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A necessary condition for H^{∞}-wellposed Cauchy problem of Schrödinger type equations with variable coefficients

By

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§ 1. Introduction

The linear partial differential operator of second order

$$L=\frac{1}{i}\partial_t-\frac{1}{2}\sum_{j,k=1}^{n}\partial_{x_j}(g^{jk}(x)\partial_{x_k})+\sum_{j=1}^{n}b^j(x)\partial_{x_j}+c(x)\qquad (t\in \boldsymbol{R}^1,\ x\in \boldsymbol{R}^n) \tag{1.1}$$

is called Schrödinger type operator with variable coefficients, where $g^{jk}(x)$, $b^j(x)$, and $c(x)\in \mathcal{B}^{\infty}(\boldsymbol{R}^n)$. We suppose that $g^{jk}(x)=g^{kj}(x)$ $(j,\ k=1,\cdots,n)$ are real valued and satisfy the uniform ellipticity

$$\delta^{-1}|p|^2\leqq\left|\sum_{j,k=1}^{n}g^{jk}(x)p_jp_k\right|\leqq\delta|p|^2,\qquad \text{for any}\quad x,\ p\in\boldsymbol{R}^n, \tag{1.2}$$

for some positive constant δ. In this paper, we study the necessary condition in order that the Cauchy problem

$$Lu(t,\ x)=f(t,\ x),\qquad u(0,\ x)=u_0(x) \tag{1.3}$$

is H^{∞}-wellposed (see Definition 1.1). In [3], W. Ichinose succeeded to find and verify a necessary condition for L^2-wellposedness of (1.3) using the Maslov's method of [6]. Here we apply his idea to H^{∞} case under an assumption below. We denote the set of all $H^{\infty}(\boldsymbol{R}^n)$ valued continuous functions in $t\in[0,\ T]$ by $\mathcal{E}_t^0([0,\ T]\,;\ H^{\infty}(\boldsymbol{R}^n))$.

Definition 1.1. We say that the Cauchy problem (1.3) is H^{∞}-wellposed on $[0,\ T_0]$ $(T_0>0)$ (resp. $[T_0,\ 0]$ $(T_0<0)$), if the following is valid for any $T\in(0,\ T_0]$ (resp. $T\in[T_0,\ 0)$). For any $u_0(x)\in H^{\infty}(\boldsymbol{R}^n)$ and any $f(t,\ x)\in\mathcal{E}_t^0([0,\ T]\,;\ H^{\infty}(\boldsymbol{R}^n))$ (resp. $\mathcal{E}_t^0([T,\ 0]\,;\ H^{\infty}(\boldsymbol{R}^n))$) there exists one and only one solution $u(t,\ x)$ of (1.3) in $\mathcal{E}_t^0([0,\ T]\,;\ H^{\infty}(\boldsymbol{R}^n))$ (resp. $\mathcal{E}_t^0([T,\ 0]\,;\ H^{\infty}(\boldsymbol{R}^n))$).

We shall define a Hamiltonian function by

$$H(x,\ p)=\frac{1}{2}\sum_{j,k=1}^{n}g^{jk}(x)p_jp_k \tag{1.4}$$

and the canonical equations for the Hamiltonian function $H(x,\ p)$ with an initial value $(x,\ p)$ at $t=0$ by

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$$(1.5)\qquad \frac{dX_j}{dt}=\frac{\partial H}{\partial p_j}(X, P), \quad \frac{dP_k}{dt}=-\frac{\partial H}{\partial x_k}(X, P), \qquad (j, k=1, 2, \cdots, n),$$
$$(X, P)|_{t=0}=(x, p).$$

By the uniform ellipticity (1.2), we can see that for each $(x, p)\in \boldsymbol{R}^{2n}$ there exists one and only one solution $(X(t, x, p), P(t, x, p))$ of (1.5) for all $t\in \boldsymbol{R}$. For the present we do not know much about the property of the solution of (1.5) in general. So we consider subsets $F\subset \boldsymbol{R}^n\times S^{n-1}$ which satisfy the following propery [A].

[A] There exist constants ε, C, $m>0$ such that

$$(1.6)\qquad \max_{|\alpha|=1}\sup_{(x,p)\in F_\varepsilon}\{|\partial_{x,p}^\alpha X(t, x, p)|+|\partial_{x,p}^\alpha P(t, x, p)|\}\leqq C(1+|t|)^m$$

for any $t\in \boldsymbol{R}$. Here we denote

$F_\varepsilon=\{(x, p)\in \boldsymbol{R}^n\times S^{n-1};$ there exists $y\in \boldsymbol{R}^n$ such that $(y, p)\in F$ and $|x-y|<\varepsilon\}$.

In Remark below, we explain a little more the reason why we consider such F.

Theorem. *If Cauchy problem* (1.3) *is H^∞-wellposed on* $[0, T_0]$ $(T_0>0)$ *or* $[T_0, 0]$ $(T_0<0)$ *for a* $T_0\neq 0$, *then for any* $F\subset \boldsymbol{R}^n\times S^{n-1}$ *which satisfies* [A], *there exists a positive constant* M *such that*

$$(1.7)\qquad \sup_{(x,p)\in F}\left|\int_0^t\sum_{j=1}^n \mathcal{R}_e b^j(X(s, x, p))P_j(s, x, p)ds\right|\leqq M\log(1+|t|)+M, \quad \textit{for any } t\in \boldsymbol{R}.$$

W. Ichinose [2] and J. Takeuchi [7] studied the necessary condition for H^∞-wellposedness of the Cauchy problem (1.3) in the constant coefficient case, that is, $g^{jk}(x)=\delta_{ik}$ (Kronecker's delta) in (1.2). Our result is a generalization of their one to the variable coefficient case (see Example (1)).

Example. (1) $H(x, p)=(1/2)\sum_{j=1}^n p_j^2$ (the constant coefficient case). Then $X(t, y, p)=y+pt$, $P(t, y, p)=p$. In this case, [A] holds for $F=\boldsymbol{R}^n\times S^{n-1}$.

(2) Let $n=2$ and $H(x, p)=(1/2)\{p_1^2+a(x_1)p_2^2\}$, where $a(x_1)\in \mathcal{B}^\infty(\boldsymbol{R})$ satisfying $\delta^{-1}\leqq a(x_1)\leqq\delta$ for some $\delta>1$. By the canonical equations (1.5), we have

$$(1.8)\qquad H(X(t, x, p), P(t, x, p))=H(x, p) \qquad \text{for any} \quad (t, x, p)\in \boldsymbol{R}\times\boldsymbol{R}^2\times S^1.$$

From (1.2) and (1.8), we get

$$(1.9)\qquad \delta^{-1}\leqq|P(t, x, p)|\leqq\delta.$$

Noting that $P_2(t, x, p)=p_2$, if we set for some positive constant $b<1$

$$F=\{(x, p)\in \boldsymbol{R}^2\times S^1;\ |p_2|\leqq\delta^{-1}\sqrt{b}\},$$

then we have from (1.9)

$$(1.10)\qquad |P_1(t, x, p)|\geqq\delta^{-1}\sqrt{1-b} \qquad \text{for any} \quad (t, x, p)\in \boldsymbol{R}\times F.$$

Let z be one of x_1, x_2, p_1 and p_2, then using (1.8) we have

$$P_1(\partial_t\partial_z X_1)-(\partial_t P_1)\partial_z X_1=\partial_z H(x, p)-\frac{\partial H}{\partial p_2}(X, P)\partial_z P_2.$$

The right side of above equation is bounded uniformly in $(t, x, p)\in \boldsymbol{R}\times\boldsymbol{R}^2\times S^1$, and we denote it by $f_z(t, x, p)$. So we have

$$\partial_z X_1(t, x, p)=P_1(t, x, p)\{p_1^{-1}\partial_z X_1(0, x, p)+\int_0^t f_z(s, x, p)P_1(s, x, p)^{-2}ds\}.$$

By (1.9) and (1.10), we obtain from this equation

$$\sup_{(x,p)\in F}|\partial_z X_1(t, x, p)|\leqq C(1+|t|) \qquad \text{for any} \quad t\in \boldsymbol{R}.$$

And we can easily prove that $\partial_z X_2$, $\partial_z P_1$ is also estimated by the polynomial of t for $(x, p)\in F$.

(3) Let $n=2$ and $H(x, p)=1/2\{p_1^2+(\psi(x_1)x_1^2+1)p_2^2\}$, where $\psi(x_1)\in C_0^\infty(\boldsymbol{R})$ satisfying $\psi\geqq 0$ and $\psi(x_1)=1$ (if $|x_1|\leqq r$) for a constant $r>0$. For any $\varepsilon\in(0, r)$ and $b\in(0, 1)$, we set

$$F=\{(x, p)\in \boldsymbol{R}^2\times S^1;\ (|x_1|+\varepsilon)^2+(p_1/p_2)^2\leqq r^2 \ \text{or} \ |p_2|\leqq\delta^{-1}\sqrt{b}\},$$

where δ is a constant satisfying (1.2) for this case. When $|p_2|\leqq\delta^{-1}\sqrt{b}$, we can adopt the argument of example (2). When $(|x_1|+\varepsilon)^2+(p_1/p_2)^2\leqq r^2$, we have

$$X_1(t, x, p)=x_1\cos(p_2 t)+(p_1/p_2)\sin(p_2 t),$$

$$X_2(t, x, p)=x_2+\frac{x_1 p_1}{2p_2}+\left(p_2+\frac{1}{2}x_1^2 p_2+\frac{p_1^2}{2p_2}\right)t+\frac{1}{4}(x_1^2-(p_1/p_2)^2)\sin(2p_2 t)$$
$$-(x_1 p_1/2p_2)\cos(2p_2 t),$$

$$P_1(t, x, p)=-x_1 p_2\sin(p_2 t)+p_1\cos(p_2 t),\ P_2(t, x, p)=p_2.$$

Hence [A] holds for this F.

We shall explain the outline of the proof. Following [3], we prove it by contradiction. At first we change a variable t to $\tau=\lambda t$ with a large parameter $\lambda\geqq 1$. Then the operater L is changed to

$$\lambda^2 L_\lambda\equiv\lambda^2\left[\frac{1}{\lambda}D_\tau+\frac{1}{2}\sum_{j,k=1}^n\frac{1}{\lambda}D_{x_j}\left(g^{jk}(x)\frac{1}{\lambda}D_{x_k}\right)-(i\lambda)^{-1}\sum_{j=1}^n b^j(x)\frac{1}{\lambda}D_{x_j}-(i\lambda)^{-2}c(x)\right], \tag{1.11}$$

where $D_\tau=(1/i)\partial_\tau$, $D_{x_j}=(1/i)\partial_{x_j}(j=1, 2, \cdots, n)$. Suppose that the Cauchy problem (1.3) is H^∞-wellposed on $[0, T_0]$, we can obtain a priori estimate by Banach's closed graph theorem. Next assuming that (1.7) is not valid, we can construct asymptotic solutions for the equation

$$L_\lambda v(\tau, x)=0, \tag{1.12}$$

which contradict the a priori estimate. Thus we obtain Theorem.

