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Functional law of the iterated logarithm for lacunary trigonometric and some gap series

By

Katusi FUKUYAMA

0. Introduction

The purpose of this paper is to prove the functional law of the iterated logarithm for lacunary trigonometric series and some gap series under probability measures with some regularity.

Let us first recall two limit theorems for a sequence of i.i.d. random variables.

Theorem A. *Let* $\{\xi_i\}$ *be a sequence of i.i.d. with mean* 0 *and variance* 1 *and put* $S_n=\xi_1+\cdots+\xi_n$. *Then it holds that*

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}}=1 \quad a.s.$$

Theorem B. *Let* $\{\xi_i\}$ *satisfy the same conditions as those of Theorem A and let us denote by* $\{X_n\}$ *a sequence of* $C[0, 1]$ *valued random variables such that* $X_n(s)=(S_{[ns]}+(ns-[ns])\xi_{[ns]+1})/\sqrt{n}$. *Then the sequence* $\{X_n/\sqrt{2\log\log n}\}$ *is relatively compact in* $C[0, 1]$ *and the set of all clusters of this sequence coincides with* K, *almost surely, where* $K=\{x\in C[0, 1]\,;\ x(0)=0,\ x$ *is absolutely continuous and* $\int_0^1 \dot{x}^2(t)dt\leqq 1\}$.

Theorem A is the classical version of the law of the iterated logarithm (LIL) due to Hartman-Wintner [9] and Theorem B is its functional version due to Strassen [23] which is called Strassen's law of the iterated logarithm or functional law of the iterated logarithm (FLIL). There are various extensions of these results to the case of dependent sequence of random variables, for example, mixing sequences, martingale difference sequences, lacunary trigonometric series, some gap series of functions and multiplicative systems. We shall here state two results due to Takahashi on lacunary trigonometric series corresponding to Theorem A and B resprctively. In the following two theorems, lacunary trigonometric series are regarded as random series on the probability space ([0, 1], dx).

Theorem C. *Suppose that sequences* $\{n_j\}$ *of integers and* $\{a_j\}$ *of real numbers satisfy following conditions.*

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$$n_{j+1}/n_j>1+cj^{-\alpha} \quad \textit{for some} \quad c>0 \textit{ and } 0<\alpha<1/2,$$

$$A_n^2=a_1^2+\cdots+a_n^2 \longrightarrow \infty \quad \textit{and} \quad a_n=O(A_n n^{-\alpha}\omega_n^{-1}) \textit{ as } n\longrightarrow\infty,$$

where $\omega_n=(\log n)^\beta(\log A_n)^4+(\log A_n)^8$ *and* $\beta>1/2$. *Then, for any sequence* $\{\gamma_j\}$ *of real numbers,*

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2A_n^2\log\log A_n}}=1 \quad a.s.$$

where $S_n=\sum_{j=1}^n a_j\sqrt{2}\cos(2\pi n_j x+\gamma_j)$.

Theorem D. *Suppose* $\{n_j\}$ *and* $\{a_j\}$ *satisfy*

$$n_{j+1}/n_j>1+cj^{-\alpha} \quad \textit{for some} \quad c>0 \textit{ and } 0<\alpha\leqq 1/2,$$

$$A_n^2=a_1^2+\cdots+a_n^2 \longrightarrow \infty \quad \textit{and} \quad a_n=O(A_n n^{-\alpha}(\log A_n)^{-\beta}) \textit{ as } n\longrightarrow\infty,$$

where $\beta>1/2$, *and* $\{\gamma_j\}$ *be arbitrary. Let* X_n *be a* $\boldsymbol{C}[0, 1]$*-valued random variable such that*

$$X_n\left(\frac{A_j^2}{A_n^2}\right)=\frac{S_j}{A_n} \quad \textit{and } X_n \textit{ is linear in } \left[\frac{A_j^2}{A_n^2}, \frac{A_{j+1}^2}{A_n^2}\right] \quad (j=1, \cdots, n). \tag{0.1}$$

Then the sequence $\{X_n/\sqrt{2\log\log A_n}\}$ *is relatively compact and the set of all clusters coincides with* K, *almost surely.*

Takahashi [27], [28] proved Theorem C using a method of multiplicative systems in essence and Theorem D was proved by Takahashi [29], who derived it as a corollary of almost sure invariance principles using a method of martingale approximation and Skorohod embedding. Theorem D was first proved by Berkes [1] under restrictive conditions.

The first aim of this paper is to extend Theorem D to the case in which the probability measure in question is not necessarily the Lebesgue measure.

Theorem 1. *Let* $\Omega=\boldsymbol{R}$ *and* P *satisfy*

$$|\hat{P}(u)|=O(|u|^{-\rho/2}) \quad as \quad u\longrightarrow\infty, \tag{0.2}$$

where $\hat{P}$ *is the characteristic function of* P. *Suppose that* $\{a_j\}$ *and* $\{\beta_j\}$ *satisfy*

$$\beta_1>0, \quad \beta_{j+1}/\beta_j>1+cj^{-\alpha} \ (j\in\boldsymbol{N}) \quad \textit{for some} \quad c>0 \textit{ and } 0\leqq\alpha<1/2, \tag{0.3}$$

$$A_n^2\longrightarrow\infty \quad \textit{and} \quad a_n=O(A_n(\log A_n)^{-8}n^{-\alpha}(1+\alpha\log n)^{-1}) \textit{ as } n\longrightarrow\infty, \tag{0.4}$$

$\{\gamma_j\}$ *be arbitrary and* $\{X_n\}$ *be defined by* (0.1) *using* $S_n(\omega)=\sum_{j=1}^n a_j\sqrt{2}\cos(\beta_j\omega+\gamma_j)$. *Then* $\{X_n/\sqrt{2\log\log A_n}\}$ *is relatively compact and the set of all clusters coincides with* K, P*-almost surely.*

Theorem 2. *Let* $\Omega=\boldsymbol{R}$, P *satisfy*

$$P\{[\omega, \omega+h]\}\leqq Mh^\rho(\omega\in\Omega,\ h>0) \quad \textit{for some} \quad M>0 \textit{ and } \rho\in(0, 1], \tag{0.5}$$

and $\{\beta_j\}$ *satisfy* (0.3). *Then, there exists a subset* E *of* $\boldsymbol{R}$ *with Lebesgue measure* 0 *such that for any* $t\in\boldsymbol{R}\backslash E$, *any* $\{a_j\}$ *satisfying* (0.4) *and any* $\{\gamma_j\}$, $\{X_n/\sqrt{2\log\log A_n}\}$ *is relatively compact and the set of all clusters coincides with* K, *P-almost surely, where* $\{X_n\}$ *is defined by* (0.1) *using* $S_n(\omega)=\sum_{j=1}^{n}a_j\sqrt{2}\cos(\beta_j t\omega+\gamma_j)$.

Theorem 3. *Let* $\Omega=\boldsymbol{R}$, P *satisfy* (0.5) *and a sequence* $\{\phi(j)\}$ *of positive numbers satisfy*

$$\phi(1)>0,\quad \phi(j+1)-\phi(j)\geqq dj^{-\alpha}\ \left(j\in\boldsymbol{N},\ \text{for some } d>0 \text{ and } 0\leqq\alpha<\frac{1}{2}\right). \tag{0.6}$$

Then, there exists a subset E *of* $(1,\infty)$ *with Lebesgue measure* 0 *such that for any* $x\in(1,\infty)\backslash E$, *any* $\{a_j\}$ *satisfying* (0.4) *and any* $\{\gamma_j\}$, $\{X_n/\sqrt{2\log\log A_n}\}$ *is relatively compact and the set of all clusters coincides with* K, *P-almost surely, where* $\{X_n\}$ *is defined by* (0.1) *using* $S_n(\omega)=\sum_{j=1}^{n}a_j\sqrt{2}\cos(x^{\phi(j)}\omega+\gamma_j)$.

We can easily prove using Fubini's theorem and Theorem 1 that there exists a subset E of $\boldsymbol{R}$ with Lebesgue measure 0 depending on $\{a_j\}$ and $\{\gamma_j\}$ such that X_n in Theorem 2 obey the FLIL. But this conclusion is weaker than Theorem 2, since in the last two theorems, the exceptional set E dose not depends on $\{a_j\}$ and $\{\gamma_j\}$.

There are, however, important singular probability measures satisfying (0.2) or (0.5). For details, we refer the reader to Wiener-Wintner [31], [32]. Under the same conditions, mean central limit theorem was proved by Fukuyama [8], and originally the central limit theorem was proved by Takahashi [30] assuming a gap condition of stronger type.

We next state the classical LIL for some gap series of functions due to Takahashi [24]. Here again, gap series are treated as a random series on a probability space $([0,1], dx)$. Let $\mathrm{Lip}\,\alpha$ be a class of α-Lipschitz continuous functions with period 2π.

Theorem E. *Let* $f\in\mathrm{Lip}\,\alpha$ $(0<\alpha\leqq 1)$ *satisfy*

$$\int_0^{2\pi}f(x)dx=0 \quad\text{and}\quad \int_0^{2\pi}f^2(x)dx=2\pi, \tag{0.7}$$

and $\{n_k\}$ *satisfy* $n_{k+1}/n_k\to\infty$ *as* $k\to\infty$. *Then,*

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log n}}=1 \quad a.s.$$

where $S_n=\sum_{k=1}^{n}f(n_k x)$.

Various papers are written on the limit theorems of gap series. For details, see Berkes [3]. The second aim of this paper is to prove the following three extensions of Theorem E.

Theorem 4. *Let* $\Omega=\boldsymbol{R}$, P *satisfy* (0.2), $\{a_k\}$ *satisfy*

$$A_n\longrightarrow\infty \quad\text{and}\quad a_n=O(A_n(\log A_n)^{-8}(\log n)^{-1}) \quad\text{as}\quad n\longrightarrow\infty, \tag{0.8}$$

$\{\gamma_k\}$ *be arbitrary,* $\{\beta_k\}$ *satisfy*

$$(0.9)\qquad \beta_1>0, \quad \beta_{k+1}/\beta_k \longrightarrow \infty \quad as \quad k \longrightarrow \infty,$$

$f\in \mathrm{Lip}\,\alpha\,(0<\alpha\leqq 1)$ satisfy (0.7) *and* $\{X_n\}$ *be defined by* (0.1) *using* $S_n(\omega)=\sum_{k=1}^n a_k f(\beta_k\omega+\gamma_k)$. *Then* $\{X_n/\sqrt{2\log\log A_n}\}$ *is relatively compact and the set of all clusters coincides with K, P-almost surely.*

Theorem 5. *Let* $\Omega=\boldsymbol{R}$, *P satisfy* (0.5), $\{\beta_k\}$ *satisfy* (0.9) *and* $f\in \mathrm{Lip}\,\alpha\,(0<\alpha\leqq 1)$ *satisfy* (0.7). *Then, there exists a subset E of* $\boldsymbol{R}$ *with Lebesgue measure* 0 *such that for any* $t\in \boldsymbol{R}\backslash E$, *and* $\{a_k\}$ *satisfying* (0.8) *and any* $\{\gamma_k\}$, $\{X_n/\sqrt{2\log\log A_n}\}$ *is relatively compact and the set of all clusters coincides with K, P-almost surely, where* $\{X_n\}$ *is defined by* (0.1) *using* $S_n(\omega)=\sum_{k=1}^n a_k f(\beta_k t\omega+\gamma_k)$.

