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On the structure of Cousin complexes

By

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0. Introduction

Throughout the paper, A will denote a commutative Noetherian ring (with non-zero identity), and M will denote an A-module. It should be noted that M need not be finitely generated. The Cousin complex $C(M)$ for M is described in [3, Section 2]: it is a complex of A-modules and A-homomorphisms

$$0 \xrightarrow{b^{-2}} M \xrightarrow{b^{-1}} B^0 \xrightarrow{b^0} B^1 \to \cdots \to B^n \xrightarrow{b^n} B^{n+1} \to \cdots$$

with the property that, for each $n \in \mathbf{N}_0$ (we use $\mathbf{N}_0$ to denote the set of non-negative integers),

$$B^n = \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Supp}(M) \\ \mathrm{ht}_M \mathfrak{p} = n}} (\mathrm{Coker}\, b^{n-2})_{\mathfrak{p}} .$$

(Here, for $\mathfrak{p} \in \mathrm{Supp}(M)$, the notation $\mathrm{ht}_M \mathfrak{p}$ denotes the *M-height of* $\mathfrak{p}$, that is the dimension of the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$; the dimension of a non-zero module is the supremum of lengths of chains of prime ideals in its support if this supremum exists and ∞ otherwise.)

Cohen-Macaulay modules can be characterized in terms of the Cousin complex: a non-zero finitely generated A-module N is Cohen-Macaulay if and only if $C(N)$ is exact [4, (2.4)]. Also, the Cousin complex provides a natural minimal injective resolution for a Gorenstein ring: see [3, (5.4)].

The Cousin complex $C(M)$ will play a major rôle in this paper. It is a special case of a more general complex which can be constructed whenever we have a filtration $\mathcal{F}$ of $\mathrm{Spec}(A)$ which admits M [8, 1.1]; this more general complex is called the Cousin complex for M with respect to $\mathcal{F}$ and is denoted by $C(\mathcal{F}, M)$. As this complex will also feature prominently in this paper, it is appropriate for us to recall the details of its construction and definition from [8, Section 1].

A *filtration* of $\mathrm{Spec}(A)$ is a descending sequence $\mathcal{F} = (F_i)_{i \in \mathbf{N}_0}$ of subsets of $\mathrm{Spec}(A)$, so that

$$\mathrm{Spec}(A) \supseteq F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots ,$$

with the property that, for each $i \in \mathbf{N}_0$, each member of $\partial F_i = F_i \backslash F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that $\mathcal{F}$ *admits* M if $\mathrm{Supp}(M) \subseteq F_0$.

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Given such a filtration $\mathscr{F}$ which admits M, the Cousin complex $C(\mathscr{F}, M)$ for M with respect to $\mathscr{F}$ has the form

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \to \cdots \to M^n \xrightarrow{d^n} M^{n+1} \to \cdots$$

with, for each $n \in \mathbf{N}_0$,

$$M^n = \bigoplus_{\mathfrak{p} \in \partial F_n} (\operatorname{Coker} d^{n-2})_{\mathfrak{p}} .$$

The homomorphisms in this complex have the following properties: for $m \in M$ and $\mathfrak{p} \in \partial F_0$, the component of $d^{-1}(m)$ in $M_{\mathfrak{p}}$ is $m/1$; and, for $n > 0$, $x \in M^{n-1}$ and $\mathfrak{q} \in \partial F_n$, the component of $d^{n-1}(x)$ in $(\operatorname{Coker} d^{n-2})_{\mathfrak{q}}$ is $\pi(x)/1$, where $\pi\colon M^{n-1} \to \operatorname{Coker} d^{n-2}$ is the canonical epimorphism. The fact that such a complex can be constructed is explained in [8, 1.3] and relies on arguments from [3, Section 2].

The Cousin complex $C(M)$ mentioned in the first paragraph of this paper is actually the Cousin complex $C(\mathscr{H}(M), M)$ for M with respect to the M-height filtration $\mathscr{H}(M) = (H_i)_{i \in \mathbf{N}_0}$ of $\operatorname{Spec}(A)$, where

$$H_i = \{\mathfrak{p} \in \operatorname{Supp}(M) \colon \operatorname{ht}_M \mathfrak{p} \geqslant i\} \qquad \text{for all } i \in \mathbf{N}_0 .$$

For clarity in this paper, we are going to call $C(\mathscr{H}(M), M)$ the *basic Cousin complex for M*. We are going to show that this basic Cousin complex does play a very 'basic' rôle in the theory of Cousin complexes, because we shall show that any other Cousin complex for M (with respect to a filtration $\mathscr{F}$ of $\operatorname{Spec}(A)$ which admits M) can be obtained from $C(\mathscr{H}(M), M)$ by a rather satisfactory quotient complex construction which has the practical effect of deleting (for each $n \in \mathbf{N}_0$) some of the direct summands $(\operatorname{Coker} b^{n-2})_{\mathfrak{p}}$ $(\mathfrak{p} \in \partial H_n)$ of B^n and leaving the others intact.

