$

Proof. First, suppose that C_1' is not simple. Then, noting that C_1' is a geodesic arc, we can devide C_1' into C_{11}', C_{12}' and C_{13}' so that $C_1' = C_{13}'^{-1} C_{12}' C_{11}'$, where C_{11}' and C_{13}' are arcs from x to some $x' \ne x$ and C_{12}' is a nontrivial loop in $\Omega(x', X)$. We may assume that $l(C_{11}') \le l(C_{13}')$. Since $C_{13}'^{-1} C_{11}'$ is shorter than C_1', it is homotopic to some C_1^n, by definition. And since $C_{11}'^{-1} C_{12}' C_{11}'$ is not a geodesic arc, there is a loop in its homotopy class whose length is strictly less than $l(C_{11}'^{-1} C_{12}' C_{11}')(\le l(C_1'))$, hence it is also homotopic to some C_1^k. But these imply that C_1' is homotopic to C_1^{n+k}, a contradiction to the definition of C_1'.

Next, suppose that C_1 and C_1' have a common point x' except x. Then C_1 and C_1' can be devided into simple arcs C_{11}, C_{12}, C_{11}' and C_{12}' from x to x' so that $C_1 = C_{12}^{-1} C_{11}$ and $C_1' = C_{12}'^{-1} C_{11}'$. We may assume that $l(C_{11}) \le l(C_{12})$ and $l(C_{11}') \le l(C_{12}')$. Since $C_{11}'^{-1} C_{11}$ is nontrivial and is not a geodesic arc, there is a loop in $\Omega(x, X)$ homotopic to $C_{11}'^{-1} C_{11}$ and strictly shorter than it. The shortestness of C_1 implies $l(C_1) < l(C_{11}'^{-1} C_{11})$, hence $l(C_{12}) < l(C_{11}')$. But then both $C_{11}'^{-1} C_{11}$ and $C_{12}'^{-1} C_{11}$ are shorter than C_1', from which it follows that C_1' is homotopic to some C_1^n, a contradiction.

Finally, we show (3.1). Take a universal covering: $\Delta \to X$ such that $z = 0$ lies over x. Let G be the covering group and let $g_1 \in G$ correspond to the lift of C_1 with initial point 0. We may assume that the fixed points of g_1 are $e^{i\theta}$ and $e^{-i\theta}$, $0 < \theta \le \pi/2$, if g_1 is hyperbolic, and that the fixed point is 1 if g_1 is parabolic. The Dirichlet fundamental domain ω with center at 0 for G, which coincides with the Ford fundamental domain (cf Maskit [5, p.72]), contains $D_\Delta(0; m_1)$ by (B_1). Set $d = \inf\{d_\Delta(0; g(0)) \colon g \in G - \langle g_1 \rangle\}$ and let $E(g)$ be the exterior of the isometric circle of g. Then we have

$$\begin{aligned} \omega &\supset D_\Delta(0; d/2) \cap E(g_1) \cap E(g_1^{-1}) \\ &\supset D_\Delta(0; d/2) \cap \{z \in \Delta \colon \operatorname{Re} z < 0,\ |\operatorname{Im} z| < \tanh m_1\}. \end{aligned}$$

Hence ω contains a hyperbolic disk with radius at least $\rho = \rho(m_1, d)$ where $\rho \to \infty$ as $d \to \infty$. Since $\rho \le M/2$ by (A_1) and $d = l(C_1')$, we obtain $l(C_1') \le M_2(m_1, M)$. Q.E.D.

A subsurface P of a Riemann surface X is called *pants* if P is of type (0,3) and if each boundary component of P is a geodesic or a puncture of X. We note that P is incompressible in X. When the length of every nonpuncture boundary component of P is neither less than l nor more than L, we say that pants P is *of size* $[l, L]$.

