Mixed problems or Cauchy problems for
semi-degenerate hyperbolic equations
of 2-nd order with a parameter

By

Reiko SAKAMOTO

Introduction.

Let us consider linear hyperbolic operators of 2-nd order with real coefficients:

\[ L = L(t, x; \partial_t, \partial_x) = \partial_t^2 - 2 \sum_{j=1}^{n} a_{j0}(t, x) \partial_j \partial_{x_j} - \sum_{j, k=1}^{n} a_{jk}(t, x) \partial_j \partial_k \]

\[ + b_0(t, x) \partial_t + \sum_{j=1}^{n} b_j(t, x) \partial_j + c(t, x) \]

in \( I \times \Omega = [0, T] \times R^n_{+} = \{ 0 \leq t \leq T, x_1 > 0, x' = (x_2, \ldots, x_n) \in R^{n-1} \} \), where \( a_{jk} = a_{kj} \) and \( \partial_j = \partial/\partial x_j \). It is well known that the mixed problem:

\[
\begin{align*}
L u &= f & \text{in } I \times \Omega , \\
 u \mid_{x_1 = 0} &= g_0 & \text{on } I \times \partial \Omega , \\
 u \mid_{t = 0} &= u_0, \quad \partial_t u \mid_{t = 0} &= u_1 & \text{on } \Omega 
\end{align*}
\]

(M.P)

is well posed, if

i) \( \inf_{t \in \Omega, \xi \in S^{n-1}} \sum_{j, k=1}^{n} a_{jk}(t, x) \xi_j \xi_k > 0 \),

ii) \( a_{jk}, b_j, c \in C^0(I \times \Omega) \)

are satisfied. How about the problem if i) and ii) are replaced by

i') \( \inf_{t \in \Omega, \xi \in S^{n-1}} \sum_{j, k=1}^{n} a_{jk}(t, x) \xi_j \xi_k > 0 \) (any \( \varepsilon > 0 \)),

ii') \( a_{jk}, b_j, c \in C^0(I \times \Omega) \) (any \( \varepsilon > 0 \)),

where \( \Omega_\varepsilon = \Omega \cap \{ x_1 > \varepsilon \} \)? In this paper, assuming i') and ii)', we consider two cases. One is a degenerate case, when i) is not satisfied, and the other is a singular case, when ii) is not satisfied. Their typical examples are as follows:

(1) \( L = \partial_t^2 - \rho \partial_t \partial_x - \partial_x^2 - (\mu + 1) \partial_1 \),

(II) \( L = \partial_t^2 - \partial_t \partial_x - \partial_x^2 - (\mu + 1) \rho \partial_1 \),

Communicated by Prof. N. Iwasaki, December 1, 1989
where $\mu$ is a real parameter, $\rho = \rho(x_i) \in \Omega^\infty(R_+)$, and $\rho = x_i$ near $x_i = 0$. We consider the mixed problem (M.P) for $\mu < 0$ and we consider the Cauchy problem:

\[
(C.\ P) \quad \begin{cases}
Lu = f & \text{in } I \times \Omega, \\
u|_{t=0} = u_0, & \partial_t u|_{t=0} = u_1 \\
\end{cases}
\]

for $\mu > 0$.

There are so many studies on semi-degenerate problems for parabolic or elliptic operators since W. Feller [1], but there are little about hyperbolic problems except for fully degenerate cases (e.g. [2], [3]). Nakaoka ([4]) considered

\[
\partial_t^2 u = \rho(x)^\alpha \partial_x^2 u \quad (0 < \alpha < 1)
\]
in $\{ t > 0, x > 0 \}$ with initial data and with zero boundary data. By the change of variables

\[
s = \beta t, \quad y = x^\beta \quad (\beta = 2 - \alpha),
\]

it is transformed into

\[
\partial_t^2 u = \rho(y)^\alpha \partial_y^2 u + (\mu + 1) \partial_y u \quad (\mu = -\frac{1}{\beta} = -\frac{1}{2 - \alpha})
\]

near $y = 0$. Therefore the result of this paper is considered as a generalization of Nakaoka's. The simple idea in this paper is to reduce $L$ to a Bessel type operator. The energy method is applicable to Bessel type operators. Examples in §6 illustrate the structure of solutions relating to a parameter $\mu$.

§ 1. Semi-degenerate problems and singular coefficient problems

Let us assume that $L$ satisfies the following Ass. I$-\mu$ or Ass. II$-\mu$ in addition to i') and ii') in Introduction. Under Ass. I$-\mu$ or Ass. II$-\mu$, $L$ is a Fuchsian on $\partial \Omega$ with characteristic roots $\{0, \mu\}$ (see [5]).

**Assumption I$-\mu$ (degenerate case).** $a_{ij}, b_j, c \in \Omega^\infty(I \times \bar{\Omega})$, and

(I-A) \quad $a_{ij} = \rho \bar{a}_{ij}$, where $\bar{a}_{ij} \in \Omega^\infty(I \times \bar{\Omega})$ ($j = 0, 1, \ldots, n$),

\[
\inf_{I \times \partial \Omega} \bar{a}_{11} > 0, \quad \inf_{I \times \partial \Omega} \sum_{j=1}^{n} a_{ij} \xi_j \xi_k > 0,
\]

(I-B) \quad $b_1 = -a_{11}(\mu + 1) + \rho \bar{b}_1$, where $\mu$ is a real constant and $\bar{b}_1 \in \Omega^\infty(I \times \bar{\Omega})$.

**Assumption II$-\mu$ (singular case).** $a_{ij}, b_j$ ($j \neq 1$) $\in \Omega^\infty(I \times \bar{\Omega})$, and

(II-A) \quad $a_{ij} = \rho \bar{a}_{ij}$, where $\bar{a}_{ij} \in \Omega^\infty(I \times \bar{\Omega})$ ($j \neq 1$),

\[
\inf_{I \times \partial \Omega} a_{11} > 0, \quad \inf_{I \times \partial \Omega} \sum_{j=1}^{n} a_{ij} \xi_j \xi_k > 0,
\]

(II-B) \quad $b_1 = -a_{11}(\mu + 1) \rho^{-1} + \bar{b}_1$, where $\mu$ is a real constant and $\bar{b}_1 \in \Omega^\infty(I \times \bar{\Omega})$

(II-C) \quad $c = \rho^{-1} \xi, \ c \in \Omega^\infty(I \times \bar{\Omega})$

We say that $(M, P)$ is *solvable* in $H^\gamma$, if there exists a unique solution $u \in H^\gamma(I \times \Omega)$ if
$(f, g_0, u_0, u_1) \in H^1(I \times \Omega) \times H^1(I \times \partial \Omega) \times H^1(\Omega) \times H^1(\Omega)$

with compatibility conditions of order $l$ for some $l$ and $l'$. Compatibility conditions will be explained later. We say that $(C, P)$ is solvable in $H^k$, if there exists a unique solution $u \in H^k(I \times \Omega)$ if

$$(f, u_0, u_1) \in H^1(I \times \Omega) \times H^1(\Omega) \times H^1(\Omega)$$

for some $l$.

Under Ass. $I-\mu$, we have

$$L = -a_{11}((\rho \partial_i) + \mu + 1) \partial_t$$

$$+ \left\{ -2(a_i \partial_t + \sum_j a_j \partial_j) + b \right\} (\rho \partial_i)$$

$$+ \{ \partial_t^2 - 2 \sum_{j,k} a_{jk} \partial_j \partial_k - \sum_j a_{jk} \partial_k \partial_j + b \partial_t + \sum_j b \partial_j + c \}$$

$$= -a_{11} \{ \Phi(\rho \partial_i) \partial_t + \Psi_t(\partial_t)(\rho \partial_i) + \Psi_{tt}(\partial_t) \}$$

$$= -a_{11} \{ \Phi(\rho \partial_i) \partial_t + \Psi_t(\partial_t, \rho \partial_i) \} = -a_{11} L'',$$

where

$$\sum_j = \sum_{j \neq i}, \quad \sum' = \sum_{j \neq i, k \neq i}, \quad \partial_t = (\partial_t, \partial_x, \ldots, \partial_{\partial_n}).$$

Under Ass. $II-\mu$, we have

$$\rho L = -a_{11}((\rho \partial_i) + \mu + 1) \partial_t$$

$$+ \left\{ -2 \rho(a_i \partial_t + \sum_j a_j \partial_j) + b \right\} (\rho \partial_i)$$

$$+ \{ \rho [\partial_t^2 - 2 \sum_{j,k} a_{jk} \partial_j \partial_k - \sum_j a_{jk} \partial_k \partial_j + b \partial_t + \sum_j b \partial_j + c] + \partial_t \}$$

$$= -a_{11} \{ \Phi(\rho \partial_i) \partial_t + [\Psi_t(\partial_t, \rho \partial_i) + \Psi_{tt}(\partial_t)] \}$$

$$= -a_{11} \{ \Phi(\rho \partial_i) \partial_t + \Psi(\partial_t, \rho \partial_i) \} = -a_{11} L'.$$

In both cases,

$$L' = \Phi(\rho \partial_i) \partial_t + \Psi(t, x; \partial_t, \rho \partial_i),$$

where $\Phi(\lambda) = \lambda + \mu + 1$ and $\Psi$ is a linear differential operator of 2nd order with respect to $\partial_t, \rho \partial_i$ with $B^m$ coefficients and

$$\Psi(t, x; \partial_t, \rho \partial_i) = \Phi(t, x; \partial_t)(\rho \partial_i) + \Psi_{tt}(t, x; \partial_t).$$

To consider $L'$ near $x_1 = 0$, we define $L_1 \equiv L_2$, if $L_1 = L_2$ near $x_1 = 0$. Moreover, we define $L_\beta$ by

$$\rho^\beta L u = L_\beta(\rho^\beta u).$$

**Lemma 1.1.** Let $L$ satisfy Ass. $I-\mu$ (resp. Ass. $II-\mu$), then $L_\mu$ satisfies Ass. $I-(\mu)$ (resp. Ass. $II-(\mu)$).

**Proof.** Let $L$ satisfy Ass. $I-\mu$, then

$$-a_{11} L \equiv \rho^{-1}(\rho \partial_i + \mu)(\rho \partial_i) + \{ \Psi_t(\rho \partial_i) + \Psi_{tt} \},$$

$$L' = \Phi(\rho \partial_i) \partial_t + \Psi(t, x; \partial_t, \rho \partial_i),$$

where $\Phi(\lambda) = \lambda + \mu + 1$ and $\Psi$ is a linear differential operator of 2nd order with respect to $\partial_t, \rho \partial_i$ with $B^m$ coefficients and

$$\Psi(t, x; \partial_t, \rho \partial_i) = \Phi(t, x; \partial_t)(\rho \partial_i) + \Psi_{tt}(t, x; \partial_t).$$

To consider $L'$ near $x_1 = 0$, we define $L_1 \equiv L_2$, if $L_1 = L_2$ near $x_1 = 0$. Moreover, we define $L_\beta$ by

$$\rho^\beta L u = L_\beta(\rho^\beta u).$$

**Lemma 1.1.** Let $L$ satisfy Ass. $I-\mu$ (resp. Ass. $II-\mu$), then $L_\mu$ satisfies Ass. $I-(\mu)$ (resp. Ass. $II-(\mu)$).