Theorem 6. *Let* $\Omega=\boldsymbol{R}$, *P satisfy* (0.5), $\{\phi(k)\}$ *satisfy*

$$(0.10)\qquad \phi(1)>0, \quad \phi(k+1)-\phi(k) \longrightarrow \infty \quad as \quad k \longrightarrow \infty$$

and $f\in \mathrm{Lip}\,\alpha(0<\alpha\leqq 1)$ *satisfy* (0.7). *Then, there exists a subset E of* $(1,\infty)$ *with Lebesgue measure* 0 *such that for any* $x\in(1,\infty)\backslash E$, *any* $\{a_k\}$ *satisfying* (0.8) *and any* $\{\gamma_k\}$, $\{X_n/\sqrt{2\log\log A_n}\}$ *is relatively compact and the set of all clusters coincides with K, P-almost surely, where* $\{X_n\}$ *is defined by* (0.1) *using* $S_n(\omega)=\sum_{k=1}^n a_k f(x^{\phi(k)}\omega+\gamma_k)$.

To prove these theorems, we use the following result for weakly multiplicative systems. Here, we prepare some notation to state our results. For a sequence $\{\xi_i\}$ of random variables, let us put

$$b_{i_1,\dots,i_r}=E(\xi_{i_1}\cdots\xi_{i_r}), \qquad b'_{i_1,\dots,i_r}=E((\xi_{i_1}^2-1)\cdots(\xi_{i_r}^2-1)),$$

$$\bar{b}_{i_1,\dots,i_r}=E((\xi_{2i_1-1}^2-1)\cdots(\xi_{2i_r-1}^2-1)), \qquad \bar{\bar{b}}_{i_1,\dots,i_r}=E((\xi_{2i_1}^2-1)\cdots(\xi_{2i_r}^2-1)),$$

$$b^*_{k;\,i_1,\dots,i_r}=E(\xi_{2i_1-1}\cdots(\xi_{2i_k-1}^2-1)\cdots\xi_{2i_r-1}), \quad b^{**}_{k;\,i_1,\dots,i_r}=E(\xi_{2i_1}\cdots(\xi_{2i_k}^2-1)\cdots\xi_{2i_r}),$$

let B_r, B'_r, $\bar{B}_r$, $\bar{\bar{B}}_r$, B^*_r, B^{**}_r be infinite vectors such that

$$B_r=(b_{i_1,\dots,i_r})_{i_1<\dots<i_r}, \quad B'_r=(b'_{i_1,\dots,i_r})_{i_1<\dots<i_r}, \quad \bar{B}_r=(\bar{b}_{i_1,\dots,i_r})_{i_1<\dots<i_r},$$

$$\bar{\bar{B}}_r=(\bar{\bar{b}}_{i_1,\dots,i_r})_{i_1<\dots<i_r}, \quad B^*_r=(b^*_{k;\,i_1,\dots,i_r})_{1\leq k\leq r,\,i_1<\dots<i_r}, \quad B^{**}_r=(b^{**}_{k;\,i_1,\dots,i_r})_{1\leq k\leq r,\,i_1<\dots<i_r},$$

and $\|\cdot\|_\delta$ denotes the l_δ-norm. For example, $\|B_r\|_\delta=\sum_{i_1<\dots<i_r}(|b_{i_1,\dots,i_r}|^\delta)^{1/\delta}$.

A sequence $\{\xi_i\}$ of random variables is called a multiplicative system or weakly multiplicative system according as $B_r=0$ or B_r is nearly 0 in some sense. For details and history of these notions, we refer the reader to the survey by Móricz-Révész [18]. Our result is as follows.

Theorem 7. *Suppose that a sequence* $\{\xi_i\}$ *of random variables and a sequence* $\{c_i\}$ *of real numbers satisfy*

$$(0.11)\qquad \|B_r\|_\delta^{1/r}=O(r^{1-1/\delta}) \ as\ r\longrightarrow\infty \quad for\ some\quad \delta\in[1,2),$$

$$(0.12)\qquad \|\bar{B}_r\|_2^{1/r},\ \|\bar{\bar{B}}_r\|_2^{1/r}=O(r^{1/2}) \ as\ r\longrightarrow\infty$$

$$(0.13)\qquad \|B_3^*\|_2,\ \|B_3^{**}\|_2<\infty,$$

$$C_n^2=c_1^2+\cdots+c_n^2 \longrightarrow \infty \quad \text{and} \quad c_n,\ c_n\|\xi_n\|_\infty=O(C_n(\log C_n)^{-(8\wedge(4/\Delta))}) \quad \text{as } n\longrightarrow\infty,$$

where $\Delta=2/\delta-1$. *Let* $\{X_n\}$ *be defined by* (0.1) *using* $S_n=\sum_{i=1}^n c_i\xi_i$ *and* $\{C_n\}$ *instead of* $\{A_n\}$. *Then* $\{X_n/\sqrt{2\log\log C_n}\}$ *is relatively compact and the set of all clusters coincides with* K, *almost surely.*

The FLIL for weakly multiplicative systems was first proved by Berkes [2] under the restrictive conditions $c_i=1$, $\|\xi_i\|_\infty\leqq B(i\in \boldsymbol{N})$, $\sum_{r=1}^\infty\|B_r\|_1<\infty$ and $\sum_{r=1}^\infty\|B'_r\|_1<\infty$. Fukuyama [6] improved this results to the case $c_n=o(C_n^{1-\varepsilon})$ as $n\to\infty$, $\|\xi_i\|_\infty\leqq B(i\in\boldsymbol{N})$, $\|B_r\|_\delta^{1/r}\leqq B$ and $\|B'_r\|_2^{1/r}\leqq B$ $(r\in\boldsymbol{N})$ for some $\delta\in[1, 2)$ and $\varepsilon>0$. These results are not included in Theorem 7, since another condition (0.13) is assumed in Theorem 7. But we can prove another version of Theorem 7 which include these results completely.

Theorem 8. *Theorem 7 remains valid if we replace* (0.13) *by*

$$E\xi_n^4\leqq B \qquad (n\in\boldsymbol{N}). \tag{0.14}$$

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1. FLIL for weakly multiplicative systems

We first prepare some lemmas. Let $\{\zeta_i\}$ be a sequence of random variables, $\{u_i\}$ a sequence of positive numbers and put $S_{n,m}=\zeta_{n+1}+\cdots+\zeta_m$, $U_{n,m}=u_{n+1}+\cdots+u_m$.

Lemma 1. *Suppose that there exist* $C>0$, $L>0$ *and* $\Lambda\geqq 1$ *such that*

$$|u_n|\leqq C^2\Lambda^{-1/2} \qquad (n\in[N, M]), \tag{1.1}$$

$$E\exp(\lambda C^{-1}S_{n,m})\leqq L\exp(L\lambda^2C^{-2}U_{n,m}) \qquad (|\lambda|\leqq\Lambda,\ n,\ m\in[N, M]). \tag{1.2}$$

Moreover suppose one of the following conditions:

$$E(|S_{n,l}|^2|S_{l,m}|^2)\leqq LU_{n,l}U_{l,m} \qquad (n,\ l,\ m\in[N, M]), \tag{1.3}$$

$$E(|S_{n,m}|^4)\leqq LU_{n,m}^2 \qquad (n,\ m\in[N, M]). \tag{1.3'}$$

Then there exists an $L'>0$ *such that, for all* $y\in(0, L\Lambda^{1/2})$,

$$P\Big(\max_{M<i\leqq M}|S_{N,i}|\geqq Cy\Big)\leqq L'\Big(\exp\Big(-\frac{C^2y^2}{L'U_{N,M}}\Big)+\frac{U_{N,M}}{\Lambda^{1/2}C^2y^4}\Big).$$

Proof. By (1.1), we can take a sequence $\{n_k\}_{0\leqq k\leqq K}$ such that

$$N=n_0<n_1<\cdots<n_K=M, \qquad C^2\Lambda^{-1/2}\leqq U_{n_{k-1},n_k}\leqq 2C^2\Lambda^{-1/2} \qquad (1\leqq k\leqq K).$$

Put $M'_{n,m}=\max_{n<i<m}(|S_{n,i}|\wedge|S_{i,m}|)$. Usual argument yields (Cf. Billingsley [4] Ch. 12),

$$\begin{aligned}\max_{n_{k-1}<i\leqq n_k}|S_{N,i}|&\leqq|S_{N,n_{k-1}}|+\max_{n_{k-1}<i\leqq n_k}|S_{n_{k-1},i}|\\ &\leqq|S_{N,n_{k-1}}|+|S_{n_{k-1},n_k}|+M'_{n_{k-1},n_k}\\ &\leqq 2|S_{N,n_{k-1}}|+|S_{N,n_k}|+M'_{n_{k-1},n_k},\end{aligned}$$

which consequently proves

$$\max_{N<i\leq M}|S_{N,i}|\leq 3\max_{1\leq k\leq K}|S_{N,n_k}|+\max_{1\leq k\leq K}M'_{n_{k-1},n_k}.$$

When $|y|\leq L\Lambda C^{-2}U_{n,m}$, putting $\lambda=yC^2/(2LU_{n,m})$, (1.2) yields

$$P(|S_{n,m}|\geq Cy)\leq 2e^{-\lambda y}E\exp(\lambda C^{-1}S_{n,m})\leq 2L\exp\Big(-\frac{C^2y^2}{4LU_{n,m}}\Big).$$

Thus if we take $|y|\leq L\Lambda^{1/2}$, using Theorem 1 of Móricz [17], we have

$$P\Big(\max_{1\leq k\leq K}|S_{N,n_k}|\geq\frac{Cy}{4}\Big)\leq L'\exp\Big(-\frac{C^2y^2}{L'U_{N,M}}\Big).$$

If we assume (1.3) or (1.3′), using Theorem 12.1 or 12.2 of Billingsley [4], we have

$$P\Big(M'_{n_{k-1},n_k}\geq\frac{Cy}{4}\Big)\leq\frac{L'U^2_{n_{k-1},n_k}}{C^4y^4}.$$

Noting $\sum_{k=1}^K U^2_{n_{k-1},n_k}\leq 2C^2\Lambda^{-1/2}U_{N,M}$, we have the conclusion. ∎