The geometric intersection number $i(C_1, C_1')$ of C_1 and C_1' is the minimum number of points that any two loops freely homotopic, respectively, to C_1 and C_1' have in common. By the above lemma, we see that $i(C_1, C_1')$ is one or zero. If $i(C_1, C_1') = 1$, then set $C_2 = C_1' C_1 C_1'^{-1}$, $C_3 = C_2 C_1^{-1}$. On the other hand, if not, then set $C_2 = C_1'$ and let C_3 be one of $C_2 C_1$ and $C_2^{-1} C_1$ which is freely homotopic to a puncture or a simple geodesic. We define $P(x)$ as the unique pants in X whose boundary is $\bigcup_{j=1}^{3} g[C_j]$. Note that

$$l(g[C_j]) \leq M_3 := 2(M + M_2) \quad \text{for } j = 1, 2, 3, \text{ and} \quad d_X(P(x), x) \leq M_1$$

by Lemma 3.2 and Lemma 3.4 below. We also note that

$$l(g[C_j]) \geq m \text{ or } = 0 \qquad \text{for } j = 1, 2, 3$$

by (A_2').

Lemma 3.4. *For a simple geodesic C in X and a puncture p of X, we have*

$$C \cap N_1(p) = \emptyset. \tag{3.2}$$

Proof. Suppose that C is a dividing loop. Let X_1 be the connected component of $X - C$ which has the puncture p, and let $\hat{X}_1 = X_1 \cup C \cup j(X_1)$ be the double of X_1 with respect to the new border C of X_1, where j is the anticonformal involution of $\hat{X}_1$ keeping every point in C fixed. By applying Lemma 2.1 to a universal covering group of $\hat{X}_1$, we have $j(N_1(p)) \cap N_1(p) = N_1(j(p)) \cap N_1(p) = \emptyset$, from which (3.2) follows.

By almost the same argument, we see that (3.2) holds also for a nondividing loop C. Q.E.D.

Suppose that pants P has three boundary components with length l_1, l_2, l_3. We denote by α_1 (resp. α_2, α_3) the hyperbolic line in Δ which connects $e^{i\theta_1}$, $0 \leq \theta_1 < \pi/2$, and i (resp. $e^{i\theta_2}$ and $e^{i\theta_3}$, $\pi/2 \leq \theta_2 < \theta_3 \leq \pi$, -1 and 1), where θ_j is determined so that

$$d_\Delta(\alpha_1, \alpha_2) = l_3/2, \; d_\Delta(\alpha_2, \alpha_3) = l_1/2 \quad \text{and} \quad d_\Delta(\alpha_3, \alpha_1) = l_2/2.$$

Take a hyperbolic or parabolic element $\gamma_j \in \text{Möb}(\Delta)$ $(j = 1, 2)$ which maps α_j onto its mirrorimage α_j' in α_3, and set $G = \langle \gamma_1, \gamma_2 \rangle$. Then

$$\alpha_j = \Delta \cap \partial E(\gamma_j), \; \alpha_j' = \Delta \cap \partial E(\gamma_j^{-1}) \qquad \text{for } j = 1, 2,$$

the Ford fundamental domain for G is

$$\omega = E(\gamma_1) \cap E(\gamma_2) \cap E(\gamma_1') \cap E(\gamma_2') \cap \Delta,$$

and G represents pants P in the sense that $P = K(G)/G$, where $K(G)$ is the Nielsen convex domain for G, that is, $K(G)$ is the (hyperbolically) convex subdomain of Δ whose closure in the closed unit disk is the convex hull of the limit set of G. We note that

$$K(G)\cap\omega = U(\alpha_1, \alpha_2)\cap U(\alpha_2, \alpha_2')\cap U(\alpha_2', \alpha_1')\cap U(\alpha_1', \alpha_1),$$

where $U(\alpha, \alpha')$ is the half-plane in Δ such that $0\in U(\alpha, \alpha')$ and $\partial U(\alpha, \alpha')$ is orthogonal to two hyperbolic lines α and α'.

Lemma 3.5. *The subsurface $P(x)\cap X^*$ is triply connected and incompressible. There is a positive constant M_4 depending only on m and M such that the diameter of $P(x)\cap X^*$ is at most M_4.*

Proof. By Lemma 3.4, $X\cap\partial P(x)$, the geodesic boundary of $P(x)$, does not meet with the cusped neighborhoods $\mathrm{Cl}(N_{1/2}(p))$ of any punctures p of X. This implies the first assertion.

We have seen that the pants $P(x)$ is of size $[m, M_3]$, thus we conclude the second assertion by using the above explicit representation of pants. Q.E.D.