**Proof.** Let $L$ satisfy Ass. $I-\mu$, then

$$-a_{11} L \equiv \rho^{-1}(\rho \partial_i + \mu)(\rho \partial_i) + \{ \Psi_t(\rho \partial_i) + \Psi_{tt} \},$$

$$L' = \Phi(\rho \partial_i) \partial_t + \Psi(t, x; \partial_t, \rho \partial_i),$$

where $\Phi(\lambda) = \lambda + \mu + 1$ and $\Psi$ is a linear differential operator of 2nd order with respect to $\partial_t, \rho \partial_i$ with $B^m$ coefficients and

$$\Psi(t, x; \partial_t, \rho \partial_i) = \Phi(t, x; \partial_t)(\rho \partial_i) + \Psi_{tt}(t, x; \partial_t).$$

To consider $L'$ near $x_1 = 0$, we define $L_1 \equiv L_2$, if $L_1 = L_2$ near $x_1 = 0$. Moreover, we define $L_\beta$ by

$$\rho^\beta L u = L_\beta(\rho^\beta u).$$
therefore
\[-a_{11}^{-1} L \equiv \rho^{-1}(\rho \partial_t)(\rho \partial_t - \mu) + \{\Psi, (\rho \partial_t - \mu)\} + \Psi,\]
\[\approx \rho^{-1}(\rho \partial_t - \mu)(\rho \partial_t) + \{\Psi, (\rho \partial_t)\} + \Psi,\}

Let \( L \) satisfy Ass. II\( - \mu \), then
\[-a_{11}^{-1} L \equiv \rho^{-1}(\rho \partial_t - \mu)(\rho \partial_t) + \rho^{-1}\{\Psi, (\rho \partial_t)\} + \Psi,\]
therefore
\[-a_{11}^{-1} L \equiv \rho^{-1}(\rho \partial_t - \mu)(\rho \partial_t) + \rho^{-1}\{\Psi, (\rho \partial_t)\} + \Psi,\]

Let \( u \) be a smooth solution of \( L'u = f \). Let
\[g_j = \partial_t u \mid_{x_1 = 0}, \quad f_j = \partial_t f \mid_{x_1 = 0},\]
then we have
\[L'u = \Phi(\rho \partial_t)\partial_t u + \Psi(\partial_t, \rho \partial_t)u\]
\[\sim \Phi(\rho \partial_t)\{g_1 + g_2 \rho/1! + g_3 \rho^2/2! + \cdots\} + \Psi(\partial_t, \rho \partial_t)\{g_1 + g_2 \rho/1! + g_3 \rho^2/2! + \cdots\}\]
\[\sim \{\Phi(0)g_1 + \Psi(\partial_t, 0)g_1\} + \{\Phi(1)g_2 + \Psi(\partial_t, 1)g_1\} \rho/1! + \cdots\]
where \( \sim \) means the asymptotic expansion at \( x_1 = 0 \). Since
\[\Psi(t, x, \partial_t, \rho \partial_t) \sim \sum_{k=0}^\infty \rho^k / k! \Psi^{(k)}(t, x'; \partial_t, \rho \partial_t),\]
we have
\[(*) \quad \Phi(j)g_{j+1} + \sum_{k=0}^j (\frac{j!}{k!})\Psi^{(j-k)}(\partial_t, k)g_k = f_j \quad (j = 0, 1, 2, \ldots).\]

Conversely, let us define \( \{g_1, \ldots, g_{l'-1}\} \) by \((*)\), making use of data \( \{f_0, \ldots, f_{l'-2}, g_0\} \),
if \( \mu = -1, -2, \ldots, -(l'-1) \). Then, we have
\[g_j \in H^{l'-l}(l \times \partial \Omega),\]
if \( f \in H^l(l \times \Omega) \) and \( g_0 \in H^l(l \times \partial \Omega) \). Let us define
\[U(t, x) = \sum_{j=0}^{l'-1} (j!)^{-1} g_j(t, x') \tilde{\rho}(x_1)^j,\]
where \( \tilde{\rho} \in \mathcal{D}'(R_+), \tilde{\rho} = x_1 \) near \( x_1 = 0 \), and \( \tilde{\rho} = 0 \) if \( x_1 > 1 \), then
\[U \in H^{l'-l}(l \times \partial \Omega) \subset H^{l'+1}\]
if \( l' \leq (l+1)/3 \).

In case when \( \mu = -\nu \ (\nu = 1, 2, \ldots) \), let \( u \) be a solution of \( L'u = f \) and
\[u = v + w \log \rho,\]
where \( v \) and \( w \) are smooth functions satisfying \( \partial_t w \mid_{x_1 = 0} = 0 \ (j = 0, 1, \ldots, \nu-1) \). Let
Semi-degenerate hyperbolic equations

\[ g_j \big|_{x_1=0} = h_j \big|_{x_1=0} = (h_0 = h_1 = \cdots = h_{n-1} = 0), \]

then we have

\[ L' v = \Phi(\rho \partial_1 \partial_v + \Psi(\partial_v, \rho \partial_1))v \]

\[ \sim \{ \Phi(0)g_1 + \Psi(\partial_v, 0)g_1 \} + \{ \Phi(1)g_2 + \Psi(\partial_v, 1)g_1 \} \rho/1 \]

\[ + \{ \Phi(2)g_3 + \Psi(\partial_v, 2)g_1 \} \rho^2/2 + \cdots \]

and

\[ L'(w \log \rho) \equiv \Phi(\rho \partial_1)(w \rho^{-1} + \partial_1 w \log \rho) + \Psi(\partial_v, \rho \partial_1)(w \log \rho) \]

\[ \equiv \{ \Phi(0)h_1 + \Psi(\partial_v, h_1) \} \rho^{-1}/ \nu \]

\[ + \{ \Phi(1)(w + 2)^{-1} + \partial_1 (w + 2) \} h_2 + \Psi(\partial_v, h_1) \rho^3/(\nu + 1)! + \cdots \]

\[ (\log \rho)^{-1} I_3 \sim \{ \Phi(0)h_2 + \Psi(\partial_v, h_2) \} \rho^{-1}/ \nu \]

\[ + \{ \Phi(1)(w + 1)^{-1} + \partial_1 (w + 1) \} h_3 + \Psi(\partial_v, h_3) \rho^3/(\nu + 1)! + \cdots \]

Hence we have

\[ \Phi(j)h_{j+1} + \sum_{k=0}^{j-1} \left( \frac{j}{k} \right) \Psi^{(j-k)}(\partial_v, k)h_k = 0 \quad (j = \nu, \nu + 1, \nu + 2, \cdots), \]

\[ h_0 + \sum_{k=0}^{\nu-2} \left( \frac{\nu-1}{k} \right) \Psi^{(\nu-k)}(\partial_v, k)h_k = f_{\nu-1} \]

\[ \Phi(j)g_{j+1} + \sum_{k=0}^{j-1} \left( \frac{j}{k} \right) \Psi^{(j-k)}(\partial_v, k)g_k = f_j \quad (j = 0, 1, 2, \cdots, \nu - 2), \]

\[ h_0 + \sum_{k=0}^{\nu-1} \left( \frac{\nu-1}{k} \right) \Psi^{(\nu-k)}(\partial_v, k)g_k = f_{\nu-1} \]

\[ \Phi(j)g_{j+1} + \sum_{k=0}^{j-1} \left( \frac{j}{k} \right) \Psi^{(j-k)}(\partial_v, k)g_k \]

Conversely, let us define \( \{ g_0, \cdots, g_{\nu-1}; h_0, h_{\nu+1}, \cdots, h_{\nu-1} \} \) by (***) and (**), making use of data \( \{ f_0, \cdots, f_{\nu-2}, g_0 \} \), if \( \mu = -\nu \) (\( \nu = 1, 2, \cdots \)). Then we have

\[ g_j, h_j \in H^{1-\mu}(I \times \partial \Omega), \]

if \( f \in H^1(I \times \Omega) \) and \( g_0 \in H^1(I \times \partial \Omega) \). Let us define

\[ U = V + W \log \rho. \]

where

\[ V(t, x) = \sum_{j=0}^{\nu-1} \left( \frac{\nu-1}{j} \right)^{-1} g_j(t, x') \rho(x_1)^j, \]
Reiko Sakamoto

\[ W(t, x) = \sum_{j=0}^{l' - 1} (j + 1)^{-1} h_j(t, x') \tilde{p}(x') \cdot \]

then

\[ V, W \in H^{l'+1} \quad (l' \leq (l+1)/3), \quad U \in H^\nu \quad (\nu \leq l'). \]

Let us define

\[ H^l_\Omega = \{ u \in H^l \mid \partial_i u \mid_{x_1=0} = 0 \quad (j=0, 1, \cdots, l-1) \}, \]

and say that data \( \{ f, g_0, u_0, u_1 \} \) satisfy compatibility conditions of order \( l' \) if

\[ \begin{cases} \bar{u}_0 = u_0 - U \mid_{x_1=0} \in H^{l'-1}_0(\Omega), \\ \bar{u}_1 = u_1 - \partial_i U \mid_{x_1=0} \in H^{l'-1}_0(\Omega). \end{cases} \]

Here we have

**Lemma 1.2.** Let

\( (f, g_0, u_0, u_1) \in H^l(I \times \Omega) \times H^{l'}(I \times \partial \Omega) \times H^{l'}(\Omega) \times H^{l'}(\Omega) \)

satisfy compatibility conditions of order \( l' \) \((l' \leq (l+1)/3)\), then

\[ \tilde{f} = f - L'U \in H^{l'-1}(l \times \Omega), \]

\[ \tilde{u}_j = u_j - \partial_i U \in H^{l'-1}(l \times \Omega) \quad (j=0, 1). \]

Let us say that (C.P) is solvable in \( H^h_\Omega \), if there exists a unique solution \( u \in H^h_\Omega(I \times \Omega) \) for any \( (f, g_0, u_0) \in H^l(I \times \Omega) \times H^h(\Omega) \times H^h(\Omega) \) with some \( l \).