By the assumption (0.11), (0.12) and (0.13), we can take $B>1$ such that

(0.11′) $$\|B_r\|_\delta^{1/r}\leq Br^{1-1/\delta}\qquad(r\in\boldsymbol{N}),$$

(0.12′) $$\|\bar{B}_r\|_2^{1/r},\ \|\bar{\bar{B}}_r\|_2^{1/r}\leq Br^{1/2}\qquad(r\in\boldsymbol{N}),$$

(0.13′) $$\|B_3^*\|_2,\ \|B_3^{**}\|_2\leq B.$$

Lemma 2. *Suppose* (0.11′) *and*

$$\sum_{i=1}^\infty\lambda_i^2=1,\quad|\lambda_i|\leq B\lambda,\quad\lambda\leq 1\quad\textit{and}\quad|\lambda_i\xi_i|\leq B\lambda.$$

Then there exists $L>0$ depending only on B such that

(1.4) $$\Big|\sum_{i=1}^\infty\lambda_iE\xi_i\Big|\leq L\lambda^\Delta$$

(1.5) $$\Big|E\sum_{i=1}^\infty(1+t\lambda_i\xi_i)\Big|\leq 2\qquad(|t|\leq(L\lambda^{(1/2)\wedge\Delta})^{-1}).$$

Moreover, if we suppose (0.12′), *there exists $L>0$ depending only on B such that*

(1.6) $$E\Big(\prod_{i=1}^\infty\lambda_i^2(\xi_i^2-1)\Big)^2\leq L\lambda^2,$$

(1.7) $$E\exp\Big(t\sum_{i\in I}\lambda_i\xi_i\Big)\leq 4\exp\Big(2t^2\sum_{i\in I}\lambda_i^2\Big)\quad(I\subset\boldsymbol{N},\ |t|\leq(L\lambda^{(1/2)\wedge\Delta})^{-1}),$$

(1.8) $$\Big|E\exp\Big(it\sum_{i=1}^\infty\lambda_i\xi_i\Big)-\exp\Big(-\frac{t^2}{2}+it\sum_{i=1}^\infty\lambda_iE\xi_i\Big)\Big|$$
$$\leq L(\lambda|t|^3+\lambda^{1\wedge(2\Delta)}|t|^2)\quad(|t|\leq(L\lambda^{(1/3)\wedge\Delta})^{-1}),$$

(1.9) $$E\Big(\sum_{i=1}^\infty\lambda_i\xi_i\Big)^2\leq L,$$

$$(1.10)\qquad \left|E\exp\left(it\sum_{i=1}^{\infty}\lambda_i\xi_i\right)-\exp\left(-\frac{t^2}{2}\right)\right|$$
$$\leqq L(\lambda|t|^3+\lambda^{1\wedge(2\Delta)}|t|^2+\lambda^{\Delta}|t|)\quad (|t|\leqq(L\lambda^{(1/3)\wedge\Delta})^{-1}).$$

Proof. We have already proved these inequalities in [8]. (1.4), (1.5), (1.6), (1.7) and (1.8) are (1.1), (1.5), (1.2), (1.7) and Lemma 6 in [8]. (1.9) is proved in the proof of Lemma 1 (1) of [8]. (1.10) is clear from (1.4) and (1.8). ∎

Let us put

$$S^*_{n,m}=\sum_{\substack{n<i\leq m\\ i:\text{ odd}}}c_i\xi_i,\quad S^{**}_{n,m}=\sum_{\substack{n<i\leq m\\ i:\text{ even}}}c_i\xi_i,\quad C^{*2}_{n,m}=\sum_{\substack{n<i\leq m\\ i:\text{ odd}}}c_i^2\ \text{ and }\ C^{**2}_{n,m}=\sum_{\substack{n<i\leq m\\ i:\text{ even}}}c_i^2.$$

Lemma 3. (1) *Under the conditions* (0.11′), (0.12′) *and* (0.13′), *it holds that*

$$E(S^{*2}_{n,l}S^{*2}_{l,m})\leqq LC^{*2}_{n,l}C^{*2}_{l,m}\qquad (n\leqq l\leqq m)$$

for some $L>0$ depending only on B. It remains true if we replace $S^*_{n,m}$ by $S^{**}_{n,m}$ and $C^*_{n,m}$ by $C^{**}_{n,m}$.

(2) *Under the conditions* (0.11′), (0.12′) *and* (0.14), *it holds that*

$$ES^{*4}_{n,m}\leqq LC^{*4}_{n,m}\qquad (n\leqq m)$$

for some $L>0$ depending only on B. It alsoremains true if we replace $S^*_{n,m}$ by $S^{**}_{n,m}$ and $C^*_{n,m}$ by $C^{**}_{n,m}$.

Proof. (1) Expanding the expectation, we have

$$E(S^{*2}_{n,l}S^{*2}_{l,m})=4\sum_{\substack{n<s<t\leq l\\ l<u<v\leq m\\ s,t,u,v:\text{ odd}}}c_sc_tc_uc_vE\xi_s\xi_t\xi_u\xi_v+2\sum_{\substack{n<s<t\leq l\\ l<u\leq m\\ s,t,u:\text{ odd}}}c_sc_tc_u^2E\xi_s\xi_t\xi_u^2$$
$$+2\sum_{\substack{n<s\leq l\\ l<u<v\leq m\\ s,u,v:\text{ odd}}}c_s^2c_uc_vE\xi_s^2\xi_u\xi_v+\sum_{\substack{n<s\leq l\\ l<u\leq m\\ s,u:\text{ odd}}}c_s^2c_u^2E\xi_s^2\xi_u^2$$
$$=4\Sigma_1+2\Sigma_2+2\Sigma_3+\Sigma_4\qquad (\text{say.}).$$

Estimation of Σ_1 follows from the bound of $\|B_4\|_2$:

$$|\Sigma_1|\leqq\Big(\sum_{\substack{n<s<t\leq l\\ l<u<v\leq m\\ s,t,u,v:\text{odd}}}c_s^2c_t^2c_u^2c_v^2\Big)^{1/2}\|B_4\|_2\leqq 16B^4C^{*2}_{n,l}C^{*2}_{l,m}.$$

Noting $E\xi_s\xi_t\xi_u^2=E\xi_s\xi_t+E\xi_s\xi_t(\xi_u^2-1)$, we shall divide Σ_2 into two summations:

$$|\Sigma_2|\leqq\Big|\sum_{\substack{n<s<t\leq l\\ l<u\leq m\\ s,t,u:\text{ odd}}}c_sc_tc_u^2E\xi_s\xi_t\Big|+\Big|\sum_{\substack{n<s<t\leq l\\ l<u\leq m\\ s,t,u:\text{ odd}}}c_sc_tc_u^2E\xi_s\xi_t(\xi_u^2-1)\Big|$$
$$\leqq\sum_{\substack{l<u\leq m\\ u:\text{ odd}}}c_u^2\Big(\sum_{\substack{n<s<t\leq l\\ s,t:\text{ odd}}}c_s^2c_t^2\Big)^{1/2}\|B_2\|_2+\Big(\sum_{\substack{l<u\leq m\\ u:\text{odd}}}c_u^4\Big)^{1/2}\Big(\sum_{\substack{n<s<t\leq l\\ s,t:\text{ odd}}}c_s^2c_t^2\Big)^{1/2}\|B_3^*\|_2$$
$$\leqq(2B^2+B)C^{*2}_{n,l}C^{*2}_{l,m}.$$

$\sum_3$ is estimated in the same way and by the same bound as $\sum_2$. Noting $E\xi_s^2\xi_u^2 = E(\xi_s^2-1)(\xi_u^2-1)+1+E(\xi_s^2-1)+E(\xi_u^2-1)$, we shall divide $\sum_4$ into four summations:

$$|\Sigma_4| \leqq \Big| \sum_{\substack{n<s\leq l\\ l<u\leq m\\ s,u:\text{odd}}} c_s^2 c_u^2 E(\xi_s^2-1)(\xi_u^2-1) \Big| + \sum_{\substack{n<s\leq l\\ l<u\leq m\\ s,u:\text{odd}}} c_s^2 c_u^2$$

$$+ \Big| \sum_{\substack{n<s\leq l\\ l<u\leq m\\ s,u:\text{odd}}} c_s^2 c_u^2 E(\xi_s^2-1) \Big| + \Big| \sum_{\substack{n<s\leq l\\ l<u\leq m\\ s,u:\text{odd}}} c_s^2 c_u^2 E(\xi_u^2-1) \Big|$$

$$\leqq \Big(\sum_{\substack{n<s\leq l\\ l<u\leq m\\ s,u:\text{odd}}} c_s^4 c_u^4 \Big)^{1/2} \|\bar{B}_2\|_2 + C_{n,l}^{*2} C_{l,m}^{*2}$$

$$+ \Big(\sum_{\substack{l<u\leq m\\ u:\text{odd}}} c_u^2 \Big) \Big(\sum_{\substack{n<s\leq l\\ s:\text{odd}}} c_s^4 \Big)^{1/2} \|\bar{B}_1\|_2 + \Big(\sum_{\substack{n<s\leq l\\ s:\text{odd}}} c_s^2 \Big) \Big(\sum_{\substack{l<u\leq m\\ u:\text{odd}}} c_u^4 \Big)^{1/2} \|\bar{B}_1\|_2$$

$$\leqq (2B^2+2B+1) C_{n,l}^{*2} C_{l,m}^{*2}.$$

Thus we have the conclusion.