For a subset Y of X and t positive, we denote the t-neighborhood of Y in X by $D_X(Y; t)$. The collar lemma and Lemma 3.4 imply

Lemma 3.6. *There is a positive constant m_2 depending only on M_3 and satisfying:*

(a) *For any simple geodesic C in X with $l(C)\le M_3$, $D_X(C; m_2)$ is an annular neighborhood of C and is contained in X^*.*

(b) *For any mutually disjoint simple geodesics C and C' in X with length at most M_3,*

$$\mathrm{Cl}(D_X(C; m_2))\cap\mathrm{Cl}(D_X(C'; m_2)) = \emptyset.$$

We sum up the result obtained in this section as a proposition.

Proposition 3.1. *There are positive constants m_3, M_3 and M_5 depending only on m and M with the following properties:*

For every $x\in X^*$, there is pants $P(x)$ of size $[m, M_3]$ such that

(a) $D_X(x; M_5)\supset P(x)\cap X^*$,

(b) $P^*(x) = P(x)\cap X^* - \mathrm{Cl}(D_X(X\cap\partial P(x); m_3))$ *is an incompressible subsurface of $P(x)$ and homeomorphic to $P(x)$, and*

(c) $P^*(x)$ *contains a univalent disk $D_X(y; m_3)$ for some $y\in P^*(x)$.*

§4. An estimate on pants

Let P be pants of size $[l, L]$. Proposition 3.1 (b) implies that there is a positive constant ε depending only on l and L such that P^*, obtained from P by deleting the closed ε-neighborhood of its geodesic boundary and the closed 1/2-neighborhood of its punctures, is incompressible and homeomorphic to P.

For given constants l, L and $\delta (0 < l\le L<\infty,\ \delta > 0)$, we denote by $\mathscr{P} = \mathscr{P}(l, L, \delta)$ the set of all pairs (G, ϕ) of a Fuchsian group G acting on Δ and nonzero $\phi\in A(\Delta)$ satisfying the following two conditions:

(1) $P(G) = K(G)/G$ is pants of size $[l, L]$.
(2) $m(K^*(G), |\phi|) \geq \delta m(K(G), |\phi|)$,

where $K^*(G) = \pi^{-1}(P^*(G))$ and $\pi: \Delta \to \Delta/G$.

When $\mathscr{P}(l, L, \delta) \neq \emptyset$, we define $\beta = \beta(l, L, \delta)$ by

$$\beta = 1 - \sup\left\{\frac{m(P^*(G), |\Theta_G \phi|)}{m(K^*(G), |\phi|)}: (G, \phi) \in \mathscr{P}\right\}.$$

Then we have

Theorem 4.1. $\beta(l, L, \delta) > 0$.

Proof. Take a sequence $\{(G_n, \phi_n)\}_{n=1}^{\infty}$ in $\mathscr{P}$ such that

$$\lim_{n\to\infty} \frac{m(K_n^* \cap \omega_n, |\Theta_n \phi_n|)}{m(K_n^*, |\phi_n|)} = 1 - \beta,$$

where $K_n^* = K^*(G_n)$, ω_n is the Ford fundamental domain for G_n, and $\Theta_n = \Theta_{G_n}$. We may assume that $m(K_n, |\phi_n|) = 1$ for all n, where $K_n = K(G_n)$. Let l_{nj}, $j = 1, 2, 3$, be the length of three boundary components of $P(G_n)$. By selecting a subsequence, we may assume that $l_{nj} \to l_j$ as $n \to \infty$. By taking a conjugate group of G_n, we may furthermore assume that every G_n is a group as in the explicit representation of pants in Section 3. Then γ_{n1} and γ_{n2}, the generators of G_n, converge to some γ_1 and γ_2 in $\text{Möb}(\Delta)$, respectively. The limit group $G = \langle \gamma_1, \gamma_2 \rangle$ is discontinuous on Δ, i.e. Fuchsian, and ω_n converges to the Ford fundamental domain ω for G (we write $\omega_n \rightrightarrows \omega$) in the sense that for every compact S and every open O with $S \subset \omega \subset \text{Cl}(\omega) \subset O$, there is an integer n_0 such that $S \subset \omega_n \subset \text{Cl}(\omega_n) \subset O$ holds for all $n \geq n_0$. Note that K/G is pants of size $[l, L]$, where $K = K(G)$. Similarly, we can see that $K_n^* \cap \omega_n \rightrightarrows K^* \cap \omega$, where $K^* = K^*(G)$, and that for an arbitrary finite subset G' of G,

