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Elliptic 3-folds and Non-Kähler 3-folds

By

Yoshinori NAMIKAWA

§ 0. Introduction

The purpose of this paper is to study the relationship between Calabi-Yau 3-folds with elliptic fibrations and compact non-Kähler 3-folds with $K=0$, $b_2=0$, $q=0$. The non-Kähler 3-folds referred to here have firstly appeared in Friedman's paper [3]. In this paper he has shown that if there are sufficiently many (mutually disjoint) $(-1, -1)$-curves on a Calabi-Yau 3-fold, then one can contract these curves and can deform the resulting variety to a smooth non-Kähler 3-fold with $K_2=0$, $b_2=0$, $q=0$. For example, in the case of a (general) quintic hypersurface in $\boldsymbol{P}^4$, one can do this procedure for two lines on it. This phenomenon is analogous to the one for (-2)-curves on a $K3$ surface. In fact a (-2)-curve on a $K3$ surface often disappears in a deformation and this fact just says that one can contract this (-2)-curve to a point and can deform the resulting variety to a (smooth) $K3$ surface. By this phenomenon, we can explain the varience of the Picard numbers of $K3$ surfaces in deformations and it is well-known that a general point of the moduli space of $K3$ surfaces corresponds to a non-projective (but Kähler) $K3$ surface on which there are no (-2)-curves. Taking such a non-projective surface into consideration, one has a famous theorem that two arbitrary $K3$ surfaces are connected by deformations. There is, however, a difference between Calabi-Yau 3-folds and $K3$ surfaces, that is, a $(-1, -1)$-curve never disappears like a (-2)-curve in deformations. This is closely related to the fact that Calabi-Yau 3-folds have a large repertory of topological Euler numbers. For the speculation around this area, one may refer to M. Reid's paper [12].

The main result of this paper is the following:

Theorem A. *Let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then the bimeromorphic class of X is obtained as a semi-stable degeneration of a compact non-Kähler 3-fold with $K=0$, $b_2=0$ and $q=0$, i.e. there is a surjective proper map f of a smooth 4-dimensional variety $\mathcal{X}$ to a 1-dimensional disc Δ such that*

1) *$f^{-1}(t)$ is a compact non-Kähler 3-fold with $K=0$, $b_2=0$, $q=0$ for $t\in\Delta^*$,*

2) *$f^{-1}(0)=\sum_{i=0}^{n}X_i$ is a normal crossing divisor of $\mathcal{X}$, and*

3) *X_0 is bimeromorphic to X and other X_i's are in the class C.*

Here we will explain the motivation of the formulation in Theorem A. If there are

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sufficiently many $(-1, -1)$-curves on X in the Friedman's sense explained in the above, one has a flat morphism f of a complex analytic variety $\mathcal{X}$ to a disc Δ whose central fibre is the variety obtained by contraction of these curves and whose general fibre is a non-Kähler 3-fold with $K=0$, $b_2=0$, $q=0$. In this situation, $\mathcal{X}_0:=f^{-1}(0)$ has a number of ordinary double points, but one may assume that the total space $\mathcal{X}$ is smooth under a suitable condition (e. g. (1.1) in this paper). Next blow up these points. Then the central fibre consists of a number of irreducible components, namely, the smooth variety $\tilde{\mathcal{X}}_0$ obtained by the blowing ups of the ordinary double points on $\mathcal{X}_0$ and the $\boldsymbol{P}^3$'s coresonding to each point blown up. However this is not yet a semi-stable degeneration because the multiplicity of each $\boldsymbol{P}^3$ is two, Hence, taking a suitable base change, one has a semi-stable degeneration. This is a typical example of Theorem A.

We shall briefly explain the construction of the paper. In §1 two matters are treated. One is the Friedman's construction of a non-Kähler 3-fold with $K=0$ and $b_2=0$. The other is the canonical resolutions of Weierstrass models (for the definition of a Weierstrass model, see (1.2)). After these preliminaries, in §2 we reduce Theorem A to Theorem A′ which is concerned with Weierstrass models. The remaining sections 3, 4 and 5 are devoted to the proof of Theorem A′.

Finally the author expresses his thanks to Professor A. Fujiki who informen him of the article of Raoult [11], of which result is used in §2.

§1. Preliminaries

In this paper, a Calabi-Yau 3-fold means a smooth projective 3-fold with π_1 finite, $q=0$ and K trivial. Since π_1 is finite, those 3-folds are excluded which are, up to étale covers, Abelian 3-folds. Here we will briefly review the Friedman's construction of a non-Kähler 3-fold with $K=0$, $b_2=0$. Assume that X is a smooth compact 3-fold with K_X trivial and that mutually disjoint $(-1, -1)$-curves $C_1, \cdots, C_n$ are given on X. Here a $(-1, -1)$-curve means a smooth rational curve $\boldsymbol{P}^1$ whose normal bundle $N_{\boldsymbol{P}^1/X}$ is ismorphic to $\mathcal{O}_{\boldsymbol{P}^1}(-1)\oplus\mathcal{O}_{\boldsymbol{P}^1}(-1)$. Then one can contract these curves to points to get a compact 3-fold $\bar{X}$ with ordinary pouble points: $\pi: X\to\bar{X}$. For simplicity we will write $P_i=\pi(C_i)$, $Z=\coprod_i P_i$ and $C=\coprod_i C_i$. We have the following exact commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H^1(\pi_*\Theta_X) & \longrightarrow & H^1(\Theta_X) & \longrightarrow & H^0(R^1\pi_*\Theta_X) & \longrightarrow & H^2(\pi_*\Theta_X) & \longrightarrow & H^2(\Theta_X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1(T^0_{\bar{X}}) & \longrightarrow & \boldsymbol{T}^1_{\bar{X}} & \longrightarrow & H^0(T^1_{\bar{X}}) & \xrightarrow{\alpha} & H^2(T^0_{\bar{X}}) & \longrightarrow & \boldsymbol{T}^2_{\bar{X}} & \longrightarrow & 0 \end{array}$$

In the above diagram, the map α is interpreted as follows: First we have an ismmorphism $\beta: H^0(T^1_{\bar{X}})\to H^2_Z(T^0_{\bar{X}})$ by using the exact sequence defined locally at each P_i:

$$0\longrightarrow T^0_{\bar{X}}\longrightarrow\Theta_{C^4|\bar{X}}\longrightarrow\mathcal{O}_{\bar{X}}\longrightarrow T^1_{\bar{X}}\longrightarrow 0.$$

Here we note that (X, P_i) can be embedded into $(\boldsymbol{C}^4, 0)$ because P_i is an ordinary double point. By the isomorphism β, α is identified with the natural map $H^2_Z(T^0_{\bar{X}})\to H^2(T^0_{\bar{X}})$. In our case it is easily shown that $\pi_*\Theta_X=T^0_{\bar{X}}$. Next using the Leray spectral sequences: $H^p_Z(R^q\pi_*\Theta_X)\Rightarrow H^{p+q}_C(\Theta_X)$ and $H^p(R^q\pi_*\Theta_X)\Rightarrow H^{p+q}(\Theta_X)$, we have $H^2_Z(T^0_{\bar{X}})\cong H^2{}_C(\Theta_X)$ and $H^2(T^0_{\bar{X}})\cong H^2(\Theta_X)$, which imply that the above map is identified with the following maps:

$$\begin{array}{ccc} H^2_C(\Theta_X) & \longrightarrow & H^2(\Theta_X) \\ \| & \theta & \| \\ H^2_C(\Omega^2_X) & \longrightarrow & H^2(\Omega^2_X), \end{array}$$

where the vertical identifications come from the fact that K_X is trivial. If the map θ is surjective, then we have $\boldsymbol{T}^2_{\bar{X}}=0$. On the other hand, $H^0(T^1_{\bar{X}})\cong H^2_Z(T^0_{\bar{X}})\cong H^2_C(\Omega^2_{\bar{X}})$ are isomorphic to an n-dimensional vector space $\bigoplus_{i=1}^{n}\boldsymbol{C}$, where each factor corresponds to C_i. θ is nothing but the map which associates each basis of the above vector space with the fundamental class of C_i in X. Summing up these results, we have the following fact (1.1):

(**1.1**) *Let X be a Calabi-Yau 3-fold and $C_1, \cdots, C_n$ mutually disjoint $(-1,-1)$-curves on X. We employ the same notation as above. Then since $H^2(\Omega^2_X)=H^4(X, \boldsymbol{C})=H_2(X, \boldsymbol{C})$ by the Hodge decomposition and the Poincare duality, the map θ can be identified with the map $i_*: \bigoplus_{i=1}^{n}H_2(C^i, \boldsymbol{C})\to H_2(X, \boldsymbol{C})$. In particular, if i_* is surjective and there is an element $(a_1, \cdots, a_n)\in\mathrm{Ker}\, i_*$ such that $a_i\neq 0$ for all i, then $\bar{X}$ is deformed to a smooth compact non-Kähler 3-fold with $K=0$, $b_2=0$ and $q=0$.*

The main points in the argument of [3] are to eliminate the second deformation object $\boldsymbol{T}^2_{\bar{X}}$ and to give a geometric interpretation of the map $H^0(T^1_{\bar{X}})\to H^2(T^0_{\bar{X}})$. Here we consider a relative situation in which a Calabi-Yau 3-fold X has a fibration $f: X\to \boldsymbol{P}^1$, Then we have:

(1.1)′ **Proposition.** *Let $f: X\to\boldsymbol{P}^0$ be as above and set $F=\{x\in X;\ f$ is not smooth at $x\}$. Assume that $\dim F=0$ ond $H^1(X_\eta, \mathcal{O}_{X_\eta})=0$, where X_η is a generic fibre of f. Moreover assume that there are mutually disjoint $(-1,-1)$-curves $C_1, \cdots, C_n$ on X such that* 1) *each C_i is mapped to a point by f, and f is a smooth map around C_i;* 2) *the following sequence is exact:*

$$0 \longrightarrow \mathrm{Ker} \longrightarrow \sum_{i=1}^{n} H^1(C_i, \Omega^1_{C_i})^* \longrightarrow H^1(X, \Omega^1{}_X)^* \longrightarrow H^1(\boldsymbol{P}^1, \Omega^1{}_{P^1})^* \longrightarrow 0$$

and 3) *there is an element $(a_1, \cdots, a_n)\in\mathrm{Ker}$ such that every a_i is non-zero. Let $\pi: X\to\bar{X}$ be the contraction of these curves, and $\bar{f}$ a natural fibration from $\bar{X}$ to $\boldsymbol{P}^1$. Then there is a flat deformation $\mathfrak{f}: \mathfrak{X}\to\Delta\times\boldsymbol{P}^1$ with Δ a sufficiently small 1-dimensional disc such that*

A) $\mathfrak{X}_0:=\mathfrak{f}^{-1}(\{0\}\times\boldsymbol{P}^1)=\bar{X}$;

B) $\mathfrak{f}_0: \mathfrak{X}_0\to\{0\}\times\boldsymbol{P}^1$ *coincides with $\bar{f}$, and*

C) $\mathfrak{X}_t$ *is smooth for $t\in\Delta^*$ (punctured disc).*

Proof. Let us denote by $T^i_{\bar{X}/P^1}$ and $\boldsymbol{T}^i_{\bar{X}/P^1}$ local and global deformation objects of $\bar{X}$ over $\boldsymbol{P}^1$, respectively. In our case, they are isomorphic to $\mathcal{E}xt_i(\Omega^1_{\bar{X}/P^1}, \mathcal{O}_{\bar{X}})$ and $\mathrm{Ext}^i(\Omega^1_{\bar{X}/P^1}, \mathcal{O}_{\bar{X}})$. Then we have the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Ker}\varphi & \longrightarrow & \operatorname{Ker}\beta & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\boldsymbol{T}^1_{\bar{X}/\boldsymbol{P}^1} & \longrightarrow & H^0(T^1_{\bar{X}/\boldsymbol{P}^1}) & \xrightarrow{\alpha} & H^2(T^0_{\bar{X}/\boldsymbol{P}^1}) & \longrightarrow & \boldsymbol{T}^2_{\bar{X}/\boldsymbol{P}^1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\theta_2} & & \\
\boldsymbol{T}^1_{\bar{X}} & \longrightarrow & H^0(T^1_{\bar{X}}) & \underset{\phi}{\longrightarrow} & H^2(T^0_{\bar{X}}) & \underset{\gamma}{\longrightarrow} & \boldsymbol{T}^2_{\bar{X}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & H^1(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1})^* & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

Here the injectivity of θ_2 and the surjectivity of θ_1 follow from the fact that $H^1(\bar{X}, \bar{f}^*\Theta_{\boldsymbol{P}^1})=0$. We see $H^1(\bar{X}, \bar{f}^*\Theta_{\boldsymbol{P}^1})=0$ by the Leray spectral sequence and the fact that $R^1\bar{f}_*\mathcal{O}_{\bar{X}}=0$ (this is because $R^1\bar{f}_*\mathcal{O}_{\bar{X}}=R^1f_*\mathcal{O}_X=R^1f_*K_X$ is torsion free and $H^1(X_\eta, \mathcal{O}_{X_\eta})=0$). We get $\operatorname{Coker}\beta=H^1(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1})^*$. considering the following exact commutative diagram:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & 0 & & & & \\
 & & & & & & & \searrow \quad \nearrow & & & & & \\
 & & & & & & & \mathrm{C} & & & & & \\
 & & & & & & \nearrow & & \searrow & & & & \\
0 & \longrightarrow & T^0_{\bar{X}/P^1} & \longrightarrow & T^0_{\bar{X}} & \longrightarrow & \bar{f}^*\Theta_{P^1} & & \longrightarrow & T^1_{\bar{X}/P^1} & \longrightarrow & T^1_{\bar{X}} & \longrightarrow 0 \\
 & & & & \searrow & \nearrow & & & & & & & \\
 & & & & & \mathrm{K} & & & & & & & \\
 & & & & \nearrow & \searrow & & & & & & & \\
 & & & & 0 & & 0 & & & & & &
\end{array}
$$

and taking cohomologies. In fact, since $\dim(\operatorname{supp} T^1_{\bar{X}/P^1})=0$, we have $\dim(\operatorname{supp}\mathrm{C})=0$, which implies that $H^2(K)=H^2(\bar{f}^*\Theta_{P^1})=H^1(\bar{f}^*\Omega_{P^1})^*=H^1(\boldsymbol{P}^1, \Omega^1_{P^1})^*$. Note that $H^2(T^0_{\bar{X}})=H^2(T^0_{\tilde{X}})$ (see [3] or the argument above (1.1) of this paper) and that the natural map: $H^2(T^0_{\tilde{X}})=H^1(\Omega^1_{\tilde{X}})^*\to H^1(\Omega^1_{P^1})^*$ is obviously surjective in our case. Hence we have $\operatorname{Coker}\beta=H^1(\boldsymbol{P}^1, \Omega^1_{P^1})^*$.

Claim 1· $\operatorname{Ker}\beta\subset\operatorname{Im}\alpha$.

Proof. This follows from the injectivity of θ_2.

Claim 2. $\operatorname{Im}\beta=\operatorname{Im}(\phi\circ\varphi)$

Proof. Since φ is surjective, this follows from a geometric interpretation of ϕ (see [3] or the argument above (1.1) in this paper) and the assumption 2).

By Claims 1 and 2, α is surjective, which implies that $\boldsymbol{T}^2_{\bar{X}/\boldsymbol{P}^1}=0$. By the snake lemma we infer that the map $\mathrm{Ker}\,\varphi \to \mathrm{Ker}\,\beta$ is surjective.

Now let us denote by $P_1, \cdots, P_n$ the ordinary double points of $\bar{X}$ and write $H^0(T^1_{\bar{X}})=\bigoplus_{i=1}^{n}\boldsymbol{C}_i$, where $\boldsymbol{C}_i$ is a copy of $\boldsymbol{C}$. Let $a=(a_1, \cdots, a_n)$ be the element in the assumption 3). Then there is an element $\tilde{a}\in H^0(T^1_{\bar{X}/\boldsymbol{P}^1})$ such that $\varphi(\tilde{a})=a$. It follows that $\alpha(\tilde{a})\in \mathrm{Ker}\,\beta$ because $\phi(a)=0$. If $\alpha(\tilde{a})$ is not zero, then $\alpha(\tilde{a}+a')=0$ for a suitable element a' in $\mathrm{Ker}\,\varphi$ because the map $\mathrm{Ker}\,\varphi\to\mathrm{Ker}\,\beta$ is surjective. Set $b=\tilde{a}+a'$. Then b comes from $\boldsymbol{T}^1_{\bar{X}/\boldsymbol{P}^1}$ and $\varphi(b)=a$. Since $\boldsymbol{T}^2_{\bar{X}/\boldsymbol{P}^1}=0$, this implies that there is a global smoothing of $\bar{X}$ which preserves a fibration over $\boldsymbol{P}^1$. Q.E.D.

A typical example of (1.1) is a general quintic hypersurface X in $\boldsymbol{P}^4$ and two lines on it. In this case, since $\mathrm{Pic}(X)=\boldsymbol{Z}$, it is rather easy to check the conditions in (1.1). But in general it is very difficult to find the curves satisfying the conditions in (1.1) even if a Calabi-Yau 3-fold X is given explicitly. In another sense, (1.1) supplies us with an interesting example where the class $\mathcal{C}$ is not stable under small deformations. In fact, $\bar{X}$ is a Moisnezon space, and hence is in the class $\mathcal{C}$. However, the non-Kähler 3-fold V obtained by a small deformation of $\bar{X}$ is not in the class $\mathcal{C}$. This is shown as follows. First one has $h^{0,2}(V)=0$, because $h^{0,1}(V)=0$ and $K_V=0$. If V is in the class $\mathcal{C}$, then it is bimeromorphic to some compact Kahler manifold Y. Since $h^{0,2}(V)=0$, we have $h^{0,2}(Y)=0$. In fact, by the desingularization theorem [4], we have a complex manifold $\tilde{V}$ which dominates both V and Y, birationally and properly. Using spectral sequences and Chow lemma [5] for $(\tilde{V}, V)$ and $(\tilde{V}, Y)$, we have the result. However, $h^{0,2}(Y)=0$ implies that Y is a projective manifold. Since the algebraic dimesion of V equals to 0, this is a contradiction. So V is not in the class $\mathcal{C}$. Since $\kappa(\bar{X})=0$, this is a counter-example to a question posed in [2].

(1.2.) Definition. A Weierstrass model $W(\mathcal{L}, a, b)$ over a variety S is a closed subvariety in $\boldsymbol{P}_S(\mathcal{O}\oplus\mathcal{L}^2\oplus\mathcal{L}^3)$ defined by the equation $Y^2Z=X^3+aXZ^2+bZ^3$, where $\mathcal{L}\in\mathrm{Pic}(S)$, $a\in H^0(S, \mathcal{L}^{-4})$, $b\in H^0(S, \mathcal{L}^{-6})$ and

$$Z:\mathcal{O}\longrightarrow\mathcal{O}\oplus\mathcal{L}^2\oplus\mathcal{L}^3$$
$$X:\mathcal{L}^2\longrightarrow\mathcal{O}\oplus\mathcal{L}^2\oplus\mathcal{L}^3$$
$$Y:\mathcal{L}^3\longrightarrow\mathcal{O}\oplus\mathcal{L}^2\oplus\mathcal{L}^3$$

are natural injections.

We denote by Σ the section of $W(\mathcal{L}, a, b)$ over S defined by $X=Z=0$ and denote by π the natural projection of $W(\mathcal{L}, a, b)$ to S.

(1.2.) Definition. With the same notation as above, consider a vector field v_X on $W=W(\mathcal{L}, a, b)$ defined by

$$X/Z\frac{\partial}{\partial(X/Z)} \qquad \text{on} \quad \{Z\neq 0\}\subset W\,.$$

$$X/Y\frac{\partial}{\partial(X/Y)} \qquad \text{on} \quad \{Y\neq 0\}\subset W\,.$$

$$-Z/X\frac{\partial}{\partial(Z/X)}-Y/X\frac{\partial}{\partial(Y/X)} \quad \text{on} \quad \{X\neq 0\}\subset W\,.$$

Set $\boldsymbol{P}=\boldsymbol{P}_S(\mathcal{O}\oplus\mathcal{L}^2\oplus\mathcal{L}^3)$. Then X is considered as an injection: $\mathcal{O}_P\rightarrow\mathcal{O}_P(1)\otimes p^*\mathcal{L}^{-2}$, where $\mathcal{O}_P(1)$ denotes the tautological line bundle of $\boldsymbol{P}$, and p is the projection of $\boldsymbol{P}$ to S. Then we can define $X^{-1}v_X\in H^0(\boldsymbol{P}, \Theta_{P/S}\otimes p^*\mathcal{L}^2)$ because $v_X\in H^0(\boldsymbol{P}, \Theta_{P/S})$. Here $\Theta_{P/S}$ is the relative tangent bundle of $p: \boldsymbol{P}\rightarrow S$. We write $\partial/\partial X$ for $X^{-1}v_X$. Similarly, we can define $\partial/\partial Y$ and $\partial/\partial Z$ for v_Y and v_Z, respectively.

Next let X be a Calabi-Yau 3-fold which has an elliptic fibration with a rational section. Then by [8] (Th. 3.4), X is birationally equivalent to a Weierstrass model $W=W(K_S, a, b)$ with only canonical singularities, where S is one of the following: $\boldsymbol{P}^2$, Σ_i $(0\leqq i\leqq 12)$. This is a starting point of the proof of Theorem A. Since W has singularities in the case $S=\Sigma_i$ $(3\leqq i\leqq 12)$ even if we take a and b generally, we must set up the following definition:

(1.3) Definition. Let $W=W(K_S, a, b)$ be a Weierstrass model over $S=\Sigma_i$ $(3\leqq i\leqq 12)$. Then W is called *general* if the following two conditions holds

(1) W has singularities only on $F=\{p\in W;\ p\in\pi^{-1}(D_0), X=Y=0\}$, where D_0 is a negative section of S.

(2) Let mD_0 and nD_0 be the fixed components of $|K_S^{-4}|$ and $|K_S^{-6}|$, respectively. Let $div(a)=G+mD_0$ and $div(b)=H+nD_0$ be the decompositions into movable parts and fixed components. Then G (resp. H) intersects D_0 transeversely. Moreover, $(G\cap D_0)\cap(H\cap D_0)=\phi$.

Let $W=W(\mathcal{L}, a, b)$ be a Weierstrass model over S. Then W is obtained as a double cover of $\boldsymbol{P}_S(\mathcal{O}\oplus\mathcal{L}^2)$ branched over $B=\{X^3+aXZ^2+bZ^3=0\}\cup\{X=Z=0\}$. If W has singularities, then we consider their resolutions

(1.4.) Canonical Resolutions. Let Y be a smooth variety and B a reduced Cartier divisor on it. Assume that $\mathcal{O}_Y(B)=L^{\otimes 2}$ for a line bundle L on Y. Then we have a double cover X of Y branched along B. To resolve the singularities on X, we consider the following process of blowing-ups.

(1) $Y_0=Y$, $B_0=B$

(2) $\nu_i: Y_{i+1}\rightarrow Y_i$ $(0\leqq i\leqq m)$: a blowing up along a smooth center $D_i\subset B_i\subset Y_i$

(3) $B_i=\nu_{i-1}^*B_{i-1}$

(4) $E_i=\nu_{i-1}^{-1}(D_{i-1})$.

Let $B_m=\bar{B}+\sum\mu_jE_j$ be the decomposition of the divisor B_m into the proper trransform $\bar{B}$ of B and other exceptional parts. Put $\bar{B}_m=\bar{B}+\sum_{\mu_j:\mathrm{odd}}E_j$. Here assume that $\bar{B}_m$ is smooth (possibly with many components). Then we have a double cover of Y_m branched along $\bar{B}_m$ and obtain a smooth variety $\tilde{X}$. Since there is a birational morphism $\pi: \tilde{X}\rightarrow X$, $\tilde{X}$ is a resolutiin of X. If we have $K_{\tilde{X}}=\pi^*K_X$, then the above process is called a *canonical resolution* of X. Note that a canonical resolution is not unique.

Let $W=W(K_S, a, b)$ be a general Weierstrass model over $S=\Sigma_i$ $(3\leqq i\leqq 12)$ in the sense of Definition (1.3). Then we can perform a canonical resolution on W. In our case, it is easily verified that $\mathrm{Sing}(W)=\{q\in \boldsymbol{P};\ q\in p^{-1}(D_0),\ X=0\ Y=0\}$, where $\boldsymbol{P}=\boldsymbol{P}_S(\mathcal{O}\oplus K_S^{\otimes 2}\oplus K_S^{\otimes 3})$, D_0: negative section, and that the singularities are locally trivial deformations of a rational double points except for a finite number of points which are so-called *dissident* points. So the problem is how to overcome the difficulties which arise at these dissident points. For example, consider the case where $i=5$. (In the case where $i=3, 4, 6, 8, 12$ there are no dissident points.) Since G and H never vanish simultaneously at a point q of $\mathrm{Sing}(W)$ in Definition (1.3), we may consider two cases: (1) only G vanishes at q and (2) only H vanishes at q. It follows that q is dissident only in the case (2). Hence we may consider the situation where $q=(0, 0, 0, 0)$, W: $y^2=x^3+t^3x+st^4$ in (x, y, s, t)-space $(=\boldsymbol{C}^4)$. Then a process of a canonical resolution will be found in (Figure 1) below (1.5).

(1.5) Proposition. *Let $W=W(K_S, a, b)$ be a Weierstrass model over S, where S is one of $\boldsymbol{P}^2$, Σ_i $(0\leqq i\leqq 12)$. Then:*

(0) $K_W=\mathcal{O}_W$

(1) *In the case $S=\boldsymbol{P}^2$ or Σ_i $(0\leqq i\leqq 2)$, a general Weierstrass model W is smooth and* $\mathrm{Pic}(W)=\pi^*\mathrm{Pic}(S)\oplus\boldsymbol{Z}[\Sigma]$. *Moreover W is simply-connected.*

(2) *In the case $S=\Sigma_i$ $(3\leqq i\leqq 12)$, a general Weierstrass model W has canonical singularities such that* $\mathrm{Sing}(W)\cong\boldsymbol{P}^1$ *and that they are locally trivial deformations of rational double points except for a finite number of points. W has a canonical resolution $\mu:\tilde{W}\to W$. In the case where $3\leqq i\leqq 8$ or $i=12$, μ has the following properties:*

a) *$\tilde{W}\to S$ is a flat morphism.*

b) $K_{\tilde{W}}=0$

c) *If we regard $\tilde{W}$ and W as fibre spaces over $\boldsymbol{P}^1$ by the ruling $S\to\boldsymbol{P}^1$, then μ_t: $\tilde{W}_t\to W_t$ is the minimal resolution of a surface with rational double points for a general point t of $\boldsymbol{P}^1$.*

In the case where $9\leqq i\leqq 11$, μ has the following properties:

a) *μ is factored through a normalv variety $\bar{\bar{W}}$ and $\bar{\bar{W}}$ has the following properties:*

a1) *$\bar{\bar{W}}\to S$ is a flat morphism.*

a2) *There are mutually disjoint rational curves C_j $(1\leqq j\leqq 12-i)$ on $\bar{\bar{W}}$ and $\bar{\bar{W}}$ has locally trivial deformations of A_1-singular points along these curves as the singularities.*

a3) *$\tilde{W}\to\bar{\bar{W}}$ is a resolution of the singularities in the trivial manner.*

a4) *If we regard $\bar{\bar{W}}$ as a fibre space over $\boldsymbol{P}^1$, then $\bar{\bar{W}}_t\to W_t$ is the minimal resolution of a surface with rational double points for a general point t of $\boldsymbol{P}^1$.*

b) $K_{\tilde{W}}=0$.