Here we have

**Lemma 1.3.** i) In case when \( \mu \neq -1, -2, \cdots \), if (C.P) is solvable in \( H^h_\Omega \), then (M.P) is solvable in \( H^h \).

ii) In case when \( \mu = -\nu \) \((\nu = 1, 2, \cdots)\) if (C.P) is solvable in \( H^h_\Omega \), then (M.P) is solvable in \( H^{\min(h, \nu)} \).

Our aim is to establish the following theorems.

**Theorem (C).** Let \( L \) satisfy Ass. I-\( \mu \) or Ass. II-\( \mu \). Let \( \mu > 0 \), then (C.P) is solvable in \( H^h \) for any \( h \).

**Theorem (M).** Let \( L \) satisfy Ass. I-\( \mu \) or Ass. II-\( \mu \). Let \( -h - 1/2 \leq \mu < -h + 1/2 \), then (M.P) is solvable in \( H^h \), where \( h = 2, 3, 4, \cdots \).

**Theorem (M').** Let \( L \) satisfy Ass. I-\( \mu \) or Ass. II-\( \mu \). (M.P) has a solution in \( H^h \), if \( -1/2 \leq \mu < 0 \). (M.P) has a solution in \( H^l \), if \( -3/2 \leq \mu < -1/2 \). More precisely, (M.P) has a unique solution satisfying
Semi-degenerate hyperbolic equations

\[ u \in H^s(I \times \Omega) , \]
\[ u \in \mathcal{D}(I; \Omega) , \]
\[ (x, \partial_t)u \in \mathcal{D}(I; \Omega) , \]
\[ (x, \partial_t)^s u \in \mathcal{D}(I; \Omega) , \]

for some \( \varepsilon > 0 . \)

§ 2. Hyperbolic operators of Bessel type

Let us define

\[ \mathcal{B}_s(I \times \Omega) = \{ u | \partial_x \partial_\alpha u \in H^s(I \times \Omega) \ (|\alpha| \leq k) \} , \]
\[ H^s(I \times \Omega) = \{ u | \rho^{-1/2} \partial_x \partial_\alpha u \in H^s(I \times \Omega) \ (|\alpha| \leq k) \} , \]

where

\[ \partial_x = \partial_x \partial_\alpha (\partial_t)^s \partial_x \partial_\alpha \cdots \partial_x \partial_\alpha . \]

Let us assume that \( L \) satisfies more general assumptions than Ass. I—\( \mu \) or Ass. II—\( \mu \). Let \( \sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n) \) satisfy

\[ \sigma_i = 1, \quad 0 < \sigma_0 \leq \sigma_j \ (j=1, \ldots, n) , \]

and let us define \( \rho_i = \rho^\sigma \), then we have

\[ \rho_i = \rho, \quad \rho_0 \leq \rho_j \ (j=1, \ldots, n) , \]

Let us define

\[ \hat{\rho}_j = \rho_j \partial_j \ (j=0, 1, \ldots, n) , \]

where \( \partial_0 = \partial_t \).

**Assumption III—\( \mu - \sigma \).** \( a_{ij}, b_j, c \in \mathcal{D}(I \times \Omega) \), and

\[ L \equiv \{ \partial_t^2 + \mu \partial_t - c \} + 2 \rho_0 \sum_j \partial_x \partial_t \partial_j \partial_t^2 \partial_j + \sum_{j,k} \partial_x \partial_j \partial_k \partial_t^2 \partial_j + \rho_0 \sum_j b_j \partial_j , \]

where \( \sum_{j+k} = \sum_j, \sum^*_{j+k} = \sum^*_{j+k} \), where

(III-A) \[ \sup_{\partial_0} a_{00} < 0 , \]

(III-D) \[ \mu^2 + 4 \inf_{\partial_0} c > 0 . \]

**Remark.** If \( L \) satisfies Ass. I—\( \mu \) with \( \mu \neq 0 \), then \( -a_{ij} \rho L \) satisfies Ass. III—\( \mu \)– \( (1/2, 1, 1/2, \ldots, 1/2) \). If \( L \) satisfies Ass. II—\( \mu \) with \( \mu \neq 0 \), then \( -a_{ij} \rho \partial_\alpha L \) satisfies III—\( \mu \)– \( (1, 1, \ldots, 1) \).

Let us define

\[ \mathcal{S}^1_{\rho, \alpha} = \{ u | \rho_\alpha u \in H^s \ (j=0, 1, \ldots, n), u \in H^s \} \ (l=0, 1, \ldots) , \]
\[ \mathcal{S}^0_{\rho, \sigma} = \{ u | \rho_\sigma u \in H^s \} , \]

then we have
Our aim in §2～§4 is to establish the following

**Theorem 1.** Let \( L \) satisfy Ass. III—\( \mu - \sigma \). Let

\[
\rho^{\mu s}(\rho_1^{-1}, u_0, u_1) \in H^1_{\rho} (I \times \Omega) \times \mathcal{K}_{\rho, \sigma}^{-1}(\Omega) \times \mathcal{K}_{\rho, \sigma}^1(\Omega) \quad (l \geq 0),
\]

then there exists a unique solution \( u \) of (C.P) satisfying

\[
\rho^{\mu s} u \in \mathcal{K}_{\rho, \sigma}^{-1}(I \times \Omega).
\]

**Lemma 2.1.** Let \( Z \) be a variable transformation in \( \Omega \):

\[
Z_1: z_1 = x^1, \quad z' = x' \quad \text{near} \quad x_1 = 0,
\]

and let \( L_\mu \) be the transformed operator of \( L \). Assume that \( L \) satisfies Ass. III—\( \mu - \sigma \), then \( L_\mu \) satisfies Ass. III—\( \mu_* - \sigma_* \), where \( \mu_* = \mu / s \) and \( \sigma_* = (\sigma_0 / s, 1, \sigma_2 / s, \ldots, \sigma_n / s) \).

**Proof.** Let \( \rho_\mu(z_1) = z_1 \) near \( z_1 = 0 \), then

\[
\rho \partial_1 = s \rho \partial_{\mu 1}
\]

near \( z_1 = 0 \), where \( \partial_{\mu 1} = \partial z_1 \). Hence we have

\[
(\rho \partial_1)^2 + \mu (\rho \partial_1) - c = (s \rho \partial_{\mu 1})^2 + s \mu (s \rho \partial_{\mu 1}) - c
\]

\[
= s^2 ((s \rho \partial_{\mu 1})^2 + s \mu (s \rho \partial_{\mu 1}) - c)
\]

near \( z_1 = 0 \), where \( \mu_* = \mu s^{-1} \), \( \sigma_* = cs^{-2} \), and

\[
\mu_*^2 + 4 \inf \sigma_* = \mu_*^2 s^{-2} + 4 \inf cs^{-2} > 0.
\]

Let us consider a transformation of dependent variables:

\[
u \longrightarrow v = \rho^\beta u,
\]

where \( \beta \) is a real number, then \( L \) is transformed to \( L_\beta \) i.e.

\[
\rho^\beta Lu = L_\beta (\rho^\beta u).
\]

**Lemma 2.2.** Assume that \( L \) satisfies Ass. III—\( \mu - \sigma \), then \( L_{\mu/2} \) satisfies Ass. III—\( 0 - \sigma \).

**Proof.** Let \( L \) satisfy Ass. III—\( \mu - \sigma \), then

\[
L = \{ \tilde{\partial}_1^2 + \mu \tilde{\partial}_1 - c \} + 2 \rho \mu L_1 (\tilde{\partial}_1 \tilde{\partial}_1 + L_2 (\tilde{\partial}_1) + \rho \Sigma b_j \tilde{\partial}_j,
\]

where

\[
L_1 = \sum_j a_{1j} \tilde{\partial}_j, \quad L_2 = \sum_{j, k} a_{2jk} \tilde{\partial}_j \tilde{\partial}_k.
\]

Then we have
Semi-degenerate hyperbolic equations

\[
L_\beta \equiv (\partial_t - \beta)^2 + \mu (\partial_t - \beta) - c + 2\rho_\beta \mathcal{L}_1 (\partial_t - \beta) + \mathcal{L}_2 \\
+ \rho_\beta \{ b_1 (\partial_t - \beta) + \sum * b_j \hat{\partial}_j \} \\
\approx \{ \partial_t^2 + (-2 \beta + \mu) \partial_t - [-\beta^2 + \mu \beta + c - \rho_0 b_1 \beta] \} \\
+ 2\rho_\beta \mathcal{L}_1 \partial_t + \mathcal{L}_2 + \rho_\beta \{ b_1 \partial_t + (\sum * b_j \hat{\partial}_j - 2\beta \mathcal{L}_1) \},
\]

therefore we have

\[
L_{m/2} \equiv \delta_t^2 - c + 2\rho_\beta \mathcal{L}_1 \partial_t + \mathcal{L}_2 + \rho_\beta \{ b_1 \partial_t + (\sum * b_j \hat{\partial}_j - \mu \mathcal{L}_1) \},
\]

where

\[
c_\beta = (\mu/2)^2 + c - \rho_0 b_1 (\mu/2).
\]

Since

\[
c_\beta |_{x_1=0} = (\mu/2)^2 + c |_{x_1=0},
\]

\(L_{m/2}\) satisfies Ass. III-0-\(\sigma\).

We say that \(L\) is a hyperbolic operator of Bessel type, if \(L\) satisfy Ass. III-0-\(\sigma\).

§ 3. Energy estimates of (C.P) for hyperbolic operator of Bessel type

Let us define

\[
(u, v) = (u, v)_{L^2(\Omega)}, \quad (u, v)_\rho = (\rho^{-1/2}u, \rho^{-1/2}v)_{L^2(\Omega)},
\]

\[
\|u\|_{H^1(\Omega)} = \left\| \sum_{j=1}^{6} \|\nabla_j u\|_\rho \right\|,
\]

\[
\|u(t)\|_{H^1(\Omega)} = \left\| \sum_{j=0}^{1} \|\nabla_j u(t)\|_\rho \right\|
\]

and

\[
\|u\|_{H^1_t(\Omega)} = \left\| \int_0^t \|u(t)\|_{H^1(\Omega)} dt \right\|
\]

Moreover, let us define

\[
\|u\|_{H^{1+1}_\rho(\Omega)} = \left\| \sum_{j=1}^{6} \|\nabla_j u\|_\rho \right\| + \|u\|_{H^1(\Omega)},
\]

\[
\|u(t)\|_{H^{1+1}_\rho(\Omega)} = \left\| \sum_{j=0}^{1} \|\nabla_j u(t)\|_\rho \right\| + \|u(t)\|_{H^1(\Omega)}
\]

and

\[
\|u\|_{H^{1+1+1}_\rho(\Omega)} = \left\| \int_0^t \|u(t)\|_{H^{1+1}_\rho(\Omega)} dt \right\|
\]

Remark. If \(u \in H^{1+1}_\rho\), then \((\rho \partial_t)^j|_{x_1=0} = 0 \quad (j=0, 1, \ldots, l)\).