Remark. W. Ichinose [3] proved that if Cauchy problem (1.3) is L^2-wellposed on $[0, T_0]$ $(T_0>0)$ or $[T_0, 0]$ $(T_0<0)$ for a $T_0\neq 0$, then

$$\sup_{\substack{(x,p)\in\boldsymbol{R}^n\times S^{n-1}\\ t\in\boldsymbol{R}}}\left|\int_0^t\sum_{j=1}^n \mathcal{R}_e b^j(X(s, x, p))P_j(s, x, p)ds\right|<+\infty$$

must be fulfilled. In its proof, it is enough to construct asymptotic solutions on a bounded interval of τ. On the other hand, in the proof of our Theorem, we must construct asymptotic solutions on $[0, \rho]$ for any $\rho>0$ (see (d) in Lemma 2.2 and (4.2) in the present paper). This leads to the difficulty, so we consider subsets $F\subset \mathbf{R}^n\times S^{n-1}$ satisfying the property $[A]$.

Through this paper we use the following notations. $\mathbf{N}$ is a set of all natural numbers. $\mathbf{R}_+=\{t\in\mathbf{R};\, t\geqq 0\}$. For $x\in\mathbf{R}^n$ and $a>0$ we set $\|x\|=\max_{1\leq i\leq n}|x_i|$, and $Q_n(x;a)=\{y\in\mathbf{R}^n;\ \|y-x\|<a\}$. For $\varphi(x)\in\mathcal{S}(\mathbf{R}^n)$ (Schwartz's rapidly decreasing functions), we denote

$$\|\varphi(\cdot)\|_0=\Big(\int_{R^n}|\varphi(x)|^2dx\Big)^{1/2},\qquad \|\varphi(\cdot)\|_s=\Big(\sum_{|\alpha|\leq s}\|\partial_x^\alpha\varphi(\cdot)\|_0^2\Big)^{1/2}$$

$$|\varphi(\cdot)|_s=\sum_{|\alpha|\leq s}\sup_{x\in R^n}|\partial_x^\alpha\varphi(x)|,$$

for each $s\in\mathbf{N}$, where α is the usual multi-index. Let $K=\{k_1, k_2, \cdots, k_l\}(1\leqq k_1<k_2<\cdots<k_l\leqq n)$ be a subset of $\{1, 2, \cdots, n\}$. We denote the complementary set of K in $\{1, 2, \cdots, n\}$ by K'. We denote $|K|=l$, and

$$x_K=(x_{k_1}, x_{k_2}, \cdots, x_{k_l}),\quad \frac{\partial}{\partial x_K}=\Big(\frac{\partial}{\partial x_{k_1}}, \frac{\partial}{\partial x_{k_2}}, \cdots, \frac{\partial}{\partial x_{k_l}}\Big),\ D_{x_K}=\frac{1}{i}\frac{\partial}{\partial x_K}.$$

Let $f(x)=(f_1(x), f_2(x), \cdots, f_m(x))$ be a C^∞ function on $\mathbf{R}^n$ with values in $\mathbf{R}^m$. We denote

$$\frac{\partial f}{\partial x}(x)=\Big(\frac{\partial f_i}{\partial x_j}(x);\ \begin{matrix} i\downarrow 1, 2, \cdots, m\\ j\rightarrow 1, 2, \cdots, n\end{matrix}\Big),\quad \frac{Df}{Dx}(x)=\det\frac{\partial f}{\partial x}(x),\quad (\text{if } m=n).$$

We set

$$h^t(x, p)=(X(t, x, p), P(t, x, p)),$$

$$F(x, p)=\sum_{j=0}^n b^j(x)p_j,$$

$$\phi(t, x, p)=\int_0^t \mathcal{R}_e F(h^\theta(x, p))d\theta.$$

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§2. Lagrangian manifold

Lemma 2.1. *For any $2n$ dimensional multi-index $\alpha(|\alpha|\geqq 1)$, there exist positive constants C_α and m_α which satisfy*

$$\text{(2.1)}\qquad \sup_{(x,p)\in F_\varepsilon}\{|\partial_{x,p}^\alpha X(t, x, p)|+|\partial_{x,p}^\alpha P(t, x, p)|\}\leqq C_\alpha(1+|t|)^{m_\alpha}$$

for any $t\in\mathbf{R}$.

Proof. We shall consider only about $\partial_{x_1}^2X_j$ and $\partial_{x_1}^2P_j$ $(j=1, 2, \cdots, n)$. Others can be proved in the similar way. For the sake of simplicity, we shall denote

$$\frac{\partial^2}{\partial x_l\partial x_m}X_k(t, x, p)=X^k_{x_l x_m}(t, x, p)\qquad k, l, m=1, 2, \cdots, n\qquad \text{etc..}$$

By (1.5), we have

$$\frac{d}{dt}X^j_{x_1}=\sum_{k=1}^n H_{p_jx_k}(X, P)X^k_{x_1}+\sum_{k=1}^n H_{p_jp_k}(X, P)P^k_{x_1},$$

$$\frac{d}{dt}P^j_{x_1}=-\sum_{k=1}^n H_{x_jx_k}(X, P)X^k_{x_1}-\sum_{k=1}^n H_{x_jp_k}(X, P)P^k_{x_1},$$

(2.2)
$$\frac{d}{dt}X^j_{x_1x_1}=\sum_{k=1}^n H_{p_jx_k}(X, P)X^k_{x_1x_1}+\sum_{k=1}^n H_{p_jp_k}(X, P)P^k_{x_1x_1}+f_j(t, x, p),$$

$$\frac{d}{dt}P^j_{x_1x_1}=-\sum_{k=1}^n H_{x_jx_k}(X, P)X^k_{x_1x_1}-\sum_{k=1}^n H_{x_jp_k}(X, P)P^k_{x_1x_1}+g_j(t, x, p),$$

$$j=1, 2, \cdots, n.$$

Here, $f_j(t, x, p)$ and $g_j(t, x, p)$ include only at most first order derivatives of X, P in x_1. So, from (1.6), we get

(2.3) $$\sup_{(x,p)\in F_\varepsilon}\{|f_j(t, x, p)|+|g_j(t, x, p)|\}\leqq C(1+|t|)^m \quad \text{for} \quad t\in \boldsymbol{R},$$

for some positive constants C, m.

In view of (2.2), we can see that $(X_{x_1x_1}, P_{x_1x_1})$ satisfies the system of ordinary differential equations which has the same principal part as one for (X_{x_1}, P_{x_1}). We set

$$z(t, x, p)=(X_{x_1x_1}, P_{x_1x_1}),$$

$$b(t, x, p)=(f_1, \cdots, f_n, g_1, \cdots, g_n),$$

$$\Psi(t, x, p)=\frac{\partial(X, P)}{\partial(x, p)}.$$

Then, from (2.2), we have

(2.4) $$z(t, x, p)=\Psi(t, x, p)\int_0^t\Psi(s, x, p)^{-1}b(s, x, p)ds.$$

From (1.6) and (2.3), there exist constants C and m independent of $(t, x, p)\in \boldsymbol{R}^1\times F_\varepsilon$ such that

(2.5) $$|\Psi(t, x, p)|,\ |b(t, x, p)|\leqq C(1+|t|)^m \quad \text{for any} \quad (t, x, p)\in \boldsymbol{R}^1\times F_\varepsilon.$$

From (2.2), we have

$$\det\Psi(t, x, p)=\det\Psi(0, x, p)=1 \quad \text{for any} \quad (t, x, p)\in \boldsymbol{R}^1\times F_\varepsilon.$$

So, we obtain

(2.6) $$|\Psi(t, x, p)^{-1}|\leqq C'(1+|t|)^{m'} \quad \text{for any} \quad (t, x, p)\in \boldsymbol{R}^1\times F_\varepsilon,$$

where constants C' and m' are independent of $(t, x, p)\in \boldsymbol{R}^1\times F_\varepsilon$. Therefore, from (2.4), (2.5) and (2.6), we obtain

$$|z(t, x, p)|\leqq C''(1+|t|)^{m''} \quad \text{for any} \quad (t, x, p)\in \boldsymbol{R}^1\times F_\varepsilon,$$

where constants C'' and m'' are independent of $(t, x, p)\in \boldsymbol{R}^1\times F_\varepsilon$. This is the estimate (2.1) for $(X_{x_1x_1}, P_{x_1x_1})$. q.e.d.

Lemma 2.2. *If the inequality* (1.7) *is not valid, then for any* $\nu\in\boldsymbol{N}$ *we have* $(\rho_{\nu}, x^{(\nu)}, p^{(\nu)})\in\boldsymbol{R}_{+}\times\boldsymbol{R}^{2n}$ *which satisfies the following.*

(a) (Case-P) $(x^{(\nu)}, p^{(\nu)})\in F$,

or

(Case-N) $(x^{(\nu)}, p^{(\nu)})=h^{-\rho_{\nu}}(y^{(\nu)}, q^{(\nu)})$ *for some* $(y^{(\nu)}, q^{(\nu)})\in F$.

(b) $\phi(\rho_{\nu}, x^{(\nu)}, p^{(\nu)})>\nu\log(1+\rho_{\nu})+\nu$.

(c) $\phi(\rho_{\nu}, x^{(\nu)}, p^{(\nu)})\geqq\phi(\tau, x^{(\nu)}, p^{(\nu)})$ *for any* $\tau\in[0, \rho_{\nu}]$.

(d) $\lim_{\nu\to+\infty}\rho_{\nu}=+\infty$.

Proof. In view of (1.4) and (1.5), we have

$$(X(-t, x, p), P(-t, x, p))=(X(t, x, -p), -P(t, x, -p)). \tag{2.7}$$

Therefore we may assume for F that

$$(x, p)\in F \quad \text{implies that } (x, -p)\in F. \tag{2.8}$$

If (1.7) is not valid, then for any $\nu\in\boldsymbol{N}$ there exists $(t, x, p)\in\boldsymbol{R}\times F$ such that

$$|\phi(t, x, p)|>\nu\log(1+|t|)+\nu. \tag{2.9}$$

Hence, if we note the following two equations;

$$\phi(t, x, p)=\phi(-t, x, -p),$$

$$\phi(t, x, p)=-\phi(t, x', p') \quad \text{for } (x', p')=h^{-t}(x, -p),$$

which are derived from (2.7), then we can obtain (a) and (b) by (2.9). In order to make (c) hold, we have only to change ρ_{ν} in (b) into $\tilde{\rho}_{\nu}$ such that

$$\phi(\tilde{\rho}_{\nu}, x^{(\nu)}, p^{(\nu)})=\max_{\tau\in[0,\rho_{\nu}]}\phi(\tau, x^{(\nu)}, p^{(\nu)}).$$

(d) is clearly obtained from (b). The proof is complete. q.e.d.