(2) Theorem 1 of Móricz [16]. ■

Now we are in a position to prove the relative compactness of $\{X_n/\sqrt{2\log\log C_n}\}$. Retaking B large enough, we have by the assumption of the theorem,

$$\max_{i\leq n} |c_i|,\ \max_{i\leq n} |c_i| \|\xi_i\|_\infty \leqq B C_n (\log C_n)^{-(8\vee(4/\Delta))}.$$

From now on, we denote by L a constant depending only on B which may be different line by line. If we put

$$\lambda_n = (\log C_n)^{-(8\vee(4/\Delta))}, \qquad \lambda_{n,i} = \begin{cases} \dfrac{c_i}{C_n} & \text{if } i\leqq n \text{ and,} \\ 0 & \text{if } i>n, \end{cases}$$

$\{\xi_i\}$, $\{\lambda_{n,i}\}_{i\in N}$ and λ_n satisfy the assumptions of Lemma 2 for each $n\in \boldsymbol{N}$. Hence by (1.7), we get

$$E\exp\Big(\frac{s}{C_n} S_{l,m}^*\Big) \leqq 4\exp\Big(\frac{2s^2 C_{l,m}^{*2}}{C_n^2}\Big),\ (l<m\leqq n,\ |s|\leqq L^{-1}(\log C_n)^4).$$

By this and Lemma 2 and 3, we see that we can use Lemma 1 for $S_{l,m}^*$ putting $C = C_n$, $u_i = c_i^2$ and $\Lambda = L^{-1}(\log C_n)^4$ and conclude that for $y\in(0, L^{-1}(\log C_n)^2)$ and $N<M\leqq n$, it holds that

$$P\Big(\max_{N<j\leq M} |S_{N,j}^*| \geqq C_n y\Big) \leqq L\Big(\exp\Big(-\frac{C_n^2 y^2}{L C_{N,M}^{*2}}\Big) + \frac{C_{N,M}^{*2}}{C_n^2 (\log C_n)^2 y^4}\Big). \tag{1.11}$$

Now take $\theta>1$ arbitrary and let $\{q(n)\}$ satisfy $C_{q(r)}^2 \leqq \theta^r < C_{q(r)+1}^2$. For given $\varepsilon>0$ and $\delta>0$, we must estimate the probability of the event

$$\Lambda_r(\varepsilon, \delta) = \Big\{ \sup_{|t-s|\leq\delta} \frac{|X_{q(r)}(t) - X_{q(r)}(s)|}{\sqrt{2\log\log C_{q(r)}}} > \varepsilon \Big\}.$$

For large enough r, we can take $\{m_j\}_{j=0,\dots,p}$ such that $0 = m_0 < m_1 < \cdots < m_p = q(r)$ and

$\delta \leqq (C_{m_j}^2 - C_{m_{j-1}}^2)/C_{q(r)}^2 \leqq 2\delta (1 \leqq j \leqq p)$. It is easily seen that $p \leqq 1/\delta$ holds.

In general, we have

$$P\Big\{\sup_{|t-s|\leq\delta}|Y(t)-Y(s)|\geqq\varepsilon\Big\}\leqq\sum_{i=1}^{p}P\Big\{\sup_{t_{i-1}\leq s\leq t_i}|Y(s)-Y(t_{i-1})|\geqq\frac{\varepsilon}{3}\Big\}$$

if $Y \in \boldsymbol{C}[0, 1]$, $0=t_0<t_1<\cdots<t_p=1$ and $t_i-t_{i-1}\geqq\delta$ $(1\leqq i\leqq p)$. Applying this and noting $|S_m-S_n|\leqq|S_{n,m}^*|+|S_{n,m}^{**}|$, we easily get

$$P(A_r(\varepsilon,\delta))\leqq\sum_{j=1}^{p}P\Big\{\max_{m_{j-1}<i\leq m_j}|S_{m_{j-1},i}^*|\geqq\frac{\varepsilon}{6}\sqrt{2C_{q(r)}^2\log\log C_{q(r)}}\Big\}$$

$$+\sum_{j=1}^{p}P\Big\{\max_{m_{j-1}<i\leq m_j}|S_{m_{j-1},i}^{**}|\geqq\frac{\varepsilon}{6}\sqrt{2C_{q(r)}^2\log\log C_{q(r)}}\Big\}.$$

This and (1.11) yield

$$P(A_r(\varepsilon,\delta))\leqq L\sum_{j=1}^{p}\Big(\exp\Big(-\frac{\varepsilon^2C_{q(r)}^2\log\log C_{q(r)}}{18LC_{m_{j-1},m_j}^{*2}}\Big)+\frac{6^4C_{m_{j-1},m_j}^{*2}}{\varepsilon^4C_{q(r)}^2(\log C_{q(r)}\log\log C_{q(r)})^2}\Big)$$

$$+L\sum_{j=1}^{p}\Big(\exp\Big(-\frac{\varepsilon^2C_{q(r)}^2\log\log C_{q(r)}}{18LC_{m_{j-1},m_j}^{**2}}\Big)+\frac{6^4C_{m_{j-1},m_j}^{**2}}{\varepsilon^4C_{q(r)}^2(\log C_{q(r)}\log\log C_{q(r)})^2}\Big)$$

$$\leqq 2L_\theta\Big(\frac{1}{\delta}(r-1)^{-\varepsilon^2/36L\delta}+\frac{6^4}{\varepsilon^4(r-1)^2}\Big),$$

where L_θ depends only on B and θ. Since the last term is summable in r if we take δ small enough, by Borel-Cantelli lemma and Ascoli-Arzera theorem, $\{X_{q(r)}/\sqrt{2\log\log C_{q(r)}}\}$ is relatively compact almost surely. On the other hand, it is easily seen that

$$\max_{q(r-1)<n\leq q(r)}\sup_{|t-s|<\delta}|X_n(t)-X_n(s)|\leqq\theta\sup_{|t-s|<\delta}|X_{q(r)}(t)-X_{q(r)}(s)|$$

holds, and these two facts lead us to the conclusion.

Next we shall argue on the set of clusters. After the method of Kôno [13], we use the following theorem due to Kuelbs [14].

Theorem F. *Assume that* $\{X_n/\sqrt{2\log\log C_n}\}$ *is relatively compact in* $\boldsymbol{C}[0, 1]$ *almost surely. If for any signed measure* ν *on* $[0, 1]$ *with bounded variation,*

$$\limsup_{n\to\infty}\frac{\int_0^1X_n(t)\nu(dt)}{\sqrt{2\log\log C_n}}=K_{\nu,1}\quad a.s.,$$

then the set of all clusters of $\{X_n/\sqrt{2\log\log C_n}\}$ *coincides with* K, *almost surely, where*

$$K_{\nu,\theta}^2=E\Big(\int_0^1W(t\wedge\theta^{-1})\nu(dt)\Big)^2=\int_0^{1/\theta}(\nu[x,1])^2dx\,.$$

($W(t)$ denotes the standard Brownian motion.)

Let ν be a signed measure on $[0, 1]$ with bounded variation and suppose that $\nu\neq0$ since there is nothing to prove in case $\nu=0$. Put $N=|\nu|([0, 1])$,

$$\phi_{n,i}(t)=\begin{cases} 0 & \text{for } t\in\left[0, \frac{C_{i-1}^2}{C_n^2}\right], \\ \frac{C_n^2}{c_i^2}\left(t-\frac{C_{i-1}^2}{C_n^2}\right) & \text{for } t\in\left[\frac{C_{i-1}^2}{C_n^2}, \frac{C_i^2}{C_n^2}\right], \\ 1 & \text{otherwise}, \end{cases}$$

$$d_{n,i}=\int_0^1 \phi_{n,i}(t)\nu(dt) \quad \text{and} \quad C_{\nu,n}^2=\sum_{i=1}^n (c_i d_{n,i})^2 .$$

We have $X_n(t)=C_n^{-1}\sum_{i=1}^n c_i\phi_{n,i}(t)\xi_i$ and $\int_0^1 X_n(t)\nu(dt)=C_n^{-1}\sum_{i=1}^n c_i d_{n,i}\xi_i$ using these notation. We have proved the following formulas in [6]:

$$\lim_{n\to\infty}\frac{C_{\nu,n}^2}{C_n^2}=K_{\nu,1}^2 \quad \text{and} \quad \lim_{r\to\infty}\frac{1}{C_{q(r+1)}^2}\sum_{i=1}^{q(r)} c_i^2 d_{q(r+1),i}^2=K_{\nu,\theta}^2 . \tag{1.12}$$

We shall here prove

$$\limsup_{r\to\infty}\frac{1}{C_{q(r)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r),i}\xi_i)^2\leqq K_{\nu,1}^2 \quad \text{a.s.} \tag{1.13}$$

If r is large enough, we have by (1.12), $C_{\nu,q(r)}^2\geqq K_{\nu,1}^2 C_{q(r)}^2/4$ since $K_{\nu,1}^2>0$. Therefore, (1.13) follows from

$$\lim_{r\to\infty}\frac{1}{C_{\nu,q(r)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r),i})^2(\xi_i^2-1)=0 \quad \text{a.s.}$$

Putting

$$\lambda_r=(\log A_{q(r)})^{-8} \quad \text{and} \quad \lambda_{r,i}=\begin{cases} C_{\nu,q(r)}^{-1}c_i d_{q(r),i} & i\leqq q(r), \\ 0 & \text{otherwise}, \end{cases} \tag{1.14}$$

we have that

$$|\lambda_{r,i}|\leqq 2N|c_i|K_{\nu,1}^{-1}C_{q(r)}^{-1}\leqq 2NBK_{\nu,1}^{-1}\lambda_r \quad \text{and} \quad |\lambda_{r,i}|\,\|\xi_i\|_\infty\leqq 2NBK_{\nu,1}^{-1}\lambda_r .$$

Thus the assumption of Lemma 2 is fulfilled if we retake B large enough. Hence we can use (1.6) and get

$$E\Big(\frac{1}{C_{\nu,q(r)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r),i})(\xi_i^2-1)\Big)^2\leqq L_\theta(r-1)^{-16}.$$

Thanks to Beppo-Levi's theorem, (1.13) follows from this. In the same way, we can also prove

$$\limsup_{r\to\infty}\frac{1}{C_{q(r+1)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r+1),i}\xi_i)^2\leqq K_{\nu,\theta}^2 \quad \text{a.s.} \tag{1.15}$$

Now we are in a position to prove the upper bound estimate. Take $\varepsilon>0$ arbitrary and put $\alpha_r=K_{\nu,1}^{-1}\sqrt{2\log\log C_{q(r)}}$. Since $\max_{i\leqq q(r)}|\alpha_r c_i d_{q(r),i}\xi_i/C_{q(r)}|\leqq 1/2$ for large enough r, we can use $\exp(x-x^2/2-|x|^2)\leqq 1+x$ ($|x|\leqq 1/2$) and get

$$E\exp\Big(\frac{\alpha_r}{C_{q(r)}}\sum_{i=1}^{q(r)} c_i d_{q(r),i}\xi_i-\frac{\alpha_r^2}{2C_{q(r)}^2}\sum_{i=1}^{q(r)} c_i^2 d_{q(r),i}^2\xi_i^2-\frac{\alpha_r^3}{C_{q(r)}^3}\sum_{i=1}^{q(r)}|c_i d_{q(r),i}\xi_i|^3\Big)$$
$$\leqq E\prod_{i=1}^{q(r)}(1+\alpha_r\lambda_{r,i}\xi_i)\leqq 2,$$

where $\lambda_{r,i}$ is defined by (1.14) and the last inequality is by (1.5). On the other hand, by (1.12), we get

$$\frac{\alpha_r^3}{C_{q(r)}^3}\sum_{i=1}^{q(r)}|c_i d_{q(r),i}\xi_i|^3 \leqq \frac{BN\alpha_r^3\lambda_r}{K_{\nu,1}C_{q(r)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r),i}\xi_i)^2 \leqq \frac{\varepsilon\alpha_r^2}{2C_{q(r)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r),i}\xi_i)^2$$

for large enough r. Thus we have

$$E\exp\left[\alpha_r^2\left(\frac{1}{\alpha_r C_{q(r)}}\sum_{i=1}^{q(r)}c_i d_{q(r),i}\xi_i - \frac{1+\varepsilon}{2C_{q(r)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r),i}\xi_i)^2 - \frac{(1+\varepsilon)K_{\nu,1}^2}{2}\right)\right]$$

$$\leqq 2\exp\left(-\frac{1+\varepsilon}{2}K_{\nu,1}^2\alpha_r^2\right)$$

$$\leqq 2((r-1)\log\theta)^{-1-\varepsilon}.$$

It implies by Beppo-Levi's theorem that

$$\lim_{r\to\infty}\alpha_r^2\left(\frac{1}{\alpha_r C_{q(r)}}\sum_{i=1}^{q(r)}c_i d_{q(r),i}\xi_i - \frac{1+\varepsilon}{2C_{q(r)}^2}\sum_{i=1}^{q(r)}(c_i d_{q(r),i}\xi_i)^2 - \frac{(1+\varepsilon)K_{\nu,1}^2}{2}\right) = -\infty \quad \text{a.s.}$$