Let us define

\[
e^{-rt} Lu = L^* (e^{-rt} u),
\]

then

\[
L^* = L(t, x; \partial_t, \partial_x) = L(t, x; \partial_t + \gamma, \partial_x).
\]
Let us define
\[
\begin{align*}
\|u(t)\|_{t, 1+1, t}^2 &= \|u(t)\|_{t, 1+1}^2 + \|u(t)\|_{t, 1+1}^2, \\
\|u(t)\|_{t, 1+1, t}^2 &= \|u(t)\|_{t, 1+1}^2 + \|u(t)\|_{t, 1+1}^2,
\end{align*}
\]
then we have

**Lemma 3.1** (basic energy estimate). Let \( L \) be of Bessel type, then there exist \( \gamma_0(>0) \) and \( C(>0) \) such that
\[
\|u(t)\|_{t, 1+1, T}^2 + \int_0^T \|u(t)\|_{t, 1+1, t}^2 \, dt 
\leq C \left\{ \|u(0)\|_{t, 1+1}^2 + \int_0^T \rho_0^{-1} L^- u(t)^2 \, dt \right\} \quad (0 \leq t \leq T)
\]
for any \( \gamma > \gamma_0 \) and any \( u \in \mathcal{K}_{l, \varepsilon} (l \times \Omega), \rho_0^{-1} L^- u \in H_{\mu}^s (l \times \Omega) \).

**Proof.** It is sufficient to prove Lemma 3.1 for \( n \) satisfying \( \text{supp } [u] \subset \{ x \leq \varepsilon \} \) (\( \varepsilon(>0) \) : small enough). Let us denote
\[
L \equiv a_{00} \rho_0^2 \partial_t^2 + 2 \rho_0 \lambda_1 \partial_t + \lambda_2 + \rho_0 \partial_t,
\]
where
\[
\lambda_1 = \rho_0 a_{00} \partial_t - \sum_j a_j \partial_j, \\
\lambda_2 = -c + \hat{\alpha}^2 + 2 \rho_0 \sum_j a_{ij} \partial_j \partial_i + \sum_j a_{ji} \partial_j \partial_i, \\
\partial_t = b_{0} \partial_t + \sum_{j=1}^n b_j \partial_j,
\]
then we have
\[
(L^- u, \partial_t u)_\rho = (a_{00} \rho_0 \partial_t^2 u, \rho_0 \partial_t u)_\rho + 2 (\lambda_1 \partial_t u, \rho_0 \partial_t u)_\rho \\
+ (\lambda_2 u, \partial_t u)_\rho + (\partial_t u, \rho_0 \partial_t u)_\rho \\
= I_1 + I_2 + I_3 + I_4 = I.
\]
We have
\[
-2 \text{Re } I_1 = (\partial_t + 2\gamma)(-a_{00} \rho_0 \partial_t u, \rho_0 \partial_t u)_\rho + R_1, \\
-2 \text{Re } I_2 = (\partial_t + 2\gamma)(c u, u)_\rho + \| \partial_t u \|^2 + 2 \sum_j (\rho_0 a_{ij} \partial_i u, \partial_j u)_\rho \\
+ \sum_j (a_{ij} \partial_i u, \partial_j u)_\rho + R_3,
\]
where
\[
| R_1 | + | R_2 | + | R_3 | + | I_4 | \leq C (\| \rho_0 \partial_t u \|^2 + \sum_{j=1}^n \| \partial_j u \|^2 + \| u \|^2).
\]
Let
\[
-2 \text{Re } I = (\partial_t + 2\gamma) E(t) + R(t),
\]
where
\[
E(t) = (-a_{00} \rho_0 \partial_t u, \rho_0 \partial_t u)_\rho + (c u, u)_\rho \\
+ 2 \sum_j (\rho_0 a_{ij} \partial_i u, \partial_j u)_\rho + \sum_j (a_{ij} \partial_i u, \partial_j u)_\rho.
\]
then we have
\[ c_1 \{ \| \rho \partial_t^2 u \|_p^p + \| u(t) \|_{L^p(x)}^p \} \leq E(t) \leq c_2 \{ \| \rho \partial_t^2 u \|_p^p + \| u(t) \|_{L^p(x)}^p \}, \quad | R(t) | \leq c_3 E(t), \]
where \( \{ c_j \} \) are positive constants independent of \( t, \gamma, u \). On the other hand, since
\[ | I | \leq C \| \rho \bar{\gamma} L^{-1} u(t) \|_\rho \nu^{-1/2}, \]
we have
\[ (\partial_t \gamma E(t) \leq C \gamma^{-1} \| \rho \bar{\gamma} L^{-1} u(t) \|_\rho \gamma > \gamma_0), \]
therefore
\[ E(t) + \gamma \int_0^t E(s) ds \leq E(0) + C \gamma^{-1} \int_0^t \| \rho \bar{\gamma} L^{-1} u(t) \|_\rho \gamma > \gamma_0). \]

Remarking
\[ \int_0^t \| \rho \partial_t^2 u \|_p^p dt = \gamma \left\{ \int_0^t \{ \| \rho \partial_t^3 u \|_p^p + \gamma^2 \| \rho \partial_t u \|_p^2 \} dt + \gamma \| \rho u(t) \|_p^2 \gamma > \gamma_0 \right\}, \]
we have Lemma 3.1. \( \square \)

**Lemma 3.2.** Let \( L \) be of Bessel type, then
\[ \hat{\partial}_t^\alpha L - \hat{\partial}_t^\alpha = P_\alpha(\hat{\partial}_t) + \rho_0 \sum_{j=1}^m Q_{\alpha j}(\hat{\partial}_t) \hat{\partial}_j, \]
where
\[ P_\alpha(\hat{\partial}_t) = \sum_{|\alpha| = 1, 2, \ldots} p_{\alpha \beta}(t, x) \hat{\partial}_\beta, \]
\[ Q_{\alpha j}(\hat{\partial}_t) = \sum_{|\beta| = 1} q_{\alpha j \beta}(t, x) \hat{\partial}_\beta, \]
where \( p_{\alpha \beta}, q_{\alpha j \beta} \in B_\rho^m. \)

**Proof.** It is proved by the mathematical induction about \( \{| \alpha | = 1, 2, \ldots \} \). Let us see the case when \( |\alpha| = 1 \). Since
\[ L = -c + \hat{\partial}_t^2 + 2 \rho_0 \hat{\partial}_t + \rho_0 \sum b_j \hat{\partial}_j \]
\[ = -c + \hat{\partial}_t^2 + 2 \rho_0 \sum a_{j1} \hat{\partial}_j + \sum a_{j1} \hat{\partial}_j \hat{\partial}_1 + \rho_0 \sum b_j \hat{\partial}_j, \]
we have for \( l = 1 \)
\[ \partial_t L - L \partial_t = -c^{(t)} + 2 \rho_0 \sum a^{(t)} \partial_j \partial_t + \sum a^{(t)} (\rho_j / \rho_0) \partial_j \partial_t + b^{(t)} \partial_t \]

where \( a^{(t)} = \partial_t a_j - a_j \partial_t, \ldots \). Let us assume that it holds for \(|\alpha| = N: \]

\[ \partial_t^\alpha L - L \partial_t^\alpha = P_\alpha + \rho_0 \sum Q_{\alpha j} \partial_j. \]

Let \( l \approx 1 \), then

\[ \partial_t^\alpha L - L \partial_t^\alpha = \partial_t (\partial_t^\alpha L - L \partial_t^\alpha) + (\partial_t L - L \partial_t) \partial_t^\alpha \]

\[ = \partial_t (P_\alpha + \rho_0 \sum Q_{\alpha j} \partial_j) + (P_\alpha + \rho_0 \sum Q_{\alpha j} \partial_j) \partial_t^\alpha \]

\[ = \{ \partial_t P_\alpha + P_\alpha \partial_t^\alpha \} + \rho_0 \sum (\partial_t Q_{\alpha j} + Q_{\alpha j} \partial_t^\alpha) \partial_j. \]

**Lemma 3.3.** Let \( L \) be of Bessel type, then there exist \( \gamma_1 (>0) \) and \( C_1 (>0) \) such that for \(|\alpha| \leq l \)

\[ \sum_{i \in \mathbb{N} \times \mathbb{N}_1} \| \partial_t^\alpha u(t) \|_{\mathbb{H}^{(p - 1), 1} \cap \mathbb{H}^{(p - 1), 1}} + \| \nabla_{\partial_t^\alpha u(t)} \|_{\mathbb{H}^{(p - 1), 1}} \int_0^t \| \partial_t^\alpha L^\alpha u(t) \|_{\mathbb{H}^{(p - 1), 1}} dt \]

\[ \leq C \left\{ \sum_{i \in \mathbb{N} \times \mathbb{N}_1} \| \nabla_{\partial_t^\alpha u(t)} \|_{\mathbb{H}^{(p - 1), 1}} + \gamma_1 \| \nabla_{\partial_t^\alpha L^\alpha u(t)} \|_{\mathbb{H}^{(p - 1), 1}} \right\} \quad (0 < t < T) \]

for any \( \gamma > \gamma_1 \) and any \( \partial_t^\alpha u \in \mathbb{H}^p \), \( \nabla_{\partial_t^\alpha L^\alpha u} \in H^p_\alpha (I \times \Omega) \) \( (|\alpha| \leq l) \).