In view of (d), we denote by ν_0 the number which satisfies $\rho_{\nu}\geqq 1$ for any $\nu\geqq\nu_0$. The reasoning in (Case-P) is a little different from that in (Case-N). To argue (Case-P) and (Case-N) in parallel, we introduce the following notations.

$$\begin{aligned}\eta_{\nu,\tau}(y)&=(X(\tau, y, p^{(\nu)}), P(\tau, y, p^{(\nu)})) \quad \text{(in Case-P)}\\&=(X(\tau-\rho_{\nu}, y, q^{(\nu)}), P(\tau-\rho_{\nu}, y, q^{(\nu)})) \quad \text{(in Case-N)}\end{aligned}$$

$$\eta_{\nu}(\tau, y)=(\tau, \eta_{\nu,\tau}(y))$$

Following [3], we denote

$$\Lambda_{\tau}^{n,\nu}=\{\eta_{\nu,\tau}(y)\in\boldsymbol{R}_{x,p}^{2n};\ y\in\boldsymbol{R}^{n}\},$$

$$\Lambda^{n+1,\nu}=\{\eta_{\nu}(\tau, y)\in\boldsymbol{R}_{\tau,x,p}^{2n+1};\ \tau\in\boldsymbol{R},\ y\in\boldsymbol{R}^{n}\}.$$

It is well known that $\Lambda_{\tau}^{n,\nu}$ is a Lagrangian manifold in $\boldsymbol{R}_{x,p}^{2n}$, that is,

$$\sum_{j=1}^{n} dp_j \wedge dx_j = 0 \qquad \text{on} \quad \Lambda_{\tau}^{n,\nu}.$$

Lemma 2.3. *There exist positive constants C and m such that for any $(t, y, p) \in \boldsymbol{R} \times F$*

$$\max_{K \subset \{1,\cdots,n\}} \left| \frac{D(X_{K'}(t, y, p), P_K(t, y, p))}{Dy} \right| \geqq C(1+|t|)^{-m}. \tag{2.10}$$

Here the constants C and m are independent of (t, y, p).

Proof. As we have seen in the proof of Lemma 2.1, we have

$$\left| \frac{D(X(t, y, p), P(t, y, p))}{D(y, p)} \right| = 1 \qquad \text{for any} \quad (t, y, p) \in \boldsymbol{R}^{2n+1}. \tag{2.11}$$

We apply Laplace expansion theorem (see e.g. [4], p. 238) to (2.11), then we have

$$\sum_{\substack{K_1, K_2 \subset \{1,\cdots,n\} \\ |K_1|+|K_2|=n}} \varepsilon_{K_1K_2} \frac{D(X_{K_1}, P_{K_2})}{Dy} \frac{D(X_{K'_1}, P_{K'_2})}{Dp} = 1, \tag{2.12}$$

where $\varepsilon_{K_1K_2}$ is 1 or -1. From (1.6) and (2.12), we can see that there exist positive constants C, m and K_1, $K_2 \subset \{1, 2, \cdots, n\}(|K_1|+|K_2|=n)$ such that

$$\left| \frac{D(X_{K_1}(t, y, p), P_{K_2}(t, y, p))}{Dy} \right| \geqq C(1+|t|)^{-m} \qquad \text{for any} \quad (t, y, p) \in \boldsymbol{R} \times F. \tag{2.13}$$

Here K_1 and K_2 may depend on (t, y, p). Let $(t, y, p) \in \boldsymbol{R} \times F$ be fixed. We set $\Lambda_{t,p}^n = \{(X(t, x, p), P(t, x, p)); x \in \boldsymbol{R}^n\}$, which is a n dimensional Lagrangian manifold in $\boldsymbol{R}_{x,p}^{2n}$, that is,

$$\sum_{j=1}^{n} dp_j \wedge dx_j = 0 \qquad \text{on} \quad \Lambda_{t,p}^n. \tag{2.14}$$

By (2.13), (x_{K_1}, p_{K_2}) becomes a local coordinate system in the neighborhood of $\xi = (X(t, y, p), P(t, y, p))$ on $\Lambda_{t,p}^n$. Now let $i \in K_1 \cap K_2$. If such i does not exist, the proof is complete. We shall substitute a pair of tangent vecters $\left(\left(\frac{\partial}{\partial x_i}\right)_\xi, \left(\frac{\partial}{\partial p_i}\right)_\xi\right)$ to (2.14), where (x_{K_1}, p_{K_2}) is now considerd as local coordinate system near ξ on $\Lambda_{t,p}^n$, then we get

$$\sum_{j \in K'_1 \cap K'_2} \left(\frac{D(x_j, p_j)}{D(x_i, p_i)} \right)_\xi + 1 = 0. \tag{2.15}$$

Therefore there exists $d \in K'_1 \cap K'_2$ such that

$$\left| \left(\frac{D(x_d, p_d)}{D(x_i, p_i)} \right)_\xi \right| \geqq \frac{1}{|K'_1 \cap K'_2|} \geqq \frac{2}{n}. \tag{2.16}$$

Hence we have

$$\max\left\{ \left| \left(\frac{\partial x_d}{\partial x_i}\right)_\xi \right|, \left| \left(\frac{\partial x_d}{\partial p_i}\right)_\xi \right|, \left| \left(\frac{\partial p_d}{\partial x_i}\right)_\xi \right|, \left| \left(\frac{\partial p_d}{\partial p_i}\right)_\xi \right| \right\} \geqq \frac{1}{\sqrt{n}}. \tag{2.17}$$

For example, if $\left|\left(\frac{\partial x_d}{\partial x_i}\right)_\xi\right| \geqq \frac{1}{\sqrt{n}}$, then we shall employ the new local coordinate system $(x_{\tilde{K}_1}, p_{K_2})$ where $\tilde{K}_1 = (K_1 \setminus \{i\}) \cup \{d\}$, then we have

$$\left|\frac{D(X_{\tilde{K}_1}(t, y, p), P_{K_2}(t, y, p))}{Dy}\right|$$

$$= \left|\left(\frac{D(x_{\tilde{K}_1}, p_{K_2})}{D(x_{K_1}, p_{K_2})}\right)_\xi\right| \cdot \left|\frac{D(X_{K_1}(t, y, p), P_{K_2}(t, y, p))}{Dy}\right| \geqq \frac{C}{\sqrt{n}}(1+t)^{-m}.$$

We can treat other cases similarly. Repeating this argument, we obtain Lemma 2.3. q.e.d.

Proposition 2.4. *For each* $\nu \geqq \nu_0$ *and* $\tau_0 \in [0, \rho_\nu]$, *take* $K \subset \{1, 2, \cdots, n\}$ *which realized the inequality* (2.10) *with* $(t, y, p) = (\tau_0, x^{(\nu)}, p^{(\nu)}) \in \boldsymbol{R}_+ \times F$ (in Case-P) $(=(\tau_0 - \rho_\nu, y^{(\nu)}, q^{(\nu)}) \in \boldsymbol{R} \times F$ (in Case-N)). *Then, there exist positive constants* a, b, d *and* C *independent of* ν *which satisfy the following* (a)-(d).

(a) *For* τ *satisfying* $|\tau - \tau_0| < (1+\rho_\nu)^{-a}$, *the mapping*

$$y \longmapsto (X_{K'}(\tau, y, p^{(\nu)}), P_K(\tau, y, p^{(\nu)})) \qquad \text{(in Case-P)}$$

$$(y \longmapsto (X_{K'}(\tau - \rho_\nu, y, q^{(\nu)}), P_K(\tau - \rho_\nu, y, q^{(\nu)})) \qquad \text{(in Case-N))}$$

becomes diffeomorphism in C^∞ *class from* $Q_n(x^{(\nu)}; (1+\rho_\nu)^{-a})$ (in Case-P) $(Q_n(y^{(\nu)}; (1+\rho_\nu)^{-a})$ (in Case-N)) into $\boldsymbol{R}^n$.

(b) *For* τ *satisfying* $|\tau - \tau_0| < (1+\rho_\nu)^{-a}$, *it holds that*

$$Q_n((X_{K'}(\tau, x^{(\nu)}, p^{(\nu)}), P_K(\tau, x^{(\nu)}, p^{(\nu)})); (1+\rho_\nu)^{-b})$$

$$\subset \{(X_{K'}(\tau, y, p^{(\nu)}), P_K(\tau, y, p^{(\nu)})); \|y - x^{(\nu)}\| < (1+\rho_\nu)^{-a}\} \quad \text{(in Case-P)},$$

$$(\subset \{(X_{K'}(\tau - \rho_\nu, y, q^{(\nu)}), P_K(\tau - \rho_\nu, y, q^{(\nu)})); \|y - y^{(\nu)}\| < (1+\rho_\nu)^{-a}\} \quad \text{(in Case-N))}.$$

(c) *For* τ *satisfying* $|\tau - \tau_0| < (1+\rho_\nu)^{-a}$, *it holds that*

$$\left|\det \frac{\partial(X_{K'}(\tau, y, p^{(\nu)}), P_K(\tau, y, p^{(\nu)}))}{\partial y}\right| \geqq (1+\rho_\nu)^{-d}$$

$$\textit{for any} \quad y \in Q_n(x^{(\nu)}; (1+\rho_\nu)^{-a}) \qquad \text{(in Case-P)},$$

$$\left|\det \frac{\partial(X_{K'}(\tau - \rho_\nu, y, q^{(\nu)}), P_K(\tau - \rho_\nu, y, q^{(\nu)}))}{\partial y}\right| \geqq (1+\rho_\nu)^{-d}$$

$$\textit{for any} \quad y \in Q_n(y^{(\nu)}; (1+\rho_\nu)^{-a}) \qquad \text{(in Case-N)}.$$

(d) *For* s *and* τ *satisfying* $0 \leqq s < \tau \leqq \rho_\nu$, *it holds that*

$$\left|\int_s^\tau \mathcal{R}_e F(\eta_{\nu,\theta}(y)) d\theta - \int_s^\tau \mathcal{R}_e F(h^\theta(x^{(\nu)}, p^{(\nu)})) d\theta\right| \leqq C$$

$$\textit{for any} \quad y \in Q_n(x^{(\nu)}; (1+\rho_\nu)^{-a}) \qquad \text{(in Case-P)}$$

$$(y \in Q_n(y^{(\nu)}; (1+\rho_\nu)^{-a}) \text{ (in Case-N)}).$$

Proof. By the inverse mapping theorem, we can see that the width of the neighborhood, on which the mapping becomes diffeomorphism, is determined by the upper estimate of the derivatives of the mapping and the inverse of Jacobian matrix of the mapping. We note that in Lemma 2.1 constants C_α and m_α are independent of $(t, x, p) \in \boldsymbol{R}^1\times F_\varepsilon$ and in Lemma 2.3 constants C and m are independent of $(t, y, p)\in \boldsymbol{R}^1\times F$. So, using Lemma 2.1 and Lemma 2.3, we can prove this proposition by the inverse mapping theorem and the direct calculation. q.e.d.