This and (1.13) prove that

$$\limsup_{r\to\infty}\frac{\int_0^1 X_{q(r)}(t)\nu(dt)}{\sqrt{2\log\log C_{q(r)}}} \leqq K_{\nu,1} \quad \text{a.s.}$$

Noting the relative compactness of $\{X_n/\sqrt{2\log\log C_n}\}$, we have (Cf. Fukuyama [5]) that

$$\limsup_{n\to\infty}\frac{\int_0^1 X_n(t)\nu(dt)}{\sqrt{2\log\log C_n}} \leqq K_{\nu,1} \quad \text{a.s.}$$

We mention here that the following (1.16) can be also proved in the same way using (1.15) instead of (1.13).

$$\text{(1.16)} \qquad \limsup_{r\to\infty}\frac{1}{\sqrt{2C_{q(r+1)}^2\log\log C_{q(r+1)}}}\sum_{i=1}^{q(r)}c_i d_{q(r+1),i}\xi_i \leqq K_{\nu,\theta} \quad \text{a.s.}$$

Let us proceed to the proof of the lower bound estimate. Here after we take $\theta>1$ so large that $K_{\nu,1}^2-K_{\nu,\theta}^2=\int_{1/\theta}^1(\nu[x,1])^2dx>0$ holds. Put $\lambda_{(n,m)}=(\log C_n)^{-(8\vee(4/\Delta))}$,

$$D_n^2=\sum_{i=q(n)+1}^{q(n+1)}c_i^2 d_{q(n+1),i}^2, \qquad \eta_n=\frac{1}{D_n}\sum_{i=q(n)+1}^{q(n+1)}c_i d_{q(n+1),i}\xi_i,$$

$$\lambda_{(n,m),i}=\begin{cases}\dfrac{s}{\sqrt{s^2+t^2}D_n}c_i d_{q(n+1),i} & q(n)<i\leqq q(n+1),\\[2mm] \dfrac{t}{\sqrt{s^2+t^2}D_{n+m}}c_i d_{q(n+m+1),i} & q(n+m)<i\leqq q(n+m+1),\\[2mm] 0 & \text{otherwise},\end{cases}$$

for given $s, t\in\boldsymbol{R}$. It is clear that $s\eta_n+t\eta_{n+m}=\sqrt{s^2+t^2}\sum_{i=1}^{\infty}\lambda_{(n,m),i}\xi_i$ holds. Since we have $\sum_{i=1}^{\infty}\lambda_{(n,m),i}^2=1$ and $|\lambda_{(n,m),i}|$, $|\lambda_{(n,m),i}|\,\|\xi_i\|_\infty\leqq 2NB(K_{\nu,1}^2-K_{\nu,\theta}^2)^{-1/2}\lambda_{(n,m)}$ by (1.12), we can use Lemma 2 for large enough n, taking $B>1$ sufficiently large. By (1.10), we have

(1.17) $$|f_{n,m}(s,t)-g(s,t)|$$
$$\leqq L(n^{-8}(|t|^3+|s|^3)+n^{-8}(t^2+s^2)+n^{-4}(|t|+|s|)) \qquad (|t|,\ |s|\leqq L^{-1}n^2)$$

where $f_{n,m}(s,t)=E\exp(is\eta_n+it\eta_{n+m})$, $g(s,t)=\exp(-(s^2+t^2)/2)$ and L is a constant not depending on n and m. We shall here prove that

(1.18) $$\sup_{x,y\in R}|F_{n,m}(x,y)-G(x,y)|\leqq Ln^{-2}\log n$$

where $F_{n,m}$ is the two dimensional distribution function of (η_n, η_{n+m}), G is that of two dimensional standard normal distribution and L is a constant not depending on n and m. After the method of Révész [21], using Corollary of Theorem 1 of Sadikova [22], we have

$$\sup_{x,y\in R}|F_{n,m}(x,y)-G(x,y)|\leqq C\Big(\int_{-T}^{T}\int_{-T}^{T}\Big|\frac{\tilde f_{n,m}(s,t)-\tilde g(s,t)}{st}\Big|dsdt$$
$$+\int_{-T}^{T}\Big|\frac{f_{n,m}(s,0)-g(s,0)}{s}\Big|ds+\int_{-T}^{T}\Big|\frac{f_{n,m}(0,t)-g(0,t)}{t}\Big|dt+\frac{1}{T}\Big)$$

for all $T>0$, where C is the absolute constant, $\tilde f_{n,m}(s,t)=f_{n,m}(s,t)-f_{n,m}(s,0)\times f_{n,m}(0,t)$ and $\tilde g(s,t)=g(s,t)-g(s,0)g(0,t)$. Put $T=L^{-1}n^2$ and we divide $[-T,T]\times[-T,T]$ into two parts U and V by $U=\{(s,t);\ L^{-1}n^{-4}\leqq|s|,\ |t|\leqq L^{-1}n^2\}$ and $V=[-T,T]\times[-T,T]\cap U^c$. Since $\tilde g(s,t)=0$ and

$$|\tilde f_{n,m}(s,t)|=|E_\omega(E_{\omega'}(e^{is\eta_n(\omega)}-e^{is\eta_n(\omega')})E_{\omega'}(e^{it\eta_{n+m}(\omega)}-e^{it\eta_{n+m}(\omega')}))|$$
$$\leqq|st|E_\omega(E_{\omega'}|\eta_n(\omega)-\eta_n(\omega')|E_{\omega'}|\eta_{n+m}(\omega)-\eta_{n+m}(\omega')|)$$
$$\leqq 4|st|E^{1/2}\eta_n^2E^{1/2}\eta_{n+m}^2$$
$$\leqq L|st|$$

(the last inequality is by (1.9)), we get $\iint_V|(\tilde f_{n,m}(s,t)-\tilde g(s,t))/st|\,ds\,dt\leqq Ln^{-2}$. Noting $|\tilde f_{n,m}(s,t)-\tilde g(s,t)|\leqq|f_{n,m}(s,t)-g(s,t)|+|f_{n,m}(s,0)-g(s,0)|+|f_{n,m}(0,t)-g(0,t)|$ and (1.17), we have

$$\iint_U\Big|\frac{\tilde f_{n,m}(s,t)-\tilde g(s,t)}{st}\Big|dsdt\leqq L\int_{L^{-1}n^{-4}}^{L^{-1}n^2}\frac{dt}{t}\int_0^{L^{-1}n^2}(n^{-8}s^2+n^{-8}s+n^{-4})ds$$
$$\leqq Ln^{-2}\log n\,.$$

Since two other integrals can be estimated by Ln^{-2} in the same same way, we have (1.18). Now take $\varepsilon>0$ arbitrary, let $\{\zeta_i\}$ be a sequence of i.i.d. with the standard normal distribution and put

$$U_n=\{\eta_n\geqq\sqrt{(2-\varepsilon)\log\log C_{q(n)}}\}\quad\text{and}\quad V_n=\{\zeta_n\geqq\sqrt{(2-\varepsilon)\log\log C_{q(n)}}\}\,.$$

We shall now show that U_n occurs infinitely often almost surely. Although $\{U_n\}$ is not a sequence of independent events, we can use the following generalization of Borel-Cantelli lemma. (Cf. Rényi [20])

Lemma G. *We have* $P(\limsup_{n\to\infty} A_n)=1$ *if*

$$\sum_{n=1}^{\infty} P(A_n)=\infty \quad \textit{and} \quad \liminf_{n\to\infty} \sum_{j=1}^{n}\sum_{k=1}^{n} P(A_j\cap A_k)\Big/\Big(\sum_{j=1}^{n} P(A_j)\Big)^2=1.$$

Since we have by (1.18) $P(V_n\cap V_m)=P(V_n)P(V_m)$, $|P(U_n\cap U_m)-P(V_n\cap V_m)|\leqq Ln^{-2}\log n$ $(n\leqq m)$ and $|P(U_n)-P(V_n)|\leqq Ln^{-2}\log n$ $(n\in\boldsymbol{N})$ by (1.18), it is easy to prove the following:

$$\sum_{n=1}^{M} P(V_n)\geqq \frac{LM^{\varepsilon/2}}{(\log M)^{1/2}}\longrightarrow\infty, \qquad \sum_{n=1}^{M} P(U_n)\Big/\sum_{n=1}^{M} P(V_n)\longrightarrow 1 \quad \text{as} \quad m\to\infty,$$

$$\sum_{n=1}^{M}\sum_{m=1}^{M}(P(U_n\cap U_m)-P(U_n)P(U_m))\leqq L(\log M)^2.$$

From these, we can prove that $\eta_n\geqq\sqrt{(2-\varepsilon)\log\log C_{q(n)}}$ infinitely often almost surely. Thus, we have by (1.12),

$$\limsup_{r\to\infty}\frac{1}{\sqrt{2C_{q(r+1)}^2\log\log C_{q(r+1)}}}\sum_{i=q(r)+1}^{q(r+1)} c_i d_{q(r+1),i}\xi_i\geqq(K_{\nu,1}^2-K_{\nu,0}^2)^{1/2} \quad \text{a.s.}$$

Since this and (1.16) prove

$$\limsup_{n\to\infty}\frac{\int_0^1 X_n(t)\nu(dt)}{\sqrt{2\log\log C_n}}=K_{\nu,1} \quad \text{a.s.}$$

by usual argument (Cf. Fukuyama [5] or [7]), we have come to the end of the proof.