$$K_n \cap \text{int}(\text{Cl}(\bigcup_{g \in G'} f_n(g)(\omega_n))) \rightrightarrows K \cap \text{int}(\text{Cl}(\bigcup_{g \in G'} g(\omega))),$$

where f_n is the group isomorphism of G onto G_n with $f_n(\gamma_1) = \gamma_{n1}$ and $f_n(\gamma_2) = \gamma_{n2}$. Furthermore, from this it follows that every compact subset of K is contained in K_n for all large n. Therefore, again by selecting a subsequence, we see that $\Theta_n \phi_n$ converges locally uniformly on K to some function Φ which is holomorphic on K and satisfies

$$g^* \Phi(z) = \Phi(z) \qquad \text{for all } z \in K \text{ and } g \in G.$$

We have

$$\begin{aligned} m(K \cap \omega, |\Phi|) &\leq \liminf_{n\to\infty} m(K_n \cap \omega_n, |\Theta_n \phi_n|) \\ &\leq \liminf_{n\to\infty} m(K_n, |\phi_n|) = 1 \end{aligned}$$

and

$$m(K^* \cap \omega, |\Phi|) \geq \limsup_{n \to \infty} m(K_n^* \cap \omega_n, |\Theta_n \phi_n|)$$

$$= (1 - \beta) \limsup_{n \to \infty} m(K_n^*, |\phi_n|)$$

$$\geq (1 - \beta)\delta > 0,$$

in particular, $\Phi \neq 0$. Now, by applying the same argument as in McMullen [7, p.114, lines 16–22 and the proof of Theorem 1.1 (nonamenable case)], we have

$$m(K_n^* \cap \omega_n, |\Theta_n \phi_n|) \leq cm(K_n^*, |\phi_n|)$$

for some constant $c < 1$ independent of n. Thus $1 - \beta \leq c < 1$, we are done.

§5. A total estimate over nicely arranged fitting pants

Let x_n, $n \in \mathbf{N}$, be points in $\mathrm{Cl}(X^*)$ such that

$$(5.1) \qquad d_X(x_n, x_k) \geq 2M_5 \quad \text{for all } n \neq k, \text{ and}$$

$$\mathrm{Cl}(X^*) \subset \bigcup_{n=1}^{\infty} D_X(x_n; 2M_5).$$

Lemma 5.1. *Such points x_n, $n \in \mathbf{N}$, exist.*

Proof. We note that for a compact subset S of X, there are the finite number of points $a_1, \ldots, a_k$ in S such that

$$d_X(a_i, a_j) \geq 2M_5 \quad \text{if } i \neq j, \text{ and} \quad S \subset \bigcup_{j=1}^{k} D_X(a_j; 2M_5).$$

Let $\{X_n\}_{n=1}^{\infty}$ be a regular exhaustion of X and $X_n^* = X^* \cap X_n$. We may assume that $\mathrm{Cl}(X_{n+1}^*) - D_X(X_n^*; 2M_5) \neq \emptyset$ for all n. First, for a compact set $\mathrm{Cl}(X_1^*)$, take the finite number of points $a_1^{(1)}, \ldots, a_{k(1)}^{(1)}$ as above. Next, for $\mathrm{Cl}(X_2^*) - \bigcup_{j=1}^{k(1)} D_X(a_j^{(1)}; 2M_5)$, take points $a_1^{(2)}, \ldots, a_{k(2)}^{(2)}$ as above, and so on. What we seek is the points $a_j^{(n)}$, $n \in \mathbf{N}$, $1 \leq j \leq k(n)$, obtained by repeating this process.

Q.E.D.

Let P_n be pants $P(x_n)$ and y_n be a point in $P_n^* = P^*(x_n)$ such that $D_X(y_n; m_3)$ is a univalent disk in P_n^* (cf. Proposition 3.1). Then we see

Lemma 5.2. $X^* \subset \bigcup_{n=1}^{\infty} D_X(y_n; 3M_5)$, *and $P_n \cap P_k = \emptyset$ for all $n \neq k$.*

Proof. The first assertion is obvious by Proposition 3.1.

Suppose that $P_n \cap P_k \neq \emptyset$ for some $n \neq k$. Since $P_n \cap P_k \cap X^* = \emptyset$ by (5.1) and Proposition 3.1, two pants P_n and P_k must have the same puncture. But this necessarily implies $P_n \cap P_k \cap X^* \neq \emptyset$, a contradiction. Q.E.D.