We shall construct the open covering of the path $\{(\tau, h^\tau(x^{(\nu)}, p^{(\nu)}))\}_{0\leq\tau\leq\rho_\nu}$ on $\Lambda^{n+1,\nu}$ for $\nu\geqq\nu_0$. We denote by s_ν the minimal positive integer such that $\rho_\nu/s_\nu\leqq(1+\rho_\nu)^{-a}$. Let $d_\nu=\rho_\nu/s_\nu$ and $\tau_j^{(\nu)}=jd_\nu$ $(j=0, 1, \cdots, s_\nu)$, then clearly $(1/2)(1+\rho_\nu)^{-a}<d_\nu\leqq(1+\rho_\nu)^{-a}$. We set

$$\Omega_j^\nu=\{\eta_\nu(\tau, y)\in\Lambda^{n+1,\nu};\ \|(\tau, y)-(\tau_j^{(\nu)}, x^{(\nu)})\|<d_\nu\} \qquad \text{(in Case-P)},$$

$$(\Omega_j^\nu=\{\eta_\nu(\tau, y)\in\Lambda^{n+1,\nu};\ \|(\tau, y)-(\tau_j^{(\nu)}, y^{(\nu)})\|<d_\nu\} \qquad \text{(in Case-N)}),$$

$(j=0, 1, \cdots, s_\nu)$. We denote $K\subset\{1, \cdots, n\}$ which realizes the inequality (2.10) for $(\tau_j^{(\nu)}, x^{(\nu)}, p^{(\nu)})\in \boldsymbol{R}_+\times F$ by K_j^ν in (Case-P), (in (Case-N) $(\tau_j^{(\nu)}-\rho_\nu, y^{(\nu)}, q^{(\nu)})\in\boldsymbol{R}\times F$). It clearly holds that

$$\text{(2.18)} \qquad \begin{cases} \Omega_j^\nu\cap(0, \Lambda_0^{n,\nu})=\emptyset & \text{for } j=1, \cdots, s_\nu \\ \Omega_j^\nu\cap(\rho_\nu, \Lambda_{\rho_\nu}^{n,\nu})=\emptyset & \text{for } j=0, 1, \cdots, s_\nu-1 \\ \Omega_i^\nu\cap\Omega_j^\nu=\emptyset & \text{for } |i-j|\geqq2. \end{cases}$$

We set

$$J_{K_j^\nu}(r)=\left|\det\frac{\partial(X_{(K_j^\nu)'}(\tau, y, p^{(\nu)}), P_{K_j^\nu}(\tau, y, p^{(\nu)}))}{\partial y}\right| \qquad \text{(in Case-P)},$$

$$\left(=\left|\det\frac{\partial(X_{(K_j^\nu)'}(\tau-\rho_\nu, y, q^{(\nu)}), P_{K_j^\nu}(\tau-\rho_\nu, y, q^{(\nu)}))}{\partial y}\right| \qquad \text{(in Case-N)}\right)$$

$$\text{for} \quad r=\eta_\nu(\tau, y)\in\Omega_j^\nu.$$

By (c) in Proposition 2.4,

$$\text{(2.19)} \qquad J_{K_j^\nu}(r)\geqq(1+\rho_\nu)^{-d} \qquad \text{for any} \quad r\in\Omega_j^\nu.$$

In view of Proposition 2.4, the mapping $r=(\tau, x, p)\in\Omega_j^\nu$ to $(\tau, x_{(K_j^\nu)'}, p_{K_j^\nu})\in\boldsymbol{R}^{n+1}$ becomes the diffeoomorphism. The inverse diffeomorphism is denoted by $r_{K_j^\nu}$, that is, $r_{K_j^\nu}(\tau, x_{(K_j^\nu)'}, p_{K_j^\nu})=r$. For $r=(\tau, x, p)\in\Lambda^{n+1,\nu}$, we denote its projection into $\boldsymbol{R}^{2n+1}$ by $(\tilde\tau(r), \tilde x(r), \tilde p(r))$.

Let $f\in C_0^\infty(\Omega_j^\nu)$. It follows from Lemma 2.1 and (2.19) that for any nonnegative integer s

$$\text{(2.20)} \quad (1+\rho_\nu)^{-m_s}|f(\eta_\nu(\tau, \cdot))|_s\leqq|f(r_{K_j^\nu}(\tau, \cdot, \cdot))|_s$$
$$\leqq(1+\rho_\nu)^{m_s}|f(\eta_\nu(\tau, \cdot))|_s, \qquad \text{for any} \quad \tau\in(\tau_j^{(\nu)}-d_\nu, \tau_j^{(\nu)}+d_\nu),$$

is valid for a constant $m_s>0$ independent of ν, j and τ.

We set

$$\tilde\Lambda^{n+1,\nu}=\{\eta_\nu(\tau, y);\ \tau\in[0, \rho_\nu],\ \|y-x^{(\nu)}\|<(1+\rho_\nu)^{-a'}\} \qquad \text{(in Case-P)},$$

$$(=\{\eta_\nu(\tau, y);\ \tau\in[0, \rho_\nu],\ \|y-y^{(\nu)}\|<(1+\rho_\nu)^{-a'}\} \qquad \text{(in Case-N))}.$$

Here a constant $a'(>a)$ independent of ν will be determined in the proof of Lemma 3.4 (see Appendix (1)), where a is a constant in Proposition 2.4. Now we can easily construct the partition of unity $\{e_j^{(\nu)}(r)\}_{j=0}^{s_\nu}$ corresponding to $\{\Omega_j^\nu\}_{j=0}^{s_\nu}$:

$$(2.21)\qquad \begin{aligned} &(1)\quad e_j^{(\nu)}(r)\in C_0^\infty(\Omega_j^\nu), \qquad 0\leqq e_j^{(\nu)}\leqq 1;\\ &(2)\quad \sum_{j=0}^{s_\nu} e_j^{(\nu)}(r)=1 \qquad \text{for any} \quad r\in\tilde{\Lambda}^{n+1,\nu},\\ &(3)\quad |e_j^{(\nu)}(\eta_\nu(\tau, \cdot))|_s\leqq(1+\rho_\nu)^{m_s} \qquad \text{for any} \quad \tau\in[0, \rho_\nu] \quad \text{and} \quad s\in\boldsymbol{N}, \end{aligned}$$

where the constant m_s is independent of ν.

Following [3], we can see that

$$dp\wedge dx-dH\wedge d\tau=0 \qquad \text{on} \quad \Lambda^{n+1,\nu}.$$

By Poincaré's lemma, we obtain real valued C^∞ function $S^{(\nu)}(r)$ on $\Lambda^{n+1,\nu}$ such that

$$(2.22)\qquad dS^{(\nu)}=pdx-Hd\tau \qquad \text{on} \quad \Lambda^{n+1,\nu}.$$

If we set

$$(2.23)\qquad S_{K_j^\nu}(r)=S^{(\nu)}(r)-\tilde{x}_{K_j^\nu}(r)\cdot\tilde{p}_{K_j^\nu}(r) \qquad \text{for} \quad r\in\Omega_j^\nu,$$

then we have

$$(2.24)\qquad \begin{aligned} &\frac{\partial S_{K_j^\nu}(r_{K_j^\nu}(\tau, x_{(K_j^\nu)'}, p_{K_j^\nu}))}{\partial x_{(K_j^\nu)'}}=\tilde{p}_{(K_j^\nu)'}(r),\\ &\frac{\partial S_{K_j^\nu}(r_{K_j^\nu}(\tau, x_{(K_j^\nu)'}, p_{K_j^\nu}))}{\partial p_{K_j^\nu}}=-\tilde{x}_{K_j^\nu}(r),\\ &\frac{\partial S_{K_j^\nu}(r_{K_j^\nu}(\tau, x_{(K_j^\nu)'}, p_{K_j^\nu}))}{\partial\tau}=-H(\tilde{x}(r), \tilde{p}(r)),\\ &\qquad\qquad \text{for any} \quad r=r_{K_j^\nu}(\tau, x_{(K_j^\nu)'}, p_{K_j^\nu})\in\Omega_j^\nu. \end{aligned}$$

§ 3. Canonical operater

We shall omit the suffix $\nu\geqq\nu_0$ up to Lemma 3.1. For $\lambda\geqq 1$ We define λ-Fourier transform and inverse λ-Fourier transform over a part of variables by

$$(3.1)\qquad \begin{aligned} &\mathcal{F}_{\lambda, x_K\to p_K}[u(x)]=(\lambda/2\pi)^{|K|/2}\int e^{-i\lambda x_K\cdot p_K}u(x)dx_K,\\ &\mathcal{F}_{\lambda, p_K\to x_K}^{-1}[v(p)]=(\lambda/2\pi)^{|K|/2}\int e_\nu^{i\lambda x_K\cdot p_K}v(p)dp_K,\\ &\qquad\qquad \text{for} \quad u(x), v(p)\in C_0^\infty(\boldsymbol{R}^n) \quad \text{and} \quad K\subset\{1, \cdots, n\}. \end{aligned}$$

Let Ω_j be a component of the open covering $\{\Omega_j^\nu\}_{j=0}^{s_\nu}$ constructed in section 2, and $(\tau, x_{K_j'}, p_{K_j})\equiv(\tau, I_{K_j})$ be a local coordinate on Ω_j. Following [3], for $\varphi\in C_0^\infty(\Omega_j)$ we define pre-canonical operater $\mathcal{K}(\Omega_j, I_{K_j})$ by

$$\mathcal{K}(\Omega_j,\ I_{K_j})\varphi(\tau,\ x)=\mathcal{F}_{\lambda, p_{K_j}\to x_{K_j}}^{-1}\left[\frac{e^{i\lambda S_{K_j}(r)}}{\sqrt{J_{K_j}(r)}}\varphi(r)|_{r=r_{K_j}(\tau, x_{K_j'}, p_{K_j})}\right].$$

Lemma 3.1. *For* $\varphi\in C_0^\infty(\Omega_j)$ *and* $\tau\in(\tau_j^{(\nu)}-d_\nu,\ \tau_j^{(\nu)}+d_\nu)$, *we have the following.*

(a) $$\|\mathcal{K}(\Omega_j,\ I_{K_j})\varphi(\tau,\ \cdot)\|_0=\|\varphi(\eta_\nu(\tau,\ \cdot))\|_0.$$

(b) *For any nonnegative integer s there exists a constant* $m_s>0$ *which depends only on s such that*

$$\|\mathcal{K}(\Omega_j,\ I_{K_j})\varphi(\tau,\ \cdot)\|_s\leqq\lambda^s(1+\rho_\nu)^{m_s}|\varphi(\eta_\nu(\tau,\ \cdot))|_s.$$