2. FLIL for lacunary trigonometric series

In this section we prove Theorem 1. Theorems 2 and 3 are proved in a similar way. For necessary techniques, see [8]. First, we cite some results of [8]. In [8] we derive the mean central ilmit theorem under the same condition except for the condition on $\{a_j\}$. In [8] we assumed

$$\mu_n\longrightarrow 0,\quad A_n\mu_n\uparrow\infty \quad \text{as} \quad n\longrightarrow\infty \quad \text{and}$$

$$\mu_n\leqq 1,\quad |a_n|\leqq C\mu_n A_n n^{-\alpha}(1+\alpha\log n)^{-1} \quad (n\in\boldsymbol{N}).$$

and we are now treating the special case $\mu_n=(\log A_n)^{-8}$. To state the necessary results in [8], let us introduce sequences $\{p(k)\}$, $\{l(k)\}$, $\{m(i)\}$ and subsets J_i of $\boldsymbol{N}$ by

$$p(k)=\begin{cases} 0 & k=0, \\ \max\{j\,;\ \beta_j\leqq 2^k\} & k\in\boldsymbol{N}, \end{cases}$$

$$l(k)=[(8/\alpha\rho)\log_2 p(k)+4+\log_2(1+1/c)],$$

$$m(i)=\begin{cases} 0 & i=0, \\ \min\{k\,;\ p(k)>0\} & i=1, \\ \max\{k\,;\ k-l(k)\leqq m(i-1)\} & i\geqq 2, \end{cases}$$

$$J_i=\{j\in\boldsymbol{N};\ p(m(i-1))<j\leqq p(m(i))\}.$$

Put $\zeta_j(\omega)=\sqrt{2}\cos(\beta_j t\omega+\gamma_j)$, $c_i=(A^2_{p(m(i))}-A^2_{p(m(i-1))})^{1/2}$ and $\xi_i=c_i^{-1}\sum_{j\in J_i}a_j\zeta_j$. We shall here show that $\{\xi_i\}$ and $\{c_i\}$ fulfill the assumption of Theorem 4 and the corresponding $\boldsymbol{C}[0,1]$-valued random variable Y_n (here we use Y instead of X to avoid confusions.) obey the FLIL. From now on we denote constant not depending on n and r by D and it may be different line by line. Rewriting (2.8) of [8] using the above notation, we have $\max_{i\leq n}|c_i|$, $\max_{i\leq n}|c_i|\,\|\xi_i\|\leqq D\mu_n A_{p(m(n))}$. In the proof of the last inequality we used

$$\max_{i\leq n}\sum_{j\in J_i}|a_j|\leqq D\mu_n A_{p(m(n))}. \tag{2.1}$$

In [8], we have proved that there exists a constant D not depending on r such that

$$\|B_r\|_1^{1/r},\quad \|\bar{B}_r\|_1^{1/r},\quad \|\bar{\bar{B}}_r\|_1^{1/r}\leqq D \qquad (r\in\boldsymbol{N}).$$

As to (0.13), we can obtain the following stronger results.

Lemma 4. *There exists a constant D not depending on r such that*

$$\|B_r^*\|_1^{1/r},\quad \|B_r^{**}\|_1^{1/r}\leqq D \qquad (r\in\boldsymbol{N}).$$

Proof. In a similar way as the proof of the estimate of $|\bar{b}_{i_1,\cdots,i_r}|$ in [8], we have

$$|b^*_{r;i_1,\cdots,i_r}|D^r\sum_{\varepsilon=1,-1}\ \sum_{\substack{j_q\in J_{2i_q-1}\\(q=1,\cdots,r-1)\\ j_r,j'_r\in J_{2i_r-1}\\ j_r\neq j'_r}}\ \sum_{\substack{\varepsilon'_q=1,-1\\(q=1,\cdots,r-1)}}|\hat{P}((\beta_{j_r}+\varepsilon\beta_{j'_r})+\varepsilon'_{r-1}\beta_{j_{r-1}}+\cdots+\varepsilon'_1\beta_{j_1})|,$$

and the case $k=1,\cdots,r-1$ is similar. We have shown in [8] the following estimate:

$$\begin{aligned}\|\bar{B}_r\|_1&\leqq D^r\sum_{i_1<\cdots<i_r}\ \sum_{\substack{j_q,j'_q\in J_{2i_q-1}\\(q=1,\cdots,r)\\ j'_r>j_r}}|(\beta_{j_r}-\beta_{j'_r})-(\beta_{j_{r-1}}+\beta_{j'_{r-1}})-\cdots-(\beta_{j_1}+\beta_{j'_1})|^{-\rho/2}\\ &\leqq D^r.\end{aligned}$$

We can easily see that a similar estimate holds, i.e.

$$\begin{aligned}\|B_r^*\|_1&\leqq rD^r\sum_{i_1<\cdots<i_r}\ \sum_{\substack{j_q,j'_q\in J_{2i_q-1}\\(q=1,\cdots,r)\\ j'_r>j_r}}|(\beta_{j_r}-\beta_{j'_r})-(\beta_{j_{r-1}}+\beta_{j'_{r-1}})-\cdots-(\beta_{j_1}+\beta_{j'_1})|^{-\rho/2}\\ &\leqq rD^r\leqq D^r.\end{aligned}$$

Since this estimate is also valid for $\|B_r^{**}\|$, by the same method as that in [8], we have the conclusion. ■

Applying Theorem 7, we can prove that $\{Y_n/\sqrt{2\log\log C_n}\}$ is relatively compact and the set of all clusters coincides with K, almost surely. Because of $\|Y_n-X_{p(m(n))}\|_{C[0,1]}\leqq\mu_{p(m(n))}$ which follows from (2.1), we can easily see that $\{X_{p(m(n))}/\sqrt{2\log\log A_{p(m(n))}}\}$ is relatively compact and the set of all clusters of $\{Y_n/\sqrt{2\log\log C_n}\}$ and that of $\{X_{p(m(n))}/\sqrt{2\log\log A_{p(m(n))}}\}$ coincide. In case $p(m(n-1))<n\leqq p(m(n))$, we have

$$X_n(t)=\frac{A_{p(m(n))}}{A_n}X_{p(m(n))}\Big(t\frac{A_n^2}{A_{p(m(n))}^2}\Big).$$

Since we have

$$1\geqq\frac{A_n}{A_{p(m(n))}}\geqq\frac{A_{p(m(n-1))}}{A_{p(m(n))}}$$

and

$$0<1-\frac{A_{p(m(n-1))}^2}{A_{p(m(n))}^2}=\frac{1}{A_{p(m(n))}^2}\Big(\sum_{j\in J_n}|a_j|\Big)^2\leqq D^2\mu_n^2\longrightarrow 0 \qquad \text{as} \quad n\to\infty,$$

we can conclude that $\{X_n/\sqrt{2\log\log A_n}\}$ is relatively compact and the set of all clusters of $\{X_{p(m(n))}/\sqrt{2\log\log A_{p(m(n))}}\}$ and that of $\{X_n/\sqrt{2\log\log A_n}\}$ coincide.

3. FLIL for gap series

In this section we prove Theorem 4. Theorems 5 and 6 are proved in a similar way. For necessary techniques, see [8].

Let us put $f(t)=\sum_{j=1}^{\infty}d_j\sqrt{2}\cos(jt+\gamma_j')$ and $\sigma_n(t)=\sum_{j\leq n}d_j\sqrt{2}\cos(jt+\gamma_j')$. Since $f\in\mathrm{Lip}\,\alpha$, there exists a constant $L>1$ such that

$$\|f-\sigma_n\|_\infty\leqq Ln^{-\alpha}\log n\,, \tag{3.1}$$

$$\sum_{j=n}^{\infty}d_j^2\leqq L^2n^{-2\alpha}\,, \tag{3.2}$$

$$\|f\|_\infty,\ \|\sigma_n\|_\infty\leqq L \qquad (n\in\boldsymbol{N}). \tag{3.3}$$

(Cf. Zygmund [33].) First we shall prove a lemma.

Lemma 5. *Under the assumption of Theorem 4, it holds that*

$$\limsup_{n\to\infty}\frac{1}{\sqrt{2A_n^2\log\log A_n^2}}\sum_{k=1}^{n}a_kf(\beta_k\omega+\gamma_k)\leqq 2 \quad a.s.$$

Proof. Here after let L denote the absolute constant which may change line by line. Let us put $\tau_k(\omega)=\sigma_{[k^{1/\alpha}]}(\beta_k\omega+\gamma_k)$ and introduce a sequence $\{p_n\}$ of integers satisfying $1\leqq p_n\leqq n$, $p_n\to\infty$ and $A_{p_n}/A_n\to 0$ as $n\to\infty$. Since we have

$$\begin{aligned}\Big|\sum_{k=1}^{n}(f(\beta_k\omega+\gamma_k)-\tau_k(\omega))\Big|&\leqq L\sum_{k\leq p_n}|a_k|k^{-1}\log k+L\sum_{p_n<k\leq n}|a_k|k^{-1}\log k\\&\leqq LA_{p_n}\Big(\sum_{k=1}^{\infty}k^{-2}(\log k)^2\Big)^{1/2}+LA_n\Big(\sum_{k=p_n}^{\infty}k^{-2}(\log k)^2\Big)^{1/2}\\&=o(A_n) \qquad \text{as} \quad n\longrightarrow\infty\end{aligned}$$

by (3.1), assertion of the lemma is equivalent to

$$\limsup_{n\to\infty}\frac{1}{\sqrt{2A_n^2\log\log A_n^2}}\sum_{k=1}^{n}a_k\tau_k\leqq 2 \quad \text{a.s.}$$

Let i_0 be the minimum integer i such that $d_i\neq 0$. Since $\tau_k=0$ if $k^{1/\alpha}<i_0$, we may

restrict our consideration on $\{\tau_k\}_{k\geqq i_0^\alpha}$. Neglecting first finitely many terms, without loss of generality, we may assume $\beta_1\geqq 1$, $\beta_{k+1}/\beta_k\geqq 2\psi(k)$ $(k\in\boldsymbol{N})$, where $\{\psi(k)\}$ is an increasing sequence satisfying $\psi(1)\geqq 1$. Let us introduce a sequence $\{m(i)\}$ of integers and a sequence $\{J_i\}$ of subsets of $\boldsymbol{N}$ by

$$m(0)=[i_0^\alpha],\quad m(i+1)=m(i)+3+[(5/\rho\alpha)\log_2 m(i)]\quad (i\in\boldsymbol{N}),$$

$$J_i=\{k\in\boldsymbol{N};\ m(i-1)<k\leqq m(i)\}\quad (i\in\boldsymbol{N}).$$

Because of $m(i+1)/m(i)\to 1$, we can take $L\geqq 1$ such that

$$m(i+1)/m(i)\leqq L\,, \tag{3.4}$$

$$|J_i|\leqq L\ \log_2 m(i)\leqq Lm(i)\quad (i\in\boldsymbol{N}), \tag{3.5}$$

where $|J|$ denotes the cardinal number of J. Put

$$c_i=\Big(\sum_{k\in J_i} a_k^2 \sum_{j\leqq k^{1/\alpha}} d_j^2\Big)^{1/2},\quad \zeta_i=\frac{1}{c_i}\sum_{k\in J_i}\tau_k\quad\text{and}\quad C_n^2=c_1^2+\cdots+c_n^2.$$