For details, see Figures 1 and 2 below.

(3) *For an arbitrary point $t\in\boldsymbol{P}^1$ except for a countable number of points, $\tilde{W}_t$ is naturally an elliptic $K3$ surface and its Mordell Weil group is trivial.*

(4) *Let E_j $(1\leqq j\leqq m)$ be μ-exceptional divisors. Then* $\mathrm{Pic}(\tilde{W})=(\pi\circ\mu)^*\mathrm{Pic}(S)\oplus\sum_{i=1}^{m}\boldsymbol{Z}[E_j]$.

(5) *$\tilde{W}$ is simply-connected.*

Proof. Since (0) and (1) were proved in [8], we will prove here (2), (3), (4) and (5). Consider the complete linear system $|\mathcal{L}|$ on $\boldsymbol{P}=\boldsymbol{P}_S(\mathcal{O}\oplus K_S^{\otimes 2}\oplus K_S^{\otimes 3})$, where $\mathcal{L}=$

$\mathcal{O}_P(3)\otimes\pi^*K_S^{-6}$ and $\mathcal{O}_P(1)$ is a tautological line bundle of $\boldsymbol{P}(\mathcal{O}\oplus K_S^{2}\oplus K_S^{3})$. Let Λ be a sublinear system of $|\mathcal{L}|$ which consists of the elements of the following form:

$$\varphi_1 Y^2 Z+\varphi_2 X^3+\varphi_3 XZ^2+\varphi_4 Z^3=0,$$

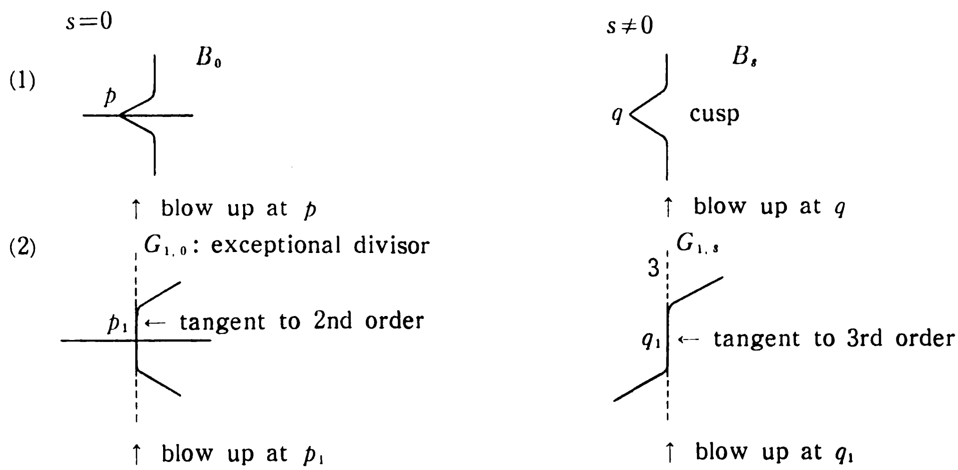
where $\varphi_1, \varphi_2\in H^0(S, \mathcal{O}_S)$, $\varphi_3\in H^0(S, K_S^{-4})$ and $\varphi_4\in H^0(S, K_S^{-6})$. An important result is that φ_3 and φ_4 always have zeros on the negative section of S as a ruled surface over $\boldsymbol{P}^1$. Then it is easily checked that the base locus $Bs(\Lambda)$ of Λ is $E=\{q\in\boldsymbol{P};\ q\in p^{-1}(D_0),\ X=0\ YZ=0\}$, where p is a projection from $\boldsymbol{P}$ to S and D_0 is a negative section of S. By the theorem of Bertini, a general element of Λ is smooth outside B. On the other hand, since $\{q\in\boldsymbol{P};\ X=Z=0\}$ is a section of a Weierstrass model $W\in\Lambda$, W is smooth on this locus. Hence a general element $W\in\Lambda$ has singularities only on $E_1=\{q\in\boldsymbol{P};\ q\ni p^{-1}(D_0),\ X=0\ Y=0\}$. Other claims in (2) follow from straightforward calculations if we consider a canonical resolution (1.4). Here we mention that the canonical resolution is not unique. Two different such resolutions are connected by a certain sequence of flops. For details see Figures 1 and 2.

Figure 1 (examples of canonical resolution)

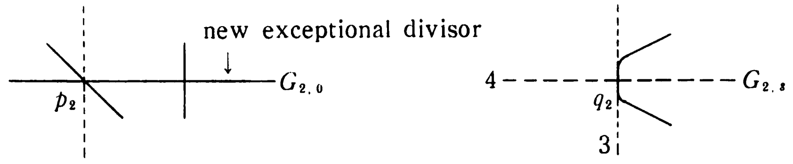
If $W=W(K_S, a, b)$, $S=\Sigma_i$ $(i\geqq 3)$ is given, then the canonica resolution in (1.5) (2) is not unique. Two different such resolutions are connected by a certain composition of flops. We will illustrate the process of one of such resolutions for E_6, E_7 and E_8-case (i.e. the cases where Sing(W) is a locally trivial deformation of a rational double point of type E_6, E_7, E_8 except for a finite number of points). In the figure, the real lines illustrate the proper transform of B and the dotted lines illustrate the exceptional divisors. The numbers associated with exceptional divisors mean the multiplicities in the total transform of B.

E_6-case

$B: X^3+t^3X+st^4=0$

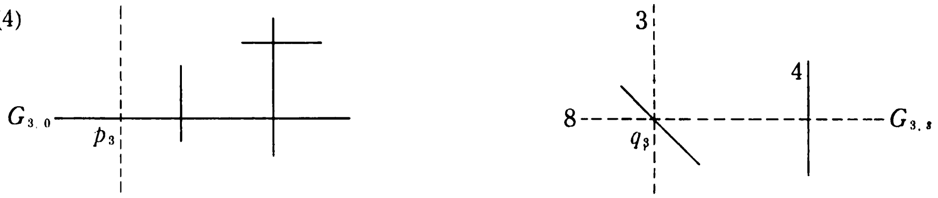


(3)



↑ blow up at p_2 ↑ blow up at q_2

(4)



↑ blow up at p_3 ↑ blow up at q_3

(5)

In (5), we may take a double cover branched along $B+G_1$.

E_7-case

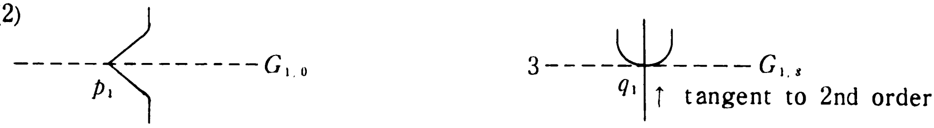
$B: X^3+st^3X+t^5=0$

$s=0$ $s\neq 0$

(1)

↑ blow up at p ↑ blow up at q

(2)

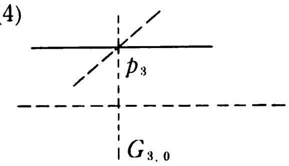


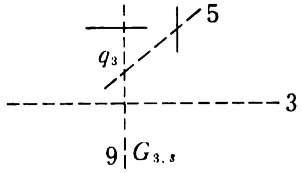
↑ blow up at p_1 ↑ blow up at q_1

(3)

↑ blow up at p_2 ↑ blow up at q_2

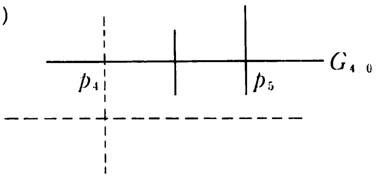
(4)

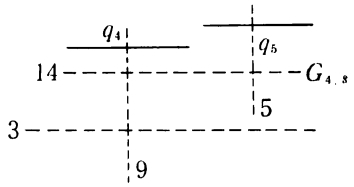




↑ blow up at p_3 ↑ blow up at q_3

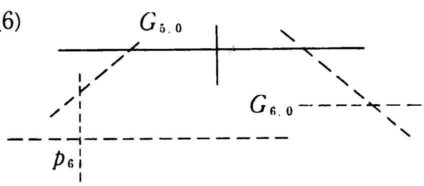
(5)

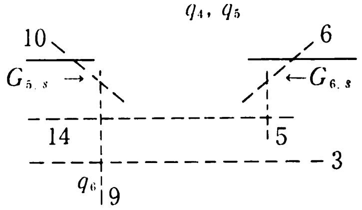




↑ blow up at in order of p_4, p_5 ↑ blow up at in order of q_4, q_5

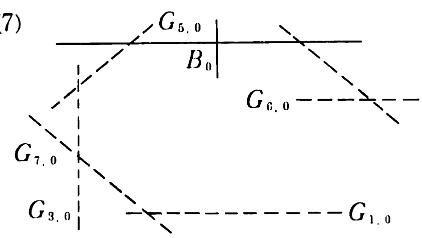
(6)

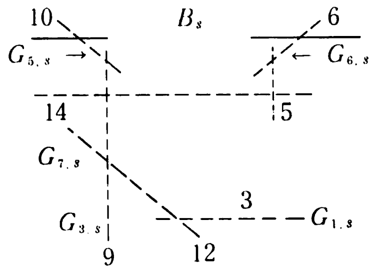




↑ blow up at p_6 ↑ blow up at q_6

(7)





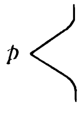
In (7), we may take a double cover branched along $B+G_1+G_2+G_3$.

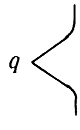
E_8-case

$B: X^3+t^4X+st^5=0$

$s=0$ $s\neq 0$

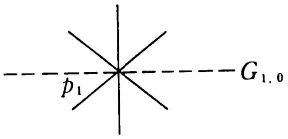
(1)

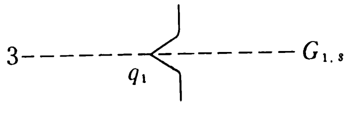




↑ blow up at p ↑ blow up at q

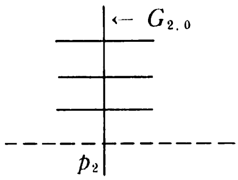
(2)

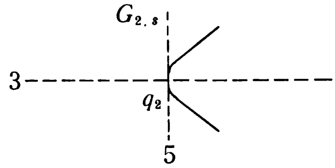




↑ blow up at p_1 ↑ blow up at q_1

(3)

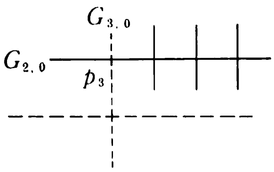


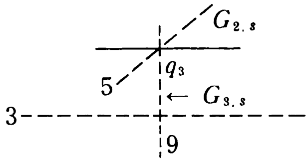


↑ blow up at p_2

↑ blow up at q_2

(4)

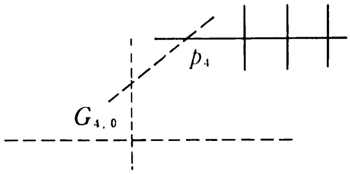


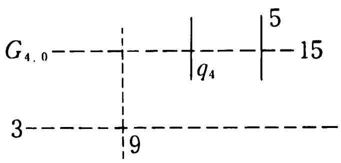


↑ blow up at p_3

↑ blow up at q_3

(5)

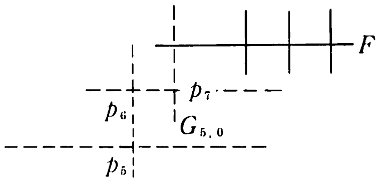


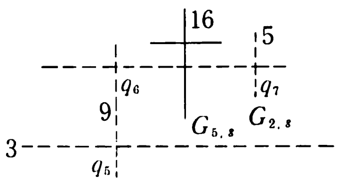


↑ blow up at p_4

↑ blow up at q_4

(6)



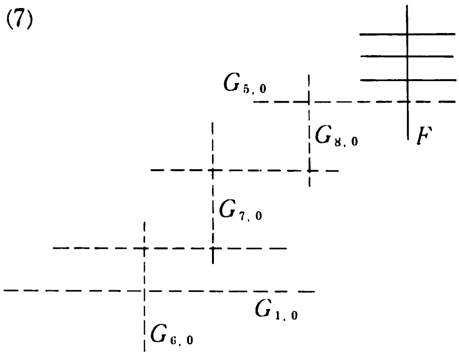


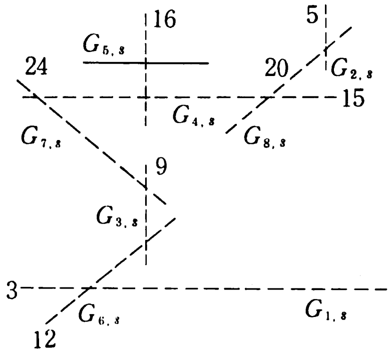
Here we note that $G_{2,0}=F+G_{5,0}$.

↑ blow up at p_5, p_6 and p_7

↑ blow up at q_5, q_6 and q_7

(7)

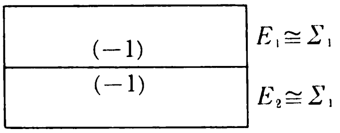




In (7), we take a double cover branched along $B+G_1+G_3+G_4+G_2$ Then we have the $\overline{\overline{W}}$ in (1.5). Since $G_{2,0}=G_{5,0}+F$ in our case, G_2 intersects B. Furthermore we can check that G_2 intersect B along F transversely. Hence it follows that $\overline{\overline{W}}$ has a locally trivial deformation of a rational double point of A_1-type as its singularities.

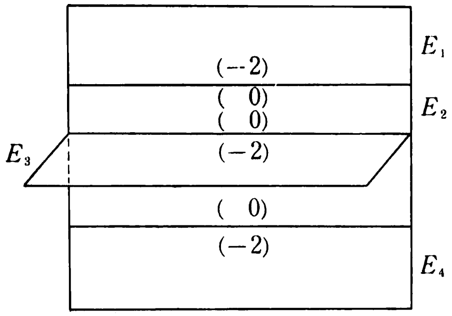
Figure 2 (Configuration of μ-exceptional divisors)

$i=3$ (A_2-type)

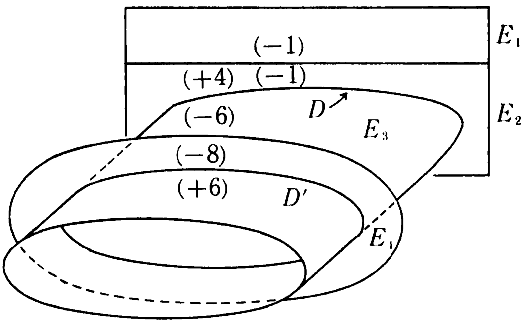


The double curve is the negative section of each E_i.

$i=4$ (D_4-type)



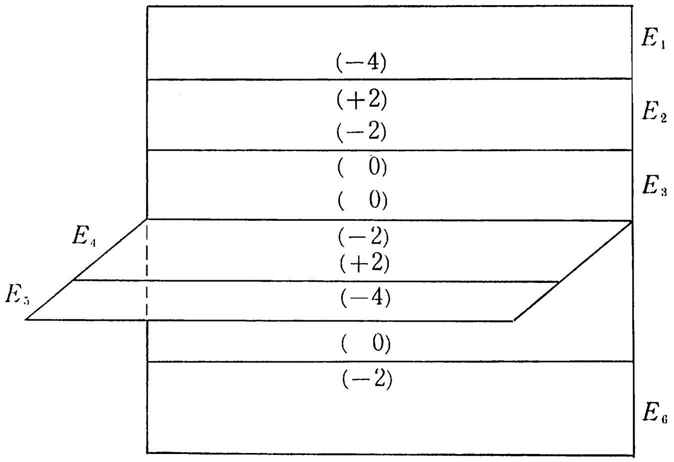
$i=5$ (E_6-type)



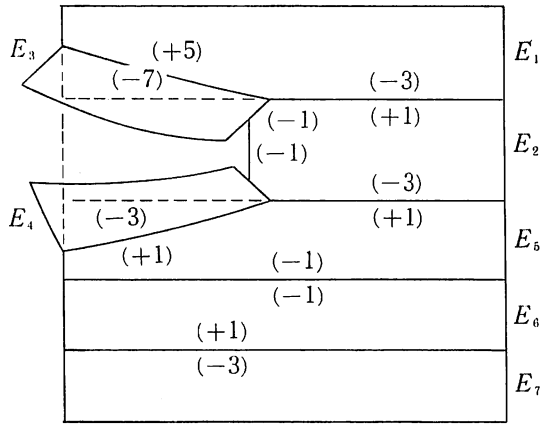
E_j's are ruled surface.

The double curve D is a multi-section of deg2 of $E_2\cong\Sigma_1$, and D' is a section of both E_3 and E_4. The branched points on D correspond to the dissident points of W.

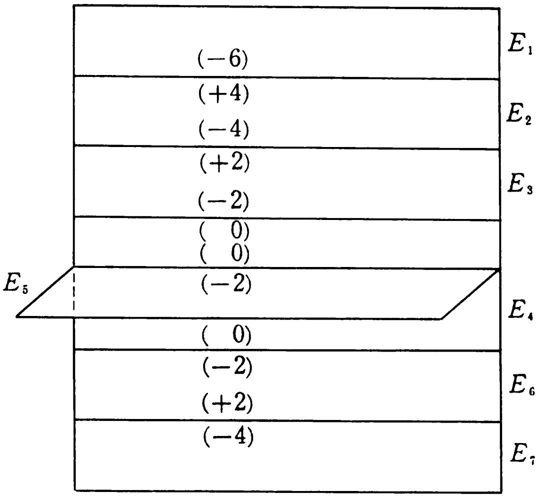
$i=6$ (E_6-type)



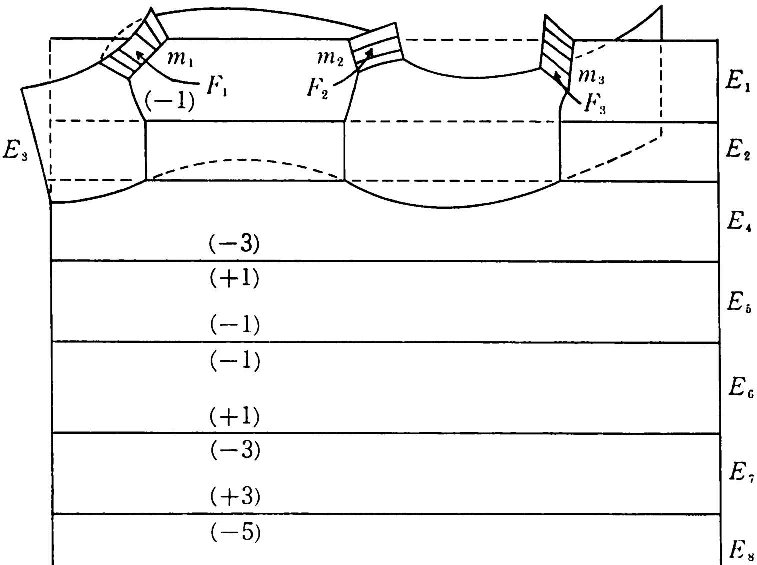
$i=7$ (E_7-type)



$i=8$ (E_7-type)

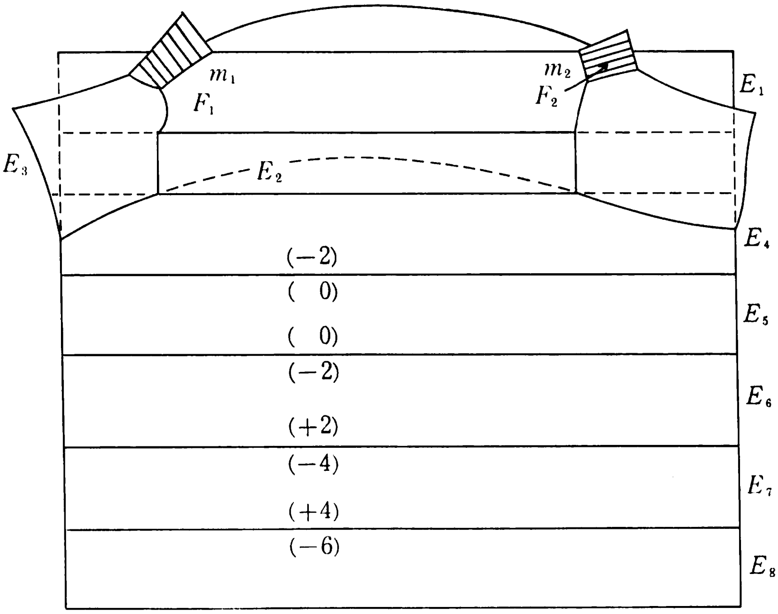


$i=9$ (E_8-type)



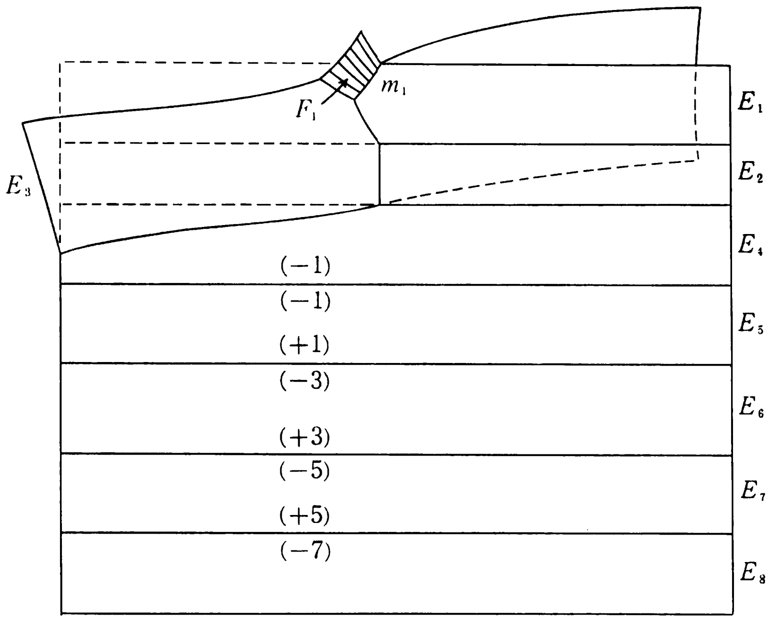
If we blow down F_i ($1\leqq i\leqq 3$) to the curves m_i, then we have $\overline{\overline{W}}$. E_3 intersect with E_1 along the curves m_i ($1\leqq i\leqq 3$).

$i=10$ (E_8-type)



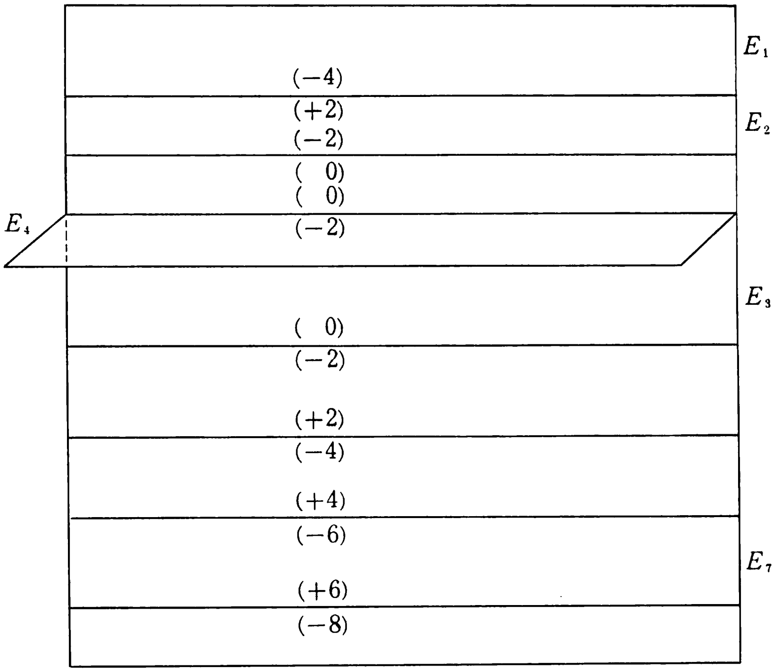
If we blow down F_i ($1 \leqq i \leqq 2$) to the curves m_i ($1 \leqq i \leqq 2$), then we have $\overline{\overline{W}}$.

$i=11$ (E_8-type)



If we blow down F_1 to the curve m_1, then we have $\overline{\overline{W}}$.

$i=12$ (E_8-type)



Proof of (5). Let us view $\widetilde{W}$ as a fibre space over $\boldsymbol{P}^1$, $\nu:\widetilde{W}\to\boldsymbol{P}^1$. Then its general fibre is a $K3$ surface. Suppose that $\widetilde{W}$ is not simply connected, then by [21], there is a non-trivial étale cover of $\widetilde{W}$, $\phi: V\to\widetilde{W}$. Taking the Stein factorization of $\nu\circ\phi$, we have a finite cover $h: C\to\boldsymbol{P}^1$. If h is an isomorphism, then we have a contradiction because a general fibre of ν is a $K3$ surface and hence is simply-connected. Hence h is a finite cover of deg$\geqq 2$. Pick up a section D of ν. Then $D\cong\boldsymbol{P}^1$, and $\phi^{-1}(D)=\coprod\boldsymbol{P}^1_i$ because ϕ is étale. Every component of $\phi^{-1}(D)$ is isomorphic to D by ϕ. On the other hand, each component of $\phi^{-1}(D)$ has a surjective map to C, which contradicts the fact that deg $h\geqq 2$. Hence $\widetilde{W}$ is simply-connected.

Proof of (3)→(4): (Case 1) $3\leqq i\leqq 8$ or $i=12$: Let $\mathcal{L}$ be a line bundle on $\widetilde{W}$. Then after tensoring with $\mathcal{O}(m\Sigma)$, $m\in\boldsymbol{Z}$, we may assume that $(\mathcal{L}.f)=1$ for a general fibre f of $\pi\circ\mu$. Then $\tilde{\pi}_*\mathcal{L}$ is a reflexive sheaf of rank 1 by [16] (Cor. 1.7), where $\tilde{\pi}:=\pi\circ\mu$. Hence $\tilde{\pi}_*\mathcal{L}$ is a line bundle on S. Since we have an injection $\tilde{\pi}^*\tilde{\pi}_*\mathcal{L}\to\mathcal{L}$, we obtain a non-zero section s of $\mathcal{L}\otimes\tilde{\pi}^*(\tilde{\pi}_*\mathcal{L})^{-1}$. By (3), $\pi\circ\mu$ has no other sections other than Σ. So we have $div(s)=\Sigma+\tilde{\pi}^*H+\sum_{i=1} m_iE_i$, $m_i\geqq 0$, where H is an effective divisor on S, which implies (4).

(Case 2): $9\leqq i\leqq 11$: We can apply the same argument as above to the $\overline{\overline{W}}$ in (2). Then we obtain the result comparing Pic$(\widetilde{W})$ with Pic$(\overline{\overline{W}})$.

Before proving (3), let us prepare four lemmas.