**Proof.** From Lemma 3.2, we have for \(|\alpha| \leq l \)

\[ L^\alpha \partial_t^\alpha u = \partial_t^\alpha L^\alpha u - P_\alpha (\partial_t^\alpha) u - \rho_0 \sum_{j=1}^N Q_{\alpha j} (\partial_t^\alpha) \partial_j u \]

\[ = F_1 + F_2 + F_3 = F, \]

where

\[ \int_0^t \| \rho_0 \partial_t^\alpha F_1(t) \|_{\mathbb{H}^{(p - 1), 1}} dt \leq C \int_0^t \| \rho_0 \nabla^\alpha L^\alpha u \|_{\mathbb{H}^{(p - 1), 1}} dt, \]

\[ \int_0^t \| \rho_0 \partial_t^\alpha F_2(t) \|_{\mathbb{H}^{(p - 1), 1}} dt \leq C \int_0^t \sum_{j \in \mathbb{N}_1} \| \nabla_{\partial_t^\alpha u(t)} \|_{\mathbb{H}^{(p - 1), 1}} dt. \]

Since

\[ \int_0^t (F_2, \partial_t^\alpha \partial_t^\alpha u) dt = \int_0^t (P_\alpha (\partial_t^\alpha) u, \partial_t^\alpha \partial_t^\alpha u) dt \]

\[ = \{ -(P_\alpha (\partial_t^\alpha) u(t), \partial_t^\alpha \partial_t^\alpha u(t)) \} dt \]

\[ + 2 \int_0^t (P_\alpha (\partial_t^\alpha) u(0), \partial_t^\alpha \partial_t^\alpha u(0)) dt \]

\[ - 2 \gamma \int_0^t (P_\alpha (\partial_t^\alpha) u, \partial_t^\alpha \partial_t^\alpha u) dt + \int_0^t (\partial_t^\alpha P_\alpha (\partial_t^\alpha) u, \partial_t^\alpha \partial_t^\alpha u) dt, \]

we have

\[ \int_0^t (F_2, \partial_t^\alpha \partial_t^\alpha u) dt \leq C \gamma^{-1} (E_{l+1}(t) + E_{l+1}(0)) + C \int_0^t E_{l+1}(t) dt. \]
where
\[ E_{1+1}(t) = \sum_{a \in A} \|D_{\xi}^a u(t)\|_{L^2}^2. \]

Here we have
\[ \left| \int_0^t (F, \partial_t D_{\xi}^a u) \, dt \right| \leq C \left\{ \int_0^t \|D_{\xi}^a L^{-1} u(t)\|_{L^2}^2 \, dt \right\}^{1/2} \left\{ \int_0^t E_{1+1}(t) \, dt \right\}^{1/2} \]
\[ + C \gamma^{-1} \left( E_{1+1}(t) + E_{1+1}(0) \right) + C \int_0^t E_{1+1}(t) \, dt. \]

Considering
\[ \sum_{a \in A} 2 \text{Re}(L^{-1}(\partial_{\xi}^a u), (\partial_t \partial_{\xi}^a u)) \]

as in the proof of Lemma 3.1, we have
\[ E_{1+1}(t) + \gamma \int_0^t E_{1+1}(t) \, dt \leq C_i \left\{ E_{1+1}(0) + \gamma^{-1} \int_0^t \sum_{a \in A} \|D_{\xi}^a L^{-1} u(t)\|_{L^2}^2 \, dt \right\} \]

for large \( \gamma \). \( \square \)

**Lemma 3.4.** Let \( L \) be of Bessel type, then
\[ \tilde{\xi}_i^{\text{re}} = M_{\text{re}}(\partial_{\xi}) L + P_{\text{re}}(\partial_{\xi}) + \sum_{j=0}^k Q_{k,j}(\partial_{\xi}) \tilde{\xi}_j \quad (k=0, 1, \ldots). \]

where
\[ M_{\text{re}} = \sum_{\beta \leq k} m_{h,\beta}(t, x) \tilde{\xi}_\beta, \]
\[ P_{\text{re}} = \sum_{\beta \leq k} p_{h,\beta}(t, x) \tilde{\xi}_\beta, \]
\[ Q_{k,j} = \sum_{\beta < k+1} q_{k,j}(t, x) \tilde{\xi}_\beta, \]

where \( m_{h,\beta}, p_{h,\beta}, q_{k,j} \in B_{\text{re}}^n \).

**Proof.** Since
\[ L \equiv -c + \partial_t^2 + 2\rho_0 L_1(\partial_{\xi}) \tilde{\xi}_1 + L_2(\partial_{\xi}) + \rho_0 \sum h_j \tilde{\xi}_j, \]
we have
\[ \tilde{\xi}_1^{\text{re}} \equiv L + c - \{ 2\rho_0 L_1(\partial_{\xi}) \tilde{\xi}_1 + L_2(\partial_{\xi}) + \rho_0 \sum h_j \tilde{\xi}_j \}, \]
\[ = L + c - \{ 2\rho_0 \sum a_{j1} \rho_0 \partial_j \tilde{\xi}_1 + \rho_0 b_1 \| \tilde{\xi}_1 \|^2 + \sum_{k} \{ \frac{m_{h,k} \rho_k \partial_k - \rho_0 b_j} \| \tilde{\xi}_j \|^2 \}, \]
\[ = L + c + \sum Q_{k,j}(\partial_{\xi}) \tilde{\xi}_j. \]

Let us assume that
\[ \tilde{\xi}_1^{\text{re}} = M_{N}(\partial_{\xi}) L + P_{N}(\partial_{\xi}) + \sum_{j=0}^k Q_{N,j}(\partial_{\xi}) \tilde{\xi}_j, \]
then
\[ \tilde{\xi}_1^{\text{re}} = \tilde{\xi}_1 M_{N}(\partial_{\xi}) L + P_{N}(\partial_{\xi}) \tilde{\xi}_1 + P_{N}'(\partial_{\xi}) \]
\[ + Q_{N,j}(\partial_{\xi}) \tilde{\xi}_j + \sum Q_{N,j}(\partial_{\xi}) \tilde{\xi}_j + \sum Q_{N,j}(\partial_{\xi}) \tilde{\xi}_j. \]
\[= \partial_t M_N(\partial_{\rho}) L + P_N(\partial_{\rho}) \partial_t + P_N'(\partial_{\rho}) \]
\[+ Q_N(\partial_{\rho}) \{ L + c + \sum Q_N(\partial_{\rho}) \partial_j \} + \sum Q_N(\partial_{\rho}) \partial_j \]
\[= (\partial_t M_N(\partial_{\rho}) + Q_N(\partial_{\rho})) L + \{ P_N(\partial_{\rho}) + Q_N(\partial_{\rho}) \} c \]
\[+ (\partial_t M_N(\partial_{\rho}) + Q_N(\partial_{\rho})) \partial_j + \sum Q_N(\partial_{\rho}) \partial_j + Q_N'(\partial_{\rho}) \partial_j \]

where \( P_N = \partial_t P_N - P_N \partial_t \).

From Lemma 3.3 and Lemma 3.4, we have

**Proposition 3.5.** Let \( L \) be of Bessel type, then there exist \( \gamma_1 (>0) \) and \( C_1 (>0) \) such that

\[
\|u(t)\|_{\rho, t+1, \gamma} + \gamma \int_0^t \|u(t)\|_{\rho, t+1, \gamma} \, dt \\
\leq C_1 \left\{ \|u(0)\|_{\rho, t+1, \gamma} + \gamma^{-1} \int_0^t \|\rho_0^{-1} L^{-1} u(t)\|_{\rho, t, \gamma} \, dt \right\} \quad (0 < t < T)
\]

for any \( \gamma > \gamma_1 \) and any \( \{ \rho \in \mathcal{H}^{t+1}_\rho, \rho_0^{-1} L^{-1} u \in H^t_\rho(I \times \Omega) \} \).

---

§ 4. **Existence theorem of (C.P) for hyperbolic operators of Bessel type**

To obtain existence theorem of (C.P) for \( L \), we construct approximate solutions for approximate problems. Let

\[ L_{(\epsilon)} = L(t, x_1 + \epsilon, x': \partial_t, \partial_x), \quad \rho_\epsilon = \rho(x_1 + \epsilon), \cdots \quad (\epsilon > 0), \]

then we have

**Lemma 4.1.** Let \( L \) be of Bessel type, then there exists \( \gamma_1 (>0) \) and \( C_1 (>0) \) such that

\[
\|u(t)\|_{\rho_\epsilon, t+1, \gamma} + \gamma \int_0^t \|u(t)\|_{\rho_\epsilon, t+1, \gamma} \, dt \\
\leq C_1 \left\{ \|u(0)\|_{\rho_\epsilon, t+1, \gamma} + \gamma^{-1} \int_0^t \|\rho_0^{-1} L_{(\epsilon)} u(t)\|_{\rho_\epsilon, t, \gamma} \, dt \right\} \quad (0 < t < T)
\]

for any \( \gamma > \gamma_1 \), any \( \{ \rho \in \mathcal{H}^{t+1}_{\rho_\epsilon}, \rho_0^{-1} L_{(\epsilon)} u \in H^t_{\rho_\epsilon}(I \times \Omega), u|_{x_1=0}=0 \} \) and any \( 0 < \epsilon < 1 \).

**Proof.** Lemma 4.1 with \( \epsilon = 0 \) is proved in the same way as in Lemma 3.1. Lemma 4.1 with \( \epsilon \geq 1 \) is proved in the same way as in Proposition 3.5, remarking that Lemma 3.2 and Lemma 3.4 are valid also when \( L, \rho, \rho_\alpha, \cdots \) are replaced by \( L_{(\epsilon)}, \rho_\epsilon, \rho_\alpha, \cdots \), where

\[ p_\alpha \beta (l, x) = p_\alpha \beta (l, x_1 + \epsilon, x'), \cdots \]

Remark that \( \rho \leq \rho_\epsilon \) and
Semi-degenerate hyperbolic equations

\[(\rho \partial_t)^j = \sum_{k=0}^j c_{jk} \rho^j \partial_t^k \quad (c_{jk} \in \mathcal{B}^n),\]

we have

**Lemma 4.2.** Let \( s \geq 1/2 \), then there exists \( C(>0) \) such that

\[\|\rho^ju\|_{\rho,1} \leq C \|\rho^ju\|_{\rho.1}\]

for any \( u \in H^1 \) and any \( 0 < \varepsilon < 1 \).

**Proposition 4.3.** Let \( L \) be of Bessel type. Let

\[\rho^{-1}_o f \in H^1_{\rho_o}(\Omega), \quad u_1 \in \mathcal{K}_{\rho_1,\sigma}(\Omega), \quad u_0 \in \mathcal{K}_{\rho_0,\sigma}(\Omega),\]

then there exists a unique solution \( u \in \mathcal{K}_{\rho_2,\sigma}(\Omega) \) of (C.P).