Proof. (a) By Parseval's formula,

$$\|\mathcal{K}(\Omega_j,\ I_{K_j})\varphi(\tau,\ \cdot)\|_0^2$$
$$=\int_{R^n}|\exp(i\lambda S_{K_j}(r))(J_{K_j}(r))^{-1/2}\varphi(r)|_{r=r_{K_j}(\tau, x_{K_j'} p_{K_j})}|^2 dx_{K_j'}dp_{K_j}$$
$$=\int_{R^n}|\varphi(r_{K_j}(\tau,\ x_{K_j'},\ p_{K_j}))|^2 J_{K_j}(r_{K_j}(\tau,\ x_{K_j'},\ p_{K_j}))^{-1}dx_{K_j'}dp_{K_j}$$
$$=\int_{R^n}|\varphi(\eta_\nu(\tau,\ y))|^2 dy.$$

(b) By Lemma 2.1, (2.19), (2.20) and (2.24), we get for any multi-index $\alpha\,(|\alpha|\leqq s)$

$$\|\partial_x^\alpha\{\mathcal{K}(\Omega_j,\ I_{K_j})\varphi(\tau,\ \cdot)\}\|_0$$
$$=\lambda^{|\tilde{\alpha}|}\left\|p_{K_j}^{\tilde{\alpha}}\partial_{x_{K_j'}}^{\tilde{\alpha}'}\left[\frac{\exp(i\lambda S_{K_j}(r))}{\sqrt{J_{K_j}(r)}}\varphi(r)|_{r=r_{K_j}(\tau, K_j', p_{K_j})}\right]\right\|_{L^2(R^n_{x_{K_j'}, p_{K_j}})}\qquad(\tilde{\alpha}=\alpha_{K_j},\ \tilde{\alpha}'=\alpha_{K_j'})$$
$$\leqq\lambda^s(1+\rho_\nu)^{m_s}|\varphi(\eta_\nu(\tau,\ \cdot))|_s$$

for some constant $m_s>0$ which depends only on s. q.e.d.

We shall define a first order linear differential operator $W_1^{(\nu)}$ on $C^\infty(\Lambda^{n+1,\nu})$ independent of λ by

$$W_1^{(\nu)}\varphi(r)=\left\{\frac{d}{d\tau}-F(\eta_{\nu,\tau}(y))\right\}\varphi(\eta_\nu(\tau,\ y))$$

at $r=\eta_\nu(\tau,\ y)\in\Lambda^{n+1,\nu}$ for $\varphi(r)\in C^\infty(\Lambda^{n+1,\nu})$. Following [3], we now introduce the canonical operator $\boldsymbol{K}_\nu$ acting on $\varphi(r)\in C_0^\infty(\tilde{\Lambda}^{n+1,\nu})$ by

$$\boldsymbol{K}_\nu\varphi(\tau,\ x)=\sum_{j=0}^{s_\nu}e^{i\sigma(\Omega_j^\nu)}\mathcal{K}(\Omega_j^\nu,\ I_{K_j^\nu})[e_j^{(\nu)}(r)\varphi(r)](\tau,\ x)$$

Here $\sigma(\Omega_j^\nu)(0\leqq j\leqq s_\nu)$ are real constants that are given by $\sigma(\Omega_j^\nu)=-\sum_{k=1}^j\sigma(\Omega_k^\nu,\ \Omega_{k-1}^\nu)$, $(1\leqq j\leqq s)$, $\sigma(\Omega_0^\nu)=0$, where real constants $\sigma(\Omega_k^\nu,\ \Omega_{k-1}^\nu)=-\sigma(\Omega_{k-1}^\nu,\ \Omega_k^\nu)$ $(k=1,\ \cdots,\ s_\nu)$ will be determined in Lemma 3.4 below.

Proposition 3.2. *For any* $\lambda\geqq1$, $\nu\geqq\nu_0$ *and* $\varphi(r)\in C_0^\infty(\tilde{\Lambda}^{n+1,\nu})$, *it holds that*

(a) $$L_\lambda\boldsymbol{K}_\nu\varphi(\tau,\ x)=\sum_{l=1}^{N-1}(i\lambda)^{-l}\boldsymbol{K}_\nu[W_l^{(\nu)}\varphi(r)](\tau,\ x)+R_N^{(\nu)}\varphi(\tau,\ x)$$

for $N=1,\ 2,\ \cdots$

where $W_l^{(\nu)}$ $(l\geqq 2)$ *are linear differential operators in* $\Lambda^{n+1,\nu}$ *of order at most* $2l$ *independent of* λ. *Furthermore for any nonnegative integer* s *and* $N\geqq 1$, *there exist constants* $m_{l,s}$ *and* $q_{l,s}$ $(2\leqq l\leqq N)$ *independent of* ν *and* λ *such that*

(b) $$|W_l^{(\nu)}\varphi(\eta_\nu(\tau,\ \cdot))|_s\leqq(1+\rho_\nu)^{m_{l,s}}|\varphi(\eta_\nu(\tau,\ \cdot))|_{q_{l,s}},$$

(c) $$\|R_N^{(\nu)}\varphi(\tau,\ \cdot)\|_s\leqq\lambda^{-N+s}(1+\rho_\nu)^{m_{N,s}}|\varphi(\eta_\nu(\tau,\ \cdot))|_{q_{N,s}}$$

$$\textit{for any}\quad \nu\geqq\nu_0,\ 2\leqq l\leqq N-1,\ \textit{and}\ \tau\in[0,\ \rho_\nu].$$

Using the following two Lemmas, we can prove this proposition by the similar argument as Proposition 3.4 in [3]. We shall give the proof in Appendix (2).

Lemma 3.3. *For any* $j\in\{0,\ 1,\ \cdots,\ s_\nu\}$ *and* $N\geqq 1$, *we can see*

(3.2) $$L_\lambda\mathcal{K}(\Omega_j^\nu,\ I_{K_j^\nu})\varphi(\tau,\ x)$$

$$=\mathcal{K}(\Omega_j^\nu,\ I_{K_j^\nu})\Big[(i\lambda)^{-1}W_1^{(\nu)}\varphi(r)+\sum_{l=2}^{N-1}(i\lambda)^{-1}\mathcal{D}^{(l)}(\Omega_j^\nu,\ I_{K_j^\nu})\varphi(r)\Big](\tau,\ x)$$

$$+R_{1,N}(\Omega_j^\nu,\ I_{K_j^\nu})\varphi(\tau,\ x)\qquad \textit{for any}\quad \varphi(r)\in C_0^\infty(\Omega_j^\nu),$$

where $\mathcal{D}^{(l)}(\Omega_j^\nu,\ I_{K_j^\nu})$ $(2\leqq l\leqq N-1)$ *are linear differential operators of order at most* l *in variables* $I_{K_j^\nu}=(x_{(K_j^\nu)'},\ p_{K_j^\nu})$ *and are determined independently of* λ. *Furthermore, for each nonnegative integer* s *and* $N\geqq 1$ *there exist constants* $m_{l,s}$ $(2\leqq l\leqq N)$ *and* $q_{N,s}$ *independent of* ν *and* λ *such that*

(3.3) $$|\mathcal{D}^{(l)}(\Omega_j^\nu,\ I_{K_j^\nu})\varphi(\eta_\nu(\tau,\ \cdot))|_s\leqq(1+\rho_\nu)^{m_{l,s}}|\varphi(\eta_\nu(\tau,\ \cdot))|_{s+l},$$

(3.4) $$\|R_{1,N}(\Omega_j^\nu,\ I_{K_j^\nu})\varphi(\tau,\ \cdot)\|_s\leqq\lambda^{-N+s}(1+\rho_\nu)^{m_{N,s}}|\varphi(\eta_\nu(\tau,\ \cdot))|_{q_{N,s}}$$

$$\textit{for any}\quad \nu\geqq\nu_0,\quad 2\leqq l\leqq N-1\ \textit{and}\ \tau\in[0,\ \rho_\nu].$$

Proof. (3.2) is the same equality as that in Proposition 4.1 of [3], which is in essential proved in the similar way to the proof of Theorem 8.4 in [6]. We can obtain the explicit forms of $\mathcal{D}^{(l)}(\Omega_j^\nu,\ I_{K_j^\nu})$ and $R_{1,N}(\Omega_j^\nu,\ I_{K_j^\nu})$ in the argument to prove (3.2) along the argument of [3] and [6]. Then we can prove the estimates (3.3) and (3.4) by using Lemma 2.1, (2.19) and (2.24). q.e.d.

Lemma 3.4. *Let* $\Omega_i^\nu\cap\Omega_j^\nu\neq\emptyset$ *and* $i\neq j$. *Then, there exist real constants* $\sigma(\Omega_i^\nu,\ \Omega_j^\nu)$ *satisfying* $\sigma(\Omega_i^\nu,\ \Omega_j^\nu)=-\sigma(\Omega_j^\nu,\ \Omega_i^\nu)$ *such that for any* $\varphi(r)\in C_0^\infty(\Omega_i^\nu\cap\Omega_j^\nu\cap\tilde{\Lambda}^{n+1,\nu})$ *and any* $N\geqq 1$ *we have*

(3.5) $$\mathcal{K}(\Omega_i^\nu,\ I_{K_i^\nu})\varphi(\tau,\ x)=e^{i\sigma(\Omega_i^\nu,\Omega_j^\nu)}\mathcal{K}(\Omega_j^\nu,\ I_{K_j^\nu})$$

$$\cdot\Big[\Big\{1+\sum_{l=1}^{N-1}(i\lambda)^{-l}V^{(l)}(I_{K_j^\nu},\ I_{K_i^\nu})\Big\}\varphi(r)\Big](\tau,\ x)+R_{2,N}(I_{K_j^\nu},\ I_{K_i^\nu})\varphi(\tau,\ x),$$

where $V^{(l)}(I_{K_j^\nu},\ I_{K_i^\nu})$, $(1\leqq l\leqq N-1)$ *are linear differential operators of order at most* $2l$ *in variables* $(x,\ p)$ *and are determined independently of* λ. *Furthermore for any nonnegative integer* s *and* $N\geqq 1$, *there exist constants* $m_{l,s}$ $(1\leqq l\leqq N)$, *and* $q_{N,s}$ *independent of* ν *and* λ *such that*

(3.6) $$|V^{(l)}(I_{K_j^\nu}, I_{K_i^\nu})\varphi(\eta_\nu(\tau, \cdot))|_s \leqq (1+\rho_\nu)^{m'_{l,s}} |\varphi(\eta_\nu(\tau, \cdot))|_{s+2l},$$

(3.7) $$\|R_{2,N}(I_{K_j^\nu}, I_{K_i^\nu})\varphi(\tau, \cdot)\|_s \leqq \lambda^{-N+s}(1+\rho_\nu)^{m''_{N,s}} |\varphi(\eta_\nu(\tau, \cdot))|_{q_{N,s}},$$

for any $\nu \geqq \nu_0$, $1 \leqq l \leqq N-1$ *and* $\tau \in [0, \rho_\nu]$.

This Lemma is also proved in the similar way to the proof of Proposition 4.2 in [3]. We shall give the proof in Appendix (1).