(1.6.) Lemma. *Let* $f: X\to C$ *be a family of* $K3$*-surfaces over a curve* C *and* $h: S\to C$

a P^1-bundle over C. Let $g: X \to S$ be an elliptic fibration with a section $\Sigma \subset X$ such that $f = h \circ g$. (Hence we can view f as a family of elliptic $K3$-surfaces.) For a given point $t_0 \in C$, we denote by $l_1, \cdots, l_r$ the reducible fibres of $g_{t_0}: X_{t_0} \to S_{t_0}$ and consider small neighborhoods of l_i in $X_{t_0}: (X_{t_0}, t_i)$. Assume that f gives a trivial deformation of each (X_{t_0}, l_i) and that the Mordell Weil group $\mathfrak{S}(X_{t_0}/S_{t_0})$ of g_{t_0} is trivial. Then $\mathfrak{S}(X_t/S_t)$ is trivial for every $t \in C$ except for a countable number of points.

Proof. Let us consider an irreducible component H of $Hilb_{X/C}$ which generically parametrizes (-2)-curves $D_t \subset X_t$ which are sections of g_t. Let $\mathcal{H}$ be a universal family over H. Assume that H dominates C and that $\mathcal{H}_h \neq \Sigma_{p(h)}$ for a general point $h \in H$, where p denotes the natural morphism of H to C and $\Sigma_{p(h)}$ denotes the restiction of Σ to $X_{p(h)}$. Since H is projective over C, there is a point $h_0 \in H$ such that $\mathcal{H}_{h_0} \subset X_{t_0}$. For every point $h \neq h_0$ sufficiently near h_0, $\mathcal{H}_h$ is a (-2) curve on $X_{(h)}$. As a cycle on X_{t_0} we have $\mathcal{H}_{h_0} = \Sigma_{t_0} + \sum_i a_i E_i$; $a_i \geqq 0$, where Σ_{t_0} is a restriction of Σ to X_{t_0} and each E_i is an effective divisor contained in a fibre of g_{t_0} because $(\mathcal{H}_{h_0}, l)_{X_{t_0}} = 1$ for a general fibre l of g_{t_0} and $\mathfrak{S}(X_{t_0}/S_{t_0}) = \{id\}$. Since $\mathcal{H}_h$ is a (-2)-section of $g_{p(h)}: X_{p(h)} \to S_{p(h)}$ which is different from $\Sigma_{p(h)}$, we have $(\mathcal{H}_{h_0}, \Sigma) \geqq 0$ and $(\mathcal{H}_{h_0}, E_i) \geqq 0$ for every i from our assumption. This contradicts, however, the fact that $(\mathcal{H}_{h_0})^2 = -2$. Hence H does not dominate C, which implies our lemma.

(1.7) Lemma. *There are elliptic $K3$-surfaces $\pi: S \to P^1$ with sections which have the following properties:*

(1) *π has only one reducible fibre l and l is of type* II* (*resp.* III*, IV*, I_0^*, IV);

(2) *The Mordell Weil group $\mathfrak{S}(S/P^1)$ is trivial.*

Proof. In the cases where l is of type II* and l is of type IV, there are examples with the properties (1) and (2), respectively ([17] §2. (I), §2. (II) 8°). As for the remaining cases we can construct the desired dxamples by deforming the above example of type II*. In the sequel we will explain this. Let $\pi: S_0 \to P^1$ be an elliptic $K3$-surface with a section and assume that π has only one reducible fibre of type II* and that $\mathfrak{S}(S_0/P^1)$ is trivial. Let $(V, 0)$ be a Kuranishi space of S_0. Here S_0 corresponds to the point 0. Let us fix an isometry

$$\phi: H^2(S_0, \mathbf{Z}) \longrightarrow L := U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8).$$

Since the Picard lattice for S_0 is isomretric to $U \oplus (-E_8)$, we may assume that for a suitable isometry ϕ, L is the direct sum of the Picard lattice and $U \oplus U \oplus (-E_8)$ ([18] (Cor. 2.6) or [19]). We remark here that the lattice $(-E_8)$ contains $(-E_7)$, $(-E_6)$, $(-D_4)$ as sublattices. Set

$$\Omega = \{[\omega] \subset P(L_C); (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}.$$

Then there is a period map p for a marked $K3$-surface (S_0, ϕ)

$$p: V \longrightarrow \Omega.$$

Let us write $[\omega_0] = p(0)$. By the local Torelli theoem, p is an isomorphism near the point 0. If we choose a suitable point $[\omega]$ near $[\omega_0]$, then the Picard lattice $\omega^{\perp}\cap H^{1,1}{}_R$ becomes $U\oplus(-E_7)$ ($U\oplus(-E_6)$ or $U\oplus(-D_4)$, respectively) by the next Lemma (1.8). Let S be a $K3$-surface corresesoponding to $[\omega]$. Then S also has an elliptic fibration with a section Σ. Moreover, the lattice U is generated by a fibre and Σ, and the lattice $(-E_7)$ ($(-E^9)$ or $(-D_4)$, resp.) is generated by the (-2)-curves which have no intersection with both Σ and a general fibre. In fact, the Picard lattice $U\oplus(-E_8)$ of S_0 is generated by a fibre, a unique section, and the (-2)-curves in a reducible fibre. U is generated by the fibre and the unique section and $(-E_8)$ is generated by the (-2)-curves in a reducible fibre which have no intersection with the unique section. Let $\mathcal{L}_0$ be a line bundle corresponding to a fibre f (i.e. $\mathcal{O}_{S_0}(f)=\mathcal{L}_0$). Then $\mathcal{L}_0$ can be extended to a line bundle $\mathcal{L}$ on S because U is invariant in the Picard lattice under the deformation of S_0 to S. By the Riemann-Roch theorem, $\chi(\mathcal{L})=\chi(\mathcal{L}_0)=2$. Since $h^2(\mathcal{L}_0)=h^0(\mathcal{L}_0^{-1})=0$, it follows that $h^1(\mathcal{L}_0)=0$, which implies that $h^0(\mathcal{L}_0)=h^0(\mathcal{L})=2$. From this, we deduce that an elliptic fibration is preserved in deformations. Next consider a (-2)-curve C on S_0. Let $\mathcal{M}_0$ be a line bundle such that $\mathcal{O}_{S_0}(C)=\mathcal{M}_0$. If $\mathcal{M}_0$ is extended to a line bundle $\mathcal{M}$ on S, then by the Riemann-Roch it follows that (-2)-curve C itself extends to a (-2)-curve on S. In our situation, we may consider as C a unique section or (-2)-curves in a reducible fibre which have no intersection with the unique section. Then using the above fact, we have the claim for (-2)-curves.

Finally we remark that $\mathfrak{S}(S/\boldsymbol{P}^1)=\{0\}$ follows from [20]. Q.E.D.

In the next lemma, we use the following notation:

$L:=\underbrace{U\oplus(-E_8)}_{L_1}\oplus\underbrace{U\oplus U\oplus(-E_8)}_{L_2}$ an Euclidian lattice

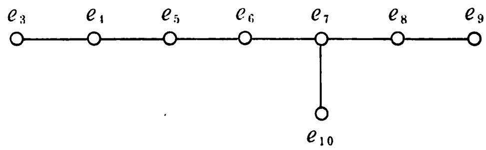
$L_C:=L\otimes_{\boldsymbol{Z}}\boldsymbol{C}$

$\Omega:=\{[\omega]\,;\,[\omega]\subset\boldsymbol{P}(L_C),\ \langle\omega,\omega\rangle=0,\ \langle\omega,\bar{\omega}\rangle>0\}$

$P_\omega:=$the $\boldsymbol{C}$-linear space spanned by ω and $\bar{\omega}$, where $\omega\subset L_C$

$P_\omega^{\perp}:=$the orthogonal space for P_ω

$e_1, e_2, e_3, \cdots, e_{10}$: a basis of L_1 such that e_1, e_2 is a basis of U and that $e_3, \cdots, e_{10}$ is a basis of $(-E_8)$ as follows:



$(-E_7)$: a sublattice of $(-E_8)$ generated by $e_4, \cdots, e_{10}$

$(-E_6)$: a sublattice of $(-E_8)$ generated by $e_5, \cdots, e_{10}$

$(-D_4)$: a sublattice of $(-E_8)$ generated by e_6, e_7, e_8, e_{00}

$e_{11}, \cdots, e_{22}$: a basis of L_2

(1.8) Lemma. *Let ω_0 be a non-zero element of L_C. Assume that $L\cap P_{\omega_0}^{\perp}=L_1$. Then there are curves Δ (i.e. 1-dim complex analytic space) in Ω which pass through $[\omega_0]$ and satisfy the following:*

(1) *For each $[\omega_t]\in\Delta$, $L\cap P_{\omega_t}^{\perp}\supset U\oplus(-E_7)$ (resp. $U\oplus(-E_6)$, $U\oplus(-D_4)$),*

(2) *$L\cap P_{\omega_t}^{\perp}=U\oplus(-E_7)$ (resp. $U\oplus(-E_6)$, $U\oplus(-D_4)$) for each point $[\omega_t]\in\Delta$ except*

for a countable number of points.

Proof. Write $\omega_0=\sum_{i=1}^{22}a_ie_i$. Let $(T_1, \cdots, T_{22})$ be a homogeneous coordinates system of $\boldsymbol{P}(L_C)$. Then $[\omega_0]=(a_1, \cdots, a_{22})=: a$. Since $[\omega_0]\subset\Omega$, we have

$$*)\qquad \sum_{i,j}\langle e_i, e_j\rangle a_ia_j=0.$$

Since $\langle e_i, e_j\rangle=0$ for $1\leqq i\leqq 10$, $11\leqq j\leqq 22$, $*$) can be written as follows:

$$*')\qquad \sum_{1\leq i,j\leq 10}\langle e_i, e_j\rangle a_ia_j+\sum_{11\leq i,j\leq 22}\langle e_i, e_j\rangle a_ia_j=0$$

The condition that $z=(z_1, \cdots, z_{12})\in P_{\omega_0}^{\perp}$ is written as follows:

$$**)\qquad \sum_{1\leq i,j\leq 10}\langle e_i, e_j\rangle a_iz_j+\sum_{11\leq i,j\leq 22}(e_i, e_j)\, a_iz_j=0$$

$$\sum_{1\leq i,j\leq 10}\langle e_i, e_j\rangle \bar{a}_iz_j+\sum_{11\leq i,j\leq 22}\langle e_i, e_j\rangle \bar{a}_iz_j=0$$

Since $L\cap P_{\omega}^{\perp}=L_1$, we have

$$\sum_{i=2}^{10}\langle e_i, e_k\rangle a=0, \qquad 1\leqq k\leqq 10$$

$$\sum_{i=1}^{10}\langle e_i, e_k\rangle \bar{a}_i=0, \qquad 1\leqq k\leqq 10.$$

Thus we have $a_1=\cdots=a_{10}=0$ because L_1 is unimodular. Choose i; $11\leqq i\leqq 22$ such that $a_i\neq 0$. Set $U_i=\{T_i\neq 0\}\subset\boldsymbol{P}(L_C)$ and consider the projection from $U_i=\boldsymbol{C}^{21}$ to $\boldsymbol{C}^{10}$ defined by $\left(\frac{T_1}{T_i}, \cdots, \frac{T_{i-1}}{T_i}, \frac{T_{i+1}}{T_i}, \cdots, \frac{T_{22}}{T_i}\right)\to\left(\frac{T_1}{T_i}, \cdots, \frac{T_{10}}{T_i}\right)$. Put $\Omega_i=U_i\cap\Omega$ and denote by $p: \Omega_i\to\boldsymbol{C}^{10}$ the restriction of the projection to Ω_i. p is a flat morphism and $p([\omega_0])=(0, \cdots, 0)$. Here choose some $(\beta_1, \cdots, \beta_{00})\in\boldsymbol{R}^{10}$ and set

$$\begin{pmatrix}a_1(t)\\ \vdots\\ a_{10}(t)\end{pmatrix}=\begin{pmatrix}\langle e_1, e_1\rangle & \cdots & \langle e_{10}, e_1\rangle\\ \vdots & & \vdots\\ \langle e_1, e_{10}\rangle & \cdots & \langle e_{10}, e_{10}\rangle\end{pmatrix}^{-1}\begin{pmatrix}\beta_1t\\ \vdots\\ \beta_{10}t\end{pmatrix}$$

where t is a parameter. Then we have

$$\sum_{i=1}^{10}\langle e_i, e_k\rangle a_i(t)=\beta_kt, \qquad 1\leqq k\leqq 10$$

$$\sum_{i=1}^{10}\langle e_i, e_k\rangle \bar{a}_i(t)=\beta_k\bar{t}, \qquad 1\leqq k\leqq 10.$$

We can modify the lattice

$$\left\{(z_1, \cdots, z_{10})\in\boldsymbol{Z}^{10};\ \sum_{k=1}^{10}\beta_ktz_k=0,\ \sum_{k=1}^{10}\beta_k\bar{t}z_k=0\right\}$$

by changing $\beta=(\beta_1, \cdots, \beta_{10})$. Let C be a curve on $\boldsymbol{C}^{10}$ defined by $(a_1(t), \cdots, a_{10}(t))$ through the origin. Then there is a curve Δ on Ω_i passing through $[\omega_0]$ such that $p(\Delta)=C$ because p is a flat morphism. This curve is a desired one in this lemma.

Q.E.D.

(1.9.) Lemma. *Let $\pi : S \to \boldsymbol{P}^1$ be an elliptic K3-surface with a section such that π has only one reducible fibre and it is of type* II*, III*, IV*, I_0^* *or* IV. *Then π has a Weierstrass model $\bar{\pi} : \bar{S} \to \boldsymbol{P}^1$ which is defined by $Y^2Z = X^3 + aXZ^2 + bZ^3$ in $\boldsymbol{P} := \boldsymbol{P}(\mathcal{O} \oplus K_{\boldsymbol{P}^1}^2 \oplus K_{\boldsymbol{P}^1}^3)$, where X, Y, Z, a, b are similar to the ones in Definition* (1.2). *Moreover we may assume that a and b have the following form according as the reducible fibre of π is of type* II*, III*, IV*, I_0^* *or* IV:

$$a = T_0^{\alpha} a' \; ; \; a'(0:1) \neq 0$$

$$b = T_0^{\beta} b' \; ; \; b'(0:1) \neq 0 ,$$

where $(T_0 : T_1)$ is a homogenous coordinates system af $\boldsymbol{P}^1$

type of a reducible fibre	*the condition for a and b*
II*	$\alpha \geqq 4, \ \beta = 5$
III*	$\alpha = 3, \ \beta \geqq 5$
IV*	$\alpha \geqq 3, \ \beta = 3$
I_0^*	$\alpha \geqq 2, \ \beta \geqq 3, \ \alpha = 2$ or $\beta = 3$
IV	$\alpha \geqq 2, \ \beta = 2$

Proof. Since S has a section Σ, we can consider the rational map ϕ of S to $\boldsymbol{P}_{\boldsymbol{P}^1}(\pi_* \mathcal{O}_S(3\Sigma))$ over $\boldsymbol{P}^1$. It follows that $\mathcal{O}_S(3\Sigma)$ is π-free and that $\pi_* \mathcal{O}(3\Sigma) = \mathcal{O} \oplus K_{\boldsymbol{P}^1}^2 \oplus K_{\boldsymbol{P}}^3$. The image of S by this map is a Weierstrass model $\bar{S}$. ϕ contracts the (-2)-curves in a reducible fibre of π which have no intersection with Σ. In our case, $\bar{S}$ has only one singular point which is a rational double point of type E_8, E_7, E_6, D_4 or A_2 according as the type of a reducible fibre is II*, III*, IV*, I_0^* or IV. Let ι be the fibre of $\bar{\pi}$ passing through the singular point.

Claim ι has a cusp singularity.

Proof. Before the proof we remark that ι has a singular point only on the singular point p of $\bar{S}$. Let $(T_0 : T_1)$ be a homogeneous coordinates pystem of $\boldsymbol{P}^1$.

Changing the coordinates: $Z \to Z$, $X \to X + cZ$ for a suitable c, we may assume that ι has a singularity on $X = Y = 0$, $T_0 = 0$ and that $\bar{S}$ is defined by $F := Y^2Z - X^3 - eX^2Z - fXZ^2 - gZ^3 = 0$. Here $Z = 0$ defines a section of $\bar{S}$. Thus we may consider the open set where $Z \neq 0$. Then $\bar{S}$ is considered as the affine variety defined by

$$F = Y^2 - X^3 - eX^2 - fX - g = 0$$

around the singular point p. Since both F and its Jacobian vanish on $X = Y = 0$, $T_0 = 0$, we have $f(0, 1) = g(0, 1) = 0$. If l has a node as the singularity, then $e(0, 1) \neq 0$, which implies that p is a rational double point of type A_n. If p is of type A_2 and l has a node, then S must have a singular fibre of type I_2. Therefore, we have the claim.

Q.E.D.

Let us return to the original situation, that is, $\bar{S}$ is defined by $Y^2Z = X^3 + aXZ^2 + bZ^3$. Let q be a singular point of $\bar{S}$. We may assume that q is contained in a fibre $\bar{\pi}^{-1}((0:1))$, where $(T_0 : T_1)$ is a homogeneous coordinates system of $\boldsymbol{P}^1$. Since the fibre

l passing through q has a cusp singularity by the Claim, it follows that $a(0:1)=b(0:1)=0$, which implies that q is defined by $X=Y=0$, $T_0=0$. Let us write $a(T_0, T_1)=T_0^a a'(T, T)$ and $b(T_0, T_1)=T_0^b b'(T_0, T_1)$, whre $a'(0:1)\neq 0$ and $b'(0:1)\neq 0$. The remaining task is to determine a and b according to the type of the singular point q. By [18] (II, 8 pp. 61~64) we have the following:

q: E_8-type $\Longleftrightarrow$ $a\geqq 4$, $b=5$

E_7-type $\Longleftrightarrow$ $a=3$, $b\geqq 5$

E_6-type $\Longleftrightarrow$ $a\geqq 3$, $b=4$

D_4-type $\Longrightarrow$ $a\geqq 2$, $b\geqq 3$, $a=2$ or $b=3$

A_2,type $\Longrightarrow$ $a\geqq 2$, $b=2$. Q.E.D.

Proof of (3) *of Proposition* (1.5) Let S be a surface which is isomorphic to Σ_i with $3\leqq i\leqq 12$. Let D_0 be a negative section of S, and let mD_0 and nD_0 be the fixed components of the linear systems $|K_S^{-4}|$ and $|K_S^{-6}|$, respectively. A general Weierstrass model W over S has singularities which are locally trivial deformation of a rational double points except for a finite number of points. We will call the type of this rational double point the "type of singularities of W". Let $\mu: \tilde{W}\to W$ be a canonical resolution. Viewing $\tilde{W}$ and W as fibre spaces over $\boldsymbol{P}^1$ via $S\to\boldsymbol{P}^1$, we have a minimal resolution of W_t, $\mu_t: \tilde{W}_t\to W_t$ for a general point $t\in\boldsymbol{P}^1$. Let C_t be a fibre of $S\to\boldsymbol{P}^1$ over t. Then $\tilde{W}_t$ has an elliptic fibration with a section over $C_t=\boldsymbol{P}^1$. This elliptic fibration has only one reducible fibre and we call the type of this reducible fibre the "type of a resolution". Then m, n, the type of singularities of W and the type of a resolution are as follows according to i; $S=\Sigma_i$.

List (1)

i	m	n	type of sing.	type of resolution
3	2	2	A_2	IV
4	2	3	D_4	I_0^*
5, 6	3	4	E_6	IV*
7, 8	3	5	E_7	III*
9, …, 12	4	5	E_8	II*

Let Y be an elliptic $K3$-surface with a section with the properties (1) and (2) in Lemma (1.7). Let $\bar{Y}$ be a Weierstrass model of Y. Note that Y is a minimal resolutiin of $\bar{Y}$. Here we consider the problem when $\bar{Y}$ is realized as a fibre of $W\to\boldsymbol{P}^1$. In our case, by Lemma (1.9), the above list and straightforward calculations, we see that:

If Y has a reductble fibre of type IV (*resp.* I_0^*, IV*, III*, II*), *then for* $i=3$ (*resp.* $i=4$, $i=5$, 6, $i=7$, 8, $i=9$, 10, 11, 12) $\bar{Y}$ *is realized as a fibre of* $W\to\boldsymbol{P}^1$, *where W is a* (*not necessarily general in the sense of Definition* (1.3)) *Weierstrass model over* $S=\Sigma_i$.

Here we remark again that $\bar{Y}$ has only one singular point and that it is a rational double point of type A_2, D_4, E_6, E_7, E_8 according as Y has a reducible fibre of type IV,

I_0^*, IV*, III*, II*.

(1.10) Lemma. *Let i be an integer such that $3 \leqq i \leqq 12$. Let $W = W(K_{\Sigma_i}, a, b)$ be a Weierstrass model over Σ_i. Let $\nu: W \to \boldsymbol{P}^1$ be a composite of $\pi: W \to \Sigma_i$ and $\Sigma_i \to \boldsymbol{P}^1$. Assume that $i=3$(resp. $i=4$, $i=5$ or 6, $i=7$ or 8, $i=9$, 10, 11 or 12) and that for a point $t \in \boldsymbol{P}^1$, W_t has only one singular point q and it is a rational double point of type A_2 (resp. D_4, E_6, E_7, E_8). Then there is a sufficiently small open neighbourhood Δ (in the usual topology) of $t \in \boldsymbol{P}^1$ such that*

(1) *the singular locus* $\mathrm{Sing}(W_\Delta)$ *of $W_\Delta := \nu^{-1}(\Delta)$ is a curve Γ passig through q which is isomorphic to Δ by ν;*

(2) *W_Δ has a trivial deformation of a rational double point $q \in W_t$ along Γ as singularities.*

Proof. It suffices to show that $\nu: W_\Delta \to \Delta$ induces a trivial deformation of the germ (W_t, q) of W_t at q because W_t has no other singularities than q. We will only prove the case $i=3$ here, but the proofs of other cases are similar. Suppose that $i=3$ and that (W_t, q) has an A_2-singularity. Then by List (1), $m=2$ and $n=2$, which implies that W_s has at least an A_2-singularity along the locus $X=Y=0$ for each point $s \in \boldsymbol{P}^1$. From this and the deformation theory of rational double points, it follows that ν induces a trivial deformation of (W_t, q). Q.E.D.

(1.11) Lemma. *Let W and W_t be the same as above. Let $\mathcal{W} \to \Xi$ be a deformation of W as a Weierstrass model over Σ_i, i.e. consider the following diagram:*

$$
\begin{array}{ccccc}
 & \boldsymbol{P}_{\Sigma_i}(\mathcal{O} \oplus K_{\Sigma_i}^{-2} \oplus K_{\Sigma_i}^{-3}) \times \Xi & \searrow & & \\
\mathcal{W} \subset & \longrightarrow & \Sigma_i \times \Xi & \xrightarrow{p_2} & \Sigma_i \\
\searrow^{g} & \boldsymbol{P}^1 \times \Xi & \swarrow & & \\
 & \downarrow & & & \\
 & \Xi & & &
\end{array}
$$

$$\mathcal{W}: Y^2 Z = X^3 + AXZ^2 + BZ^3 \qquad A \in H^0(\Sigma_i \times \Xi, p_2^* K_{\Sigma_i}^{-4})$$

$$\mathcal{W}_0 := g^{-1}(0) = W \quad \text{for } 0 \in \Xi \qquad B \in H^0(\Sigma_i \times \Xi, p_2^* K_{\Sigma}^{-6})$$

Then $\mathcal{W} \to \boldsymbol{P}^1 \times \Xi$ induces a trivial deformation of (W^t, q) near $q \in \mathcal{W}$.

Proof. The proof is quite similar to Lemma (1.10). The main point is that the indices m and n of List (1) are restricted.

Let Y and $\bar{Y}$ be the same as above and let q be a rational double point on $\bar{Y}$. Assume that $\bar{Y}$ is realized as a fibre W_t of $\nu: W \to \boldsymbol{P}^1$ where W is a Weierstrass model $W(K_{\Sigma_i}, a, b)$ over Σ_i for some i; $3 \leqq i \leqq 12$. Consider a deformation of W as a Weierstrass model over Σ_i, $\mathcal{W} \to \Xi$ like in Lemma (1.11) such that $\mathcal{W}_s$ is a general Weierstrass mocel over Σ_i in the sense of Definition (1.3) for each point $s \neq 0$ of Ξ which is sufficiently near 0. We will employ here the diagram and the notation of Lemma (1.11). Then by Lemma (1.11). $g: \mathcal{W} \to \boldsymbol{P}^1 \times \Xi$ induces a trivial deformation of (W_t, q). Note that

$g^{-1}(t, 0)=W_t$. We choose a point (t', s) of $\boldsymbol{P}^1\times\Xi$ sufficiently near $(t, 0)$. Consider a curve C passing through $(t, 0)$ and (t', s) in $\boldsymbol{P}^1\times\Xi$. Then we have a family of surfaces over C. At every point near $(t, 0)$ in C, the member of this family has only one singular double point of the same type by Lemma (1.11). Thus if we shrink C around $(t, 0)$, we have a simultaneous resolution of these surfaces. Let $f: X\to C$ be such one. Here $f^{-1}(t, 0)=X_{(t,0)}$ is a minimal resolution of W_t. On the other hand, W_t coincides with $\bar{Y}$ and Y is a minimal resolution of $\bar{Y}$. Hence $X_{(t,0)}$ coincides with Y. Since the elliptic surface Y has trivial Mordell Weil group, we can apply Lemma (1.6) to $f: X\to C$. Since $X_{(t',s)}$ is a minimal resolution of $g^{-1}(t', s)$, we conclude that:

Consider a general Weierstrass model $\mathcal{W}_s$ *and let* $\nu: \mathcal{W}_s\to\boldsymbol{P}^1$ *be a natural fibration which is the composite of* $\mathcal{W}_s\to\Sigma_i$ *and* $\Sigma_i\to\boldsymbol{P}^1$. *Then the fiber* $\mathcal{W}_{(t',s)}$ *of* ν *over* t' *has the following properties.*

(1) *By the map* $\mathcal{W}_s\to\Sigma_i$, $\mathcal{W}_{(t',s)}$ *has an elliptic fibration with a section.*

(2) *A minimal resolution of* $\mathcal{W}_{(t',s)}$ *has also an elliptic fibration and its Mordell Weil group is trivial.*

(3) *Let* q *be a rational double point on* $\mathcal{W}_{(t',s)}$. *Then* $\mathcal{W}_s$ *has a trivial deformation of this rational double point around* q *as the singularities.*

We can consider a canonical resolution of $\mathcal{W}_s$ by (2) of Proposition (1.5) and again apply Lemma (1.6) to $\tilde{\mathcal{W}}_s\to\boldsymbol{P}^1$. Then we have the claim of Proposition (1.5) (3).

Q.E.D.