**Proof.** Let

\[\rho^{-1}_o f \in H^1_{\rho_o}(\Omega), \quad u_1 \in \mathcal{K}_{\rho_1,\sigma}(\Omega), \quad u_0 \in \mathcal{K}_{\rho_0,\sigma}(\Omega),\]

and set

\[f_s = f(t, x_i + \varepsilon, x'), \quad u_{j1} = u_j(x_1 + \varepsilon, x') \quad (j = 0, 1),\]

then there exists \( u \in H^{1+1} \) satisfying

\[
\begin{cases}
L(t)u_s = f_s, \\
u_s \big|_{x_1=0} = 0, \\
\partial_t^j u_s \big|_{t=0} = u_{j1} \quad (j = 0, 1).
\end{cases}
\]

Moreover, from Lemma 4.1,

\[\|u_s\|_{\mathcal{K}^{1+1}_{\rho_1,\sigma}(\Omega)} \leq C \left\{ \|\rho^{-1}_o f\| H^1_{\rho_o}(\Omega) + \|u_1\|_{\mathcal{K}^1_{\rho_1,\sigma}(\Omega)} + \|u_0\|_{\mathcal{K}^1_{\rho_0,\sigma}(\Omega)} \right\}.
\]

On the other hand, we have

\[\|\rho^{-1}_o f\| H^1_{\rho_o}(\Omega) = \|\rho^{-1}_o f\| H^1_{\rho_o}(\Omega) \leq \|\rho^{-1}_o f\| H^1_{\rho_o}(\Omega)
\]

and so on. Let

\[s = \max (\sigma_0, \sigma_1, \ldots, \sigma_n) \quad (\geq 1),\]

then, remarking Lemma 4.2, we have

\[c \sum_{j} \|\rho^j u_s\| H^1_{\rho_o}(\Omega) + \|\rho^j u_{s_1}\| H^1_{\rho_o}(\Omega) + \|\rho^j u_{s_2}\| H^1_{\rho_o}(\Omega) \leq C \left\{ \|\rho^{-1}_o f\| H^1_{\rho_o}(\Omega) + \|u_1\|_{\mathcal{K}^1_{\rho_1,\sigma}(\Omega)} + \|u_0\|_{\mathcal{K}^1_{\rho_0,\sigma}(\Omega)} \right\} \equiv K.
\]

Hence, there exists \( \{ u_{s_k} \} \) such that

\[v_k = \rho^s u_{s_k} \longrightarrow v \quad \text{in} \quad H^1_{\rho},\]

where \( u = \rho^{-s} v \) satisfies (C.P) and
Moreover, let $\alpha > 0$ and $|\alpha| \leq l$ then we have
\[
\| \rho \partial_x^{l+\alpha} \partial_x^\alpha \| + \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \\
\leq \lim_{k} \{ \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho + \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \} \\
\leq \lim_{k} \{ \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho + \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \} \leq CK,
\]
because
\[
\| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho + \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \\
\leq C \{ \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho + \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \} \\
\leq C \{ \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho + \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \} \\
= C \{ \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho + \| \partial_x^\alpha \|_\rho \} \leq CK.
\]
Since
\[
| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \leq | \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \] as $\alpha \to 0$, we have
\[
\rho \partial_x^{l+\alpha} \partial_x^\alpha \rho \partial_x^{l+\alpha} \partial_x^\alpha \in H^0_\rho,
\]
therefore it holds that
\[
\| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho + \| \rho \partial_x^{l+\alpha} \partial_x^\alpha \|_\rho \leq K.
\]
Here we have $u = \rho \partial x^{l+\alpha} \partial_x^\alpha u$.  

Owing to Lemma 2.2, Theorem 1 follows from Proposition 4.3 as its corollary.

§ 5. Problems (C.P) or (M.P) for $L$ under Ass. I—$\mu$ or Ass. II—$\mu$

Lemma 5.1.
\[
u \in H^1_\rho \iff \rho^{-1+1/2} u \in H^0_\rho.
\]

Proof. \( \Rightarrow \) Let $u \in H^1_\rho$ and let $|\alpha| \leq l$. Since
\[
\partial_x^\alpha u(x) = \{l - |\alpha| - 1\}^{-1} x^{l+1/2} \int_0^1 (1 - \theta)^{l+1/2} \partial_x^{l+1/2} \partial_x^{l+1/2} u(x, \theta, x') d\theta,
\]
we have
\[
\| x^{l+1/2} \partial_x^{l+1/2} u \| \leq \int_0^1 \| (\partial_x^{l+1/2} \partial_x^{l+1/2} u(x, \theta, x') \| d\theta,
\]
where $\| \| \leq \| L^2(0,1)^{n-1}$, Hence we have
\[
\rho^{-1+1/2} \partial_x^{l+1/2} u(x) \in H^0_\rho.
\]

\( \Leftarrow \) Let $\rho^{-1+1/2} u \in H^0_\rho$. Since it is easy to see that $u \in H^1_\rho$, let us show that $\partial_x^\alpha u |_{x=0} = 0$ \( (k \leq l-1) \). We remark that $\langle \rho \partial_x \rangle \partial_x^\alpha |_{x=0} (k \leq l-1)$ if $v \in H^0_\rho$. Therefore it holds that $\partial_x^\alpha v |_{x=0}$ if $\rho^{-1} v \in H^0_\rho (k \leq l-1)$, because
\[
\partial_t^2 v = \rho^{-1}(\rho \partial_t) \cdots \rho^{-1}(\rho \partial_t) v = (\rho \partial_t + 1)(\rho \partial_t + 2) \cdots (\rho \partial_t + k)(\rho^{-k} v),
\]
Since \( \rho^{-t+1} u \in H^1_{\rho} \), it holds that
\[
\partial_t^k u \big|_{x_1=0} \quad (k \leq t-1).
\]

Let us assume that \( L \) satisfy Ass. I-\( \mu \) or Ass. II-\( \mu \) in § 5, then
\[
L' = (\rho \partial_t + \mu + 1) \partial_t + \Psi(t, x; \partial_t, \rho \partial_t)
\]
is a Fuchsian, therefore regularity theorem holds [5]. Here we see it in our situation.

**Lemma 5.2.** Let \( \beta < \mu \). Assume that
\[
\rho^{\beta+t} u \in H^1_{\rho}, \quad \rho^{\beta+t} f \in H^{1-\varepsilon}_{\rho} \quad (\text{any } \varepsilon > 0),
\]
and \( L' u = f \), then
\[
\rho^{\beta+t} \partial_t u \in H^{1-\varepsilon}_{\rho} \quad (\text{any } \varepsilon > 0).
\]

**Proof.** Let \( \chi(x_1) \in B^m(R) \) such that \( \chi(x_1) = 1 \) near \( x_1 = 0 \) and \( \chi(x_1) = 0 \) for \( x_1 \geq 1 \). Multiplying both sides of \( L' u = f \) by \( \chi \), we have
\[
(x_1 \partial_t + \mu + 1) \partial_t v = g,
\]
where \( v = \chi u \) and
\[
x^\beta \partial_t v \in H^1_{\rho}, \quad x^\beta g \in H^{1-\varepsilon}_{\rho}.
\]
Multiplying both sides by \( x_t^\beta \), we have
\[
\partial_t(x_t^\beta \partial_t v) = x_t^\beta g.
\]
Since
\[
x_t^\beta \partial_t v = x_t^{-\beta-t}(x_t^{\beta+t} \partial_t v) = x_t^{-\beta-t} w, \quad w \in H^{1-\varepsilon}_{\rho},
\]
we have from Lemma 5.1
\[
x_t^\beta \partial_t v \big|_{x_1=0} = 0.
\]
taking \( \varepsilon \) small enough to satisfy \( \mu - \beta - \varepsilon > 0 \). Hence we have
\[
x_t^\beta \partial_t v = \int_0^{x_t} x_t^\beta g \, dx_1
\]
\[
= \int_0^{x_t} (x_t^{\beta+1-\beta-t})(x_t^{\beta+t} g) x_t^{-1} dx_1,
\]
therefore
\[
|x_t^\beta \partial_t v| \leq C x_t^{\beta+1-\beta-t} \left( \int |x_t^{\beta+t} g|^{1} x_t^{-1} dx_1 \right)^{1/\beta},
\]
that is,
\[
|x_t^{\beta+t} \partial_t v| \leq C x_t^{\beta-t} \left( \int |x_t^{\beta+t} g|^{1} x_t^{-1} dx_1 \right)^{1/\beta}.
\]
Taking \( 0 < \varepsilon < \varepsilon' \) for any \( \varepsilon' > 0 \), we have
\[
x_t^{\beta+t} \partial_t v \in H^0_{\rho} \quad (\text{any } \varepsilon' > 0).
\]
In the same way, we have

\[ x_i^{\beta + \epsilon} \partial_i v \in H_{p}^{\beta} \quad (\text{any } \epsilon > 0). \]

**Lemma 5.3.** Let \(0 < \beta < \mu\) and let \(l \geq [\beta] + 1\). Assume that

\[ \rho^a u \in H_p^\beta(I \times \Omega), \quad f \in H^\beta(I \times \Omega), \]

and \(L' u = f\), then

\[ u \in H^{l-\epsilon [\beta] - 1}(I \times \Omega). \]

**Proof.** Multiplying both sides of \(L' u = f\) by \(\chi\), defined in Lemma 5.2, we have

\[ (x_i \partial_i + \mu + 1) \partial_i v + \mathcal{P}(t, x; \partial_r, x_i \partial_i) v = g, \]

where \(v = \chi u\),

\[ x_i^{\beta} v \in H_p^{\beta}, \quad g \in H^{\beta+1}. \]

We have only to prove \(v \in H^{l-\epsilon [\beta] - 1}(I \times \Omega)\).

i) From Lemma 5.2, we have

\[ x_i^{\beta + \epsilon} \partial_i v \in H_p^{\beta} \quad (\text{any } \epsilon > 0). \]

Since

\[ |\partial_i^\alpha v| = \left| \int_1^x \partial_i \partial_i^\alpha v dx_1 \right| \]

\[ \leq C x_i^{-\beta + \epsilon} \left( \int_0^1 x_i^{\beta} |\partial_i \partial_i^\alpha v|^2 x_i^{-1} dx_1 \right)^{1/2} \]

(\(|\alpha| \leq 2(l-1)\)) if \(\beta > 1\), we have

\[ x_i^{\beta - 1 + \epsilon} \partial_i v \in H_p^{\beta} \quad (\text{any } \epsilon > 0), \]

therefore, from Lemma 5.2,

\[ x_i^{\beta - 1 + \epsilon} v \in H_p^{\beta+1}, \quad x_i^{\beta - 1 + \epsilon} \partial_i v \in H_p^{\beta+2}. \]

In this way, step by step, we have

\[ x_i^{\beta - [\beta] + \epsilon} v \in H_p^{\beta+1-\beta}, \]

\[ x_i^{\beta - [\beta] + \epsilon} \partial_i v \in H_p^{\beta+2-\beta}. \]

ii) We have

\[ |\partial_i^\alpha v| = \left| \int_1^x \partial_i \partial_i^\alpha v dx_1 \right| \]

\[ \leq C \left( \int_0^1 x_i^{\beta - [\beta] + \epsilon} |\partial_i \partial_i^\alpha v|^2 x_i^{-1} dx_1 \right)^{1/2} \]

(\(|\alpha| \leq 2(l-[\beta]-1)\)), taking \(\epsilon\) small enough to satisfy \(-2(\beta-[\beta] + \epsilon) > -2\), therefore

\[ x_i \partial_i^\alpha v \in H_p^{\beta} \quad (\text{any } \epsilon > 0). \]

Hence we have

\[ x_i v \in H_p^{\beta+1-\beta}, \quad x_i^{\beta + \epsilon} \partial_i v \in H_p^{\beta+2-\beta}. \]
iii) Multiplying both sides of $L'v = g$ by $\partial_1$, we have

$$(x_1\partial_1 + \mu + 2)\partial_1^tv = -\Psi(t, x_1, x_2, \partial_2, x_1\partial_1)v - \Psi_x(t, x_1, x_2, x_1\partial_1)v + \partial_1 g = g_2,$$

where

$$x_1^i g_2 \in H_t^{\alpha(t - [\beta] - 1)}\bigcap H^\alpha_{t}(l \times \Omega),$$

hence we have

$$x_1^i \partial_1 v \in H_t^{\alpha(t - [\beta] - 1)}\bigcap H^\alpha_{t}(l \times \Omega).$$

In the same way, we have

$$x_1^i \partial_1 v \in H_t^{\alpha(t - [\beta] - 1)}\bigcap H^\alpha_{t}(l \times \Omega),$$

therefore

$$x_1^i \partial_1 v \in H_t^{\alpha(t - [\beta] - 1)} \bigcap H^\alpha_{t}(l \times \Omega),$$

therefore $v \in H^{1 - [\beta]}_\Omega$.