§ 4. Proof of Theorem

Following [3], we prove Theorem by contradiction. Namely we shall assume that

(As. 1) the Causchy problem (1.3) is H^∞-wellposed on $[0, T_0]$ $(T_0>0)$ or $[T_0, 0]$ $(T_0<0)$ for a $T_0 \neq 0$,
and

(As. 2) the inequality (1.7) is not valid.
Then, we may assume without loss of generality,

(As. 1)′ The Cauchy problem (1.3) is H^∞-wellposed on $[0, T_0]$ $(T_0>0)$, instead of (As. 1).

Similarly to the proof of Lemma 3.1 in [3], we get

Lemma 4.1. *Assume (As. 1)′. Then there exist constants* $C(T_0) \geqq 1$ *and nonnegative integer* q *such that for each* $\lambda \geqq 1$ *and each* $T \in (0, \lambda T_0]$ *we have the following. If* $v_\lambda(\tau, x) \in \mathcal{E}_\tau^0([0, T]; H^\infty(\boldsymbol{R}^n))$ *and* $L_\lambda v_\lambda(\tau, x) \in \mathcal{E}_\tau^0([0, T]; H^\infty(\boldsymbol{R}^n))$, *then the inequality*

(4.1) $$\max_{0 \leqq \tau \leqq T} \|v_\lambda(\tau, \cdot)\|_0 \leqq C(T_0)(\|v_\lambda(0, \cdot)\|_q + \lambda^2 \max_{0 \leqq \tau \leqq T} \|L_\lambda v_\lambda(\tau, \cdot)\|_q)$$

is valid.

We shall construct asymptotic solutions of equation (1.3) with $f(t, x)=0$ in the form

(4.2) $$v_\lambda^{(\nu)}(\tau, x) = \boldsymbol{K}_\nu \Big[\sum_{j=0}^{q+1} (i\lambda)^{-j} \varphi_j^{(\nu)}(r)\Big](\tau, x) \qquad (\tau \in [0, \rho_\nu], \quad x \in \boldsymbol{R}^n)$$

where $\varphi_j^{(\nu)}(r) \in C_0^\infty(\tilde{\Lambda}^{n+1,\nu})$ $(0 \leqq j \leqq q+1, \nu \geqq \nu_0)$ and q is a constant determined in Lemma 4.1. We shall determine $\varphi_j^{(\nu)}(r)$ by using (a) in Proposition 3.2. We obtain

(4.3) $$\begin{aligned} L_\lambda v_\lambda^{(\nu)}(\tau, x) &= (i\lambda)^{-1} \boldsymbol{K}_\nu [W_1^{(\nu)} \varphi_0^{(\nu)}(r)](\tau, x) \\ &+ (i\lambda)^{-2} \boldsymbol{K}_\nu [W_1^{(\nu)} \varphi_1^{(\nu)}(r) + W_2^{(\nu)} \varphi_0^{(\nu)}(r)](\tau, x) \\ &\cdots\cdots\cdots \\ &+ (i\lambda)^{-(q+2)} \boldsymbol{K}_\nu [W_1^{(\nu)} \varphi_{q+1}^{(\nu)}(r) + \cdots + W_{q+2}^{(\nu)} \varphi_0^{(\nu)}(r)](\tau, x) \\ &+ \sum_{j=0}^{q+1} (i\lambda)^{-j} R_{q+3-j}^{(\nu)} \varphi_j^{(\nu)}(\tau, x). \end{aligned}$$

We determine $\varphi_j^{(\nu)}(r)$ $(0 \leqq j \leqq q+1)$ by the solutions of

(4.4) $$W_1^{(\nu)} \varphi_0^{(\nu)}(r) = 0,$$

$$W_1^{(\nu)} \varphi_1^{(\nu)}(r) + W_2^{(\nu)} \varphi_0^{(\nu)}(r) = 0,$$

....

$$W_1^{(\nu)}\varphi_{q+1}^{(\nu)}(r)+\cdots+W_{q+2}^{(\nu)}\varphi_0^{(\nu)}(r)=0, \qquad \text{for any} \quad r\in\tilde{\Lambda}^{n+1,\nu},$$

with the initial conditions

$$\begin{aligned}(4.5)\qquad &\varphi_0^{(\nu)}(\eta_\nu(0, y))=g^{(\nu)}(y),\\ &\varphi_1^{(\nu)}(\eta_\nu(0, y))=\cdots=\varphi_{q+1}^{(\nu)}(\eta_\nu(0, y))=0, \qquad \text{for any} \quad y\in\boldsymbol{R}^n,\end{aligned}$$

where

$$(4.6)\qquad g^{(\nu)}(y)=\prod_{i=1}^{n} G((1+\rho_\nu)^{a'}(y_i-x_i^{(\nu)})) \qquad (\text{in Case-P}),$$

$$\left(=\prod_{i=1}^{n} G((1+\rho_\nu)^{a'}(y_i-y_i^{(\nu)})) \qquad (\text{in Case-N})\right)$$

Here a' is a constant in the definition of $\tilde{\Lambda}^{n+1,\nu}$, and $G\in C_0^\infty((-1, 1))$ is not identically zero.

Lemma 4.2.

$$(4.7)\qquad \varphi_0^{(\nu)}(\eta_\nu(\tau, y))=g^{(\nu)}(y)\exp\int_0^\tau F(\eta_{\nu,\theta}(y))d\theta,$$

$$(4.8)\qquad \varphi_j^{(\nu)}(r)\in C_0^\infty(\tilde{\Lambda}^{n+1,\nu}) \qquad \textit{for } j=0, 1, \cdots, q+1.$$

Furthermore for each $j=0, 1, \cdots, q+1$ and any nonnegative integer s, there exists a positive constant $m_{j,s}$ independent of ν and τ such that

$$(4.9)\qquad |\varphi_j^{(\nu)}(\eta_\nu(\tau, \cdot))|_s\leqq(1+\rho_\nu)^{m_{j,s}}e^{\psi(\tau, x^{(\nu)}, p^{(\nu)})}, \qquad \textit{for any } \tau\in[0, \rho_\nu] \textit{ and any } \nu\geqq\nu_0.$$

Proof. From (4.4) and (4.5), we can easily obtain (4.7) and

$$(4.10)\qquad \varphi_j^{(\nu)}(\eta_\nu(\tau, y)) \\ =-\int_0^\tau\sum_{l=0}^{j-1} W_{j-l+1}^{(\nu)}\varphi_l^{(\nu)}(\eta_\nu(\sigma, y))\exp\left\{\int_\sigma^\tau F(\eta_{\nu,\theta}(y))d\theta\right\}d\sigma$$

for each $j=1, 2, \cdots, q+1$. In view of (4.6), (4.7), and (4.10), we see that (4.8) holds. We shall prove (4.9) by induction. By (d) in Proposition 2.4 and (4.6), we obtain (4.9) for $j=0$. Suppose that (4.9) are valid up to $j-1$. Then, using (d) in Proposition 2.4, (b) in Proposition 3.2, and (4.10), we can see that there exist positive constants $m'_{j,s}$ and $m_{j,s}$ independent of ν such that

$$\begin{aligned}&|\varphi_j^{(\nu)}(\eta_\nu(\tau, \cdot))|_s\\ &\leqq(1+\rho_\nu)^{m'_{j,s}}\sum_{l=0}^{j-1}\int_0^\tau|\varphi_l^{(\nu)}(\eta_\nu(\sigma, \cdot))|_s\exp\left\{\int_\sigma^\tau \mathfrak{Re}F(h^\theta(x^{(\nu)}, p^{(\nu)}))d\theta\right\}d\sigma\\ &\leqq(1+\rho_\nu)^{m_{j,s}}e^{\psi(\tau, x^{(\nu)}, p^{(\nu)})}.\end{aligned}$$

This completes the proof. q.e.d.

By (4.3) and (4.4), we have

$$L_\lambda v_\lambda^{(\nu)}(\tau, x)=\sum_{j=0}^{q+1}(i\lambda)^{-j}R_{q-j+3}\varphi_j^{(\nu)}(\tau, x).$$

Noting (c) in Proposition 3.2, we have

$$\|L_\lambda v_\lambda^{(\nu)}(\tau, \cdot)\|_q \leqq \sum_{j=0}^{q+1} \lambda^{-j} \|R_{q-j+3}\varphi_j^{(\nu)}(\tau, \cdot)\|_q$$

$$\leqq \lambda^{-3}(1+\rho_\nu)^m \sum_{j=0}^{q+1} |\varphi_j^{(\nu)}(\eta_\nu(\tau, \cdot))|_\mu ,$$

where m and μ are positive constants independent of ν, τ and λ. Hence, by (c) in Lemma 2.2 and (4.9), there exists a constant $m^{(1)}>0$ independent of ν, τ and λ such that

(4.11) $\|L_\lambda v_\lambda^{(\nu)}(\tau, \cdot)\|_q \leqq \lambda^{-3}(1+\rho_\nu)^{m^{(1)}} e^{\psi(\rho_\nu, x^{(\nu)} p^{(\nu)})}$ for any $\tau \in [0, \rho_\nu]$, $\nu \geqq \nu_0$ and $\lambda \geqq 1$.

By (4.5), we have

$$v_\lambda(0, x) = \mathcal{K}(\Omega_0^\nu, I_{K_0^\nu})[\varphi_0^{(\nu)}(r)](0, x).$$

Hence, by (b) in Lemma 3.1 and (4.6), there exists a constant $m^{(2)}>0$ independent of ν and λ such that

(4.12) $\|v_\lambda^{(\nu)}(0, \cdot)\|_q \leqq \lambda^q (1+\rho_\nu)^{m^{(2)}}$ for any $\nu \geqq \nu_0$ and $\lambda \geqq 1$.