(1.12) **Proposition.** *Let* S *be a surface isomorphic to* Σ_i $(3\leqq i\leqq 12)$ *and* C *a curve. Consider the following flat family of Weierstrass models over* S:

$$\begin{array}{ccccc} \boldsymbol{P}_S(\mathcal{O}\oplus K_S^2\oplus K_S^3)\times C & & & & \\ \cup & \searrow & & & \\ \mathcal{W} & \longrightarrow & S\times C & & \\ & g\searrow & \swarrow p_1 \quad p_2\searrow & & \\ & C & & S & \end{array}$$

$$\mathcal{W}: Y^2Z=X^3+aXZ^2+bZ^3,\quad a\in H^0(S\times C,\ p_2^*K_S^{-4}),$$
$$b\in H^0(S\times C,\ p_2^*K_S^{-6})$$

Assume that $\mathcal{W}_t$ *is general for every* $t\in C$ *except for a finite number of points* $\{t_1, \cdots, t_n\}$. *Then there is a projective resolution* $\mu: \tilde{\mathcal{W}}\to\mathcal{W}$ *such that* $\mu_t: \tilde{\mathcal{W}}_t\to\mathcal{W}_t$ *becomes the resolution in Proposition* (1.5) *for every* $t\notin\{t_1, \cdots, t_n\}$.

Proof. We may do the same thing as (1.4) for

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \boldsymbol{P}_S(\mathcal{O}\oplus K_S^2)\times C \\ & \searrow\ \swarrow & \\ & \boldsymbol{P}^1\times C & \end{array} \quad \text{instead for} \quad \begin{array}{ccc} W & \longrightarrow & \boldsymbol{P}_S(\mathcal{O}\oplus K_S^2) \\ & \searrow\ \swarrow & \\ & \boldsymbol{P}^1 & \end{array}$$

Here we follow each step in the case $S=\Sigma_9$. In Figure 1, this case corresponds to the E_8-case. Set $\mathcal{B}=\{X^3+aXZ^2+bZ^3=0\}\cup\{X=Z=0\}\subset \boldsymbol{P}_S(\mathcal{O}\oplus K_S^{\otimes 2})\times C$. Let $\mathcal{B}_1$ denote the first component of $\mathcal{B}$. According to Figure 1, we may blow up the $\boldsymbol{P}_S(\mathcal{O}\oplus K_S^{\otimes 2})\times C$ in order of the following:

(1) Blow up at $\mathcal{P}$, where $\mathcal{P}$ is the irreducible component with the reduced structure of Sing$(\mathcal{B}_1)$ which dominates C by $p:\boldsymbol{P}_S(\mathcal{O}\oplus K_S^{\otimes 2})\times C\to C$. In the fiber (of p) level, this step corresponds to (1) in Figure 1. In the remaining, the index (j) for each blow up corresponds to the blow up from (j) to $(j+1)$ in Figure 1.

(2) Blow up at $\mathcal{P}_1:=\mathcal{G}_1\cap\mathcal{B}_1$, where $\mathcal{G}_1$ is a exceptional divisor which dominates C, and $\mathcal{B}_1$ denotes the proper transform of $\mathcal{B}_1$ by the blow up in (1). Here $\mathcal{P}_1$ is given the reduced structure as a scheme.

(3) Blow up at $\mathcal{P}_2:=\mathcal{G}_1\cap\mathcal{B}_1$. Here $\mathcal{B}_1$ is the proper transform of the $\mathcal{B}_1$ in (2), and $\mathcal{G}_1$ is the proper transform of $\mathcal{G}_1$ in (2) In the remaining, we employ the similar notation by abuse of notation.

(4) Blow up at $\mathcal{P}_3:=\mathcal{G}_3\cap\mathcal{B}_1$

(5) Blow up at $\mathcal{P}_4:=\mathcal{G}_4\cap\mathcal{B}_1$

(6) Blow up at $\mathcal{P}_5:=\mathcal{G}_1\cap\mathcal{G}_3$
Blow up at $\mathcal{P}_6:=\mathcal{G}_3\cap\mathcal{G}_4$
Blow up at $\mathcal{P}_7:=\mathcal{G}_2\cap\mathcal{G}_4$

Finally we blow up at $\mathcal{G}_2\cap\mathcal{B}_1$. In this situation, we can construct a suitable double cover, and obtain $\tilde{\mathcal{W}}$. Then $\tilde{\mathcal{W}}$ has no singularities over a general point of C. Note that the above procedure induces a canonical resolution for a general fibre of $g:\mathcal{W}\to C$. Therefore, if we resolve the singularities of $\tilde{\mathcal{W}}$, then have the result.

§2. Reduction of Theorem A to Theorem A′

In this section we will show that Theorem A is reduced to the following Theorem A′.

Theorem A′. *Let W and $\tilde{W}$ be a general Weiertsrass model and its resolution as above. If we choose a and b generally, then we have:*

(1) *In the case $S=\boldsymbol{P}^2$ or Σ_i $(0\leqq i\leqq 2)$, there are mutually disjoint $(-1,-1)$-curves $C_1,\cdots,C_4$ on W such that $i_*:\bigoplus_{i=1}^{4}H_2(C_i,\boldsymbol{C})\to H_2(W,\boldsymbol{C})$ is surjective and that one can obtain, by the procedure of* (1.1), *a smooth compact non-Kähler* 3*-fold with $K=0$, $b_2=0$ and $q=0$.*

(2) *In the case $S=\Sigma_i$ $(3\leqq i\leqq 12)$, there is a smooth Moishezon* 3*-fold $\tilde{W}'$ birational to $\tilde{W}$ which has the following properties:*

a) *$\tilde{W}'$ is obtained from $\tilde{W}$ by a succession of flops of $(-1,-1)$-curves.*

b) *There are mutually disjoint $(-1,-1)$-curves C_j $(1\leqq j\leqq n(i))$ with $n(i)$ a number which depends on i such that $i_*:\bigoplus_{j=1}^{n(i)}H_2(C_j,\boldsymbol{C})\to H_2(\tilde{W}',\boldsymbol{C})$ is surjective.*

c) *One can obtain a smooth compact non-Kähler* 3*-fold with $K=0$, $b_2=0$ and $q=0$ from $\tilde{W}'$ by the procedure of* (1.1).

In the remaining of this section we assume the theorem above. Let X be a Calabi-

Yau 3-fold which has an elliptic fibration with a rational section. Then, as is mentioned before, X is birationally equivalent to a Weierstrass model W with only canonical singularities. W is not general in the sense of Definition (1.3). Though W has only canonical singularities, its singularities are possiblly worse than the ones described in Definition (1.3). Setting $\mathcal{L}=\mathcal{O}_P(3)\otimes\pi^*K_S^{-6}$ with $\mathcal{O}_P(1)$ the tautological line bundle of $\boldsymbol{P}(\mathcal{O}\oplus K_S^{2}\oplus K_S^{3})$, let us consider the linear system $|\mathcal{L}|$ on $\boldsymbol{P}(\mathcal{O}\oplus K_S^{2}\oplus K_S^{3})$. Let Λ be a linear subsystem of $|\mathcal{L}|$ which consists of the elements of the following form:

$$\hat{\varphi}_1 Y^2 Z+\varphi_2 X^3+\varphi_3 XZ^2+\varphi_4 Z^3=0\,,$$

where $\varphi_1,\ \varphi_2\in H^0(S,\ \mathcal{O}_S)$, $\varphi_3\in H^0(S,\ K_S^{-4})$ and $\varphi_4\in H^0(S,\ K_S^{-6})$. Then consider the universal family over $T=\boldsymbol{P}(\Lambda)$, $g:\mathcal{W}\to T$. Assume that $g^{-1}(t_0)=W$. If we choose a general point t on T, then $\mathcal{W}_t=g^{-1}(t)$ has the property in Theorem A$'$. Let C be a curve in T passing through t_0 and t. Then we haue a family of Weierstrass models over C, which we denote again by $g:\mathcal{W}\to C$. In the case where $S=\boldsymbol{P}^2$ or Σ_i $(0\leqq i\leqq 2)$, a general fibre of g is smooth. If $S=\Sigma_i$ $(3\leqq i\leqq 12)$, then a general fibre has singularities by Proposition (1.5) (2). In this case we can use (1.12).

Then we have a flat projective morphism $\tilde{g}:\tilde{\mathcal{W}}\to C$ whose general fibre is smooth. For a general point $t\in C$, $\tilde{\mathcal{W}}_t$ satisfies Theorem A$'$, (2), that is, there is a sequence of flops of $(-1,-1)$-curves $D_j\subset\mathcal{W}_t^{(j)}$:

$$\begin{array}{ccccccc}
\mathcal{W}_t^{(0)} & \dashrightarrow & \mathcal{W}_t^{(1)} & \dashrightarrow & \mathcal{W}_t^{(2)} \cdots\cdots & \dashrightarrow & \mathcal{W}_t^{(m)} \\
\| & & & & & & \| \\
\tilde{\mathcal{W}}_t & & & & & & \tilde{\mathcal{W}}_t'
\end{array}$$

and there are $(-1,-1)$-curves on $\tilde{\mathcal{W}}_t'$ to be contracted. Let us consider the irreducible component H of $Hilb_{\tilde{\mathcal{W}}/C}$ which contains $[D_0]$. Note that $Hilb_{\tilde{\mathcal{W}}/C}$ is ètale over C at $[D_0]$ because D_0 is a $(-1,-1)$-curve on $\tilde{\mathcal{W}}_t$. Hence H is determined uniquely and H is etale over C at $[D_0]$. Taking a suitable finite cover of C, we may assume that H_{red} is birational to C. Then we have the following diagram:

$$\begin{array}{ccc}
\mathcal{D} & \subset & \tilde{\mathcal{W}} \\
& \searrow \quad \swarrow & \\
& C &
\end{array}$$

where $\mathcal{D}_t$ is a $(-1,-1)$-curve on $\tilde{\mathcal{W}}_t$ for every point $t\in C^*$: a Zariski open subset of C. Restrict $\tilde{g}:\tilde{\mathcal{W}}\to C$ to C^* and consider $\tilde{g}^*:\tilde{\mathcal{W}}^*\to C^*$. Then by [1] (Cor. 6.10), we can perform a flop of $\mathcal{D}^*$ relatively over C^* and get $\tilde{g}^{(1)*}:\tilde{\mathcal{W}}^{(1)*}\to C^*$. Here $\tilde{\mathcal{W}}^{(1)*}$ is in general not a scheme, but an algebraic space. We can compactify $\tilde{\mathcal{W}}^{(1)*}$ by [11] and have a proper surjective map $\tilde{g}^{(1)}:\tilde{\mathcal{W}}^{(1)}\to C$. $\tilde{\mathcal{W}}^{(1)}$ is assumed to be smooth by [4] and $\tilde{\mathcal{W}}^{(1)}$ is birational to $\tilde{\mathcal{W}}$ over C. Since $\tilde{\mathcal{W}}_{t_0}$ contains an irreducible component birational to W and both $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}^{(1)}$ are smooth, $\tilde{\mathcal{W}}_{t_0}^{(1)}$ also contains an irreducible component birational to W. As a consequence, by repeating this process, we may assume from the first that there are $(-1,-1)$-curves to be contracted on $\tilde{\mathcal{W}}_t$. In the case where $S=\Sigma_i$ $(3\leqq i\leqq 12)$, we consider $\tilde{g}:\tilde{\mathcal{W}}\to C$ which is obtained by repeating above process. In the case $S=\boldsymbol{P}^2$ or Σ_i $(0\leqq i\leqq 2)$, we consider the original $g:\mathcal{W}\to C$. Then we come to Theorem A by using the following, after the base change by a finite cover of C if it is necessary.

(2.1.) Proposition. *Let $h: \mathcal{Y}\to C$ be a proper flat morphism with connected fibres of an irreducible smooth 4-dimensional algebraic space $\mathcal{Y}$ to a smooth curve C. Let $t_0\in C$ be a fixed point and W an irreducible component of $\mathcal{Y}_{t_0}$. Assume that there is a proper flat family of curves in $\mathcal{Y}$:*

$$\begin{array}{ccc} C_i & \subset & \mathcal{Y} \\ & \text{flat proper}\searrow \quad \swarrow & \\ & C & \qquad (1\leqq i\leqq n) \end{array}$$

such that for a general point $t\in C$, (1) C_i $(1\leqq i\leqq n)$ are mutually disjoint $(-1,-1)$-curves on $\mathcal{Y}_t$, (2) these curve satisfy the condition in (1.1) and (3) we can obtain from $\mathcal{Y}_t$ a non-Kähler 3-fold with $K=0$, $b_2=0$ and $q=0$ by the process in (1.1). Then there is a proper surjective morphism of a 4-dimensional complex manifold $\mathfrak{X}$ to a 1-dimensional disc Δ such that

1) *$f^{-1}(t)$ is a compact non-Kähler 3-fold with $K=0$, $b_2=0$ and $q=0$, for $t\in\Delta^*$,*
2) *$f^{-1}(0)=\sum_{i=1} W_i$ is a normal crossing divisor of $\mathfrak{X}$,*
3) *W_0 is bimeromorphic to W, and*
4) *each W_i is in the class C.*

Proof. Let C^* be a suitable Zariski open subset in C. Then by [1] (Cor. 6.10), we can contract C_i^*'s on $\mathcal{Y}^*$ relatively over C^* and obtain $\bar{\mathcal{Y}}^*$. We can compactify $\bar{\mathcal{Y}}^*$ and have proper flat morphism of normal algebraic space $\bar{\mathcal{Y}}$ to C. Then $\bar{\mathcal{Y}}$ is birational to $\mathcal{Y}$ over C. Consider the function field K of $\mathcal{Y}$ and let v be a discrete valuation ring which corresponds to W. Let L be a suitable Galois extension of K. Then the normalizations of $\mathcal{Y}$ and $\bar{\mathcal{Y}}$ in L become schemes by the argument of [11] (Proposition 1). Denote them by $\mathfrak{X}$ and $\bar{\mathfrak{X}}$, respectively. Then $\mathcal{Y}$ (resp. $\bar{\mathcal{Y}}$) is the quotient of $\mathfrak{X}$ (resp. $\bar{\mathfrak{X}}$) by the Galois group $G=Gal(L/K)$. Let $v_1, \cdots, v_k$ be the extensions of v in L. Then each element $g\in G$ induces a permutation of v_i's. If g sends v_i to v_j, then we will write $j=g(i)$. By [7] (p. 153), for v_1, there is a variety $\bar{\mathfrak{X}}_{v_1}$ birational to $\bar{\mathfrak{X}}$ such that (1) $\bar{\mathfrak{X}}_{v_1}$ is projevtive over $\bar{\mathfrak{X}}$, (2) $\bar{\mathfrak{X}}_{v_1}\supset\{z\in\bar{\mathfrak{X}}\,;\,\bar{\mathfrak{X}}$ is isomorphic to $\mathfrak{X}$ at $z\}$, and (3) if v_1 dominates a point y of $\bar{\mathfrak{X}}_{v_1}$ and a point y' on $\mathfrak{X}$, then $\mathcal{O}_y$ dominates $\mathcal{O}_{y'}$. In this case we may assume that $\bar{\mathfrak{X}}_{v_1}$ and $\bar{\mathfrak{X}}$ are isomorphic at every point except for points over $t_0\in C$. We denote $\bar{\mathfrak{X}}_{v_1}$ by $\bar{\mathfrak{X}}_1$ and define $\bar{\mathfrak{X}}_g$ for each $g\in G$ by the following fibre product:

$$\begin{array}{ccc} \bar{\mathfrak{X}}_1 & \xleftarrow{\ \sim\ } & \bar{\mathfrak{X}}_g \\ \downarrow & & \downarrow \\ \bar{\mathfrak{X}} & \xleftarrow[g_*^{-1}]{\ \sim\ } & \bar{\mathfrak{X}} \end{array}$$

On the other hand, viewing $\bar{\mathfrak{X}}_g$'s as $\bar{\mathfrak{X}}$-schemes, we have:

$$(*)\qquad \begin{array}{ccccc} \mathfrak{X}_1 & \dashrightarrow & \mathfrak{X}_{g_1} & \overset{\text{birat'}}{\dashrightarrow \cdots\cdots \dashrightarrow} & \mathfrak{X}_{g_l} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathfrak{X} & & G=\{1, g_1, \cdots, g_l\}\,. \end{array}$$

Take a closure $\tilde{\mathfrak{X}}$ of the *graph* of (*) in $\prod_g \mathfrak{X}_g$ and embed $\mathfrak{X}$ into $\prod_g \mathfrak{X}$ in such a way that $z \to \prod_g z$. Then the natural prolection from $\prod_g \mathfrak{X}_g$ to $\prod_g \mathfrak{X}$ induces a pojective morphism of $\tilde{\mathfrak{X}}$ to $\mathfrak{X}$. Consider the action of G on $\prod_g \mathfrak{X}$ defined in such a way that g sends g_i-th factor $\mathfrak{X}$ of $\prod_g \mathfrak{X}$ to gg_i-th factor $\mathfrak{X}$ of $\prod_g \mathfrak{X}$ and that this map of $\mathfrak{X}$ to itself coincides with the natural g-action on $\mathfrak{X}$. Clearly $\tilde{\mathfrak{X}}$ is stable by this G-action and this action coincides with the original G-action on $\mathfrak{X}$. If we take a normalization of $\tilde{\mathfrak{X}}$, then the action of G naturally extends to that on it. Hence we may assume $\tilde{\mathfrak{X}}$ is normal. Then the quotient $\tilde{\mathcal{Y}}$ of $\tilde{\mathfrak{X}}$ by G is an algebraic space by [6] (p. 183, 1.8) and we have a birational morphism of $\tilde{\mathcal{Y}}$ to $\mathcal{Y}$. This morphism is an isomorphism over a general point $t \in C$. By the construction, $\tilde{\mathcal{Y}}_{t_0}$ contains an irreducible component birational to W. Thus, from the first, we may assume that $\mathcal{Y}_{t_0}$ has an irreducible component birational to W. Now let us consider the Kuranishi space $(\mathcal{U}, u_1)$ of $\mathcal{Y}_{t_0}$, which is a complex space and has the versal property at every point u near u_0 [13, 14]. On the other hand, $\mathcal{Y}_t$ can be deformed to a non-Kähler 3-fold with $K=0$, $b_2=0$ and $q=0$ for every point t near t_0, which implies that there is a flat deformation $f: \mathcal{X} \to \Delta$ such that $f^{-1}(0)=\mathcal{Y}_{t_0}$ and that $f^{-1}(t)$ is a non-Kähler 3-fold with $K=0$, $b_2=0$ and $q=0$ for a point t of Δ^*. Then the semi-stable reduction for f is a desired one.

§ 3. Rational curves on Weierstrass models

Let $W=W(K_S, a, b)$ be a Weierstrass model over S, where $S=\Sigma_i$, $0 \leqq i \leqq 12$ or $S=\boldsymbol{P}^2$. In this section we will study the rational curves C in W such that $(C.\Sigma)=0$ or 1, which will be needed to prove Theorem A$'$.

(3.1) Proposition. *Let W and S be as above. Let D be a smooth rational curve on S such that $(D^2_S \geqq 0$ and let $C \subset W_D$ be a section of $\pi|_{W_D}: W_D=\pi^{-1}(D) \to D$. Assume that the Kodaira-Spencer map*:

$$\phi: T_{[D], \boldsymbol{P}(H^0(\mathcal{O}_S(D))^*)} \longrightarrow H^1(C, T_{W_D}|_C)$$

is injective, where $\boldsymbol{P}(H^0(\mathcal{O}_S(D))^*)$ is considered as a parameter space of the linear system $|D|$ on S and where $T_{[D], \boldsymbol{P}(H^0(\mathcal{O}_S(D)))^*}$ is the tangent space at $[D]$. Then we have

$$N_{C/W}=\mathcal{O}_{\boldsymbol{P}^1}(-1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-1)\,.$$

Let $\mathcal{U}$ be the universal family over $\boldsymbol{P}(H^0(\mathcal{O}_S(D))^*)$ and consider the following diagram.

$$\begin{array}{ccccc}
C\subset W_D & \subset & \mathcal{W} & \xrightarrow{\tau} & W \\
\downarrow & & \downarrow \nu & & \downarrow \pi \\
D & \subset & \mathcal{U} & \longrightarrow & S \\
 & & \downarrow & & \\
 & & [D]\in \boldsymbol{P}(H^0(\mathcal{O}_S(D))^*)=\boldsymbol{P}^{r+1}, & & r=(D^2)_S
\end{array}$$

where $\mathcal{W}$ is the fibre product of $\mathcal{U}$ and W over S. Before the proof of (3.1) we will prove two lemmas.

(3.2) Lemma. *Notation being as above, the following are equivalent.*

(1) $N_{C/W}=\mathcal{O}_{P^1}(-1)\oplus\mathcal{O}_{P^1}(-1)$

(2) $N_{C/\mathcal{W}}=\underbrace{\mathcal{O}_{P^1}(-1)\oplus\cdots\oplus\mathcal{O}_{P^1}(-1)}_{r+2}$

Proof. Let us consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N_{C/W_D} & \longrightarrow & N_{C/W} & \longrightarrow & \mathcal{O}_{P^1}(r) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N_{C/W_D} & \longrightarrow & N_{C/\mathcal{W}} & \longrightarrow & N_{W_D/\mathcal{W}|C}=\mathcal{O}_{P^1}^{\oplus r+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \mathrm{Ker} & \longrightarrow & \underbrace{\mathcal{O}_{P^1}(-1)\oplus\cdots\oplus\mathcal{O}_{P^1}(-1)}_{r} & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

(1)$\Rightarrow$(2):

Since $N_{C/W_D}=\mathcal{O}_{P^1}(-r-2)$, $\deg N_{C/W}=-2$ and $\deg N_{C/\mathcal{W}}=-r-2$. If $N_{C/\mathcal{W}}$ is not isomorphic to $\mathcal{O}_{P^1}(-1)\oplus\cdots\oplus\mathcal{O}_{P^1}(-1)$, then $N_{C/\mathcal{W}}$ contains a line bundle isomorphic to $\mathcal{O}_{P^1}(a)$; $a\geqq 0$ as a direct factor. Then the composite of the homomorphisms $\mathcal{O}_{P^1}(a)\subset N_{C/\mathcal{W}}$ and $N_{C/\mathcal{W}}\to N_{C/W}$ is a zero-map by (1). So we have $\mathcal{O}_{P^1}(a)\subset\mathrm{Ker}$. On the other hand, the horizontal map at the bottom of the diagram is an inclusion into $\mathcal{O}_{P^1}(-1)\oplus\cdots\oplus\mathcal{O}_{P^1}(-1)$, which is a contradiction. Q. E. D.

(2)$\Rightarrow$(1):

If $N_{C/W}$ is not isomorphic to $\mathcal{O}_{P^1}(-1)\oplus\mathcal{O}_{P^1}(-1)$. then since $\deg N_{C/W}=-2$, $N_{C/W}$ contains a line bundle $\mathcal{O}_{P^1}(a)$; $a<-1$ as a direct factor. Then since $N_{C/\mathcal{W}}=\mathcal{O}_{P^1}(-1)\oplus\cdots\oplus\mathcal{O}_{P^1}(-1)$, $N_{C/\mathcal{W}}\to N_{C/W}$ is not surjective, which is a contradiction. Q. E. D.

(3.3) Lemma. *Notation being as above, assume that* $N_{C/\mathcal{W}}\not\cong\mathcal{O}_{P^1}(-1)\oplus\cdots\oplus\mathcal{O}_{P^1}(-1)$. *Then there is a 1-dimensional subfamily* $\beta:\mathcal{F}\to\Delta$ *of* $\mathcal{W}\to\boldsymbol{P}(H^0(\mathcal{O}_S(D)^*)$:

$$\begin{array}{ccc}
W_D\subset\mathcal{F} & \subset & \mathcal{W} \\
\downarrow & & \downarrow \\
[D]\in\Delta & \subset & \boldsymbol{P}(H^0(\mathcal{O}_S(D)^*),
\end{array}$$

where Δ is a 1-dimensional smooth curve defined around $[D]$ which passes through $[D]$ and where $\mathcal{T}$ is the restriction of $\mathcal{W}$ over Δ. Moreover, for the $\mathcal{T}$, the exact sequence

$$0 \longrightarrow N_{C/W_D} \longrightarrow N_{C/\mathcal{T}} \longrightarrow N_{W_D/\mathcal{T}}|_C \longrightarrow 0$$

is a trivial extension.

Proof. Since $N_{C/W_D}=\mathcal{O}_{P^1}(-r-2)$, we have an inclusion

$$H^0(N_{C/\mathcal{W}}) \xrightarrow{\phi} H^0(N_{W_D/\mathcal{W}}|_C).$$

Since $N_{C/\mathcal{W}}\not\cong\mathcal{O}_{P^1}(-1)\oplus\cdots\oplus\mathcal{O}_{P^1}(-1)$ and $\deg N_{C/\mathcal{W}}+-r-2$, we have a non-zero element $\eta\in H^0(N_{C/\mathcal{W}}$ and hence a non-zero element $\phi(\eta)$ of $H^0(N_{W_D/\mathcal{W}}|_C)$. By the natural identification of $H^0(N_{W_D/\mathcal{W}}|_C)$ with $H^0(D, N_{D/S})=T_{[D],P(H^0(\mathcal{O}_S(D))^*)}$, we have an element $\theta\neq0$ of $T_{[D],P(H^0(\mathcal{O}_S(D))^*)}$ corresponding to $\phi(\eta)$. Let $\beta:\mathcal{T}\to\Delta$ be a subfamily of $\mathcal{W}$ with respect to θ. By the construction, we have $H^0(N_{C/\mathcal{T}})\neq0$. Consider the exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{C/W_D} & \longrightarrow & N_{C/\mathcal{T}} & \longrightarrow & N_{W_D/\mathcal{T}}|_C & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathcal{O}_{P^1}(-r-2) & & & & \mathcal{O}_{P^1}. & & \end{array}$$

Then $H^0(N_{C/\mathcal{T}})\neq0$ implies that the sequence is a trivial extension. Q.E.D.

Proof of (**3.1**). Assume that C is not a $(-1,-1)$-curve. Then by (3.2) and (3.3), we have a 1-dimensional family $\mathcal{T}$. Let $\theta\in H^1(T_{W_D})$ be the Kodaira-Spencer class of $\mathcal{T}$. Then for a contraction morphism $\pi: W_D\to\overline{W}_D$ of C:

$$\psi: H^1(T_{W_D}) \longrightarrow H^0(R^1\pi_*T_{W_D}),$$

we have $\psi(\theta)=0$ by (3.3). Moreover, we have $H^0(R^1\pi_*T_{W_D})=H^1(T_{W_D}|_C)$. In fact, first by the formal function theorem, we have

$$H^0(R^1\pi_*T_{W_D})=\varprojlim_n H^1(T_{W_D}\otimes\mathcal{O}_{W_D}/I^n),$$

where I is the defining ideal of C in W_D. Next consider the exact sequence

$$0 \longrightarrow T_{W_D}\otimes I^n/I^{n+1} \longrightarrow T_{W_D}\otimes\mathcal{O}/I^{n+1} \longrightarrow T_{W_D}\otimes\mathcal{O}/I^n \longrightarrow 0.$$

To prove that $H^0(R^1\pi_*T_{W_D})=H^1(T_{W_D}|_C)$, it suffices to show that $H^1(T_{W_D}\otimes I^nI^{n+1})=0$ for each $n\geq1$. This is easily checked using the fact that $I/I^2=\mathcal{O}_{P^1}(r+2)$ and the exact sequence:

$$0 \longrightarrow T_C \longrightarrow T_{W_D}|_C \longrightarrow N_{C/W_D} \longrightarrow 0.$$

Consequently, we have $\psi(\theta)=0$ in $H^1(T_{W_D}|_C)$, which contradicts the assumption of (3.1). Therefore C is a $(-1,-1)$-curve on W. Q.E.D.