Set $\beta = \mu/2$ in Lemma 5.3, then we have

**Theorem 2.** Let $L$ satisfy Ass. $I - \mu$ or Ass. $II - \mu$ with $\mu > 0$. Let $t \geq [\mu/2] + 1$ and $L'u = f$, where

$$\rho^\mu u \in H_t^{\alpha(t - [\beta] - 1)}\bigcap H^\alpha_{t}(l \times \Omega), \quad f \in H^{\alpha(t - [\beta] - 1)}(l \times \Omega),$$

then $u \in H^{1 - [\beta]}_\Omega$.

Owing to Theorem 2, Theorem (C) follows from Theorem 1.

**Theorem 3.** Let $L$ satisfy Ass. $I - \mu$ or Ass. $II - \mu$ with $\mu < 0$. For any $N$, there exists a solution of (C, P) satisfying $\rho^\mu u \in H^N$, if

$$f \in H^\alpha_{t}(l \times \Omega), \quad u_0 \in H^\alpha_{t}(\Omega), \quad u_1 \in H^\alpha_{t}(\Omega),$$

for some $l$.

**Proof.** Owing to Lemma 1.1, it follows from Theorem (C) that there exists a solution $v \in H^N$ of the problem:

$$\begin{cases}
L_\mu v = \rho^\mu f, \\
v|_{t=0} = \rho^\mu u_0, \quad \partial_1 v|_{t=0} = \rho^\mu u_1,
\end{cases}$$

because

$$\rho^\mu(f, u_0, u_1) \in H^l$$

from Lemma 5.1. Set $u = \rho^{-\mu} v$, then $u$ satisfies

$$\begin{cases}
L u = f, \\
u|_{t=0} = u_0, \quad \partial_1 u|_{t=0} = u_1.
\end{cases}$$

Remember Lemma 1.2, then Theorem (M) and Theorem (M)' follow from Theorem 3.
§ 6. Examples

Let us consider examples in one-dimensional x-space, whose solutions can be constructed exactly. First, let us consider

\[
\begin{cases}
    u_{tt} = xu_{xx} + 1/2u_x & (t > 0, x > 0), \\
    u \big|_{t=0} = u_0, \quad u_t \big|_{t=0} = u_1 & (x > 0),
\end{cases}
\]

where \( u_0, u_1 \in C^N(R_+) \), and \( u_0, u_1 = O(x^N) \) as \( x \to +0 \) (\( N: \text{large} \)).

Set

\[ \phi_+(x) = u_1(x) \pm \sqrt{x} u'_0(x), \]

then we have

\[ \phi_+ \in C^{N-1}(R_+), \quad \phi^{(k)}_+ = O(x^{N-k-1/2}) \quad \text{as} \quad x \to 0 \ (k \leq N-1). \]

Moreover, set

\[ \phi_+(t, x) = \begin{cases} 
\phi_+(t/2 + \sqrt{x} \theta)^2 \pm \sqrt{x} u_0(t/2 - \sqrt{x} \theta)^2 \big/ 2 & \text{if } t/2 - \sqrt{x} > 0, \\
\phi_+(t/2 + \sqrt{x} \theta)^2 + \phi_+(t/2 - \sqrt{x} \theta)^2 \big/ 2 & \text{if } t/2 - \sqrt{x} \leq 0,
\end{cases} \]

and \( \phi_-(t, x) = \sqrt{x} \Phi_0(t, x) \).

Lemma 6.1.

\[ \Phi_0, \ \Phi_+ \in C^M(\bar{R}_+ \times \bar{R}_+) \quad (M \leq (N-2)/3). \]

Proof. It is easy to see

\[ \Phi_+ \in C^{N-1}(\bar{R}_+ \times \bar{R}_+). \]

In the following, we shall see the regularity near \( x=0 \).

i) Regularity of \( \Phi_+ \). Let \( t/2 - \sqrt{x} > 0 \), then

\[
\begin{align*}
\Phi_+(t, x) &= \{ \phi_+(t/2 + \sqrt{x} \theta)^2 \pm \sqrt{x} u_0(t/2 - \sqrt{x} \theta)^2 \big/ 2 \\
&= \{ \phi_+(t^2/4 + x) + \phi_+(t^2/4 + x)(t\sqrt{x}) + \cdots \\
&\quad + \phi_+^{(2M-1)}(t^2/4 + x)(t\sqrt{x})^{2M-1}/(2M-1)! \\
&\quad + (t\sqrt{x})^{2M}/(2M-1)! \int_0^1 (1 - \theta)^{2M-1} \phi_+^{(2M)}(t^2/4 + x + t\sqrt{x} \theta) d\theta \big/ 2 \\
&\quad + \{ \phi_+(t^2/4 + x) + \phi_+(t^2/4 + x)(-t\sqrt{x}) + \cdots \\
&\quad + \phi_+^{(2M-1)}(t^2/4 + x)(-t\sqrt{x})^{2M-1}/(2M-1)! \\
&\quad + (-t\sqrt{x})^{2M}/(2M-1)! \int_0^1 (1 - \theta)^{2M-1} \phi_+^{(2M)}(t^2/4 + x - t\sqrt{x} \theta) d\theta \} \big/ 2 \\
&= \phi_+(t^2/4 + x) + \phi_+^{(2M)}(t^2/4 + x)t^2 x/21 + \cdots \\
&\quad + \phi_+^{(2M-2)}(t^2/4 + x)t^{2M-2} x^{M-1}/(2M-2) \\
&\quad + t^M x^M / [2(2M-1)!] \int_0^1 (1 - \theta)^{2M-1} \phi_+^{(2M)}(t^2/4 + x + t\sqrt{x} \theta) \\
&\quad + \phi_+^{(2M)}(t^2/4 + x - t\sqrt{x} \theta) d\theta.
\end{align*}
\]
Here we can see that $\phi^\star$ is differentiable up to order $M$ at $(t,0)$ ($t>0$) and

$$\partial^j\partial_t^k \phi^\star(t, x) \rightarrow 0 \quad \text{as} \quad (t, x) \rightarrow (0,0) \quad \text{in} \quad \{t/2-\sqrt{x} > 0\} \quad (j+k \leq M),$$

if $3M \leq N-1$.

Let $t/2-\sqrt{x} \leq 0$ and $M \leq N-1$, then we have

$$\sum_{j+k \leq M} |\partial^j\partial_t^k \phi^\star((t/2 \pm \sqrt{x}, t^2/4))| \leq C \sum_{k \leq M} |\phi^\star((t/2 \pm \sqrt{x}, t^2/4))| x^{-M+h} \leq C x^{N-1/2-M},$$

therefore

$$\partial^j\partial_t^k \phi^\star(t, x) \rightarrow 0 \quad \text{as} \quad (t, x) \rightarrow (0,0) \quad \text{in} \quad \{t/2-\sqrt{x} \leq 0\} \quad (j+k \leq N-1).$$

ii) Regularity of $\phi_\sigma$. Let $t/2-\sqrt{x} > 0$, then

$$\Phi^\star(t, x) = \frac{1}{2} \left[ \phi^\star((t/2+\sqrt{x}, t^2/4)) - \phi^\star((t/2-\sqrt{x}, t^2/4)) \right]$$

$$\approx \frac{1}{2} \left[ \phi^\star(t^2/4+x) + \phi^\star(t^2/4+x)(t\sqrt{x}) + \cdots ight.$$

$$\cdots + \phi^{(2M)}(t^2/4+x)(t\sqrt{x})^{2M}/(2M)!$$

$$\left. + \left( t\sqrt{x} \right)^{2M+1}/(2M)! \right] \int_0^1 (1-\theta)^{2M} \phi^\star((t^2/4+x+t\sqrt{x}\theta)) d\theta \right)/2$$

$$- \left[ \phi^\star(t^2/4+x) + \phi^\star(t^2/4+x)(-t\sqrt{x}) + \cdots ight.$$

$$\cdots + \phi^{(2M)}(t^2/4+x)(-t\sqrt{x})^{2M}/(2M)!$$

$$\left. + \left( -t\sqrt{x} \right)^{2M+1}/(2M)! \right] \int_0^1 (1-\theta)^{2M} \phi^\star((t^2/4+x-t\sqrt{x}\theta)) d\theta \right)/2$$

$$= t\sqrt{x} \left[ \phi^\star(t^2/4+x) + \phi^\star(t^2/4+x)t^2x/3! + \cdots ight.$$

$$\cdots + \phi^{(2M-1)}(t^2/4+x)t^{2M-2}x^{-1}/(2M-1)!$$

$$\left. + t^{2M}x^M/[2(2M)!] \right] \int_0^1 (1-\theta)^{2M} \left[ \phi^\star((t^2/4+x+t\sqrt{x}\theta)) \right.$$

$$- \phi^\star((t^2/4+x-t\sqrt{x}\theta)) \left. d\theta \right]$$

$$= \sqrt{x} \Phi^\star(t, x).$$

Here we can see that $\phi_\sigma$ is differentiable up to order $M$ at $(t,0)$ ($t>0$) and

$$\partial^j\partial_t^k \phi_\sigma(t, x) \rightarrow 0 \quad \text{as} \quad (t, x) \rightarrow (0,0) \quad \text{in} \quad \{t/2-\sqrt{x} > 0\} \quad (j+k \leq M),$$

if $3M \leq N-2$.