On the other hand, noting (2.18) and (a) in Lemma 3.1, for any $\nu \geqq \nu_0$ and $\lambda \geqq 1$ we have

$$\|v_\lambda^{(\nu)}(\rho_\nu, \cdot)\|_0 \geqq \|\boldsymbol{K}_\nu[\varphi_0^{(\nu)}(r)](\rho_\nu, \cdot)\|_0 - \sum_{j=1}^{q+1} \lambda^{-j} \|\boldsymbol{K}_\nu[\varphi_j^{(\nu)}(r)](\rho_\nu, \cdot)\|_0$$

$$= \|\varphi_0^{(\nu)}(\eta_\nu(\rho_\nu, \cdot))\|_0 - \sum_{j=1}^{q+1} \lambda^{-j} \|\varphi_j^{(\nu)}(\eta_\nu(\rho_\nu, \cdot))\|_0 .$$

By (d) in Proposition 2.4, (4.6) and (4.7), we obtain

$$\|\varphi_0^{(\nu)}(\eta_\nu(\rho_\nu, \cdot))\|_0 = \|g^{(\nu)}(\cdot) \exp \int_0^{\rho_\nu} F(\eta_{\nu,\theta}(\cdot)) d\theta\|_0$$

$$\geqq (1+\rho_\nu)^{-m} \exp(\psi(\rho_\nu, x^{(\nu)}, p^{(\nu)})),$$

and by (4.9) for each $j=1, 2, \cdots, q+1$

$$\|\varphi_j^{(\nu)}(\eta_\nu(\rho_\nu, \cdot))\|_0 \leqq (1+\rho_\nu)^{m_j} \exp(\psi(\rho_\nu, x^{(\nu)}, p^{(\nu)})),$$

with some positive constants m and m_j independent of ν. Hence, there exists a constant $m^{(3)}>0$ independent of ν and λ such that

(4.13) $\|v_\lambda(\rho_\nu, \cdot)\|_0 \geqq \{(1+\rho_\nu)^{-m^{(3)}} - \lambda^{-1}(1+\rho_\nu)^{m^{(3)}}\} \exp(\psi(\rho_\nu, x^{(\nu)}, p^{(\nu)}))$

for any $\nu \geqq \nu_0$, and $\lambda \geqq 1$. Inserting (4.11)-(4.13) into (4.1), we get

(4.14) $(1+\rho_\nu)^{-m^{(3)}}$

$$\leqq \lambda^{-1}(1+\rho_\nu)^{m^{(3)}} + C(T_0)\{\lambda^q(1+\rho_\nu)^{m^{(2)}} e^{-\psi(\rho_\nu, x^{(\nu)}, p^{(\nu)})} + \lambda^{-1}(1+\rho_\nu)^{m^{(1)}}\}.$$

By (b) in Lemma 2.2, we have

(4.15) $e^{-\psi(\rho_\nu, x^{(\nu)}, p^{(\nu)})} \leqq (1+\rho_\nu)^{-\nu}$ for any $\nu \geqq \nu_0$.

Now, we set $\lambda = (1+\rho_\nu)^{m^{(1)}+2m^{(3)}+1}$. Then, noting (d) in Lemma 2.2 and (4.15), we can see that (4.14) cannot hold, if ν is large. This completes the proof of Theorem.

q. e. d.

Appendix

(1) *Proof of Lemma* 3.4. Let a and b be constants in Proposition 2.4. For $j=0, 1, \cdots, s_\nu$ and $\tau\in(\tau_j^{(\nu)}-d_\nu, \tau_j^{(\nu)}+d_\nu)$, we set

$$E_{K_j^\nu,\tau}=Q_n((X_{(K_j^\nu)'}(\tau, x^{(\nu)}, p^{(\nu)}), P_{K_j^\nu}(\tau, x^{(\nu)}, p^{(\nu)})); (1+\rho_\nu)^{-b}),$$

$$\tilde{E}_{K_j^\nu,\tau}=Q_n((X_{(K_j^\nu)'}(\tau, x^{(\nu)}, p^{(\nu)}), P_{K_j^\nu}(\tau, x^{(\nu)}, p^{(\nu)})); \frac{2}{3}(1+\rho_\nu)^{-b}),$$

$$\tilde{\tilde{E}}_{K_j^\nu,\tau}=Q_n((X_{(K_j^\nu)'}(\tau, x^{(\nu)}, p^{(\nu)}), P_{K_j^\nu}(\tau, x^{(\nu)}, p^{(\nu)})); \frac{1}{3}(1+\rho_\nu)^{-b}),$$

We determine a constant $a'(>a>0)$ in the definition of $\tilde{\Lambda}^{n+1,\nu}$ which is independent of ν, so that $(\tau, x, p)\in\tilde{\Lambda}^{n+1,\nu}$ implies $(x_{(K_j^\nu)'}, p_{K_j^\nu})\in\tilde{\tilde{E}}_{K_j^\nu,\tau}$. This is possible by Lemma 2.1. We can easily construct the cutoff function $\chi_{K_j^\nu,\tau}(x_{(K_j^\nu)'}, p_{K_j^\nu})\in C_0^\infty(E_{K_j^\nu,\tau})$ satisfying that

$$\chi_{K_j^\nu,\tau}\equiv 1 \text{ on } \tilde{E}_{K_j^\nu,\tau} \quad\text{and}\quad |\chi_{K_j^\nu,\tau}(\cdot, \cdot)|_s\leqq(1+\rho_\nu)^{m_s} \tag{A.1}$$

for any nonnegative integer s, where the constants m_s is independent of τ and ν.

In what follows, we shall fix τ satisfying $|\tau-\tau_i^{(\nu)}|$, $|\tau-\tau_j^{(\nu)}|<d_\nu$ $(i, j=0, 1, \cdots, s_\nu, |i-j|=1)$. For simplicity, we set $\Omega=\Omega_i^\nu$, $\hat{\Omega}=\Omega_j^\nu$, $K=K_i^\nu$, $\hat{K}=K_j^\nu$, and $a=K\cap\hat{K}$, $b=K\cap\hat{K}'$, $c=K'\cap\hat{K}$, $d=K'\cap\hat{K}'$. We have for $\varphi(r)\in C_0^\infty(\Omega\cap\hat{\Omega})$

$$\begin{aligned}
\mathcal{K}(\Omega, l_K)\varphi(\tau, x) \tag{A.2}
&=\mathcal{F}_{\lambda,p_{\hat{K}}\to x_{\hat{K}}}^{-1}\circ\mathcal{F}_{\lambda,x_{\hat{K}}\to p_{\hat{K}}}\circ\mathcal{F}_{\lambda,p_K\to x_K}^{-1}\left[\frac{e^{i\lambda S_K(r)}}{\sqrt{J_K(r)}}\varphi(r)|_{r=r_K(\tau,x_{K'},p_K)}\right]\\
&=\mathcal{F}_{\lambda,p_{\hat{K}}\to x_{\hat{K}}}^{-1}\circ\mathcal{F}_{\lambda,x_c\to p_c}\circ\mathcal{F}_{\lambda,p_b\to x_b}^{-1}\left[\frac{e^{i\lambda S_K(r)}}{\sqrt{J_K(r)}}\varphi(r)|_{r=r_K(\tau,x_{K'},p_K)}\right]\\
&=\left(\frac{\lambda}{2\pi}\right)^{(|b|+|c|)/2}\mathcal{F}_{\lambda,p_{\hat{K}}\to x_{\hat{K}}}^{-1}\Big[\chi_{\hat{K},\tau}(x_{\hat{K}'}, p_{\hat{K}})\iint e^{i\lambda\Phi(x_c,p_b;\tau,x_{\hat{K}'},p_{\hat{K}})}\frac{\varphi(r)}{\sqrt{J_K(r)}}\\
&\qquad\cdot|_{r=r_K(\tau,x_{K'},p_K)}dp_b dx_c\Big]\\
&\quad+\left(\frac{\lambda}{2\pi}\right)^{(|b|+|c|)/2}\mathcal{F}_{\lambda,p_{\hat{K}}\to x_{\hat{K}}}^{-1}\Big[\{1-\chi_{\hat{K},\tau}(x_{\hat{K}'}, p_{\hat{K}})\}\iint e^{i\lambda\Phi(x_c,p_b;\tau,x_{\hat{K}'},p_{\hat{K}})}\\
&\qquad\cdot\frac{\varphi(r)}{\sqrt{J_K(r)}}|_{r=r_K(\tau,x_{K'},p_K)}dp_b dx_c\Big]\\
&\equiv I_1(\tau, x)+I_2(\tau, x),
\end{aligned}$$

where

$$\Phi(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})=-p_c\cdot x_c+x_b\cdot p_b+S_K(r_K(\tau, x_{K'}, p_K)).$$

At first we shall estimate $I_2(\tau, x)$. By (2.24), we have for any $(x_{K'}, p_K)\in \operatorname{supp}\varphi(r_K(\tau, \cdot, \cdot))$

$$\left|\frac{\partial\Phi}{\partial x_c}(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})\right|+\left|\frac{\partial\Phi}{\partial p_b}(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})\right|$$

$$=|-p_c+\tilde{p}_c(r_K(\tau, x_{K'}, p_K))|+|x_b-\tilde{x}_b(r_K(\tau, x_{K'}, p_K))|$$

$$=|-p_{\hat{K}}+\tilde{p}_{\hat{K}}(r_K(\tau, x_{K'}, p_K))|+|x_{\hat{K}'}-\tilde{x}_{\hat{K}'}(r_K(\tau, x_{K'}, p_K))|.$$

The last equality follows from the property of the projection $(\tilde{\tau}(r), \tilde{x}(r), \tilde{p}(r))$. Noting that $\mathrm{supp}\,\varphi\subset\tilde{\Lambda}^{n+1,\nu}$ and the definitions of $\tilde{\Lambda}^{n+1,\nu}$, $\tilde{E}_{\hat{K},\tau}$ and $\tilde{\tilde{E}}_{\hat{K},\tau}$, for $(x_{K'}, p_K)\in\mathrm{supp}\,\varphi(r_K(\tau, \cdot, \cdot))$ and $(x_{\hat{K}'}, p_{\hat{K}})\in\boldsymbol{R}^n\setminus\tilde{E}_{\hat{K},\tau}$ we have

$$\|(\tilde{x}_{\hat{K}'}(r_K(\tau, x_{K'}, p_K)), \tilde{p}_{\hat{K}}(r_K(\tau, x_{K'}, p_K)))$$
$$-(X_{\hat{K}'}(\tau, x^{(\nu)}, p^{(\nu)}), P_{\hat{K}}(\tau, x^{(\nu)}, p^{(\nu)}))\|>\frac{1}{3}(1+\rho_\nu)^{-b},$$

$$\|(x_{\hat{K}'}, p_{\hat{K}})-(X_{\hat{K}'}(\tau, x^{(\nu)}, p^{(\nu)}), P_{\hat{K}}(\tau, x^{(\nu)}, p^{(\nu)}))\|\geqq\frac{2}{3}(1+\rho_\nu)^{-b}.$$

Hence, there exists a constant $m>0$ independent of ν and τ such that

$$\left|\frac{\partial\Phi}{\partial x_c}(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})\right|+\left|\frac{\partial\Phi}{\partial p_b}(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})\right|$$
$$\geqq(1+\rho_\nu)^{-m}\{1+|(x_{\hat{K}'}, p_{\hat{K}})-(X_{\hat{K}'}(\tau, x^{(\nu)}, p^{(\nu)}), P_{\hat{K}}(\tau, x^{(\nu)}, p^{(\nu)})|\}, \tag{A.3}$$

for any $(x_{\hat{K}'}, p_{\hat{K}})\in\boldsymbol{R}^n\setminus\tilde{E}_{\hat{K},\tau}$ and any $(x_{K'}, p_K)\in\mathrm{supp}\,\varphi(r_K(\tau, \cdot, \cdot))$. Set

$$T=\left(\left|\frac{\partial\Phi}{\partial x_c}\right|^2+\left|\frac{\partial\Phi}{\partial p_b}\right|^2\right)^{-1}\left(\frac{\partial\Phi}{\partial x_c}\cdot D_{x_c}+\frac{\partial\Phi}{\partial p_b}\cdot D_{p_b}\right),$$

and we have $Te^{i\lambda\Phi}=\lambda e^{i\lambda\Phi}$. Using this formula, we take integral by parts in $I_2(\tau, x)$. Then we obtain the following estimates from Lemma 2.1, (2.19), (2.20), (A.1) and (A.3). For any nonnegative integer s, N' and any $\nu\geqq\nu_0$, we have

$$\|I_2(\tau, \cdot)\|_s\leqq\lambda^{-N'}(1+\rho_\nu)^{m_{s,N'}}|\varphi(\eta_\nu(\tau, \cdot))|_{q_{s,N'}}, \tag{A.4}$$

where the constants $m_{s,N'}$, $q_{s,N'}>0$ are independent of $\nu\geqq\nu_0$, $\lambda\geqq1$, and $\tau\in[0, \rho_\nu]$.