(**3.4**) **Proposition.** *Let $X\subset Y$ be a projective 3-fold locally of complete intersection in a smooth projective variety Y. Let C be a smooth rational curve on X such that $(K_X.C)=0$ and that X is smooth around C. Let α and β denote the natural maps $H^0(N_{X/Y})\to H^0(N_{X/Y}|_C)$ and $H^0(\Theta_Y|_C)\to H^0(N_{X/Y}|_C)$. Assume that*

(1) $H^1(N_{C/Y})=0$;

(2) *the Hilbert scheme* $Hilb_Y$ *is smooth at* $[X]$;

(3) C *is isolated in* X;

(4) $H^0(N_{X/Y}|_C)$ *is generated by* $\operatorname{Im}\alpha$ *and* $\operatorname{Im}\beta$.

Then there is a pair of small deformations (displacements): (X_t, C_t) *of* (X, C) *in* Y *such that* C_t *is a* $(-1, -1)$*-curve on* X_t.

Proof. The idea of the proof is due to L. Ein [22].

We use the following notation:

I: the irreducible component of $Hilb_Y$ which contains $[X]$,

f: $\mathscr{X}\to I$: the universal family over I,

B: an irreducible component of $Hlb_{\mathscr{X}/I}$ with the reduced structure which contains $[C]$.

Then the natural map $B\to I$ dominates I because $\chi(N_{C/X})=1$ (which follows from $(K_X.C)=0$ and $C=\boldsymbol{P}^1$) and C is isolated in X. In fact, if we choose a smooth curve Δ in I passing through $[X]$, then we obtain a 1-dimentional family of 3-folds $\mathscr{X}_\Delta\to\Delta$. Then we have

$$0\longrightarrow N_{C/X}\longrightarrow N_{C/\mathscr{X}_\Delta}\longrightarrow \mathcal{O}_C\longrightarrow 0,$$

from which we deduce $\chi(N_{C/\mathscr{X}_\Delta})=1$. Since C does not move in X, this implies that C goes out of X in $\mathscr{X}_\Delta$. As a consequence B dominates I.

Since C is a smooth rational curve, a general point of B parametrizes a smooth rational curve. Let H be an open subset of B which parametrizes smooth rational curves. Consider the following diagram:

$$\begin{array}{ccc} \mathscr{X} & \longleftarrow & \mathscr{X}\times_I H\supset\mathscr{K} \\ \downarrow & p & \downarrow \\ I & \longleftarrow & H \end{array}$$

: the universal family over H

Note that p dominates I. Let t be a general point of I. Let H_t denote $p^{-1}(t)$. Choose $z\in H_t$ such that $z\in H-\operatorname{Sing}(H)$. Then we have a surjective map:

$$dp: T_{z,H}\longrightarrow T_{t,I}=H^0(N_{X_t/\mathscr{X}}),$$

where X_t is the 3-fold which corresponds to t. Let C_t be a smooth rational curve which corresponds to z. We may assume that t is sufficiently near $[X]$, and that X_t is smooth around C_t. On the other hand, since $[X]$ is a smooth point of $I, h^0(N_{X_t/Y})$ is constant at every point t around $[X]$. Let $[C]$ denote the point of H which corresponds to C. Then we have

$$(f_H)_*N_{\mathscr{X}\times_I H/Y\times H}\otimes k([C])=H^0(X, N_{X/\mathscr{X}}).$$

Moreover, $(f_H)^*N_{\mathscr{X}\times_I H/Y\times H}\otimes k([C])$ is a locally free sheaf at $[C]$. Next consider $(f_H)_*\Theta_{Y\times H}|_{\mathscr{K}}$. Since $H^1(N_{C/Y})=0$ by (1), we have $H^1(\Theta_Y|_C)=0$ by the exact sequence:

$$0\longrightarrow\Theta_C\longrightarrow\Theta_Y|_C\longrightarrow N_{C/Y}\longrightarrow 0.$$

Hence we have $(f_H)_*\Theta_{Y\times H}|_{\mathscr{K}}\otimes k([C])=H(\Theta_Y|_C)$ and $(f_H)_*\Theta_{Y\times H}|_{\mathscr{K}}$ is a locally free sheaf at $[C]$. Therefore α and β are factored as follows:

$$
\begin{array}{lcl}
(f_H)_*N_{\mathcal{X}\times_H H/Y\times H}\otimes k([C]) & \searrow^{\alpha} & \\
(f_H)_*(N_{\mathcal{X}\times_H H/Y\times H}|_{\mathcal{X}})\otimes k([C]) & \xrightarrow[\varphi]{} & H^0(N_{X/Y}|_C) \\
(f_H)_*\Theta_{Y\times H}|_{\mathcal{X}}\otimes k([C]) & \nearrow_{\beta} &
\end{array}
$$

Hence the assumption (4) implies that φ is surjective. Therefore φ is an isomorphism. From the above considerations, it follows that the assumptions (1), $\cdots$, (4) are valid for X_t and C_t.

For C_t, consider the following exact commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta_{C_t} & \longrightarrow & \Theta_{\mathcal{X}}|_{C_t} & \longrightarrow & T_{t,H}\otimes\mathcal{O}_{C_t} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \Phi & & \\
0 & \longrightarrow & \Theta_{C_t} & \longrightarrow & \Theta_Y|_{C_t} & \longrightarrow & N_{C_t Y} & \longrightarrow & 0
\end{array}
$$

Then we obtain:

$$
\begin{array}{ccccccccc}
 & & & & T_{t,H}\otimes\mathcal{O}_{C_t} & \xrightarrow{dp} & H^0(N_{X_t/Y})\otimes\mathcal{O}_{C_t} & & \\
 & & & & \downarrow \Phi & & \downarrow & & \\
0 & \longrightarrow & N_{C_t/Y_t} & \longrightarrow & N_{C_t/Y} & \longrightarrow & N_{X_t/Y}|_{C_t} & \longrightarrow & 0
\end{array}
$$

Taking H^0 of the above sequence, we have

$$
\begin{array}{ccccccc}
T_{t,H} & \xrightarrow{dp} & H^0(N_{X_t/Y}) & & & & \\
\downarrow & & \downarrow \alpha_t & & & & \\
H^0(N_{C_t/Y}) & \xrightarrow{\lambda} & H^0(N_{X_t/Y}|_{C_t}) & \longrightarrow & H^1(N_{C_t/X_t}) & \longrightarrow & 0 \\
\uparrow & \nearrow_{\beta_t} & & & & & \\
H^0(\Theta_Y|_{C_t}) & & & & & &
\end{array}
$$

Since $\mathrm{Im}\,\alpha_t$ and $\mathrm{Im}\,\beta_t$ generate $H^0(N_{X_t/Y}|_{C_t})$, and dp is surjective, we conclude that λ is surjective. Hence $H^1(N_{C_t/X_t})=0$. Since $(K_{X_t}.C_t)_{X_t}=0$, $\deg N_{C_t/X_t}=-2$, which implies that C_t is a $(-1,-1)$-curve on X_t. Q.E.D.

(**3.5**) **Corollary.** *Let $W=W(K_S, a, b)$ be a Weierstrass model over S. Let C be an isolated smooth rational curve on W. Assume that W is smooth around C. If S and C have one of the following properties* (1), (2) *and* (3), *then there is a pair of small deformations (W_t, C_t) of (W, C) in $\boldsymbol{P}=\boldsymbol{P}_S(\mathcal{O}\oplus K_S^{2}\oplus K_S^{3})$ such that C_t is a $(-1,-1)$-curve on W_t.*

(1) *$S=\Sigma_i$ $(0\leqq i\leqq 12)$ and there is a section D of $S\to\boldsymbol{P}^1$ with $(D)^2=i$ such that C is contained in $W_D:=\pi^{-1}(D)$ as a section of $\pi|_D: W_D\to D$. Moreover, $(\Sigma.C)=0$ or* 1.

(2) *$S\to\boldsymbol{P}^2$ and there is a smooth rational curve D on S with $(D)^2=k$; $k\leqq 4$ such that C is contained in W_D as a section of $\pi|_D: W_D\to D$. Moreover, $(\Sigma.C)=0$.*

(3) $S=\Sigma_i$ $(0\leqq i\leqq 12)$ *and there is a fibre* l *of* $S\to\boldsymbol{P}^1$ *such that* C *is contained in* W_l *as a section of* $\pi|_l: W_l\to l$. *Moreover* $(\Sigma.C)=0$ *or* 1.

Proof. In Proposition (3.4), put $X=W$, $Y=\boldsymbol{P}$. We may prove that the conditions (1), ⋯, (4) are satisfied in each case. First we will prove in the cases (1), (2).

Case (a): S and C satisfies (1) or (2), and $(\Sigma.C)=0$.

Condition (1): By the exact sequence:

$$0\longrightarrow\Theta_C\longrightarrow\Theta_P|_C\longrightarrow N_{C/P}\longrightarrow 0,$$

we may prove that $H^1(\Theta_P|_C)=0$. This follows from the following exact sequences:

$$0\longrightarrow\mathcal{O}_P|_C\longrightarrow\pi^*(\mathcal{E}^*)\otimes\mathcal{O}_P(1)|_C\longrightarrow\Theta_{P/S}\longrightarrow 0$$

$$0\longrightarrow\Theta_{P/S}|_C\longrightarrow\Theta_P|_C\longrightarrow\pi^*\Theta_S|_C\longrightarrow 0$$

where $\mathcal{E}=\mathcal{O}\oplus K_S^2\otimes K_S^3$ and $\mathcal{E}^*$ is its dual. First we have $H^1(\pi^*\Theta_S|_C)=0$ because $H^1(\Theta_D)=0$ and $H^1(N_{D/S})=0$. Next we have $H^1(\Theta_{P/S}|_C)=0$ because we have $H^1(C, \pi^*(\mathcal{E}^*)\otimes\mathcal{O}_P(1)|_C)=H^1(\pi^*(\mathcal{E}^*)|_C)=H^1(D, \mathcal{E}^*|_C)=0$ by the fact that $(\Sigma.C)=0$ and $\mathcal{O}_P(1)|_W=\mathcal{O}_W(3\Sigma)$. As a consequence we have $H^1(\Theta_P|_C)=0$. Condition (2) is satisfied because W is a Cartier divisor on $\boldsymbol{P}$ and $H^1(\mathcal{O}_P)=0$. Condition (3) is contained in the assumption. Condition (4) is satisfied because α is already surjective in our case. In fact, $N_{W/P}=\mathcal{O}_W(3\Sigma)\otimes\pi^*K_S^{-6}$.

Case (b): S and C satisfies (1), and $(\Sigma.C)=1$.

Let us denote by J the natural map of $H^0(\Theta_{P/S}|_C)$ to $H^0(N_{W/P}|_C)$. From now on, we will prove that $\operatorname{Im}\alpha$ and $\operatorname{Im}J$ generate $H^0(N_{W/P}|_C)$. If this is shown, then clearly $\operatorname{Im}\alpha$ and $\operatorname{Im}\beta$ generate $H^0(N_{W/P}|_C)$. In the remainings, we shall write $N_{W/P}=\mathcal{O}_W(W)$.

Let $F=Y^2Z-(X^3+aXZ^2+bZ^3)$ be the defining equation of W in $\boldsymbol{P}$. We write $\mathcal{O}_P(1)|_C=\mathcal{O}_C(1)$ and

$$H^0(\pi^*\mathcal{E}^*\otimes\mathcal{O}_C(1))=H^0(\mathcal{O}_C(1))\oplus H^0(\mathcal{O}_C(1)\otimes\pi^*K_S^{-2})\oplus H^0(\mathcal{O}_C(1)\otimes\pi^*K_S^{-3}).$$

Then J is given by

$$(l_1, l_2, l_3)\longrightarrow l_1(\partial F/\partial Z|_C)+l_2(\partial F/\partial X|_C)+l_3(\partial F/\partial Y|_C)H^0(\mathcal{O}_C(W)),$$

where

$$l_1\in H^0(\mathcal{O}_C(1)),$$

$$l_2\in H^0(\mathcal{O}_C(1)\otimes\pi^*K_S^{-2}),$$

$$l_3\in H^0(\mathcal{O}_C(1)\otimes\pi^*K_S^{-3}).$$

Note that $C\subset W_D\subset W$. In this situation, $\{\partial F/\partial H|_{W_D}=0\}$ is a divisor on W_D which has no intersections with $\Sigma|_{W_D}$. Here $\Sigma|_{W_D}$ denotes the restriction of Σ to W_D. Hence $\{\partial F/\partial Z|_C=0\}$ consists of $18+6i$ points (which may contain multiple points.). Let V_Z denote the image of $H^0(\mathcal{O}_C(1))$ in $H^0(\mathcal{O}_C(W))=H^0(\mathcal{O}_{P^1}(21+6i))$ under J. Then the linear system defined by V_Z consists of $18+6i$ fixed points and 3 points which move freely. In particular, $\dim V_Y=4$. On the other hand, we have

$$\{\partial F/\partial Y|_{W_D}=0\}=3\Sigma|_{W_D}+(\text{effective divisors which have no intersections with }\Sigma|_{W_D}).$$

Let V_Y denote the image of $H^0(\mathcal{O}_C(1)\otimes\pi^*K_S^{-3})$ in $H^0\mathcal{O}_C(W))$ under J. Then $\dim V_Y=$

$10+3i$. Finally let U be the subspace of $\operatorname{Im} H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W))$ which defines the linear system on C of the following type:

$$9\Sigma|_C+(12+6i \text{ points which move freely}).$$

We have $\dim U=13+6i$. We shall consider the intersection of V_Y and U. $\{\partial F/\partial Y|_C=0\}$ consists of $12+3i$ points; $3\Sigma|_C$ (one point with multiplicity 3) and $9+3i$ points $R_1, \cdots, R_{9+3i}$ none of which lies on Σ. Hence every section $s \in V_Y \subset H^0(\mathcal{O}_C(W))$ must be zero at these $12+3i$ points. On the other hand, we deduce from the definition of U that $s^+ \in U \subset H^0(\mathcal{O}_C(W))$ must be zero at $\Sigma|_C$ and that its multiplicity is at least 9. Therefore, we conclude that for every section $s \in V_Y \cap U$, $\{p \in C;\ s \text{ is zero at } p\}=9\Sigma|_C+R_1+\cdots+R_{9+3i}+$others. This imdlies that $\dim(V_Y \cap U) \leqq 4+3i$. Since $\dim V_Y=10+3i$ and $\dim U=13+6i$, we have $\dim(V_Y \cap U) \geqq 19+6i$. On the other hand, every section of V_Y+U must be zero at $\Sigma|_C$ and the multiplicity$\geqq 3$. Thus we obtain $\dim(V_Y+U) \leqq 19+6i$. Consequently, $\dim(V_Y+U)=19+6i$ and $\dim(V_Y \cap U)=4+3i$.

Next we shall consider the intersection of V_Z and (V_Y+U). Every section in V_Z is zero at $\{\partial F/\partial Z|_C=0\}$ (which consist of $18+6i$ points different from $\Sigma|_C$). On the other hand, every section in V_Y+U is zero at $\Sigma|_C$ with the multiplicity$\geqq 3$. Thus we conclude that $\dim V_Z \cap (V_Y+U)=1$. From this we obtain

$$\begin{aligned}\dim(V_Z+V_Y+U)&=\dim V_Z+\dim(V_Y+U)-\dim(V_Z \cap (V_Y+U))\\ &=4+19+6i-1\\ &=22+6i.\end{aligned}$$

Since $\dim(V_Z+V_Y+U)=\dim H^0(\mathcal{O}_C(W))=\dim H^0(\mathcal{O}_{P^0}(21+6i))=22+6i$, V_Z+V_Y+U coincides with $H^0(\mathcal{O}_C(W))$. This implies that $\operatorname{Im} J$ and $\operatorname{Im}(H^0(\mathcal{O}_W(W)) \subset H^0(\mathcal{O}_C(W))$ generate $H^0(\mathcal{O}_C(W))$. Therefore we have proved that Condition (4) is satisfied. Condition (1) is verified in the same way as case (a).

case (3): Let us consider the ruling $S \to \boldsymbol{P}^1$, and define the linear subspace $V_{(a,b)}$ of $H(N_{W/P})$ as follows:

$$V_{(a,b)}=\{s \in H^0(N_{W/P});\ s=S_0^a\alpha(T,S)XZ^2+S_0^b\beta(T,S)Z^3\},$$

where $T=(T_0, T_1)$ is a homogeneous coordinates system of $\boldsymbol{P}^1$, $S=(S_0, S_1)$ are natural injections:

$$S_0: \mathcal{O}_{P^1} \longrightarrow \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-i)$$

$$S_1: \mathcal{O}_{P^1}(-i) \longrightarrow \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(-i),$$

a, b are integers with $1 \leqq a \leqq 4$, $1 \leqq b \leqq 5$ and neither $\alpha(T,S)$ nor $\beta(T,S)$ is not identically zero on $D_0:=\{S_0=0\}$. We will prove that the image of $V_{(4,5)}$ under α and $\operatorname{Im}\beta$ generate $H^0(N_{W/P}|_C)$ in the case (3) of this collorary. Remark here that by List (1), $V_{(4,5)}$ is necessarily a linear subspace of $H^0(N_{W/P})$ for each i; $(0 \leqq i \leqq 12)$. We will use the following notation:

$$V_1=\{s \in V_{(4,5)};\ s=S_0^4\alpha(S,T)XZ^2\}$$

$$V_2=\{s \in V_{(4,5)};\ s=S_0^5\beta(S,T)Z^3\}$$

$$V_X=\{s\in H^0(N_{W/P}|_C);\ s=\ell_2\partial F/\partial X|_C,\ \ell_2\in H^0(\mathcal{O}_C(1)\otimes\pi^*K_{\bar{S}}^2)\}$$

$$V_Y=\{s\in H^0(N_{W/P}|_C);\ s=\ell_3\partial F/\partial X|_C,\ \ell_3\in H^0(\mathcal{O}_C(1)\otimes\pi^*K_{\bar{S}}^3)\}$$

$$V_Z=\{s\in H^0(N_{W/P}|_C);\ s=\ell_1\partial F/\partial X|_C,\ \ell_1\in H^0(\mathcal{O}_C(1))\}$$

$$P_0=C\cap\pi^{-1}(D_0),\qquad D_0:\text{ negative section of } S$$

In order to prove that $\alpha(V_{(4,5)})$ and $\mathrm{Im}\,\beta$ generate $H^0(N_{W/P}|_C)$, we may prove that $\alpha(V_1)+\alpha(V_2)+V_X+V_Z+V_Z=H^0(N_{W/P}|_C)$.

Case (c): $(\Sigma.C)=0$

Conditions (1), $\cdots$, (3) of Proposition (3.4) are valid in this case. First note that if $i\leqq 2$, then $H^0(N_{W/P})\to H^0(N_{W/P}|_C)$ is surjective because $H^0(K_{\bar{S}}^r|_l)$ is surjective for every positive integer r and for a fibre l. Hence we may assume that $3\leqq i\leqq 12$. Then the zero locus $\{\partial F/\partial X=-3X^2-aZ^2=0\}$ never contains C. In fact since $i\geqq 3$, $a\in H^0(S, K_{\bar{S}}^4)$ has zeros along D_0. W always have singularities on $\{X=Y=0\}\cap\pi^{-1}(D_0)$. Since C does not pass through any singular points, either $X\neq 0$ or $Y\neq 0$ must holds at P_0. If C is contained in $\{\partial F/\partial X=-3X^2-aZ^2=0\}$, then $X=0$ at P_0 because a vanishes at D_0. On the other hand, W is defined by $F=y^2Z-X^3-aXZ^2-bZ^3=0$, and b also vanishes at D_0. If we replace X^2 by $-1/3\,aZ^2$, then we have $Z(Y^2-2/3\,aXZ-bZ^2)=0$. Since $\{Z=0\}$ on W does not dontain C, $\{(Y^2-2/3\,aXZ-bZ^2)=0\}$ must contain C. So we have $Y=0$ at P_0 because both a and b vanish at D_0. Consequently, we have $X=Y=0$ at P_0, which is a contradiction. Therefore, C is not contained in $\{\partial F/\partial X=-3X^2-aZ^2=0\}$. From $(\Sigma.C)=0$, we have $N_{W/P}|_C=\mathcal{O}_{P^1}(12)$, $\mathcal{O}_C(1)\otimes\pi^*K_{\bar{S}}^2=\mathcal{O}_{P^1}(4)$,

$$\alpha(V_2)=\{s\in H^0(\mathcal{O}_{P^1}(12));\ (s)_0=5P_0+(7\text{ points which move freely})\}$$

$$V_X=\{s\in H^0(\mathcal{O}_{P^1}(12));\ (s)_0=(8\text{ fixed points apart from }P_0)+(4\text{ points which move freely})\}.$$

Let us consider an element s of $\alpha(V_2)\cap V_X$. Then s must have a zero at P_0 at least of order 5 and have at least 8 points apart from P_0 as its zero locus. Since $s\in H^0(\mathcal{O}_{P^1}(12))$, this implies that $s=0$. Hence it follows that $\alpha(V_2)\cap V_X=0$. Since $\dim\alpha(V_2)=8$ and $\dim V_X=5$, we have $H^0(N_{W/P}|_C)=\alpha(V_2)+V_X$. Q.E.D.

Case (d): $(\Sigma.C)=1$

If $0\leqq i\leqq 2$, then we can use the same argument in case (b), that is, the linear space generated by $\mathrm{Im}\,\alpha$, V_Z and V_Y coincides with $H^0(N_{W/P}|_C)$. Therefore, assume that $3\leqq i\leqq 12$. Then we can prove that C is not contained in $\{\partial F/\partial X=-3X^2-aZ^2=0\}$ in the same way as above. Conditions (1), $\cdots$, (3) of Prop. (3.4) are valid in this case. From $(\Sigma.C)=1$, we have the following:

$$N_{W/P}|_C=\mathcal{O}_{P^1}(21),\ \mathcal{O}_C(1)\otimes\pi^*K_{\bar{S}}^r=\mathcal{O}_{P^1}(2r+3)$$

$$\alpha(V_1)=\{s\in H^0(\mathcal{O}_{P^1}(21));\ (s)_0=7\Sigma|_C+4P_0+\text{ 6-fixed points which are apart from }\Sigma|_C\text{ and }P_0+\text{ 4 points which move freely}\}$$

$$\alpha(V_2)=\{s\in H^0(\mathcal{O}_{P^1}(21));\ (s)_0=9\Sigma|_C+5P_0+\text{ 7 points which move freely}\}$$

$$V_X=\{s\in H^0(\mathcal{O}_{P^1}(21));\ (s)_0=2\Sigma|_C+\text{ 12-fixed points which are apart from }\Sigma|_C+7\text{ points which move freely}\}$$

$V_Y=\{s\in H^0(\mathcal{O}_{P^1}(21))\,;\ (s)_0=3\Sigma|_C+$ 9-fixed points which are apart from $\Sigma|_C$ and P_0 + 9 points which move freely$\}$

$V_Z=\{s\in H^0(\mathcal{O}_{P^1}(21))\,;\ (s)_0=$18-fixed points which are apart from $\Sigma|_C+$3 points which move freely$\}$

Note that our argument below is valid even though $\Sigma|_C$ coincides with P_0. We define:

$$U=\{s\in H^0(\mathcal{O}_{P^1}(21))\,;\ (s)_0=9\Sigma|_C+4P_0+\text{ 8 points which move freely}\}\,.$$

Then U is contained in $\alpha(V_1+V_2)$. Indeed, $\dim U=9$ and $\dim\alpha(V_2)=8$. On the other hand, there is an element $s\in V_1$ such that $(s^*)_0=9\Sigma|_C+4P_0+8$ points which are apart from $\Sigma|_C$ and P_0. Clearly $\boldsymbol{C}s^*+\alpha(V_2)\subset U$. Since s^* is not contained in $\alpha(V_2)$, the dimension of $\boldsymbol{C}s^*+\alpha(V_2)$ is 9, which implies that $\boldsymbol{C}s+\alpha(V_2)=U$. Hence U is contained in $\alpha(V_1+V_2)$. From now on, we will prove that U, V_X, V_Y and V_Z generate $H^0(N_{W/P}|_C)=H^0(\mathcal{O}_{P^1}(21))$. First we have:

$$\dim(U+V_Y)=\dim U+\dim V_Y-\dim(U\cap V_Y)$$
$$=9+10-0=19$$

In fact, let s be an element of $U\cap V_Y$. Then $(s)_0$ must have $9\Sigma|_C+4P_0+$(9-fixed points which are apart from $\Sigma|_C$ and P_0, as its zeros. This implies that $s=0$ because $s\in H^0(\mathcal{O}_{P^1}(21))$. Hence $\dim(U\cap V_Y)=0$. Noxt we have:

$$\dim(U+V_Y+V_X)\geqq\dim(U+V_Y)+1=20\,.$$

This is because there is an element s of V_X such that $(s)_0=2\Sigma|_C+19$ points apart from $\Sigma|_C$, which is not contained in $U+V_Y$. Finally we have:

$$\dim((U+V_X+V_Y)\cap V_Z)\leqq 2$$

In fact, for every element s of $U+V_Y+V_X$, $(s)_0$ has $2\Sigma|_C$ as its fixed components, and for every element s of V_Z, $(s)_0$ has 18 fixed points which are apart from $\Sigma|_C$. Since $\dim(U+V_Y+V_X)\geqq 20$, $\dim V_Z=\mathbf{4}$, it follows from the above inequality that

$$\dim(U+V_Y+V_X+V_Z)\geqq 22\,.$$

Since $U+V_Y+V_X+V_Z\subset H^0(\mathcal{O}_{P^1}(21))$, this implies that the inclusion is in fact an equality.
Q. E. D.

(3.6) **Corollary.** *Let Λ be a linear subsystem of $|\mathcal{L}|$ on $\boldsymbol{P}=\boldsymbol{P}_S(\mathcal{O}\oplus K_S^2\oplus K_S^3)$ which consists of the elements of the following form:*

$$\varphi_1Y^2Z+\varphi_2X^3+\varphi_3XZ^2+\varphi_4Z^3=0\,,$$

where $S=\Sigma_i$ $(0\leqq i\leqq 12)$, $\mathcal{L}=\mathcal{O}_P(3)\otimes p^*K_S^{-6}$, $p:\boldsymbol{P}\to S$ and X, Y, Z are the same as in (1.2). *Consider the family $\pi:\mathcal{W}\to\boldsymbol{P}(\Lambda)$ of Weierstrass models over $\boldsymbol{P}(\Lambda)$. Then there is a dense subset T of $\boldsymbol{P}(\Lambda)$ which is obtained by excluding a countable number of proper closed subsets* (*in the sense of Zariski topology*) *from $\boldsymbol{P}(\Lambda)$, and T has the following property:*

Let C be a smooth rational curve in W_t, $t\in T$ which satisfies:

(1) *W_t is smooth around C;*

(2) *$(\Sigma . C)=0$ or 1, where Σ is the canonical section of W;*

(3) *$\pi_t(C)$ is a fibre l of $S \to \boldsymbol{P}^1$ and C is contained in $\pi_t^{-1}(l)$ as a section of $\pi_t^{-1}(l) \to l$.*
Then C is always a $(-1,-1)$-curve in W_t.