Set

$$V_n = \pi^{-1}(P_n), \qquad V_n^* = \pi^{-1}(P_n^*),$$

$$V = \sum_{n=1}^{\infty} V_n, \qquad V^* = \sum_{n=1}^{\infty} V_n^*$$

and $\qquad S = \pi^{-1}(\{y_n : n \in \mathbf{N}\}),$

where π is the covering projection: $\Delta \to X = \Delta/\Gamma$. Then we see

(5.2) $$D_\Delta(\zeta; m_3) \cap D_\Delta(\zeta'; m_3) = \emptyset \qquad \text{for distinct } \zeta, \zeta' \in S, \text{ and}$$

$$\Delta^* \subset \bigcup_{\zeta \in S} D_\Delta(\zeta; 3M_5).$$

Furthermore, by (5.2) we have

$$\sup_{z \in \Delta} \#\{\zeta \in S : D_\Delta(\zeta; 6M_5) \ni z\} < \infty.$$

Hence, by the same argument as in the proof of Theorem 2 in [10], we obtain

Lemma 5.3. *There is a positive constant m_4 depending only on m and M in Assumption such that*

$$m(\sum_{\zeta \in S} D_\Delta(\zeta; m_3), |\phi|) \geq m_4 \qquad \text{for all } \phi \in A(\Delta; 1, 1/3),$$

in particular,

(5.3) $$m(V^*, |\phi|) \geq m_4 \qquad \text{for all } \phi \in A(\Delta; 1, 1/3).$$

Let

$$V_n = \sum_{k=1}^{\infty} V_{nk} \qquad \text{and} \qquad V_n^* = \sum_{k=1}^{\infty} V_{nk}^*$$

be decomposition into connected components such that $V_{nk}^* \subset V_{nk}$. Take an arbitrary element ϕ in $A(\Delta; 1, 1/3)$. Then, by using Lemma 5.3 above and Lemma 1 in [10], we can find a subset I of $\mathbf{N}^2$ depending on ϕ as well as m and M in Assumption such that

(5.4) $$\frac{m_4}{2} m(V_{nk}, |\phi|) \leq m(V_{nk}^*, |\phi|) \qquad \text{for all } (n, k) \in I \text{ and}$$

(5.5) $$\sum_{(n,k) \in I} m(V_{nk}^*, |\phi|) \geq \frac{1}{2} \sum_{(n,k) \in \mathbf{N}^2} m(V_{nk}^*, |\phi|) = \frac{1}{2} m(V^*, |\phi|).$$

Let ω be a fundamental domain for Γ. We may assume that $\omega_n^* = \omega \cap V_n^*$ is contained in V_{n1}^*. Choose $\eta_{nk} \in \Gamma$ so that $\omega_{nk}^* = \eta_{nk}(\omega_n^*) \subset V_{nk}^*$, and let G_{nk} be the stabilizer subgroup of V_{nk} in Γ, then

$$\Gamma = \sum_{k=1}^{\infty} \eta_{nk} G_{n1} = \sum_{k=1}^{\infty} G_{nk} \eta_{nk}.$$

Noting that pants $P_n = K(G_{nk})/G_{nk}$ is of size $[m, M_3]$, $V_{nk} = K(G_{nk})$ and $V_{nk}^* = K^*(G_{nk})$, we get from (5.4)

$$(G_{nk}, \phi) \in \mathscr{P}(m, M_3, m_4/2) \qquad \text{for } (n, k) \in I.$$

Thus, by the definition of $\beta = \beta(m, M_3, m_4/2)$, we have

$$m(\omega_{nk}^*, |\Theta_{nk}\phi|) \leq (1 - \beta) m(V_{nk}^*, |\phi|) \qquad \text{for } (n, k) \in I,$$