Clealy $A_{m(i)}d_{i_0}\leqq C_i\leqq A_{m(i)}$. Since we have by (3.3) and (3.5) that

$$\lim_{i\to\infty}\frac{C_i}{A_{m(i)}}=1,\qquad \lim_{k\to\infty}\sum_{j\leqq k^{1/\alpha}} d_j^2=1$$

and

$$\Big\|\max_{n\in J_i}\sum_{m(i-1)<k\leqq n} a_k\tau_k\Big\|_\infty\leqq L\sum_{k\in J_i}|a_k|=O(A_{m(i)}(\log A_{m(i)})^{-8}),$$

assertion of the lemma reduced to

$$\limsup_{n\to\infty}\frac{1}{\sqrt{2C_{2n}^2\log\log C_{2n}}}\sum_{i=1}^n c_{2i}\zeta_{2i}\leqq 1\quad\text{a.s.} \tag{3.6}$$

and

$$\limsup_{n\to\infty}\frac{1}{\sqrt{2C_{2n-1}^2\log\log C_{2n-1}}}\sum_{i=1}^n c_{2i-1}\zeta_{2i-1}\leqq 1\quad\text{a.s.} \tag{3.7}$$

We shall here prove (3.6) using the method of weakly multiplicative systems. (3.7) can be proved in the same way. The above argument proves

$$|c_n|,\ |c_n|\,\|\zeta_n\|_\infty=O(C_n(\log C_n)^{-8})\qquad\text{as}\quad n\longrightarrow\infty. \tag{3.8}$$

We need an inequality similar to (1.7). First we shall show $\sum_{i_1<\cdots<i_r}|E\zeta_{2i_1}\cdots\zeta_{2i_r}|\leqq L^r$. ($L$ does not depend on r.) Using Hölder's inequality, we can easily have

$$\begin{aligned}|E\zeta_{2i_1}\cdots\zeta_{2i_r}|\leqq&\frac{\sqrt{2^r}}{c_{2i_1}\cdots c_{2i_r}}\sum_{\substack{k_q\in J_{2i_q}\\(q=1,\cdots,r)}}\ \sum_{\substack{j_q\leqq k_q^{1/\alpha}\\(q=1,\cdots,r)}}\\ &|a_{k_1}\cdots a_{k_r}d_{j_1}\cdots d_{j_r}\|E\cos(\beta_{k_r}j_r+\gamma''_{k_r,j_r})\cdots\cos(\beta_{k_1}j_1+\gamma''_{k_1,j_1})|\\ \leqq&\frac{\sqrt{2^r}}{c_{2i_1}\cdots c_{2i_r}}\Big(\sum_{\substack{k_q\in J_{2i_q}\\(q=1,\cdots,r)}}\ \sum_{\substack{j_q\leqq k_q^{1/\alpha}\\(q=1,\cdots,r)}} a_{k_1}^2\cdots a_{k_r}^2 d_{j_1}^2\cdots d_{j_r}^2\Big)^{1/2}\end{aligned}$$

$$\times\Big(\sum_{\substack{k_q\in J_{2iq}\\(q=1,\cdots,r)}}\sum_{\substack{j_q\le k_q^{1/\alpha}\\(q=1,\cdots,r)}} E^2\cos(\beta_{k_r}j_r+\gamma''_{k_r,j_r})\cdots\cos(\beta_{k_1}j_1+\gamma''_{k_1,j_1})\Big)^{1/2}$$

$$\le\frac{\sqrt{2^r}}{2^{r-1}}\sum_{\substack{k_q\in J_{2ij}\\(q=1,\cdots,r)}}\sum_{\substack{j_q\le k_q^{1/\alpha}\\(q=1,\cdots,r)}}\sum_{\substack{\varepsilon_q=\pm1\\(p=1,\cdots,r-1)}}$$

$$|E\cos(\{\beta_{k_r}j_r+\varepsilon_{r-1}\beta_{k_{r-1}}j_{r-1}+\cdots+\varepsilon_1\beta_{k_1}j_1\}\omega$$

$$+\{\gamma''_{k_r,j_r}+\varepsilon_{r-1}\gamma''_{k_{r-1},j_{r-1}}+\cdots+\varepsilon_1\gamma''_{k_1,j_1}\})|,$$

where $\gamma''_{k,j}=j\gamma_k+\gamma'_j$. Because of $|E\cos(\beta\omega+\gamma)|\le|\hat{P}(\beta)|\le L|\beta|^{-\rho/2}$, the summand is estimated from above by $L|\beta_{k_r}j_r+\varepsilon_{r-1}\beta_{k_{r-1}}j_{r-1}+\cdots+\varepsilon_1\beta_{k_1}j_1|^{-\rho/2}$. Thanks to $\beta_{m(i)}m^{1/\alpha}(i)\le\beta_{m(i+1)}2^{-m(i+1)+m(i)+(1/\alpha)\log m(i)}\le\beta_{m(i+1)}/4$, we get

$$\begin{aligned}&\beta_{k_r}j_r+\varepsilon_{r-1}\beta_{k_{r-1}}j_{r-1}+\cdots+\varepsilon_1\beta_{k_1}j_1\\&\quad\ge\beta_{m(2i_r-1)+1}-(\beta_{m(2i_{r-1}+1)}+\cdots+\beta_{m(2i_1+1)})/4\\&\quad\ge2\beta_{m(2i_r-1)}-\beta_{m(2i_{r-1}+1)}(1+4^{-1}+\cdots)/4\\&\quad\ge\beta_{m(2i_r-1)}\\&\quad\ge2^{2i_r-2}(m(2i_r-2)m(2i_r-3)\cdots m(1))^{5/\rho\alpha}\end{aligned}$$

by the definition of $\{m(i)\}$. Thus we have

$$\begin{aligned}|E\zeta_{2i_1}\cdots\zeta_{2i_r}|\le L^r|J_{2i_r}||J_{2i_{r-1}}|\cdots|J_{2i_1}|(m(2i_r)m(2i_{r-1})\cdots m(2i_1))^{1/\alpha}\\\times(2^{2i_r}(m(2i_r-2)m(2i_r-3)\cdots m(1))^{5/\rho\alpha})^{-\rho/2}.\end{aligned}$$

Because of $m(2i_r)\le Lm(2i_r-3)$ and (3.5), we get $|E\zeta_{2i_1}\cdots\zeta_{2i_r}|\le L^r2^{-\rho i_r}$. From this estimate, we have finally proved

$$\sum_{i_1<\cdots<i_r}|E\zeta_{2i_1}\cdots\zeta_{2i_r}|\le L^r\frac{1}{(r-1)!}\sum_{i_r=r}^{\infty}i_r^{r-1}2^{-\rho i_r}\le L^r.$$

Noting this and (3.8), we can apply (1.5) and have

$$\Big|E\prod_{i\in I}\Big(1+\frac{tc_{2i}}{C_{2n}}\zeta_{2i}\Big)\Big|\le2\qquad(I\subset[1,n],\ |t|\le L^{-1}(\log C_n)^4).$$

Using $\exp(x-x^2)\le1+x(|x|\le1/2)$, for $I\subset[1,n]$ and $|t|\le L^{-1}(\log C_n)^4$ we have

$$\text{(3.9)}\qquad E\exp\Big(\frac{t}{C_{2n}}\sum_{i\in I}c_{2i}\zeta_{2i}\Big)$$

$$\begin{aligned}&=E\exp\Big(\frac{t}{C_{2n}}\sum_{i\in I}c_{2i}\zeta_{2i}-\frac{2t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\zeta_{2i}^2\Big)\exp\Big(\frac{2t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\zeta_{2i}^2\Big)\\&=E^{1/2}\exp\Big(\frac{2t}{C_{2n}}\sum_{i\in I}c_{2i}\zeta_{2i}-\frac{4t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\zeta_{2i}^2\Big)E^{1/2}\exp\Big(\frac{4t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\zeta_{2i}^2\Big)\\&\le E^{1/2}\prod_{i\in I}\Big(1+\frac{2tc_{2i}}{C_{2n}}\zeta_{2i}\Big)E^{1/2}\exp\Big(\frac{4t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\zeta_{2i}^2\Big)\\&\le\sqrt{2}E^{1/2}\exp\Big(\frac{4t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\zeta_{2i}^2\Big).\end{aligned}$$

Next we shall estimate the last expectation. We must divide ζ_{2i}^2 into three parts:

$$\begin{aligned}\zeta_{2i}^2=&\frac{1}{C_{2i}^2}\sum_{\varepsilon=\pm 1}\sum_{\substack{k_q\in J_{2i}\\(q=1,2)}}\sum_{\substack{j_q\leq k_q^{1/\alpha}(q=1,2)\\|\beta_{k_1}j_1+\varepsilon\beta_{k_2}j_2|\geq\beta_{m(2i-1)}}}\\&\times a_{k_1}a_{k_2}d_{j_1}d_{j_2}\cos(\{\beta_{k_1}j_1+\varepsilon\beta_{k_2}j_2\}\omega+\{\gamma''_{k_1,j_1}+\varepsilon\gamma''_{k_2,j_2}\})\\&+\frac{2}{C_{2i}^2}\sum_{\substack{k_q\in J_{2i}(q=1,2)\\k_1<k_2}}\sum_{\substack{j_q\leq k_q^{1/\alpha}(q=1,2)\\|\beta_{k_1}j_1-\beta_{k_2}j_2|<\beta_{m(2i-1)}}}\\&\times a_{k_1}a_{k_2}d_{j_1}d_{j_2}\cos(\{\beta_{k_1}j_1-\beta_{k_2}j_2\}\omega+\{\gamma''_{k_1,j_1}-\gamma''_{k_2,j_2}\})\\&+\frac{1}{C_{2i}^2}\sum_{k\in J_{2i}}\sum_{j\leq k^{1/\alpha}}a_k^2d_j^2\\=&\xi_i+\eta_i+1\qquad\text{(say)}.\end{aligned}$$

Here η_i is small in the following sense. Since the condition $|\beta_{k_1}j_1-\beta_{k_2}j_2|<\beta_{m(2i-1)}$ implies $|j_1-j_2\beta_{k_2}/\beta_{k_1}|<\beta_{m(2i-1)}/\beta_{k_1}<1/2$, introducing a notation $[x]^*=\{n\in N;\ n-1/2\leq x<n+1/2\}$, we have

$$\begin{aligned}|\eta_i|&\leq\frac{2}{C_{2i}^2}\sum_{\substack{k_q\in J_{2i}(q=1,2)\\k_1<k_2}}\sum_{j_2\leq k_2^{1/\alpha}}|a_{k_1}a_{k_2}d_{[j_2\beta_{k_2}/\beta_{k_1}]^*}d_{j_2}|\\&\leq\frac{2}{C_{2i}^2}\sum_{\substack{k_q\in J_{2i}(q=1,2)\\k_1<k_2}}|a_{k_1}a_{k_2}|\Big(\sum_{j_2\leq k_2^{1/\alpha}}d^2_{[j_2\beta_{k_2}/\beta_{k_1}]^*}\Big)^{1/2}\Big(\sum_{j_2\leq k_2^{1/\alpha}}d_{j_2}^2\Big)^{1/2}\end{aligned}$$