Proof. By Proposition (1.5), (3), we conclude that for a general Weierstrass model W, the curve $C \subset W$ with the above properties (1), (2), (3) is always isolated. Since the numerical condition: $(\Sigma . C)=0$ or 1 is fixed, we have finite number of such curves in W. Let us denote this number by $n_0(W)$ or $n_1(W)$ according as $(\Sigma . C)=0$ or 1. Running W in a family $\mathcal{W}$, we take its maximal value which is written n_1 or n_2. Then for a general point $t \in \boldsymbol{P}(\Lambda)$, $n_i(W_t)=n_i$. Thus from the first we may assume that $n_i(W)=n_i$ By (3.5), if we deform W to a suitable direction, these curves are all assumed to be $(-1,-1)$-curves. Therefore, we have the result. Q.E.D.

§4. Examples

In this section we will give some examples of $(-1,-1)$-curves on a Weierstrass model $W=W(K_S, a, b)$ over a surface $S=\Sigma_i$ $(0 \leqq i \leqq 12)$ or $\boldsymbol{P}^2$. To prove Theorem A′, we must find a number of $(-1,-1)$-curves $C_1, \cdots, C_n$ on $\widetilde{W}'$ such that they span $H_2(\widetilde{W}', \boldsymbol{C})$. But at the moment, the existence of such curves depends on the examples in this section. At the first reading, one had better skip this section and go to §5.

(4.1) We employ the same notation as in (3.1). Let us consider the same diagram as in (3.1):

$$
\begin{array}{ccccc}
 & & \boldsymbol{P}' & \longrightarrow & \boldsymbol{P} := \boldsymbol{P}_S(\mathcal{O}_S \oplus K_S^{2} \oplus K_S^{3}) \\
 & & \cup & & \cup \\
W_D & \subset & \mathcal{W} & \longrightarrow & W \\
 & & \downarrow \ \swarrow f & & \downarrow \\
 & & \mathcal{U} & \longrightarrow & S \\
 & & \downarrow & & \\
 & & [D] \in \boldsymbol{P}(H^0(\mathcal{O}_S(D))^*) = \boldsymbol{P}^{r+1}, & & (D^2)_S = r
\end{array}
\tag{4.1.1}
$$

Here $\boldsymbol{P}'$ is the fibre product of $\mathcal{U}$ and $\boldsymbol{P}$ over S. It can be easily shown that $f: \boldsymbol{P}' \to \mathcal{U}$ is a fibre bundle with $\boldsymbol{P}_D(\mathcal{O}_D \oplus K_S^{2}|_D \oplus K_S^{3}|_D)$ as a typical fibre. We denote by $\boldsymbol{P}_D$ this typical fibre for short. Let Δ^{r+1} be a small neighbourhood of $[D]$ in $\boldsymbol{P}^{r+1}$, and take one trivialization $\boldsymbol{P}_D \times \Delta^{r+1}$ of $\boldsymbol{P}'$ over Δ^{r+1}. Then $\widetilde{\mathcal{W}}$ is considered as a subvariety of $\boldsymbol{P}_D \times \Delta^{r+1}$ over Δ^{r+1}. In this situation, we obtain the following diagram:

$$
\begin{array}{ccccc}
 & & & & H^0(\Theta_{P_D}|_C) \\
 & & & & \downarrow \beta \\
T_{\Delta^{r+1},[D]} & \xrightarrow{\varphi} & H^0(N_{W_D/P_D}) & \xrightarrow{\alpha} & H^0(N_{W_D/P_D}|_C) \\
 & \searrow & & & \downarrow \\
 & & H^1(T_{W_D}) & \longrightarrow & H^1(T_{W_D}|_C) \\
 & & \phi & \nearrow & \downarrow \\
 & & & & H^1(\Theta_{P_D}|_C)
\end{array}
$$

Here the vertical sequence of the right hand side is exact and ϕ is the Kodaira-Spencer map in (3.1). We have $H^1(T_{Y_D}|_C)=\boldsymbol{C}^{r+1}$ from the exact sequence:

$$0 \longrightarrow T_C \longrightarrow T_{Y_D}|_C \longrightarrow N_{C/Y_D} \longrightarrow 0\,,$$

and the fact that $N_{C/Y_D}\cong\mathcal{O}_{P^1}(-r-2)$. If $H^2(\Theta_{P_D}|_C)=0$, then the injectivity of ϕ is equivalent to the statement that the linear subspace V of $H^0(N_{Y_D/P_D}|_C)$ generated by $\mathrm{Im}(\alpha\circ\varphi)$ and $\mathrm{Im}(\beta)$ coincides with $H^0(N_{Y_D/P_D}|_C)$. We will apply (3.1) in this form, as a criterion for $(-1\,'-1)$-curves, to each example in this section.

(4.2) **Example.** Consider a Weiertrass model $W_D=W(\mathcal{O}_{P^1}(-2-i),\ a_0,\ b_0)$ over $D=\boldsymbol{P}^1$. Take a homogeneous coordinates system $(T_0:T_1)$ of $\boldsymbol{P}^1$. By the definition of Weierstrass models we have $a_0\in H^0(\boldsymbol{P}^1,\ \mathcal{O}_{P^1}(4(2+i)))$ and $b_0\in H^0(\boldsymbol{P}^1,\ \mathcal{O}_{P^1}(6(2+i)))$. W_D is defined by the equation

$$F=Y^2Z-X^3-a_0XZ^2-b_0Z^3=0$$

in $\boldsymbol{P}_D=\boldsymbol{P}_{P^1}(\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(-2(2+i))\oplus\mathcal{O}_{P^1}(-3(2+i)))$

Set

$$
\begin{aligned}
&a_0=T_0^{4(2+i)}\\
&b_0=\bar{b}_0^{\,2}\,,\\
&\bar{b}_0=(T_0^{3(2+i)}-T_1^{3(2+i)})^2\,,\\
&C=\{w\in W_D;\ X=0,\ Y=\bar{b}Z\}\,.
\end{aligned}
$$

Then C is a section of $W_D\to D$ and, in particular, it is a smooth rational curve. C does not intersect with the canonical section of W_D. By the ruling $g:S\to\boldsymbol{P}^1$, we consider the homogeneous coordinates system $(T_0:T_1)$ of the base space $\boldsymbol{P}^1$ as that of C. Since $(\Sigma.C)=0$, we have $\mathcal{O}_C(1)\cong\mathcal{O}_C$, where $\mathcal{O}_C(1)$ denotes $\mathcal{O}_{P_D}(1)|_C$. ($\mathcal{O}_{P_D}(1)$ is the tautological line bundle of $\boldsymbol{P}_D$.) If we choose a suitable isomorphism between $\mathcal{O}_C(1)$ and $\mathcal{O}_C$, then we have $Z|_C=1$. Then we have:

$$\left.\frac{\partial F}{\partial Z}\right|_C=Y^2-2a_0XZ-3b_0Z^2|_C=-2b_0Z^2|_C=-2b_0$$

$$\left.\frac{\partial F}{\partial X}\right|_C=-3X^2-a_0Z^2|_C=-a_0Z^2|_C=-a_0$$

$$\left.\frac{\partial F}{\partial Y}\right|_C=2YZ|_C=2\bar{b}_0Z|_C=2\bar{b}_0$$

Since a_0 and b_0 has no common zeros, we infer that C does not pass through any singular points of W_D. In our case it is shown that $H^1(\Theta_{P_D}|_C)=0$.

Set $S=\Sigma_i$ $(0\leqq i\leqq 12)$. We put:

$$U: \mathcal{O}_{P^1} \longrightarrow \mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(-i) \tag{4.2.1}$$
$$V: \mathcal{O}_{P^1}(-i) \longrightarrow \mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(-i),$$

where U is a natural injection (id, 0) and V is a natural injection (0, id). Then define:

$$A\in H^0(S, K_S^{-4})=H^0(S, \mathcal{O}_S(8D_0+(8+4i)l))$$
$$A=a_0(t)U^8+a_1(t)U_L V+\cdots+a_n(t)U^{8-n}V^n \tag{4.2.2}$$
$$t=(T_0:T_1)$$
$$n=\begin{cases}4 & \text{if } i=9, \cdots, 12\\ 5 & \text{if } i=5, \cdots, 8\\ 6 & \text{if } i=3, 4\\ 8 & \text{if } i=0, 1, 2\end{cases}$$

$$B\in H^0(S, K_S^{-6})=H^0(S, \mathcal{O}_S(12D_0+6i)l))$$
$$B=b_0(t)U^{12}+b_1(t)U^{11}V+\cdots+b_m(t)U^{12-m}V^m \tag{4.2.3}$$
$$n=\begin{cases}7 & \text{if } i=7, \cdots, 12\\ 8 & \text{if } i=5, 6\\ 9 & \text{if } i=4\\ 10 & \text{if } i=3\\ 12 & \text{if } i=0, 1, 2,\end{cases}$$

where D_0 is the negative section of S, and l is a fibre. We set $W=W(K_S, A, B)$ and $D=\{x\in S;\ V=0\}$. Then we have the commutative diagram (4.1.1). In this case we have $r=i$. From now on, we will use the same notation as in (4.1). Let $(S_{-1}:S_0:S_1\cdots:S_{i+1})$ be a homogenous coordinates system of $\boldsymbol{P}(H^0(\mathcal{O}_S(D))^*)$ which parametrizes the elements of the linear system in such a way that $S_{-1}V-S_0T_0^iU-S_1T_0^{i-1}T_1-\cdots-S_iT_1^i=0$. With respect to this coordinates system, $[D]$ corresponds to $(1:0\cdots 0)$. Therefore, we employ the coordinate $(s_0, \cdots, s_i)$; $s_k=S_k/S_{-1}$ as a local coordinate of a neighborhood of $[D]$. Let us denote by Δ^{i+1} a polydisc with the coordinates $(s_0, \cdots, s_i)$ and with $[D]=(0, 0, \cdots, 0)$. We restrict $\mathcal{W}$ and $\mathcal{V}$ to $\mathcal{W}_{\Delta^{i+1}}=\tilde{\mathcal{W}}\times_{P^{i+1}}\Delta^{i+1}$ and $\mathcal{V}_{\Delta^{i+1}}=\mathcal{V}\times_{P^{i+1}}\Delta^{i+1}$, respectively. We denote $\boldsymbol{P}'\times_{P(H^0(\mathcal{O}_S(D))^*)}\Delta^{i+1}$ by $\boldsymbol{P}'_\Delta$. We want to give an explicit trivialization:

$$\boldsymbol{P}_D\times\Delta\cong\boldsymbol{P}'_\Delta.$$

In order to do that, first we give the following trivialization of $\mathcal{V}_\Delta$:

$$\begin{array}{ccc} D\times\Delta & \xrightarrow{\ p\ } & \mathcal{V}_\Delta \\ \Cup & & \Cup \\ (x, s) & \longrightarrow & ((g|_{D_s}^{-1}\circ g|_D)(x), s), \end{array}$$

where $g: S \to \boldsymbol{P}^1$ is the fibration as a ruled surface and D_s denotes the section of g which corresponds to $s \in \Delta$.

We denete by $q: D\times\Delta \to S$ the composition of the maps: $D\times\Delta \subset S\times\Delta$, $S\times\Delta \to S$ and denote by $r: \mathcal{U}_\Delta \to S$ the composition of the maps: $\mathcal{U}_\Delta \subset S\times\Delta$, $S\times\Delta \to S$. Let $\mathcal{O}_S(1)$ be the tautological line bundle of $g: S=\boldsymbol{P}_{P^1}(\mathcal{O}\oplus\mathcal{O}(-i)) \to \boldsymbol{P}^1$. Let L be a line bundle on S. Then we can write $L \cong \mathcal{O}_S(n) \otimes g^*\mathcal{O}_{P^1}(m)$ for suitable integers n and m. It is casily shown that p^*r^*L and q^*L are (non-canonically) isomorphic to each other. If $L = g^*\mathcal{O}_{P^1}(m)$, we have a canonical isomorphism between p^*r^*L and q^*L because $g\circ q = g\circ r\circ p$. Therefore, to give an isomorphism between p^*r^*L and q^*L, we may give an isomorphism between $p^*r^*\mathcal{O}_S(1)$ and $q^*\mathcal{O}_S(1)$. We have given an injection: $U: \mathcal{O}_S \to \mathcal{O}_S(1)$ in (4.2.1). Pulling back this injection by $r\circ p$ and q, we have:

$$\begin{array}{ccc} & \xrightarrow{\lambda} & p^*r^*\mathcal{O}_S(1) \\ \mathcal{O}_{D\times\Delta} & & \wr\| \\ & \xrightarrow[\eta]{} & q^*\mathcal{O}_S(1) \end{array}$$

Since D is disjoint from $D_0 = \{x \in S;\ U=0\}$ and Δ is sufficiently small, both λ and η are isomorphisms. Therefore, we have an isomorphism ξ between $p^*r^*\mathcal{O}_S(1)$ and $q^*\mathcal{O}_S(1)$ such that the above diagram is commutative. From this ξ, we obtain isomorphisms:

$$K_{S\times\Delta}^{-2}|_{D\times\Delta} \cong p^*(K_{S\times\Delta}^{-2}|_{\mathcal{U}_\Delta})$$
$$K_{S\times\Delta}^{-3}|_{D\times\Delta} \cong p^*(K_{S\times\Delta}^{-3}|_{\mathcal{U}_\Delta}).$$

This gives rise to a trivialization

$$\boldsymbol{P}_D \times \Delta \cong \boldsymbol{P}'_\Delta.$$

Let us prove that $H^0(N_{W_D/P_D}|_C)$ coincides with the linear subspace generated by $\mathrm{Im}(\alpha\circ\varphi)$ and $\mathrm{Im}\,\beta$ for suitable A and B. We employ the same notation as in the proof of Corollary (2.5), case (3). (Replace W by W_D, and $\boldsymbol{P}$ by $\boldsymbol{P}_D$) Then we have

$$V_Y = \{v \in H^0(\boldsymbol{P}^1, \mathcal{O}_{P^1}(6(2+i)));\ v \text{ is of the following form:}$$
$$\sum_{\substack{k>j \\ k+j=6(2+i)}} \alpha_{3(2+i)-j,j} T_0^k T_1^j + (\alpha_{0,3(2+i)} - \alpha_{3(2+i),0}) T_0^{3(2+i)} T_1^{3(2+i)}$$
$$+ \sum_{\substack{k<j \\ k+j=6(2+i)}} (-\alpha_{k,3(2+i)-k}) T_0^k T_1^j,\ \alpha_{k,j} \in \boldsymbol{C}\}$$

$$V_X = \{v \in H^0(\boldsymbol{P}^1, \mathcal{O}_{P^1}(6(2+i)));\ v \text{ is of the following form:}$$
$$\sum_{\substack{k\geq 4(2+i) \\ k+j=6(2+i)}} \beta_{k,j} T_0^k T_1^j,\ \beta_{k,j} \in \boldsymbol{C}\}$$

Therefore, we have:

$$V_X + V_Y = \sum_{j=0}^{2(2+i)} \boldsymbol{C} T_0^{6(2+i)-j} T_1^j \oplus \sum_{j=3(2+i)}^{5(2+i)} \boldsymbol{C} T_0^{6(2+i)-j} T_1^j \tag{4.2.4}$$
$$\oplus \sum_{j=2(2+i)+1}^{3(2+i)-1} \boldsymbol{C}(T_0^{6(2+i)-j} T_1^j - T_0^{3(2+i)-j} T_1^{3(2+i)+j}).$$

Since $\mathrm{codim}_{H^0(N_{W_D/P_D}|_C)}(V_X + V_Y) = i+1$, $H^1(\Theta_{P_D}|_C)=0$ and $H^1(T_{W_D}|_C) = \boldsymbol{C}^{i+1}$, we conclude that $\mathrm{Im}\,\beta = V_X + V_Y$.

Next consider the map $(\alpha\circ\varphi)$. Let T_k be the 1-dimensional subspace of $T_{\Delta i+1,[D]}$ which is generated by $\partial/\partial s_k$. We denote by $(\alpha\circ\varphi)_k$ the restriction of $(\alpha\circ\varphi)$ to T_k, and write $T_k=\{u_k(\partial/\partial s_k),\ u_k\in\boldsymbol{C}\}$.

With respect to the trivialization explained in the above, we have:

$$(\alpha\circ\varphi)_k\Big(\frac{\partial}{\partial s_k}\Big)=a_1(t)T_0^{i-k}T_1^k(XZ^2|_C)+b_1(t)T_0^{i-k}T_1^k(Z^3|_C)$$

$$=b_1(t)T_0^{i-k}T_1^k(Z^3|_C),$$

where $a_1(t)$ (resp. $b_1(t)$) is the one in (4.2.2) (resp. (4.2.3)). Hence, for example, if we put $b_1(t)=T_0^{3i+7}T_1^{2i+5}$, then by (4.2.4), we infer that

$$H^0(N_{W_D/P_D}|_C)=\mathrm{Im}\,\beta+\sum_k(\alpha\circ\varphi)_k(T_k)$$

$$=\mathrm{Im}\,\beta+\mathrm{Im}\,(\alpha\circ\varphi).$$

As a consequence, C is a $(-1.-1)$-curve on W such that (1) $(C.\Sigma)=0$ (2) $D=\pi(C)$ is a section of S with $(D^2)_S=i$, where $S=\Sigma_i$, and (3) C is a section of $W_D\to D$.

(4.3) Example. With the same notation as in (4.2), we set

$$F=Y^2Z-X^3-a_0XZ^2-b_0Z^3=0$$

$$a_0=2\bar{b}_0T_1^{i+3}$$

$$b_0=\bar{b}_0^2T_0^2$$

$$\bar{b}_0=T_0^{3i+5}.$$

We denote by W_D^0 (resp. W_D^1) the open set of W_D defined by $T_0\neq0$ (resp. $T_1\neq0$).

Let C be a smooth rational curve which is a section of $W_D\to D$, and is defined by the following equations:

On W_D^0

$$T_0^2X=T_1^{2i+6}Z,\qquad T_0^3Y=(T_1^{3i+9}+\bar{b}_0T_0^4)Z$$

On W_D^1

$$T_0^2X=T_1^{2i+6}Z,\qquad T_0T_1^{2i+6}Y=(T_1^{3i+9}+\bar{b}_0T_0^4)X.$$

By the ruling $g: S\to\boldsymbol{P}^1$, we consider the system of homogeneous coordinates $(T_0:T_1)$ of the base space $\boldsymbol{P}^1$ as that of $C\cong\boldsymbol{P}^1$. This is equivalent to taking an identification of L with $g^*\mathcal{O}_{P^1}(1)|_C$, where L is the line bundle on C which is a positive generator of $\mathrm{Pic}(C)=\boldsymbol{Z}$. Hence the identification is unique up to constants. If we choose a suitable (non-zero) constant, then we may assume that $(X|_C)=T_1^{2i+6}T_0$, $(Y|_C)=(T_1^{3i+9}+\bar{b}_0T_0^4)$, $(Z|_C)=T_0^3$. Then we have:

$$\frac{\partial F}{\partial X}\Big|_C=-T_0^2(3T_1^{4i+12}+2\bar{b}_0T_0^4T_1^{i+3})$$

$$\frac{\partial F}{\partial Y}\Big|_C=2T_0^3(T_1^{3i+9}+\bar{b}_{00}T_0^4)$$

$$\frac{\partial F}{\partial Z}\Big|_C=T_1^{6i+18}-2\bar{b}_0T_0^4T_1^{3i+9}-2\bar{b}_0^2T_0^8$$

It can be checked that C does not pass through any singular points of W_D and that $(\Sigma . C)_{W_D}=1$ for the canonical section Σ of W_D.

In the same way as (4.2), we construct a Weierstrass model W and consider the diagram in (4.1). We employ the same trivialization

$$\boldsymbol{P}_D\times\Delta\cong\boldsymbol{P}'_\Delta$$

as (4.2). The similar calculations to (4.2) show that

$$H^0(C,\ N_{W_D/\boldsymbol{P}_D}|_C)/\operatorname{Im}\alpha=\sum_{k=2}^{i+2}\boldsymbol{C}[T_0^{6i+21-k}T_1^k]$$

On the other hand, we have

$$\begin{aligned}(\alpha\circ\varphi)_k\Big(\frac{\partial}{\partial s_k}\Big)&=a_1(t(T_0^{i-k}T_1^k(XZ^2|_C)+b_1(t)T_0^{i-k}T_1^k(Z^3|_C)\\&=a_1(t)T_0^{i-k+7}T_1^{2i+k+6}+b_1(t)T_0^{i-k+9}T_1^k\end{aligned}$$

Therefore, for example, if we put $a_1(t)=0$ and $b_1(t)=T_0^{5i+10}T_1^2$, then we infer that

$$\begin{aligned}H^0(N_{W_D/\boldsymbol{P}_D}|_C)&=\operatorname{Im}\beta+\sum_k(\alpha\circ\varphi)_k(T_k)\\&=\operatorname{Im}\beta+\operatorname{Im}(\alpha\circ\varphi).\end{aligned}$$

As a consequence, C is a $(-1,-1)$-curve on W such that (1) $(C.\Sigma)=1$, (2) $D=\pi(C)$ is a section of $S=\Sigma_i$ with $(D^2)_S=i$, and (3) C is a section of $W_D\to D$.

(4.4) **Example.** Consider a Weierstrass model $W_D=W(K_{\boldsymbol{P}^1},0,\beta)$ over $D=\boldsymbol{P}^1$. Take a homogeneous coordinates system $(T_0:T_1)$ of $\boldsymbol{P}^1$. W_D is defined by the equation

$$F=Y^2Z-X^3-\beta Z^3=0 \quad \text{in} \quad \boldsymbol{P}_D:=\boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{O}_{\boldsymbol{P}^1}\oplus K_{\boldsymbol{P}^1}^2\oplus K_{\boldsymbol{P}^1}^3).$$

Let C be a smooth rational curve on $\boldsymbol{P}_D$ which is defined by

$$X-b(t)Z=0,\qquad Y-a(t)Z=0,$$

where

$$\begin{aligned}a(t)&=3T_0^4T_1^2+T_1^6\\b(t)&=2T_0^4+T_1^4\end{aligned}$$

Here if we set $\beta=-8T_0^{12}-3T_0^8T_1^4$, then C is contained in W_D as a section of $W_D\to D$. It is shown that $(\Sigma.C)=0$ and that C does not pass through any singular ponints of W_D.

Set $S=\Sigma_i$ $(0\leqq i\leqq 12)$. We put U and V in the same way as (4.2). Furthermore let $s=(S_0:S_1)$ be a homogeneous coordinate of the base space $\boldsymbol{P}^1$ with respect to the ruling $g:S\to\boldsymbol{P}^1$. Then define:

$$\begin{aligned}&A\in H^0(S,\ K_S^{-4})=H^0(S,\ \mathcal{O}_S(8D_0+(8+4i)l))\\(4.4.1)\qquad &A=a_0(s)U^8+a_1(s)U^7V+\cdots+a_n(s)U^{8-n}V^n\\&s=(S_0:S_1)\end{aligned}$$

$$n=\begin{cases} 4 & \text{if } i=9, \cdots, 12 \\ 5 & \text{if } i=5, \cdots, 8 \\ 6 & \text{if } i=3, 4 \\ 8 & \text{if } i=0, 1, 2 \end{cases}$$

$$B\in H^0(S, K_{\bar{S}})=H^0(S, \mathcal{O}_S(12D_0+(12+6i)l))$$

(4.4.2) $$B=b_0(s)U^{12}+b_1(s)U^{11}V+\cdots+b_m(s)U^{12-m}V^m$$

$$n=\begin{cases} 7 & \text{if } i=7, \cdots, 12 \\ 8 & \text{if } i=5, 6 \\ 9 & \text{if } i=4 \\ 10 & \text{if } i=3 \\ 12 & \text{if } i=0, 1, 2, \end{cases}$$

and we consider the Weierstrass model $W=W(K_S, A, B)$ over S. Let us define $f=g\circ\pi$. Then f is a $K3$-fibration over $\boldsymbol{P}^1$. We will investigate how to define the coefficients A and B of W in order that W_D is realized as a fibre of f (i.e. D is a fibre of g and $\pi^{-1}(D)\cong W_D$).

(4.4.3) Let W be as above and set $\boldsymbol{P}(K_S):=\boldsymbol{P}_S(\mathcal{O}_S\oplus K_S^{\frac{2}{S}}\oplus K_S^{\frac{3}{S}})$. Let p be the projection of $\boldsymbol{P}(K_S)$ to S. Denote by S_Δ (resp. $\boldsymbol{P}_\Delta$) the inverse image $g^{-1}(\Delta)$ (resp. $(g\circ p)^{-1}(\Delta)$), where Δ is a sufficiently small neighborhood of $x\in\boldsymbol{P}^1$. We want to give an explicit trivialization:

$$(g\circ p)^{-1}(x)\times\Delta\cong\boldsymbol{P}_\Delta$$

First we can define the natural trivialization between $g^{-1}(x)\times\Delta$ and S_Δ by using U and V. In fact we can define the isomorphism:

$$\mathcal{O}_\Delta\oplus\mathcal{O}_{P^1}(-i)|_\Delta \overset{j}{\longrightarrow} \mathcal{O}_\Delta\oplus\mathcal{O}_\Delta\,;\ \mathcal{O}_{P^1}|_\Delta=\mathcal{O}_\Delta$$

such that $(j\circ U)(1)=(1, 0)$, $(j\circ V)(1)=(0, 1)$. This isomorphism induces the trivialization $\eta: g^1(x)\times\Delta\to S_\Delta$. Let q be the composition of the maps: $g^{-1}(x)\times\Delta\subset S\times\Delta$ and $S\times\Delta\to S$. Let r be the map $S_\Delta\subset S$. Let $\mathcal{O}_S(1)$ be the tautological line bundle of $S=\boldsymbol{P}_{P^1}(\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(-i))$. Let L be a line bundle on S. Then we can write $L\cong\mathcal{O}_S(n)\otimes g^*\mathcal{O}_{P^1}(m)$ for suitable integers n and m. We want to give an isomorphism between η^*r^*L and q^*L. By the above homomorphism j, we have a natural isomorphism between them if L is of the form $\mathcal{O}_S(n)$. Say $x=(x_0: x_1)$ with respect to the homogeneous coordinates system $s=(S_0: S_1)$ of $\boldsymbol{P}^1$. Then $x_0\neq0$ or $x_1\neq0$ must hold. Assume, for instance, $x_0\neq0$. Consider the injection

$$S_0: \mathcal{O}_S \longrightarrow g^*\mathcal{O}_{P^1}(1),$$

and pull back the injection by $(r\circ\eta)$ and q, respectively. Then we have:

$$\mathcal{O}_{g^{-1}(x)\times\Delta} \begin{array}{l} \xrightarrow{\ \kappa\ } \eta^*r^*g^*\mathcal{O}_{P^1}(1) \\ \qquad\qquad \wr\| \\ \xrightarrow[\ \omega\ ]{} q^*g^*\mathcal{O}_{P^1}(1) \end{array}$$

Since $x_0\neq 0$ and Δ is sufficiently small, both κ and ω are isomorphisms. Hence there is an isomorphism between $\eta^*r^*g^*\mathcal{O}_{P^1}(1)$ and $q^*g^*\mathcal{O}_{P^1}(1)$ such that the above diagram commutes. As a consequence we have an isomorphism between η^*r^*L and q^*L for every $L\in \mathrm{Pic}(S)$, which, of course, induces an trivialization of $\boldsymbol{P}_\Delta$.