where Θ_{nk} is the Poincaré series operator for G_{nk}. Therefore

$$\begin{aligned} m\Big(\sum_{n=1}^{\infty} P_n^*, |\Theta_\Gamma \phi|\Big) &= \sum_{n=1}^{\infty} m(\omega_n^*, |\Theta_\Gamma \phi|) \\ &= \sum_{n=1}^{\infty} m\Big(\omega_n^*, \Big|\sum_{k=1}^{\infty} \eta_{nk}^* \Theta_{nk}\phi\Big|\Big) \\ &\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} m(\omega_{nk}^*, |\Theta_{nk}\phi|) \\ &\leq \sum_{(n,k)\notin I} m(V_{nk}^*, |\phi|) + (1 - \beta) \sum_{(n,k)\in I} m(V_{nk}^*, |\phi|) \\ &\leq \left(1 - \frac{\beta}{2}\right) m(V^*, |\phi|), \end{aligned}$$

here we use (5.5) in the last inequality. Consequently, by (5.3),

$$\begin{aligned} \|\Theta_\Gamma \phi\|_\Gamma &= m\Big(\sum_{n=1}^{\infty} P_n^*, |\Theta_\Gamma\phi|\Big) + m\Big(X - \sum_{n=1}^{\infty} P_n^*, |\Theta_\Gamma\phi|\Big) \\ &\leq \left(1 - \frac{\beta}{2}\right) m(V^*, |\phi|) + m(\Delta - V^*, |\phi|) \\ &\leq \left(1 - \frac{\beta m_4}{2}\right) \|\phi\|. \end{aligned}$$

This completes the proof of the necessity part of Theorem 1.

§6. An application to Teichmüller theory

For a Fuchsian group Γ acting on Δ, we denote by $Q(\Gamma)$ the set of all quasiconformal self-mappings f of Δ such that f leaves ± 1, i fixed and $\Gamma_f = f\Gamma f^{-1}$ is Fuchsian. For $f \in Q(\Gamma)$, we set

$$k(f) = \operatorname*{ess\,sup}_{z\in\Delta} |f_{\bar z}(z)/f_z(z)| \qquad \text{and} \qquad K(f) = \frac{1 + k(f)}{1 - k(f)}.$$

We said that two elements f, g in $Q(\Gamma)$ is equivalent if $f = g$ on $\partial\Delta$, and denote by $[f]_\Gamma$ the equivalence class containing f. We remark that $\Gamma_f = \Gamma_g$ for equivalent f and g in $Q(\Gamma)$. The Teichmüller space $T(\Gamma)$ is the set of these equivalence classes, and the Teichmüller distance $\tau_\Gamma(\cdot, \cdot)$ on $T(\Gamma)$ is defined by

$$\tau_\Gamma([f]_\Gamma, [g]_\Gamma) = \frac{1}{2}\log\inf\{K(h): h \in Q(\Gamma_f),\ h\circ f = g \text{ on } \partial\Delta\}.$$

We first show

Proposition 6.1. *The conditions (O_1) and (O_2) are quasiconformally invariant, that is, if a Fuchsian group Γ satisfies one of them, then, for any $f \in Q(\Gamma)$, so does Γ_f. In particular, if $\|\Theta_{\Gamma_f}\| = 1$ for some $[f]_\Gamma \in T(\Gamma)$, then $\|\Theta_{\Gamma_g}\| = 1$ for all $[g]_\Gamma \in T(\Gamma)$.*

Proof. First, suppose that a hyperbolic disk D in Δ with radius ρ is a partial fundamental set for Γ, i.e., a subset of a fundamental set. A distortion theorem of hyperbolic distances (cf. Lehto and Virtanen [4, p.65]) implies that $f(D)$ contains a hyperbolic disk whose radius is at least $r = r(\rho, K(f))$, where $r \to \infty$ as $\rho \to \infty$. Obviously, $f(D)$ is a partial fundamental set for Γ_f. Hence, if Γ satisfies (O_1), then so does Γ_f.

Next, γ be a hyperbolic element in Γ with multiplier λ. Since the multiplier is quasiinvariant, $f\gamma f^{-1}$ has the multiplier at most $\lambda^{K(f)}$ (cf. Gardiner [2, p.159]). Thus, if (O_2) holds for Γ, then so does for Γ_f, The last assertion immediately follows from Theorem 1. Q.E.D.

Theorem 2 is a consequence of Theorem 1, the above proposition and the following result.