Let $t/2-\sqrt{x} \leq 0$ and $M \leq N-1$, then we have

$$\sum_{j+k \leq M} |\partial^j\partial_t^k \phi_\sigma((t/2 \pm \sqrt{x}, t^2/4))\sqrt{x}| \leq C \sum_{k \leq M} |\phi^\star((t/2 \pm \sqrt{x}, t^2/4))| x^{-M+h-1/2} \leq C x^{N-1-M},$$

therefore

$$\partial^j\partial_t^k \phi_\sigma(t, x) \rightarrow 0 \quad \text{as} \quad (t, x) \rightarrow (0,0) \quad \text{in} \quad \{t/2-\sqrt{x} \leq 0\} \quad (j+k \leq N-2).$$

□
Now, let us define

\[ u_+(t, x) = u_0(x) + \int_0^t \Phi_+(t, x) dt, \]

then we have

**Lemma 6.2.**

i) \( u_- \) is a solution of (P) satisfying

\[
\begin{align*}
&u \in C^t(\bar{\mathbb{R}}_+ \times \mathbb{R}_+), \\
&(u, u_t) \in C^t(\bar{\mathbb{R}}_+ \times \mathbb{R}_+), \\
&u|_{x=0} = 0.
\end{align*}
\]

Conversely, let \( u \) be a solution of (P) satisfying (\( \ast \)), then \( u = u_- \).

ii) \( u_+ \) is a solution of (P) satisfying

\[
\begin{align*}
&u \in C^t(\bar{\mathbb{R}}_+ \times \mathbb{R}_+).
\end{align*}
\]

Conversely, let \( u \) be a solution of (P) satisfying (\( \ast \ast \)), then \( u = u_+ \).

**Proof.** Let \( u \) satisfy

\[ u_{tt} = xu_{xx} + 1/2 u_x, \]

that is,

\[ (\partial_t - \sqrt{x} \partial_x) \partial_t + \sqrt{x} \partial_x) u = 0, \]

then we have

\[ (\partial_t \pm \sqrt{x} \partial_x) u = \text{const.} \quad \text{on } t \pm 2\sqrt{x} = \text{const.}. \]

Since

\[ (\partial_t \pm \sqrt{x} \partial_x) u \big|_{t=0} = \phi_\pm, \]

we have

\[ (\partial_t \pm \sqrt{x} \partial_x) u = \phi_\pm(\xi) \quad \text{on } t \pm 2\sqrt{x} = \pm 2\sqrt{\xi}. \]

Here we remark that

\[ (\partial_t + \sqrt{x} \partial_x) u \big|_{(t, x) = (2\sqrt{\xi}, 0)} = \phi_+(\xi). \]

In case of (i), since

\[ \partial_t u \longrightarrow 0 \quad \text{as } (t, x) \rightarrow (2\sqrt{\xi}, 0), \]

we have

\[ \sqrt{x} \partial_x u \longrightarrow \phi_+(\xi) \quad \text{as } (t, x) \rightarrow (2\sqrt{\xi}, 0). \]

On the other hand, since

\[ (\partial_t - \sqrt{x} \partial_x) u = \text{const.} \quad \text{on } t - 2\sqrt{x} = 2\sqrt{\xi}, \]

we have

\[ (\partial_t - \sqrt{x} \partial_x) u = -\phi_+(\xi) \quad \text{on } t - 2\sqrt{x} = 2\sqrt{\xi}. \]

Here we have

\[ (\partial_t \pm \sqrt{x} \partial_x) u = \pm \phi_+(t/2 \pm \sqrt{x})^2 \quad (t > 2\sqrt{x}). \]

Therefore we have

\[ \partial_t u(t, x) = \phi_+(t/2 + \sqrt{x})^2 - \phi_+(t/2 - \sqrt{x})^2)/2, \]
Semi-degenerate hyperbolic equations

\[ \sqrt{x} \partial_x u(t, x) = \frac{\phi_+(t/2 + \sqrt{x}) + \phi_+(t/2 - \sqrt{x})}{2} \]

for \( t > 2\sqrt{x} \).

In case of (ii), since

\[ \sqrt{x} \partial_x u \rightarrow 0 \quad \text{as} \quad (t, x) \rightarrow (2\sqrt{\xi}, 0), \]

we have

\[ \partial_t u \rightarrow \phi_+(\xi) \quad \text{as} \quad (t, x) \rightarrow (2\sqrt{\xi}, 0). \]

On the other hand, since

\[ (\partial_t - \sqrt{x} \partial_x)u = \text{const.} \quad \text{on} \quad t - 2\sqrt{x} = \frac{2\xi}{\sqrt{T}}, \]

we have

\[ (\partial_t - \sqrt{x} \partial_x)u = \phi_+(\xi) \quad \text{on} \quad t - 2\sqrt{x} = \frac{2\xi}{\sqrt{T}}. \]

Here we have

\[ (\partial_t \pm \sqrt{x} \partial_x)u = \pm \phi_+(t/2 \pm \sqrt{x}) \quad (t > 2\sqrt{x}), \]

therefore we have

\[ \partial_t u(t, x) = \frac{\phi_+(t/2 + \sqrt{x}) + \phi_+(t/2 - \sqrt{x})}{2}, \]

\[ \sqrt{x} \partial_x u(t, x) = \frac{\phi_+(t/2 + \sqrt{x}) - \phi_+(t/2 - \sqrt{x})}{2} \]

for \( t > 2\sqrt{x} \).

In both cases, we have

\[ (\partial_t \pm \sqrt{x} \partial_x)u = \pm \phi_+(t/2 \pm \sqrt{x}) \quad (t \leq 2\sqrt{x}), \]

therefore we have

\[ \partial_t u(t, x) = \frac{\phi_+(t/2 + \sqrt{x}) + \phi_+(t/2 - \sqrt{x})}{2}, \]

\[ \sqrt{x} \partial_x u(t, x) = \frac{\phi_+(t/2 + \sqrt{x}) - \phi_+(t/2 - \sqrt{x})}{2} \]

for \( t \leq 2\sqrt{x} \). Here we have

\[ u = u_1 = u_0(x) + \int_0^t \Phi(t, x)dt. \]

Since \( \Phi \) is defined by initial data \( \{u_0, u_1\} \), we also use the notations:

\[ \Phi_0(t, x) = \Phi_0(t, x; u_0, u_1). \]

Next, we consider

\[ (P)_{\mu=\kappa-1/2} \]

\[ \begin{cases} u_{tt} = xu_{xx} + (k + 1/2)u_x & \quad (t > 0, x > 0), \\ u_{|t=0} = u_0, \quad u_t |_{t=0} = u_0 & \quad (x > 0), \end{cases} \]

where \( u_0, u_1 \in C^N(R_+), u_0, u_1 = O(x^N) \) as \( x \to +0 \) (\( N: \) large), and \( k \) is a positive integer.

Let us define

\[ \partial_x^{-1}u \left( = \int_0^x u(x)dx \right), \quad \partial_x^{-k}u = (\partial_x^{-1})^k u, \]

and

\[ U_{k-1/2} = u_0(x) + \int_0^t \partial_x^k \Phi(t, x; u_0, \partial_x^{-k}u_1)dt, \]

then we have
Proposition 6.3. $U_{k+1/2}^{\pm}$ are solutions of (P)$_{p=-k+1/2}$, and

$$ U_{k+1/2}^{\pm} \in C^4(\mathbb{R}_+ \times \mathbb{R}_+), $$

$$ U_{k+1/2}^{\pm} = O(x^{k-1/2}) \text{ as } x \to 0. $$

Proof. Let us consider

$$(P)_{p=-1/2}$$

$$ \begin{cases} v_{tt} = x v_{xx} + 1/2 v_x & (t > 0, x > 0), \\ v|_{t=0} = v_0, \\ v|_{x=0} = v_1 & (x > 0), \end{cases} $$

where $v_0 = \partial_{x}^{-k} u_0, v_1 = \partial_{x}^{-k} u_1$, then there exist solutions:

$$ v = v_+ = v_0(x) + \int_0^t \Phi_x(t, x; v_0, v_1) dt. $$

It is easy to see that $\partial_{x}^{k} v_+$ satisfies $(P)_{p=-k+1/2}$.  \qed

Finally, we consider

$$(P)_{p=-k+1/2}$$

$$ \begin{cases} u_{tt} = x u_{xx} + (-k + 3/2) u_x & (t > 0, x > 0), \\ u|_{t=0} = u_0, \\ u|_{x=0} = u_1 & (x > 0), \end{cases} $$

where $u_0, u_1 \in C^4(\mathbb{R}_+), u_0, u_1 = O(x^N)$ as $x \to +0$ ($N$: large), and $k$ is a positive integer. Let us define

$$ U_{k+1/2}^{\pm} = u_0(x) + \int_0^t x^{-k+1/2} \partial_{x}^{k} \Phi_x(t, x; x^{-k+1/2} u_0, \partial_{x}^{-k}(x^{-k+1/2} u_1)) dt. $$

then we have

Proposition 6.4. $U_{k+1/2}^{\pm}$ are solutions of $(P)_{p=-k+1/2}$, and

$$ U_{k+1/2}^{\pm} \in C^4(\mathbb{R}_+ \times \mathbb{R}_+), $$

$$ U_{k+1/2}^{\pm} = O(x^{k-1/2}) \text{ as } x \to 0. $$

Proof. Let us consider

$$(P)_{p=-k+1/2}$$

$$ \begin{cases} w_{tt} = x w_{xx} + (k + 1/2) w_x & (t > 0, x > 0), \\ w|_{t=0} = w_0, \\ w|_{x=0} = w_1 & (x > 0), \end{cases} $$

where

$$ w_0 = x^{-k+1/2} u_0, \quad w_1 = x^{-k+1/2} u_1, $$

then there exist solutions of $(P)_{p=-k+1/2}$:

$$ w = w_+ = w_0(x) + \int_0^t x^{k} \partial_{x}^{k} \Phi_x(t, x; \partial_{x}^{k} w_0, \partial_{x}^{k} w_1) dt. $$

It is easy to see that $x^{k-1/2} w_+$ satisfies $(P)_{p=-k+1/2}$. Moreover, since
Semi-degenerate hyperbolic equations

\[ u_- = u_0(x) + x^{k-1/2} \partial_x^k \left\{ \sqrt{x} \int_0^t \Phi(t, x) dt \right\} \]

\[ = u_0(x) + x^{k-1/2} \sum_{j=0}^k c_{k,j} x^{j-1/2} \int_0^t \partial_x^{k-j} \Phi(t, x) dt , \]

\( u_- \) is smooth up to the boundary. \( \Box \)

DEPARTMENT OF MATHEMATICS
NARA WOMEN’S UNIVERSITY

References