(4.4.4) Let $x=(x_0: x_1)\in \boldsymbol{P}^1$, $x_0\neq 0$. Let Δ be a small neighborhood of x, and we employ $s=S_1/S_0$ as a local coordinate for Δ at x. Then the trivialization of $\boldsymbol{P}_\Delta$ in (4.4.3) induces an isomorphism between $W_\Delta:=W\times_{P^1}\Delta$ and $\boldsymbol{W}$ which is defined by

$$\begin{aligned} Y^2Z=X^3&+(a_0(1:s)U^8+a_1(1:s)U^7V+\cdots+a_n(1:s)U^{8-n}V^n)XZ^2 \\ &+(b_0(1:s)U^{12}+b_1(1:s)U^{11}V+\cdots+b_m(1:s)U^{12-m}V^m)Z^3 \end{aligned}$$

in $\boldsymbol{P}_{P^1}(\mathcal{O}_{P^1}\oplus K_{P^1}^2\oplus K_{P^1}^3)\times\Delta$, where X, Y and Z are injections:

$$X:\ K_{P^1}^2 \longrightarrow \mathcal{O}_{P^1}\oplus K_{P^1}^2\oplus K_{P^1}^3$$

$$Y:\ K_{P^1}^3 \longrightarrow \mathcal{O}_{P^1}\oplus K_{P^1}^2\oplus K_{P^1}^3$$

$$Z:\ \mathcal{O}_{P^1} \longrightarrow \mathcal{O}_{P^1}\oplus K_{P^1}^2\oplus K_{P^1}^3,$$

and $(U: V)$ is the relative homogeneous coordinates of $\boldsymbol{P}^1\times\Delta$. For example, if $a_j(1: x_1/x_0)=0$ for every j, and $b_0(1: x_1/x_0)=-8$, $b_4(1: x_1/x_0)=-3$, $b_k(1: x_1/x_0)=0$ for every $k\neq 0, 4$, then we know that $f^{-1}(x)\cong W_D$. We can prove the following fact:

(**4.4.5**) (1) *In the above situation, if we choose suitable A and B, then W_D is realized as a fibre of $f: W=W(K_S, A, B)\to \boldsymbol{P}^1$ over at least* $12+2i$ *points:* $\{P_1, \cdots, P_{12+2i}\}\subset \boldsymbol{P}^1$. *(i.e. $f^{-1}(P_j)\cong W_D$).*

(2) *Let C_j denote the smooth rational curve on $f^{-1}(P_j)$ which corresponds to C on W_D. Then we may assume that every C_j is a $(-1,-1)$-curve on W.*

Proof of (1): Since W_D is defined by

$$Y^2Z=X^3+(-8T_0^{12}-3T_0^8T_1^4)Z^3,$$

we may put

$$a_j(s)=0 \qquad \text{for every } j,$$

$$b_k(s)=0 \qquad \text{for every } k\geqq 5,$$

$$b_k(s) \text{ has zeros at } \{P_1, \cdots, P_{12+2i}\}$$

$$b_0(1: x_1^{(j)}/x_0^{(j)})=-8$$

$$b_4(1: x_1^{(j)}/x_0^{(j)})=-3,$$

where $(x_0^{(j)}: x_1^{(j)})=P_j$. We can easily check from the condition for the degree of each $b_k(s)$ that there are such b_k's. Q.E.D.

Proof of (2): We apply the criterion of $(-1,-1)$-curve in (4.1). Of course we mploy the trivialization of $\boldsymbol{P}_\Delta$ explained in (4.4.3). First, by a straightforward calculation, we have for each C_j:

$$\operatorname{Im}\beta=\sum_{\substack{k\geq i\\ k\neq 3,7,11}}^{12} CU^kV^{12-k}\oplus C(4U^{11}V+U^7V^5)\oplus C(12U^{11}V-U^3V^9)$$

Next we have for each C_j:

$$(\alpha\circ\varphi)\Big(\frac{\partial}{\partial s}\Big)=\sum_k\Big(\frac{db_k}{ds}(1:s)|_{s=x_1^{(j)}/x_0^{(j)}}\Big)U^{12-k}V^k$$

Therefore, if we choose b_1 such that b_1 has a zero of order 1 at each P_j, then we have $\operatorname{Im}\beta+\operatorname{Im}(\alpha\circ\varphi)=H^0(N_{W_D/P_D}|_{C_j})$ for each j. Note that it follows from the condition for the degree of b_1 that such a b_1 exists. Q.E.D.

(4.5) **Example.** Consider the Weierstrass model $W_D=W(K_{P^1}, a_0, b_0)$ which is defined by

$$F=Y^2Z-X^3-a_0XZ^2-b_0Z^3$$

in $\boldsymbol{P}_D:=\boldsymbol{P}_{P^1}(\mathcal{O}_{P^1}\oplus K_{P^1}^2\oplus K_{P^1}^3)$. Take a homogeneous coordinates system $(T_0:T_1)$ of $\boldsymbol{P}^1$ and we set:

$$a_0=2\bar{b}_0T_1^3$$
$$b_0=\bar{b}_0^2T_0^2$$
$$\bar{b}_0=T_0^5.$$

We denote by W_D^0 (resp. W_D^1) the open set of W_D defined by $T_0\neq 0$ (resp. $T_1\neq 0$). Let C be a smooth rational curve which is a section of $W_D\to D$ and is defined by the following equations:

on W_D^0

$$T_0^2X=T_1^6Z,\qquad T^3Y_0=(T_1^9+\bar{b}_0T_0^4)Z$$

on W_D^1

$$T_0^2X=T_1^6Z,\qquad T_0T_1^6Y=(T_1^9+\bar{b}_0T_0^4)X.$$

It can be checked that C does not pass through any singular points of W_D and that $(\Sigma.C)_{W_D}=1$ for the canonical section Σ of W_D. As in (4.4), we consider a Weierstrass model $W=W(K_S, A, B)$ on $S=\Sigma_i$ $(0\leqq i\leqq 12)$. Let us define $f=g\circ\pi$. Note that a general fibre of f is an elliptic fibration. Then if we choose suitable A and B, then W_D is realized as a fibre of f (i.e. $\pi(D)$ coincides with $g^{-1}(x)$ for some $x\in\boldsymbol{P}^1$ and $W_D\cong f^{-1}(x)$). Moreover, we may assume that C is a $(-1,-1)$-curve on W. The proof is done by using the criterion (4.1) and the trivialization described in (4.4.3). Then the calculation is almost similar to that of (4.3) because the defining equations of C in W_D is almost of the same form.

(4.6) **Conclusion from** (4.2), (4.3), (4.4), (4.5). In the above examples, we have constructed four $(-1,-1)$-curves of different numerical types on special Weierstrass model W over $S=\Sigma_i$ $(0\leqq i\leqq 12)$. Since a $(-1,-1)$-curve is stable in deformation of the ambient 3-fold, we know the following.

Let $W=W(K_S, a, b)$ be a Weierstrass model over $S=\Sigma_i$ $(0\leqq i\leqq 12)$. If we choose a *and b generally, then W contains* $(-1,-1)$*-curves* C_1, C_2, $C_3^{(1)}, \cdots, C_3^{(12+2i)}$, C_4 *of the following numerical classes:*

C	$(\Sigma . C)$	$(H_1 . C)$	$(H_2 . C)$
C_1	0	0	1
C_2	1	0	1
$C_3^{(j)}$	0	1	0
C_4	1	1	0

where H_1 *denotes* $\pi^* D_0$ (D_0 *is the negative section of* $S=\Sigma_i$) *and* H_2 *denotes* $\pi^* l$ (l *is a fiber of* $g: S \to \boldsymbol{P}^1$).

The above examples give, however, no informations about the arrangements of the curves on W (e.g. the problem whether these curves can be chosen such that they are mutually disjoint). The next example and Proposition (4.8) are designed for the purpose.

(4.6.1) **Definition.** A $(-1,-1)$-curve C on $W=W(K_S, a, b)$ with $S=\Sigma_i$ $(0 \leqq i \leqq 12)$ is called of type I (resp. II, III, IV) if C has the same numerical class as that of C_1 (resp. C_2, C_3, C_4).

(4.7) **Example.** This example will assure that we can take two $(-1.-1)$-curves of type I and II mutually disjoint. We use the notation of (4.2). Define a Weierstrass model $W=W(K_S, A, B)$ over $S=\Sigma_i$ $(0 \leqq i \leqq 12)$ by setting

$$A=T_0^{4i8}U^8+(-T_0^{3i+8}+2T_0^3T_1^{3i5})U^7V$$

$$B=(T_0^{3i+6}-T_1^{3i+6})^2U^{12}+T_0^{3i+7}T_1^{2i+5}U^{11}V+(2T_0^{i+6}T_1^{3i+6}-T_0^{2i+7}T_1^{2i+5}+T_0^{4i+12})U^{10}V^2$$

Let $D_1=\{x \in S ; V=0\}$ and $D_2=\{x \in S ; V=T_0^iU\}$. Then $W_{D_1}:=\pi^{-1}(D_1)$ is isomorphic to the Weierstrass model W_1 over $\boldsymbol{P}^1$ which is defined by

$$Y^2Z=X^3+T_0^{4i+8}XZ^2+(T_0^{3i+6}-T_1^{3i+6})^2Z^3$$

in $\boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-2(2+i)) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-3(2+i)))$. Similarly W_{D_2} is isomorphic to the Weierstrass model W_2 over $\boldsymbol{P}^1$ defined by

$$Y^2Z=X^3+2T_0^{i+3}T_1^{3i+5}XZ^2+T_1^{6i+12}Z^3.$$

in $\boldsymbol{P}_{\boldsymbol{P}^1}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(-2(2+i)) \oplus \mathcal{O}_{\boldsymbol{P}^1}(-3(2+i)))$. Let us consider the smooth rational curves C_+ (resp. C_-) on W_1 defined by

$$X=0, \quad Y=(T_0^{3i+6}-T_1^{3i+6})Z \quad (\text{resp. } -(T_0^{3i+6}-T_1^{3i+6})Z).$$

By abuse of notation, we denote by C_+ (resp. C_-) the rational curve on W_{D_1} correspond ing to C_+ (resp. C_-). Then by (4.2), C_+ is a $(-1,-1)$-curve on W. Since W has a involution with respect to Σ and since C_+ and C_- are conjugate with respect to this involution, C_- is also a $(-1,-1)$-curve on W. Remark that both C_+ and C_- are of type I (see (4.6.1)).

Next consider the smooth ratoinal curve C' on W_2 defined by

$$T_1^2X=T_0^{2i+6}Z\,,\qquad T_1^3Y=(T_0^{3i+9}+\tilde{b}_0T_1^4)Z \qquad \text{on } W_2^1\,,$$

$$T_1^2X=T_0^{2i+6}Z\,,\qquad T_1T_0^{2i+6}Y=(T_0^{3i+9}+\tilde{b}_0T_1^4)W \qquad \text{on } W_2^0\,,$$

where W_2^0 (resp. W_2^1) denotes the open set of W_2 defined by $T_0\neq 0$ (resp. $T\neq 0$).

Here we remark that W_2 is the Weierstrass model obtained from the example in (4.3) by exchanging T_0 and T_1, and that C' is obtained from C in (4.3) by the same way. By abuse of notation, we denote by C' the rational curve on W_{D_2} corresponding to C'. Then by the remark, in order to check whether C' is a $(-1,-1)$-curve on W, first we may replace

$$\begin{aligned} &T_0 \text{ by } T_1\,,\\ &T_1 \text{ by } T_0\,,\\ &V \text{ by } V+T_1^4U\,,\\ &U \text{ by } U \end{aligned}$$

in the defining eqnations of W and C' and next we may apply the calculation of (4.3) directly. Then it follows that C' is not $(-1,-1)$-curve on W. We can, however, prove that there is a small deformation (W_t, C_t') of (W, C') in $\boldsymbol{P}_S(\mathcal{O}_S\oplus K_S^2\oplus K_S^3)$ such that C_t' is a $(-1,-1)$-curve on W_t. This small deformation induces a small deformation $(W_t, C_{-,t})$. Since C_- are $(-1,-1)$-curves on W, $C_{-,t}$ are also $(-1,-1)$-curves on W_t. On the other hand, it is easily checked that $C_-\cap C'=\emptyset$ in W. Hence we have $C_{-,t}\cap C_t'=\emptyset$ in W_t. Therefore, we infer that W_t contains mutually disjoint $(-1,-1)$-curves $C_{-,t}$ and C_t'. Moreover, $C_{-,t}$ is of type I and C_t' is of type II.

(4.8) **Proposition.** *Let $W=W(K_S, a, b)$ be a Weierstrass model over $S=\Sigma_i$ $(0\leqq i\leqq 12)$. Let D be a section of $g: S\to \boldsymbol{P}^1$ with $(D)_S^2=i$ and let l be a fiber of g. Let C be a smooth rational curoe on $W_D:=W\times_S D$ which is a section of $W_D\to D$. Let C' be a smooth rational curve on $W_l:=W\times_S l$ which is a section of $W_l\to l$. Assume that*

(1) *both C and C' are $(-1,-1)$-curves on W,*

(2) *$(C.\Sigma)=0$ or 1*

(3) *$C\cap C'\neq\emptyset$*

Then there is a small deformation (W_t, C_t, C_t') of (W, C, C') in $\boldsymbol{P}=\boldsymbol{P}_S(\mathcal{O}_S\oplus K_S^2\oplus K_S^3)$ such that $C_t\cap C_t'=\emptyset$.

Proof. The case where $(C.\Sigma)=0$: Since $(\Sigma.C)=0$, the natural map: $H^0(N_{W/\boldsymbol{P}})\to H^0(N_{W/\boldsymbol{P}}|_C)$ is surjective. On the other hand, since $H^0(N_{C/W})=H^1(N_{C/W})=H^1(N_{C/W})=0$ by (1), we have an isomorphism $H^0(N_{C/\boldsymbol{P}})\cong H^0(N_{W/\boldsymbol{P}}|_C)$. Therefore, we have a map $\theta: H^0(N_{W/\boldsymbol{P}})\to H^0(N_{C/\boldsymbol{P}})$. If we note that C is stable under deformations of W, then θ can be interpretted geometrically as a correspondence between an infinitesimal displacement of W in $\boldsymbol{P}$ and an infinitesimal displacement of C in $\boldsymbol{P}$.

Here first we consider the following commutative diagram:

$$\begin{array}{ccc} H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(W)) & \longrightarrow & H^0(W, N_{W/\boldsymbol{P}}) \\ \uparrow & & \downarrow j \\ H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(W)\otimes\mathcal{J}_{\pi^{-1}(l)}) & \xrightarrow[\phi]{} & H^0(W, N_{W/\boldsymbol{P}}\otimes\mathcal{O}_W(-\pi^{-1}(l())), \\ \uparrow & & \\ H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(W-p^{-1}(l))) & & \end{array}$$

where p is the projection of $\boldsymbol{P}$ to S and where $\mathcal{J}_{\pi^{-1}(l)}$ denotes the defining ideal of $\pi^{-1}(l)$ in $\boldsymbol{P}$. Note that $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(W)\otimes\mathcal{J}_{\pi^{-1}(l)})$ corresponds to the linear subsystem of $|\mathcal{O}_{\boldsymbol{P}}(W)|$ which consists of the elements W_t such that W_t contains $\pi^{-}(l)$. ϕ is surjective because we have $H^1(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-p^{-1}(l)))=0$, which follows from the spectral sequences of Leray:

$$0 \longrightarrow H^1(S, \mathcal{O}_S(-l)) \longrightarrow H^1(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(-p^{-1}(l)) \longrightarrow H^0(S, R^1p_*\mathcal{O}_{\boldsymbol{P}}\otimes\mathcal{O}_S(-l))$$

$$0 \longrightarrow H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-1))\ H^1(S, \mathcal{O}_S(-l)) \longrightarrow H^0(\boldsymbol{P}^1, R^1g_*\mathcal{O}_S\otimes\mathcal{O}_{\boldsymbol{P}^1}(-1))$$

Next let us write $q=C\cap C'$ and consider the map $k: H^0(N_{C/\boldsymbol{P}})\rightarrow N_{C/\boldsymbol{P}}\otimes k(q)$. Then k is surjective because $N_{C/\boldsymbol{P}}$ is generated by its global sections. In fact, this follows from $H^1(N_{C/W}\otimes\mathcal{O}_{\boldsymbol{P}^1}(-1))=0$ because C is a rational curve and $N_{C/W}$ is a direct sum of line bundles. $H^1(N_{C/W}\otimes\mathcal{O}_{\boldsymbol{P}^1}(-1))$ can be computed by using the exact sequence which are obtained from the following by tensoring $\mathcal{O}_{\boldsymbol{P}^1}(-1)$:

$$0 \longrightarrow \Theta_C \longrightarrow \Theta_{\boldsymbol{P}}|_C \longrightarrow N_{C/\boldsymbol{P}} \longrightarrow 0$$

$$0 \longrightarrow \Theta_{\boldsymbol{P}/S}|_C \longrightarrow \Theta_{\boldsymbol{P}}|_C \longrightarrow \pi^*\Theta_S|_C \longrightarrow 0$$

$$0 \longrightarrow \Theta_D \longrightarrow \Theta_{S/D} \longrightarrow N_{D/S} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \pi^*(\mathcal{E}^*)\otimes\mathcal{O}_{\boldsymbol{P}}(1)|_C \longrightarrow \Theta_{\boldsymbol{P}/S}|_C \longrightarrow 0,$$

where $\mathcal{E}:=\mathcal{O}_S\oplus K_S^2\oplus K_S^3$, and $\mathcal{E}^*$ is its dual sheaf. Then since the restriction $H^0(N_{W/C}\otimes\mathcal{O}_W(-\pi^{-1}(l)))\rightarrow H^0(N_{W/\boldsymbol{P}}|_C\otimes\mathcal{O}_C(-\pi^{-1}(l))\backslash$ is surjective, we have:

$$\begin{array}{ccccc} H^0(N_{W/\boldsymbol{P}}\otimes\mathcal{O}_W(-\pi^{-1}(l))) & \subset & H^0(N_{W/\boldsymbol{P}}) & & \\ \downarrow & & \downarrow & & \\ H^0(N_{W/\boldsymbol{P}}|_C\otimes\mathcal{O}_C(-\pi^{-1}(l))) & \underset{\text{codim } 1}{\subset} & H^0(N_{W/\boldsymbol{P}}|_C) & \twoheadrightarrow & N_{C/W}\otimes k(q) \cong \boldsymbol{C}^3 \end{array}$$

Let V be the image of $\Theta_{C'}\otimes k(q)$ by the composition of the maps $\Theta_{C'}\otimes k(q)\subset\Theta_{\boldsymbol{P}}\otimes k(q)\rightarrow N_{C/\boldsymbol{P}}\otimes k(q)$. Then $\dim_{\boldsymbol{C}} V=1$. By the above diagram, we infer that there is an element $\alpha\in H^0(N_{W/\boldsymbol{P}}\otimes\mathcal{O}_W(-\pi^{-1}(l)))$ which is mapped to an element $\beta\in N_{C/W}\otimes k(q)$ with $\beta\in V$. On the other hand, since ϕ is surjective, we can take a flat family in $\boldsymbol{P}\times\Delta$, $f:\mathcal{W}\rightarrow\Delta^1=\{t\in\boldsymbol{C}\,;\,|t|<\varepsilon\}$ such that (2) $f^{-1}(0)=W$, (2) $\varphi(\partial/\partial t)=\alpha$, where $\varphi: T_{[0],\Delta^1}\rightarrow H^0(N_{W/\boldsymbol{P}})$ is the Kodaira-Spencer map with respect to φ and (3) for every $t\in\Delta^1$, W_t contains $\pi^{-1}(l)$. This implies that on W_t $(t\neq0)$, C_t and $C_t'(=C')$ are mutually disjoint.

THE CASE WHERE $(\Sigma.C)=1$: Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & H^0(N_{C/P}) & \simeq & H^0(N_{W/P|C}) \\
 & & & & \uparrow\gamma & & \| \\
H^0(p^*\mathcal{E}^*\otimes\mathcal{O}_C(1)) & \xrightarrow{\alpha} & H^0(\Theta_{P/S|C}) & \xrightarrow{\beta} & H^0(\Theta_{P|C}) & \longrightarrow & H^0(N_{W/P|C}) \\
 & & & J & & & \uparrow j_1 \\
 & & & & & & H^0(N_{W/P}) \\
 & & & & & & \uparrow j_2 \\
 & & H^0(\boldsymbol{P}, \mathcal{O}_P(W)\otimes\mathcal{I}_{\pi^{-1}(l)}) & & \xrightarrow{\phi} & & H^0(N_{W/P}\otimes\mathcal{O}_W(-\pi^{-1}(l)))
\end{array}$$

We will find an element $\theta\in H^0(N_{W/P}\otimes\mathcal{O}_W(-\pi^{-1}(l)))$ such that 1) there is an element $\tau\in H^0(p^*\mathcal{E}^*\otimes\mathcal{O}_C(1))$ such that $J(\tau)=j_1\circ j_2(\theta)$, 2) $(\beta\circ\alpha)(\tau)\in H^0(\Theta_P|_C)$ is not zero at $q=C\cap C'$. If we find such an element, then we have our assertion. In fact, consider the diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & N_{C/P}\otimes k(q) & & & & \\
 & & & & \uparrow & & & & \\
0 & \longrightarrow & \Theta_{P/S}\otimes k(q) & \xrightarrow{\beta(q)} & \Theta_P\otimes k(q) & \xrightarrow{h} & p^*\Theta_S\otimes k(q) & \longrightarrow & 0 \\
 & & & \nearrow i_2 & \uparrow i_1 & & & & \\
 & & \Theta_{C'}\otimes k(q) & & \Theta_C\otimes k(q) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Here note that $\mathrm{Im}(h\circ i_1)\cap\mathrm{Im}(h\circ i_2)=0$ in $p^*\Theta_S\otimes k(q)$. Since now we have $\beta(q)\circ\alpha(q)(\theta)\neq 0$ and $h\circ\beta(q)\circ\alpha(q)(\theta)=0$, we infer that $\beta(q)(\theta)$ is not contained in the vector subspace $\Theta_{C'}\otimes k(q)+\Theta_C\otimes k(q)$ of $\Theta_P\otimes k(q)$. Since the map ϕ is surjective, there is a flat family $f:\mathcal{W}\to\Delta^1=\{t\in\boldsymbol{C}\,;\,|t|<\varepsilon\}$ in $\boldsymbol{P}\times\Delta$ such that (1) $f^{-1}(0)=W$, (2) $\varphi(\partial/\partial t)=\theta$ and (3) for every $t\in\Delta^1$, W_t contains $\pi^{-1}(l)$, where $\varphi: T_{[0],\Delta^1}\to H^0(N_{W/P}$ is the Kodaira-Spencer map with respect to f. This implies our result.

Let us start the proof of the existence of θ with the properties we want. Note that $N_{W/P}\otimes\mathcal{O}_W(-\pi^{-1}(l))\cong\mathcal{O}_W(9\Sigma)\otimes\pi^*\mathcal{O}_S(12D_0+(6i+11)l)$. Here we define the vector subspace V of $H^0(W, N_{W/P}\otimes\mathcal{O}_W(-\pi^{-1}(l)))\subset H^0(W, N_{W/P})$ as follows:

$$V:=\{s\in H^0(N_{W/P})\,;\,(s)_0=9\Sigma+\pi^{-1}(12D_0)+\pi^{-1}(l)+\sum_{k=1}^{6i+11}\pi^{-1}(l_k)$$
$$\text{each } l_k \text{ is an arbitrary fiber of } g: S\to\boldsymbol{P}_1\}$$

We write $V_C=j_1(V)$ for simplicity. We will prove in each case of

(Case 1) $q\notin\Sigma$,

(Case 2) $q\in\Sigma$

Proof in (Case 1): Write $q_0=C\cap\Sigma$. Then we have:

$$V_C=\{s\in H^0(N_{W/P}|_C)\,;\,(s)_0=q+9q_0+(6i+11)\text{-points which move freely})\}.$$

J and $(\beta\circ\alpha)$ are defined by

$$H^0(\mathcal{O}_{P^1}(3)\oplus\mathcal{O}_{P^1}(7+2i)\oplus\mathcal{O}_{P^1}(9+3i)) \xrightarrow{J} H^0(N_{W/P}|_C)$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$(\ell_1, \ell_2, \ell_3) \longmapsto \ell_1\Big(\frac{\partial F}{\partial Z}\Big|_C\Big)+\ell_2\Big(\frac{\partial F}{\partial X}\Big|_C\Big)+\ell_3\Big(\frac{\partial F}{\partial Y}\Big|_C\Big),$$

where $F=Y^2-X^3-aXZ^2-bZ^3$, and

$$H^0(\mathcal{O}_{P^1}(3)\oplus\mathcal{O}_{P^1}(7+2i)\oplus\mathcal{O}_{P^1}(9+3i)) \xrightarrow{\beta\circ\alpha} H^0(\Theta_P|_C)$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$(\ell_1, \ell_2, \ell_3) \longmapsto \ell_1\Big(\frac{\partial}{\partial Z}\Big|_C\Big)+\ell_2\Big(\frac{\partial}{\partial X}\Big|_C\Big)+\ell_3\Big(\frac{\partial}{\partial Y}\Big|_C\Big)$$

For the notation $\partial/\partial Z$, $\partial/\partial X$, $\partial/\partial Y$, see (1.2.1). Therefore, it suffces to find an element $\tau=(\ell_1, \ell_2, \ell_3)$ such that

(a) $\ell_1\Big(\frac{\partial}{\partial Z}\Big|_C\Big)+\ell_2\Big(\frac{\partial}{\partial X}\Big|_C\Big)+\ell_3\Big(\frac{\partial}{\partial Y}\Big|_C\Big)\neq 0$ at q,

(b) $\ell_1\Big(\frac{\partial F}{\partial Z}\Big|_C\Big)+\ell_2\Big(\frac{\partial F}{\partial X}\Big|_C\Big)+\ell_3\Big(\frac{\partial F}{\partial Y}\Big|_C\Big)$

has a zero of order$\geqq 1$ at q, and has a zero of order$\geqq 9$ at q_0.