Since (3.2) gives $\sum_{j_2\leq k_2^{1/\alpha}}d^2_{[j_2\beta_{k_2}/\beta_{k_1}]^*}\leq L^2(2\phi(m(2i-1))^{-2\alpha(k_2-k_1)}$, we get

$$\begin{aligned}|\eta_i|&\leq L\frac{1}{C_{2i}^2}\sum_{r=1}^{m(2i)-m(2i-1)}(2\phi(m(2i-1)))^{-\alpha r}\sum_{k=m(2i-1)+1}^{m(2i)-r}|a_ka_{k+r}|\Big(\sum_{j\leq(k+r)^{1/\alpha}}d_j^2\Big)^{1/2}\\&\leq L\frac{1}{C_{2i}^2}\sum_{r=1}^{m(2i)-m(2i-1)}(2\phi(m(2i-1)))^{-\alpha r}\\&\quad\times\Big(\sum_{k=m(2i-1)+1}^{m(2i)-r}a_k^2\Big)^{1/2}\Big(\sum_{k=m(2i-1)+1}^{m(2i)-r}a_{k+r}^2\sum_{j\leq(k+r)^{1/\alpha}}d_j^2\Big)^{1/2}\\&\leq L\phi^{-\alpha}(m(2i-1)).\end{aligned}$$

As to ξ_i, we shall prove

$$\sum_{i_1<\cdots<i_r}|E\xi_{i_1}\cdots\xi_{i_r}|\leq L^r.$$

Using $|\beta_{k_1}j_1+\varepsilon\beta_{k_2}j_2|\geq\beta_{m(2i_q-1)}$ and $\beta_{k_1}j_1+\varepsilon\beta_{k_2}j_2\leq 2\beta_{m(2i_q)}m^{1/\alpha}(2i_q)\leq\beta_{m(2i_q+1)}/2$, estimating in a similar way as before, we have

$$\begin{aligned}|E\xi_{i_1}\cdots\xi_{i_r}|&\leq L^r|J_{2i_r}|^2|J_{2i_{r-1}}|^2\cdots|J_{2i_1}|^2(m(2i_r)m(2i_{r-1})\cdots m(2i_1))^{2/\alpha}\\&\qquad\times(2^{2i_r}(m(2i_r-2)m(2i_r-3)\cdots m(1)^{5/\rho\alpha})^{-\rho/2}.\\&\leq L^r2^{-i_r\rho}.\end{aligned}$$

This yields the above estimation as before.

Because of $|\xi_i|\leqq 1+|\eta_i|+\zeta_{2i}^2$, we have c_{2i}^2, $c_{2i}^2\|\xi_i\|_\infty=O((\log C_{2i})^{-16})$. Using (1.5) again, we get

$$\left|E\prod_{i\in I}\left(1+\frac{sc_{2i}^2}{C_{2n}^2}\xi_i\right)\right|\leqq 2 \quad (I\subset[1,\ n],\ |s|\leqq L^{-1}(\log C_{2n})^8).$$

This yields

$$E\exp\left(\frac{s}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\xi_i-\frac{s^2}{C_{2n}^4}\sum_{i\in I}c_{2i}^4\xi_i^2\right)\leqq E\prod_{i\in I}\left(1+\frac{sc_{2i}^2}{C_{2n}^2}\xi_i\right)\leqq 2.$$

Noting this and

$$\frac{|s|c_{2i}^2}{C_{2n}^2}\xi_i^2\leqq L(\log C_{2n})^{-8}|\xi_i|\leqq(\log C_{2n})^{-8}(\xi_i+2|\eta_i|+2),$$

we have

$$E\exp\left((1-o(1))\frac{s}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\xi_i\right)\leqq 2 \quad (I\subset[1,\ n],\ |s|\leqq L^{-1}(\log C_{2n})^8).$$

This and (3.9) gives, for large enough n,

$$\text{(3.10)}\quad E\exp\left(\frac{t}{C_{2n}}\sum_{i\in I}c_{2i}\zeta_{2i}\right)\leqq\sqrt{2}\exp\left(\frac{2t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2(1+\|\eta_i\|_\infty)\right)E^{1/2}\exp\left(\frac{4t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\xi_i\right)$$
$$\leqq 2\exp\left(\frac{4t^2}{C_{2n}^2}\sum_{i\in I}c_{2i}^2\right)\quad (I\subset[1,\ n],\ |t|\leqq L^{-1}(\log C_{2n})^4).$$

Let us next put $S_{n,m}=\zeta_{2n+2}+\cdots+\zeta_{2m}$ and $U_{n,m}=c_{2n+2}^2+\cdots+c_{2m}^2$. We shall prove (1.3). We can easily get

$$\begin{aligned}ES_{n,m}^2&\leqq 2\sum_{n<i_1<i_2\leq m}c_{2i_1}c_{2i_2}E\zeta_{2i_1}\zeta_{2i_2}+\sum_{n<i\leq m}c_{2i}^2E\zeta_{2i}^2\\ &\leqq 2\sum_{n<i\leq m}c_{2i}^2\left(\sum_{i_1<i_2}E^2\zeta_{2i_1}\zeta_{2i_2}\right)^{1/2}+\sum_{n<i\leq m}c_{2i}^2(E\xi_i+|\eta_i|+1)\\ &\leqq 2\sum_{n<i\leq m}c_{2i}^2\sum_{i_1<i_2}|E\zeta_{2i_1}\zeta_{2i_2}|+L\sum_{n<i\leq m}c_{2i}^2+\left(\sum_{n<i\leq m}c_{2i}^4\right)^{1/2}\left(\sum_{n<i\leq m}E^2\xi_i\right)^{1/2}\\ &\leqq L\sum_{n<i\leq m}c_{2i}^2+\sum_{n<i\leq m}c_{2i}^2\sum_i|E\xi_i|\\ &\leqq LU_{n,m},\end{aligned}$$

and consequently we can obtain

$$ES_{n,m}^2S_{m,l}^2\leqq ES_{n,m}^2\left(\sum_{n<i\leq l}c_{2i}^2(1+\|\eta_i\|_\infty)\right)$$
$$+ES_{n,m}^2\left(2\sum_{m<i_1<i_2\leq l}c_{2i_1}c_{2i_2}\zeta_{2i_1}\zeta_{2i_2}+\sum_{m<i\leq l}c_{2i}^2\xi_i\right).$$

The first term is trivially estimated by $LU_{n,m}U_{m,l}$ and the second term is also estimated by the same bound by expanding and estimating in a similar way as before.

(3.10) and (1.3) enable us to use Lemma 1 and prove the relative compactness of $\{Y_n/\sqrt{2\log\log C_n}\}$ where Y_n is defined by (0.1) using $S_n=\sum_{i=1}^n c_i\zeta_i$ and C_n instead of A_n. Since we only need the upper bound estimate, we can go through the proof of upper bound estimate in Theorem 7 in case $\nu(dx)=\delta_1(dx)$ in the same way except for the proof of (1.13). Here we prove (1.13) in case $d_{n,i}=1$. Since $\{\xi_i\}$ is weakly multiplicative, easy calculation gives

$$E\Big(\frac{1}{C_{q(r)}^2}\sum_{2i\leq q(r)} c_{2i}^2\xi_i\Big)^2=O((\log C_{q(r)})^{-8}) \qquad \text{as} \quad r \longrightarrow \infty .$$

This yields the following by Beppo-Levi's theorem:

$$\lim_{r\to\infty}\frac{1}{C_{q(r)}^2}\sum_{2i\leq q(r)} c_{2i}^2\xi_i=0 \qquad \text{a.s.}$$

Thus we have

$$\begin{aligned}&\limsup_{r\to\infty}\frac{1}{C_{q(r)}^2}\sum_{2i\leq q(r)} c_{2i}^2\zeta_i^2\\ &\leq\limsup_{r\to\infty}\frac{1}{C_{q(r)}^2}\sum_{2i\leq q(r)} c_{2i}^2+\limsup_{r\to\infty}\frac{1}{C_{q(r)}^2}\sum_{2i\leq q(r)} c_{2i}^2\eta_i+\limsup_{r\to\infty}\frac{1}{C_{q(r)}^2}\sum_{2i\leq q(r)} c_{2i}^2\xi_i\\ &\leq 1,\end{aligned}$$

because of $\lim_{i\to\infty}\|\eta_i\|_\infty=0$. Thus we have proved the conclusion. ■

In the proof of this lemma, we have also proved the relative compactnes of $\{X_n/\sqrt{2\log\log C_n}\}$ where X_n is defined by $S_n=\sum_{i=1}^n c_i\zeta_i$. In the same way as before, we can prove that $\{X_n^{(f)}/\sqrt{2\log\log A_n}\}$ is also relatively compact where $X_n^{(f)}$ is defined by $S_n=\sum_{k=1}^n a_k f(\beta_k\omega+\gamma_k)$. By lemma 2, we can easily prove

$$\limsup_{n\to\infty}\frac{1}{\sqrt{2A_n\log\log A_n}}\max_{i\leq n}\Big|\sum_{k=1}^{i} a_k f(\beta_k\omega+\gamma_k)\Big|\leq 2 \qquad \text{a.s.}$$

Thus we have $\limsup_{n\to\infty}\|\phi_n X_n^{(f)}\|_{C[0,1]}\leq 2\|f\|_2$ a.s. for $f\in \mathrm{Lip}\,\alpha$ with $\int_0^{2\pi} f(t)dt=0$ and $\int_0^{2\pi} f^2(t)dt>0$, where $\phi_n=1/\sqrt{2\log\log A_n}$. Consequently, if we take $m\in \boldsymbol{N}$ arbitrary, the set of all clusters of $\{\phi_n X_n^{(f-S_m)}\}$ is included in the centered ball with radius $2\|f-S_m\|_2$, and by Theorem 1, the set of all clusters of $\{\phi_n X_n^{(S_m)}\}$ coincides with K almost surely. Because of $X_n^{(f)}=X_n^{(S_m)}+X_n^{(f-S_m)}$ and $\lim_{m\to\infty}\|f-S_m\|_2=0$, we have the conclusion.

INSTITUTE OF MATHEMATICS,
UNIVERSITY OF TSUKUBA

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