Theorem 6.1. *Let Γ be a Fuchsian group and G be a subgroup of Γ. If $[f]_\Gamma$ and $[g]_\Gamma \in T(\Gamma)$ satisfy $\min(\|\Theta_{G_f\backslash\Gamma_f}\|, \|\Theta_{G_g\backslash\Gamma_g}\|) < 1$, then we have*

$$\tau_G([f]_G, [g]_G) < \tau_\Gamma([f]_\Gamma, [g]_\Gamma). \tag{6.1}$$

Moreover, for a fixed $[g]_\Gamma$, we have

$$\frac{\tau_G([f]_G, [g]_G)}{\tau_\Gamma([f]_\Gamma, [g]_\Gamma)} \le \|\Theta_{G_g\backslash\Gamma_g}\| + o(1) \qquad \text{as } \tau_\Gamma([f]_\Gamma, [g]_\Gamma) \longrightarrow 0, \text{ and} \tag{6.2}$$

$$\tau_G([f]_G, [g]_G) \le \tau_\Gamma([f]_\Gamma, [g]_\Gamma) + \frac{1}{2}\log\frac{1 + \|\Theta_{G_g\backslash\Gamma_g}\|}{2} + o(1) \tag{6.3}$$

$$\text{as } \tau_\Gamma([f]_\Gamma, [g]_\Gamma) \longrightarrow \infty.$$

Proof. We may assume that $\|\Theta_{G_g\backslash\Gamma_g}\| < 1$, and by changing the origin of the Teichmüller spaces, we may also assume that g is the identity mapping. Moreover, we can assume that f is extremal, i.e.,

$$K(f) = \inf\{K(h): h = f \text{ on } \partial\Delta,\ h \in Q(\Gamma)\},$$

or, equivalently,

$$\frac{1}{2}\log K(f) = \tau_\Gamma([f]_\Gamma, [\mathrm{id}]_\Gamma).$$

Take sequences of finitely generated subgroups G_n of G and closed subsets E_n of $\partial\Delta$ with the following properties:

(1) $G_n < G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$,
(2) E_n is invariant under G_n and $(E_n - \Lambda(G_n))/G_n$ is a finite set, and
(3) $E_n \subset E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n$ is dense in $\partial\Delta$.

Then, for each n, in the set

$$\{h \in Q(G_n) \colon h = f \text{ on } E_n\},$$

there is the unique extremal mapping h_n, whose Beltrami coefficient is of the form $k_n \overline{\phi_n}/|\phi_n|$, where $k_n = k(h_n)$ and $\phi_n \in A(\Delta, G_n)$ with $\|\phi_n\|_{G_n} = 1$. By the main inequality of Reich and Strebel, we get

$$K_n \leq \iint_{\Delta/G_n} |\phi_n| \frac{|1 + \kappa\phi_n/|\phi_n||^2}{1 - |\kappa|^2} dx\, dy,$$

where $K_n = K(h_n)$ and κ is the Beltrami coefficient of f (cf Strebel [11, p. 437]). Set $k = k(f)$, then the right hand side of the above inequality is dominated by

$$\frac{1}{1-k^2}\left\{1 + k^2 + 2\,\mathrm{Re}\iint_{\Delta/G_n} \kappa\phi_n\, dx\, dy\right\}$$

$$= \frac{1}{1-k^2}\left\{1 + k^2 + 2\,\mathrm{Re}\iint_{\Delta/\Gamma} \kappa\Theta_{G_n\backslash\Gamma}\phi_n\, dx\, dy\right\}.$$

Since $\|\Theta_{G_n\backslash\Gamma}\|$ is uniformly bounded by $d = \|\Theta_{G\backslash\Gamma}\|$, we obtain

$$\log K_n \leq \log K(f) + \log F(k, d),$$

where

$$F(k, d) = \frac{1 + k^2 + 2kd}{(1+k)^2}.$$

By applying a usual normal family argument to $\{h_n\}$ (cf. Gardiner [2, p.145]), we see

$$\frac{1}{2}\log K_n \longrightarrow \tau_G([f]_G, [\mathrm{id}]_G) \qquad \text{as } n \longrightarrow \infty.$$

Thus

$$\tau_G([f]_G, [g]_G) \leq \tau_\Gamma([f]_\Gamma, [g]_\Gamma) + \frac{1}{2}\log F(K, d),$$

from which (6.3) follows immediately. By noting $\log F(k, d) < 0$, we get (6.1). Since

$$\lim_{k \to 0} \frac{\log F(k, d)}{\log K(f)} = d - 1,$$

we obtain (6.2). Q.E.D.

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