First we will investigate the condition (b). For example, set $\tau=(\ell_1, 0, \ell_3)$. On one hand, since $((\partial F/\partial Y)|_C)$ has a zero of order 3 at q_0 and $((\partial F/\partial Z)|_C)$ is not zero at q_0, ℓ_1 must have a zero of order$\geqq 3$ at q_0. On the other hand, since $\ell\in H^0(\mathcal{O}_{P^1}(3))$, ℓ_1 has no zeros other than q_0. Fix such an ℓ_1. Then the condition (b) for ℓ_3 is represented by at most 7 equations of degree$=1$ in the vector space $H^0(\mathcal{O}_{P^1}(3i+9))$. Therefore, we find a non-zero ℓ_3 which satisfies (b). Next we investigate the condition (a). Since $q\notin\Sigma$, we have $X\neq 0$ or $Z\neq 0$ at q. If $X\neq 0$ holds at q, then we have:

$$(\beta\circ\alpha)(\tau)=\ell_1\frac{\partial}{\partial Z}\Big|_C+\ell_3\frac{\partial}{\partial Y}\Big|_C\neq 0 \quad \text{at } q.$$

In fact, if $X\neq 0$ at q, then we can consider $(Y/X, Z/X)$ as local coordinates at q. Then $(\beta\circ\alpha)(\tau)=(\ell_1/X)(\partial/\partial(Z/X))+(\ell_3/X)(\partial/\partial(Y/X))$ at q. Since $\ell_1(q)\neq 0$, $(\beta\circ\alpha)(\tau)\neq 0$ at q. In the case where $X=0$, $Z\neq 0$ at q, we consider $\tau'=(0, \ell_2, \ell_3)$. We will divide the problem into 2 cases: (case i) $X=0$, $Y\neq 0$, $Z\neq 0$ at q, (case ii) $X=0$, $Y=0$, $Z\neq 0$. In (case i), first we take an ℓ_2 such that ℓ_2 is not zero at q and has a zero of order 3 at q_0. Since $Y\neq 0$ and $Z\neq 0$ at q, we have $(\partial F/\partial Y)(q)\neq 0$. Next, for the ℓ_2, we take an ℓ_3 such that $(\beta\circ\alpha)(\tau')=\ell^2(\partial F/\partial X)|_C+\ell_3(\partial F/\partial Y)|_C$ has a zero of order$\geqq 1$ at q and has a zero of order $\geqq 9$ at q_0. This is possible. In fact, both ℓ_2 and $(\partial F/\partial Y)|_C$ already have zeros of order 3 at q_0. Therefore, the condition for ℓ_3 in $H^0(\mathcal{O}_{P^1}(9+3i))$ can be represented by 7 linear equations (6 equations for q_0 and one equation for q). Hence we can find such an ℓ_3. We take $(X/Z, Y/Z)$ as local coordinates at q. Then since $\ell_2(q)\neq 0$, we have

$$(\beta\circ\alpha)(\tau')=\ell_2\frac{\partial}{\partial X}\Big|_C+\ell_3\frac{\partial}{\partial Y}\Big|_C=(\ell_2/Z)\frac{\partial}{\partial(X/Z)}+(\ell_3/Z)\frac{\partial}{\partial(Y/Z)}\neq 0$$

at q. This completes the proof of (case i).

In (case ii), first we take an ℓ_2 which has a zero of order 1 at q and has a zero of order 3 at q_0. Next, for the ℓ_2, we take an ℓ_3 with $\ell_3(q)\neq 0$ such that $(\beta\circ\alpha)(\tau')=\ell_2(\partial F/\partial X)|_C+\ell_3(\partial F/\partial Y)|_C$ has a zero of order 1 at q and has a zero of order$\geqq 9$ at q_0. This is possible. In fact, since $(\partial F/\partial Y)(q)=0$, we have $(\beta\circ\alpha)(\tau_{\prime})(q)=0$ however $\ell_3(q)\neq 0$. Then we can use the same argument as above. Consequently, we have our assertion in this case.

Proof in (CASE 2) Take $(X/Y, Z/Y)$ as local coordinates at q. Then from the fact that $X=Z=0$ at q, it follows that

$$(\beta\circ\alpha)(\tau)(q)=\ell_1 Y\frac{\partial}{\partial(Z/Y)}+\ell_2/Y\frac{\partial}{\partial(X/Y)}\Big|_q \tag{4.8.1}$$

First we take an ℓ_2 such that $\ell_2(q)\neq 0$. Since $(\partial F/\partial Z)(q)\neq 0$, we can find an ℓ_1 such that $\ell_1((\partial F/\partial Z)|_C)+\ell_2((l\partial F/\partial X)|_C)$ has a zero of order$\geqq 3$ at q. This is possible because $\ell_2\in H^0(\mathcal{O}_{P^1}(3))$. Next for the ℓ_1, ℓ_2 we take an ℓ_3 such that $\ell_1((\partial F/\partial Z)|_C)+\ell_2((\partial F/\partial X)|_C)+\ell_3((\partial F/\partial Y)|_C)$ has a zero of order$\geqq 10$ at q. This is possible because $((\partial F/\partial Y)|_C)$ has a zero of order 3 at q and the condition for ℓ_3 is represented by 7 linear equations in $H^0(\mathcal{O}_{P^1}(9+3i))$. If we take the triple (ℓ_1, ℓ_2, ℓ_3) then we have a required element of $H^0(N_{W/P}\otimes\mathcal{O}_W(-\pi^{-1}(l))$. Q.E.D.

(4.9) Examle. Let $W=W(K_S, A, B)$ be a Weierstrass model over $S=\boldsymbol{P}^2$. Define A and B as follows:

$$A=T_0T_1f(T_0:T_1:T_2)+2(T_0^4+T_1^4)T_2^8;$$

$$B=T_0T_1g(T_0:T_1:T_2)+T_2^{18},$$

where $(T_0:T_1:T_2)$ is homogenous coordinates of $\boldsymbol{P}^2$ and where f (resp. g) is a homogenous polynomial of deg 10 (resp. 16). Set $D_i=\{T_i=0\}\subset S$. Then $W_{D_0}:=W\times_S D_0$ is isomorphic to the Weierstrass model $W(\mathcal{O}_{P^1}(3), 2T_1^4T_2^8, T_2^{18})$ over $\boldsymbol{P}^1$. Similarly W_{D_1} is isomorphic to the Weierstrass model $W(\mathcal{O}_{P^1}(3), 2T_0^4T_2^8, T_2^{18})$ over $\boldsymbol{P}^1$. We have smooth rational curves C_0 and C_1 on W_{D_0} and W_{D_1}, respectively, which are defined by:

C_0: on $W_{D_0}^0$,

$$T_2^2X=T_1^8Z,\quad T_2^3Y=(T_1^{12}+T_2^{12})Z \quad \text{on } W_{D_0}^1,$$

$$T_2^2X=T_1^8Z,\quad T_2T_1^8Y=(T_1^{12}+T_2^{12})X$$

C_1: on $W_{D_1}^0$,

$$T_2^2X=T_0^8Z,\quad T_2^3Y=-(T_0^{12}+T_2^{12})Z \quad \text{on } W_{D_1}^1,$$

$$T_2^2X=T_0^8Z,\quad T_2T_0^8Y=-(T_0^{12}+T_2^{12})X.$$

Note that the above (W_{D_i}, C_i) are essentially the same as (W_D, C) in (4.3). In fact, if we replace T_0 by T_2 in (4.3), we get (W_{D_0}, C_0). If we replace T_0 by T_2, T_1 by T_0 and Y by $-Y$, then we get (W_{D_1}, C_1). Therefore, we can use the calculations in (4.3) for C_0 and C_1 on W. As a consequence, we see that if we take the suitable polynomials f and g, then both C_0 and C_1 are $(-1,-1)$-curve on W. By the construction, we have $C_0\cap C_1=\emptyset$ and $(\Sigma.C_i)=1$ $(i=0, 1)$.

(4.10) Example. Let $W=W(K_S, A, B)$ be a Weierstrass model over $S=\boldsymbol{P}^2$. We define A and B as follows:

$$A=T_0^{11}T_1$$
$$B=(T_0^{11}-T_1^{11})T_2^7+T_2^1{}_8$$

Write ℓ_j for the line on S defined by $T_0-\mu^jT_1=0$, $\mu=e^{2\pi i/11}$. Set $W_j:=W\times_S\ell_j$. Then we have a smooth rational curve C_j on each $W_j\subset W$ which is defined by:

$$T_0-\mu^jT_1=0,\quad X=0,\quad Y=T_2^9Z.$$

Let $\Delta_{(j)}$ be a sufficiently small neighborhood of $[\ell_j]$ in $\boldsymbol{P}(H(\mathcal{O}_{P^2}(1))^*)$ with local coordinates (s_1, s_2). The in the way similar to (4.2), $W_{\Delta_{(j)}}:=\mathcal{W}\times_{P(H^0(\mathcal{O}_{P^2}(1))^*)}\Delta_{(j)}$ is isomorphic to the Weierstrass model $\mathcal{W}_j=W(\mathcal{O}_{P^1}(3), A_j, B_j)$ with

$$A_j=A((\mu^j+s_1)T_1+s_2T_2,\ T_1,\ T_2)$$
$$B_j=B((\mu^j+s_1)T_1+s_2T_2,\ T_1,\ T_2).$$

Using the same notation as in (4.1), we have

$$H^0(N_{W_j/P}|_{C_j})/\mathrm{Im}\,\beta=\boldsymbol{C}[T_1^{10}T_2^8]\oplus\boldsymbol{C}[T_1^{11}T_2^7].$$

Moreover, we have

$$(\alpha\circ\varphi)_1\Big(\frac{\partial}{\partial s_1}\Big)=11\mu^{10j}T_1^{11}T_2^7$$

$$(\alpha\circ\varphi)_2\Big(\frac{\partial}{\partial s_2}\Big)=11\mu^{10j}T_1^{10}T_2^8.$$

Therefore, we infer that

$$H^0(N_{W_j/P}|_{C_j})=\mathrm{Im}\,\beta+\mathrm{Im}\,(\alpha\circ\varphi).$$

This shows that each C_j is a $(-1,-1)$-curve on W. But note that C_j's intersect at one point on W. Here we use the following for each pair (C_j, C_k), $j\neq k$.

(4.10.1) **Proposition.** *Let $W=W(K_{P^2}, a, b)$ be a Weierstrass model over $\boldsymbol{P}^2$. Let D_1 and D_2 be distinct lines on $\boldsymbol{P}^2$. Let C_1 ((resp. C_2) be a smooth rational curve on $W_{D_1}:=W\times_{P^2}D_1$ (resp. W_{D_2}) such that C_1 (resp. C_2) is a section of $W_{D_1}\to D_1$ (resp. $W_{D_2}\to D_2$). Assume that*

(1) *both C_1 and C_2 are $(-1,-1)$-curves on W,*

(2) $(C_1.\Sigma)=0$,

(3) $C_1\cap C_2\neq\emptyset$.

Then there is a small deformation $(W_t, C_{1,t}, C_{2,t})$ of (W, C_1, C_2) in $\boldsymbol{P}=\boldsymbol{P}_{P^2}(\mathcal{O}_{P^2}\oplus K_{P^2}^2\oplus K_{P^2}^3)$ such that $C_{1,t}\cap C_{2,t}=\emptyset$.

Proof. The proof is quite similar to that of (case $(C.\Sigma)=0$) in (4.8). Hence we omit the proof.

The we have mutually disjoint 11 $(-1,-1)$-curves on W.

(4.11) **Conclusion from** (4.9), (4.10). By (4.9), $W=W(K_{P^2}, A, B)$ contains at least two $(-1,-)$-curves C_1 and C_2 which are mutually disjoint and $(C_i.\Sigma)=0$ ($i=1, 2$) if we take general A and B. Similarly, by (4.10), we may assume that W contains at least

11 $(-1,-1)$-curves $C_3, \cdots, C_{14}$ which are mutually disjoint and $(C_i.\Sigma)=1$ $(i=3, \cdots, 14)$. Let us write D_i for $\pi(C_i)$. Then D_1 and D_2 are mutually distinct lines on $\boldsymbol{P}^2$. On the other hand, $D_3, \cdots, D_{14}$ are mutually distinct lines on $\boldsymbol{P}^2$. Therefore we can find D_j and D_k from $D_3, \cdots, D_{14}$ such that D_1, D_2, D_j, D_k are mutually distinct lines on $\boldsymbol{P}^2$. Here we apply (4.10.1) to the pairs (C_1, C_j), (C_1, C_k), (C_2, C_j), (C_2, C_k). Therefore, if we take general A and B, then $W=W(K_{P^2}, A, B)$ contains mutually disjoint $(-1,-1)$-curves C_1, C_2, C_3, C_4 such that $(C_i.\Sigma)=0$ $(i=1, 2)$ and $(C_i.\Sigma)=1$ $(i=3, 4)$.

§5. Proof of Theorem A′

(5.1) **(the Case $S=\boldsymbol{P}^2$)** In this case, we take the four curves $C_1, \cdots, C_4$ in (4.11). By Proposition (1.5) (1), $\mathrm{Pic}(W)=\boldsymbol{Z}[\pi^*\mathcal{O}_{P^2}(1)]\oplus\boldsymbol{Z}[\Sigma]$. Write H for $\pi^*\mathcal{O}_{P^2}(1)$. Then the intersections of the generator of $\mathrm{Pic}(W)$ with the above curves are as follows:

	H	Σ
C_1	1	0
C_2	1	0
C_3	1	1
C_4	1	1

Therefore, the conditions in (1.1) are satisfied.

(5.2) **(the Case $S=\Sigma_i$ with $0\leqq i\leqq 2$)** In this case, we take four curves $C_1, \cdots, C_4$ in (4.6). Note here that we may assume that these curves are mutually disjoint. In fact, first we choose the four curves in (4.6) such that C_3 and C_4 do not intersect. This is possible because we have at least $(12+2i)$-curves of type III and we may pick the suitable one as C_3 from these curves. Next from (4.7) and (4.8), it follows that if we deform W to W_t, then we may assume that the four curves are mutually disjoint. By ɖroposition (1.5), (2), we have $\mathrm{Pic}(W)=\boldsymbol{Z}[H_1]\oplus\boldsymbol{Z}[H_2]\oplus\boldsymbol{Z}[\Sigma]$, where H_1 denotes the pull-back of the negative section on S and H_2 denotes the pull back of a fibre of $g: S\to\boldsymbol{P}^1$. Then the table of intersections is as follows.

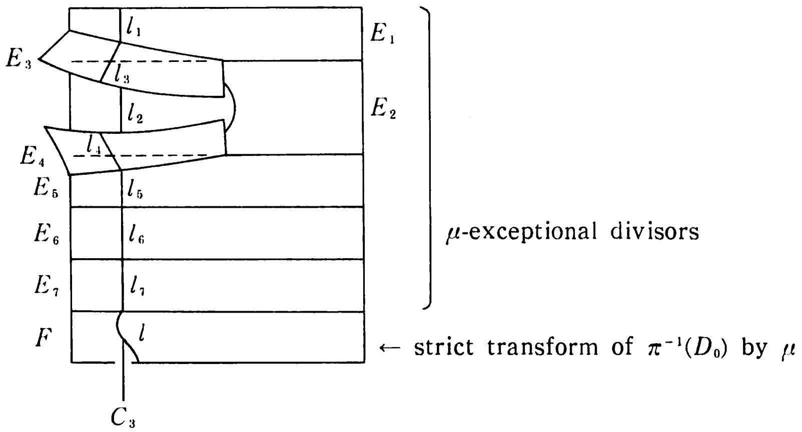
	Σ	H_1	H_2
C_1	0	0	1
C_2	1	0	1
C_3	0	1	0
C_4	1	1	0

Therefore, the conditions in (1.1) are satisfied.

(5.3) **(the Case $S=\Sigma_i$ with $3\leqq i\leqq 8$)**

(5.3.1) We will explain the case $i=7$ here. Other cases can be treated in the same way. Let $\mu:\widetilde{W}\to W$ be a resolution in (1.5). Let D_0 be the negative section of

$g:S\to\boldsymbol{P}^1$. Then $(\pi\circ\mu)^{-1}(D_0)$ is illustrated in the following picture. Here the self-intersection number of the double curves in each irreducible component is omitted because our argument does not need such informations.



In the figure, C_3 denote the one in (4.6), and ℓ denotes the fibre of F passing through the point $q:=C_3\cap F$. ℓ_7 is the fibre of E_7 which intersects ℓ. The other ℓ_j's are determined in a similar way. First we perform the flop of C_3. Then the strict transform of ℓ becomes a (-1)-curve on F (exactly speaking, the strict transform of F). Since $K_{\tilde{W}}\cong\mathcal{O}_{\tilde{W}}$, this implies that it is a $(-1,-1)$-curve on the new 3-folc obtained by the flop from $\tilde{W}$. From now on, by abuse of notation, we will use the same notation for the ℓ, ℓ_j, E_j and their strict transforms by a certain flop. Next preform the flop of ℓ. Then ℓ_7 becomes a (-1)-curve on E_7, which implies that ℓ_7 is a $(-1,-1)$-curve on the ambient 3-fold. We can continue the similar process. Let E_j be an arbitrary μ-exceptional divisor. Then, from the above observation, we know the following.

If we perform a suitable composition of flops of $(-1,-1)$-curves, then we have a $(-1,-1)$-curve C on the new 3-fold $\tilde{W}'$ obtained by the composition of flops and C satisfies $(C, E_j)=-1$.

(**5.3.2**) A general Weierstrass model W has canonical singularities and they are locally trivial deformation of a certain rational double point except for a finite number of points. We call these points dissident. In our case $S=\Sigma_7$, there is only one dissident point. When we construct $\tilde{W}$ from W, the situation of μ-exceptional divisors changes over the point. Let p be a dissident point of W. Then we can find 9 $(-1,-1)$-curves $C_3^{(0)},\cdots,C_3^{(8)}$ such that $(g\circ\pi)(C_3^{(j)}\neq(g\circ\pi)(p)$, where $\pi:W\to S$ and $g:S\to\boldsymbol{P}^1$ are the natural projections. This follows from the fact that there are at least 26 $(-1,-1)$-curves on W of type III (See (4.6)). On the other hand, we can find $(-1,-1)$-curves C_1, C_2, C_4 of other types that C_1, C_2, $C_3^{(0)},\cdots,C_3^{(8)}$, C_4 are mutually disjoint (See (4.6), (4.7), (4.8)). Since the resolution μ of W changes nothing around these curves, we may assume that they are curves on $\tilde{W}$.

(5.3.3) Here recall (5.3.1). First, we put $C_3=C_3^{(0)}$, and perform a flop of C_3. Then ℓ becomes a $(-1, -1)$-curve. Next we consider $C_3^{(1)}$ as the C_3 in (5.3.1). This time, we perform flops in the order of C_3, ℓ. Then ℓ_7 becomes a $(-1, -1)$-curve. We continue these operations for $C_3^{(j)}$'s $(j\leqq 7)$ so that. for $C_3^{(j)}$, ℓ_j becomes a $(-1, -1)$-curve. For $C_3^{(8)}$, we leave as it is. Then we have 12 mutually diajoint $(-1, -1)$-curves on the new Moishezon 3-fold:

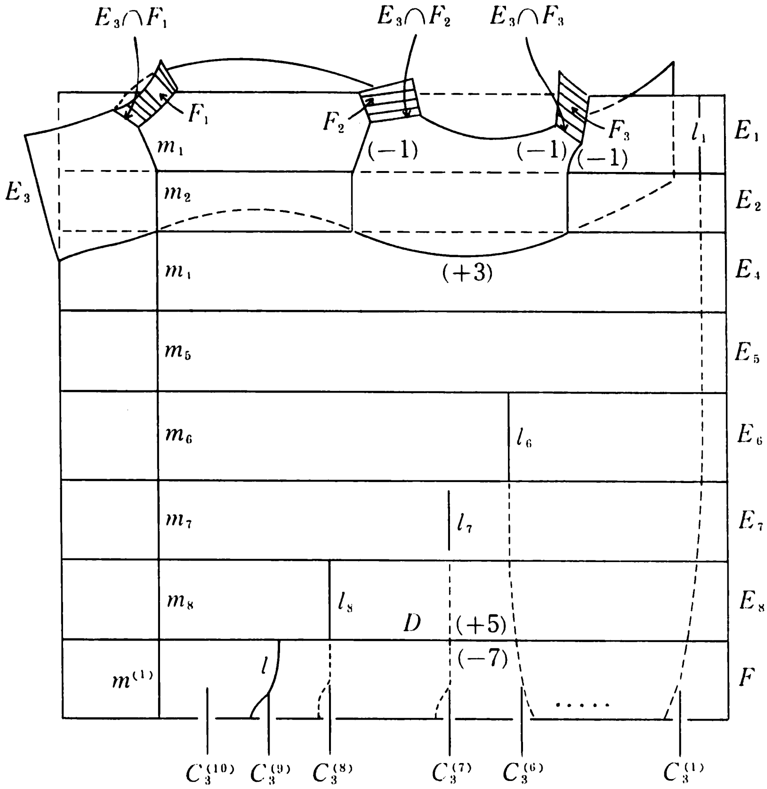
$$C_1,\ C_2,\ C_3^{(8)},\ C_4,\ \ell,\ \ell_7,\ \ell_6,\ \cdots,\ \ell_1\,.$$

We denote by $\widetilde{W}'$ the new 3-fold. Then we have that $\mathrm{Pic}(\widetilde{W})=\boldsymbol{Z}[\Sigma]\oplus\boldsymbol{Z}[H_1]\oplus\boldsymbol{Z}[H_2]\oplus\sum_{j=1}^{7}\boldsymbol{Z}[E_j]$, where E_j's are the μ-exceptional divisors in (5.3.1) and $H_1=\mu^*\pi^*D_0$, $H_2=\mu^*\pi^*l$ (D_0 is the negative section and l is a fiber). It follows that $\mathrm{Pic}(\widetilde{W}')$ is generated by the strict transforms of Σ, H_1, H_2, $E_1, \cdots, E_7$. By abuse of notation, we denote them by the same symbol as the original ones. Then the intersection numbers between these curves and these divisors are as follows:

	Σ	H_1	H_2	E_1	E_2	E_3	E_4	E_5	E_6	E_7
C_1	0	0	1	0	0	0	0	0	0	0
C_2	1	0	1	0	0	0	0	0	0	0
$C_3^{(8)}$	0	1	0	0	0	0	0	0	0	0
C_4	1	1	0	0	0	0	0	0	0	0
ℓ	1	1	0	0	0	0	0	0	0	1
ℓ_7	1	1	0	0	0	0	0	0	0	-1
ℓ_6	1	1	0	0	0	0	0	0	-1	0
ℓ_5	1	1	0	0	0	0	0	-1	0	0
ℓ_4	1	1	0	0	1	0	-1	0	0	0
ℓ_3	1	1	0	0	0	-1	1	0	0	0
ℓ_2	1	1	0	1	-1	0	1	0	0	0
ℓ_1	1	1	0	-1	0	1	1	0	0	0

Then it is checked that the curves generate $H_2(\widetilde{W}'\,;\boldsymbol{C})$. Moreover we can find the element $\theta=C_1-C_2-C_3+2C_4+\ell-\ell_1-\ell_2-\ell_3-2\ell_4+\ell_5+\ell_6+\ell_7$ in $\mathrm{Ker}\, i_*$. Here i_* is the same one in (1.1). Therefore, the conditions in (1.1) are satisfied.

(5.4) **(the Case where** $S=\Sigma_i$ **with** $9\leqq i\leqq 12$**)** We will explain the case $i=9$ here. The cases where $i=10$, 11 can be treated in the same way. Since W has no dissident points in the case where $i=12$, we can treat this case in the same way as the case $3\leqq i\leqq 8$. A difference between the cases $9\leqq i\leqq 11$ and the cases $3\leqq i\leqq 8$ is that $g\circ\pi$: $\widetilde{W}\to S$ is not flat if $9\leqq i\leqq 11$. In other words, there are μ-exceptional divisors which are contracted to points by $g\circ\pi$. $(\pi\circ\mu)^{-1}(D_0)$ is illustrated as follows when $i=9$.



Since W has 3 dissident points when $i=3$, we have 3 bad μ-exceptional divisors F_1, F_2, F_3. They are contracted to points by $g\circ\pi$. We perform flops in the order of m_1, m_2, m_4, m_5, $\cdots$, m_8. Then $m^{(1)}$ becomes a $(-1, -1)$-curve. We do the same procedure for F_2 and F_3. The we have $(-1, -1)$-curves $m^{(2)}$ and $m^{(3)}$. They are also (-1)-curves on F. We note that, after these flops, the double curve D has self-intersection number -10 in F. Next we choose 10 $(-1, -1)$-curves $C_3^{(1)}, \cdots, C_3^{(10)}$ of type III in the same way as (5.3). We leave $C_3^{(10)}$ as it is. For each $C_3^{(j)}$; $1\leqq j\leqq 8$, we perform a suitable composition of flops whchi starts from $C_3^{(j)}$, as we have done in (5.3). Then $\ell_1, \cdots, \ell_7, \ell_5$ in the figure become $(-1, -1)$-curve. For $C_3^{(9)}$, we perform the flops in the order of $C_3^{(9)}$, ℓ (see the figure). We note that the self-intersection number of D in F becomes -1 in this situation. This implies that D becomes a $(-1, -1)$-curves. On the other hand, we can choose $(-1, -1)$-curves C_1, C_2, C_4 of type I, II, IV such that C_1, C_2, $C_3^{(1)}, \cdots, C_3^{(10)}$, C_4 are mutually disjoint. As a consequence we have mutually disjoint $(-1, -1)$-curves on the new Moishezon 3-fold $\widetilde{W}'$:

$$C_1,\ C_2,\ C_3^{(10)},\ C_4,\ D,\ m^{(1)}, \cdots, m^{(3)},\ \ell_1, \cdots, \ell_8\,.$$

On the other hand, since $\mathrm{Pic}(\widetilde{W})=\boldsymbol{Z}[\Sigma]\oplus\boldsymbol{Z}[H_1]\oplus\boldsymbol{Z}[H_2]\oplus\sum_{j=1}^{8}\boldsymbol{Z}[E_j]\oplus\sum_{j=1}^{9}\boldsymbol{Z}[F_j]$ (H_1 and H_2 are the same as in (5.3.3)), $\mathrm{Pic}(\widetilde{W}')$ is generated by the strict transforms of Σ, H_1, H_2, $E_1, \cdots, E_8$, $F_1, \cdots, F_9$. By abuse of notation, we denote them by the same symbols as the original ones. Then the table of intersection numbers are as follows:

	Σ	H_1	H_2	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	F_1	F_2	F_3
C_1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
C_2	1	0	1	0	0	0	0	0	0	0	0	0	0	0
$C_3^{(10)}$	0	1	0	0	0	0	0	0	0	0	0	0	0	0
C_4	1	1	0	0	0	0	0	0	0	0	0	0	0	0
D	9	-3	0	-1	0	0	0	0	0	0	-1	0	0	0
$m^{(1)}$	1	-1	0	0	0	0	0	0	0	0	0	1	0	0
$m^{(2)}$	1	-1	0	0	0	0	0	0	0	0	0	0	1	0
$m^{(3)}$	1	-1	0	0	0	0	0	0	0	0	0	0	0	1
ℓ_8	1	1	0	0	0	0	0	0	0	1	-1	0	0	0
ℓ_7	1	1	0	0	0	0	0	0	1	-1	0	0	0	0
ℓ_6	1	1	0	0	0	0	0	1	-1	0	0	0	0	1
ℓ_5	1	1	0	0	0	0	1	-1	0	0	0	0	0	0
ℓ_4	1	1	0	0	1	1	-1	0	0	0	0	0	0	0
ℓ_3	1	1	0	0	1	-1	0	0	0	0	0	0	0	0
ℓ_2	1	1	0	1	-1	1	0	0	0	0	0	0	0	0
ℓ_1	1	1	0	-1	0	1	0	0	0	0	0	0	0	0

It is checked that the curves generate $H_2(\tilde{W}'; \boldsymbol{C})$. Moreover we can find the element

$$\theta = 2C_1 - 2C_2 - 8C_3 - 2C_4 - D + m^{(1)} + m^{(2)} + m^{(3)} + \ell_8 + \ell_7 + \ell_6 + \ell_5 - \ell_4 + 3\ell_3 + 2\ell_2 + 2\ell_1$$

in $\operatorname{Ker} i_*$. Here i_* is the same one in (1.1). Therefore, the conditions of (1.1) are satisfied.

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SOPHIA UNIVERSITY

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