For $I_1(\tau, x)$, we may consider that $(x_{\hat{K}'}, p_{\hat{K}})\in\mathrm{supp}\,\chi_{\hat{K},\tau}\subset E_{\hat{K},\tau}$. Set $r=r_{\hat{K}}(\tau, x_{\hat{K}}, p_{\hat{K}})\in\Omega\cap\hat{\Omega}$. Following [3], we can easily obtain the following results. For any $(x_{\hat{K}'}, p_{\hat{K}})\in E_{\hat{K},\tau}$ fixed, $\Phi(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})$ has unique stationary point $(\tilde{x}_c(r), \tilde{p}_b(r))$, and we have

$$\Phi(\tilde{x}_c(r), \tilde{p}_b(r); \tau, x_{\hat{K}'}, p_{\hat{K}})=S_{\hat{K}}(r). \tag{A.5}$$

Moreover, setting

$$A(r)=\begin{pmatrix}\dfrac{\partial^2\Phi}{\partial x_c^2}(\tilde{x}_c(r), \tilde{p}_b(r); \tau, x_{\hat{K}'}, p_{\hat{K}}) & \dfrac{\partial^2\Phi}{\partial x_c\partial p_b}(\tilde{x}_c(r), \tilde{p}_b(r); \tau, x_{\hat{K}'}, p_{\hat{K}})\\ \dfrac{\partial^2\Phi}{\partial p_b\partial x_c}(\tilde{x}_c(r), \tilde{p}_b(r); \tau, x_{\hat{K}'}, p_{\hat{K}}) & \dfrac{\partial^2\Phi}{\partial p_b^2}(\tilde{x}_c(r), \tilde{p}_b(r); \tau, x_{\hat{K}'}, p_{\hat{K}})\end{pmatrix},$$

we have

$$|\det A(r)|=\frac{J_{\hat{K}}(r)}{J_K(r)}. \tag{A.6}$$

Now we apply the stationary phase method to the integral in $I_1(\tau, x)$. Since, we need the explicit form of each term in the asymptotic expansion to obtain (3.6) and (3.7), we will use the argument in page 70 to 73 of [1]. Then, using (A.4) and (A.5), we get for any $N\geqq 1$

(A.7) $I_1(\tau, x)$

$$=e^{i\sigma(\Omega,\hat{\Omega})}\mathcal{K}(\hat{\Omega}, I_{\hat{K}})\Big[\varphi(r)+\sqrt{J_K(r)}\sum_{l=1}^{3N-1}\frac{1}{l!}(2i\lambda)^{-l}(D_{x_c,p_b}A(r)\cdot D_{x_c,p_b})^l$$

$$\cdot\Big\{e^{i\lambda\tilde{\Phi}(x_c,p_b;\tau,x_{\hat{K}'},p_{\hat{K}})}\frac{\varphi}{\sqrt{J_K}}(r_K(\tau, x_{K'}, p_K))\Big\}\big|_{x_c=\tilde{x}_c(r),p_b=\tilde{p}_b(r)}\Big]+R'_{3N}(\tau, x; \lambda),$$

where $\sigma(\Omega, \hat{\Omega})=(\pi/4)\operatorname{sgn}A(r)$ ($\operatorname{sgn}A(r)$ denotes the sygnature of matrix $A(r)$), and

$$\tilde{\Phi}(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})=\Phi(x_c, p_b; \tau, x_{\hat{K}'}, p_{\hat{K}})-S_{\hat{K}}(r)$$
$$-\frac{1}{2}(x_c-\tilde{x}_c(r), p_b-\tilde{p}_b(r))A(r)\cdot(x_c-\tilde{x}_c(r), p_b-\tilde{p}_b(r)).$$

The remainder term $R'_{3N}(\tau, x; \lambda)$ has the following estimate. For any nonnegative integer s there exist constants $m_{N,s}$, $q_{N,s}$ independent of $\nu\geqq\nu_0$, $\lambda\geqq 1$, and τ such that

(A.8) $$\|R'_{3N}(\tau, \cdot; \lambda)\|_s\leqq\lambda^{-N+s}(1+\rho_\nu)^{m_{N,s}}|\varphi(\eta_\nu(\tau, \cdot))|_{q_{N,s}}.$$

From (A.2), (A.4), (A.7) and (A.8), we can obtain Lemma 3.4. q.e.d.

(2) *Proof of Proposition* 3.2. Here we omit the suffix ν. Using $\mathcal{D}^{(l)}(\Omega_j, I_{K_j})$ in Lemma 3.3 and $V^{(l)}(I_{K_j}, I_{K_i})$ in Lemma 3.4, we set

$$\mathcal{D}_N(\Omega_j, I_{K_j})=(i\lambda)^{-1}W_1+\sum_{l=1}^{N-1}(i\lambda)^{-l}\mathcal{D}^{(l)}(\Omega_j, I_{K_j}),$$

$$V_N(I_{K_j}, I_{K_i})=1+\sum_{l=1}^{N-1}(i\lambda)^{-l}V^{(l)}(I_{K_j}, I_{K_i}),$$

for $N=1, 2, \cdots$, and $i, j=0, 1, \cdots, s_\nu$, $|i-j|\leqq 1$. By Lemma 3.3, we have

(A.9) $L_\lambda(\boldsymbol{K}\varphi)(\tau, x)$

$$=\sum_{j=0}^{s_\nu}e^{i\sigma(\Omega_j)}\mathcal{K}(\Omega_j, I_{K_j})\mathcal{D}_N(\Omega_j, I_{K_j})e_j\varphi+\sum_{j=0}^{s_\nu}e^{i\sigma(\Omega_j)}R_{1,N}(\Omega_j, I_{K_j})e_j\varphi.$$

Now, it is easily seen that there exist differential operators

$$\tilde{V}^{(l)}(\Omega_j, I_{K_j})(l=1, 2, \cdots, N-1) \quad\text{and}\quad \tilde{\tilde{V}}^{(l')}(\Omega_j, I_{K_j})(l'=N, \cdots, 2(N-1))$$

acting on $C_0^\infty(\Omega_j)$ such that for any $f(r)\in C_0^\infty(\Omega_j)$ we have

(A.10) $$\sum_{m=0}^{s_\nu}V_N(I_{K_j}, I_{K_m})e_m\tilde{V}_N(\Omega_j, I_{K_j})f=\Big(1+\sum_{l=N}^{2(N-1)}(i\lambda)^{-l}\tilde{\tilde{V}}^{(l)}(\Omega_j, I_{K_j})\Big)f,$$

where we set

$$\tilde{V}_N(\Omega_j, I_{K_j})=1+\sum_{l=1}^{N-1}(i\lambda)^{-l}\tilde{V}^{(l)}(\Omega_j, I_{K_j}).$$

From (3.6), we can easily see that there exist constants $m'_{l,s}$ and $q'_{l,s}$ independent of ν such that

(A.11) $$|\tilde{V}^{(l)}(\Omega_j, I_{K_j})\varphi(\eta_\nu(\tau, \cdot))|_s\leqq(1+\rho_\nu)^{m'_{l,s}}|\varphi(\eta_\nu(\tau, \cdot))|_{q'_{l,s}},$$

(A.12) $$|\tilde{\tilde{V}}^{(l)}(\Omega_j, I_{K_j})\varphi(\eta_\nu(\tau, \cdot))|_s \leqq (1+\rho_\nu)^{m'_l \cdot s}|\varphi(\eta_\nu(\tau, \cdot))|_{q'_l, s},$$

where $\varphi \in C_0^\infty(\Omega_j \cap \tilde{\Lambda}^{n+1,\nu})$, $j=0, 1, \cdots, s_\nu$ and $l=1, 2, \cdots$. Using (A.10) and Lemma 3.4, we obtain from (A.9)

(A.13) $$L_\lambda(K\varphi)(\tau, x) = \sum_{l=1}^{N-1}(i\lambda)^{-l}K[W_l\varphi](\tau, x) + R_N\varphi(\tau, x),$$

where for $l=2, 3, \cdots, N-1$

(A.14) $$W_l\varphi = \sum_{j=0}^{s_\nu}\sum_{m=1}^{l-1}\tilde{V}^{(m)}(\Omega_j, I_{K_j})\mathcal{D}^{(l-1)}(\Omega_j, I_{K_j})e_j\varphi,$$

(A.15) $$R_N\varphi = \sum_{l=N}^{2(N-1)}(i\lambda)^{-l}\sum_{j=0}^{s_\nu}\sum_{m=l-N+1}^{N-1}K\tilde{V}^{(m)}(\Omega_j, I_{K_j})\mathcal{D}^{(l-m)}(\Omega_j, I_{K_j})e_j\varphi$$
$$-\sum_{j,m=0}^{s_\nu}e^{i\sigma(\Omega_m)}R_{2,N}(I_{K_j}, I_{K_m})e_m\tilde{V}_N(\Omega_j, I_{K_j})\mathcal{D}_N(\Omega_j, I_{K_j})e_j\varphi$$
$$-\sum_{l=N}^{2(N-1)}\sum_{j=0}^{s_\nu}(i\lambda)^{-l}e^{i\sigma(\Omega_j)}\mathcal{K}(\Omega_j, I_{K_j})\tilde{\tilde{V}}^{(l)}(\Omega_j, I_{K_j})\mathcal{D}_N(\Omega_j, I_{K_j})e_j\varphi$$
$$+\sum_{j=0}^{s_\nu}e^{i\sigma(\Omega_j)}R_{1,N}(\Omega_j, I_{K_j})e_j\varphi.$$

Here, we denote for $\varphi \in C_0^\infty(\Omega_j \cap \tilde{\Lambda}^{n+1,\nu})$

$$\tilde{V}^{(0)}(\Omega_j, I_{K_j})\varphi = \varphi \quad \text{and} \quad \mathcal{D}^{(1)}(\Omega_j, I_{K_j})\varphi = W_1\varphi.$$

From this, we get (a). Estimates (b) and (c) also follow from this, using (2.21), (b) in Lemma 3.1, (3.3), (3.4), (3.6) and (3.7). q.e.d.

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