To go into a little more detail about our main results, let us say, for a prime ideal $\mathfrak{p}$ of A, that $\mathfrak{p}$ is *significant*, or *of significance*, for $C(\mathscr{F}, M)$ if there exists $i \in \mathbf{N}_0$ for which $\mathfrak{p} \in \partial F_i$ and the direct summand $(\operatorname{Coker} d^{i-2})_{\mathfrak{p}}$ of the term M^i in $C(\mathscr{F}, M)$ is non-zero; otherwise, we say that $\mathfrak{p}$ is *insignificant*, or *of no significance*, for $C(\mathscr{F}, M)$. (By [8, 1.5], any significant prime for $C(\mathscr{F}, M)$ must lie in $\operatorname{Supp}(M)$.)

We shall show that if $\mathfrak{p}$ is a significant prime for $C(\mathscr{F}, M)$, and $i \in \mathbf{N}_0$ is such that $\mathfrak{p} \in \partial F_i$, then ($\mathfrak{p} \in \operatorname{Supp}(M)$ and) $i = \operatorname{ht}_M \mathfrak{p}$ and

$$(\operatorname{Coker} d^{i-2})_{\mathfrak{p}} \cong H^i_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) ,$$

the 'top' local cohomology module for the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. In particular, for $\mathfrak{q} \in \operatorname{Supp}(M)$ of M-height j, the '$\mathfrak{q}$-part' of the basic Cousin complex for M satisfies $(\operatorname{Coker} b^{j-2})_{\mathfrak{q}} \cong H^j_{\mathfrak{q}A_{\mathfrak{q}}}(M_{\mathfrak{q}})$. These results show that, for each $i \in \mathbf{N}_0$, the i-th term in $C(\mathscr{F}, M)$ is isomorphic to a direct summand of the i-th term in $C(M) = C(\mathscr{H}(M), M)$, the basic Cousin complex for M. However, we can actually say more, as it turns out that there is a morphism of complexes

$$\Omega = (\omega^i)_{i \geqslant -2} \colon C(\mathscr{H}(M), M) \to C(\mathscr{F}, M)$$

such that $\omega^{-1}: M \to M$ is the identity mapping on M and, for each $n \in \mathbf{N}_0$ and each prime ideal $\mathfrak{p} \in \partial F_n$ that is significant for $C(\mathscr{F}, M)$ (so that $\mathfrak{p} \in \operatorname{Supp}(M)$ and $n = \operatorname{ht}_M \mathfrak{p}$), the restriction of ω^n to the direct summand $(\operatorname{Coker} b^{n-2})_{\mathfrak{p}}$ of B^n provides an isomorphism of $(\operatorname{Coker} b^{n-2})_{\mathfrak{p}}$ onto $(\operatorname{Coker} d^{n-2})_{\mathfrak{p}}$; but for $\mathfrak{p} \in \operatorname{Supp}(M)\backslash\partial F_n$ having $\operatorname{ht}_M \mathfrak{p} = n$, the restriction of ω^n to $(\operatorname{Coker} b^{n-2})_{\mathfrak{p}}$ is zero. Thus each ω^n $(n \in \mathbf{N}_0)$ is an epimorphism whose kernel is a direct summand of B^n.

We are also able to deduce from this structure theory that if $C(\mathscr{F}, M)$ is exact, then so too is $C(\mathscr{H}(M), M)$ and these two complexes are isomorphic. This gives another way in which $C(\mathscr{H}(M), M)$ is 'basic', because it is, up to isomorphism, the only possible candidate for an exact Cousin complex for M.

The impetus for this work came from study of a balanced big Cohen-Macaulay module (see [7, (1.4)] for the definition of this concept) over a local ring. Suppose, temporarily, that A is local with maximal ideal $\mathfrak{m}$. In [8, 4.1], a Cousin complex characterization of balanced big Cohen-Macaulay A-modules was given: let $d := \dim A$, and let $\mathscr{D}(A) = (D_i)_{i \in \mathbf{N}_0}$ be the *dimension filtration* of $\operatorname{Spec}(A)$ given by

$$D_i = \{\mathfrak{p} \in \operatorname{Spec}(A): \dim A/\mathfrak{p} \leqslant \dim A - i\} \qquad \text{for all } i \in \mathbf{N}_0 ;$$

then M is a balanced big Cohen-Macaulay A-module if and only if $C(\mathscr{D}(A), M)$ is exact and $H^d_{\mathfrak{m}}(M) \neq 0$. The work in this paper arose out of a desire to compare, for a balanced big Cohen-Macaulay A-module M, the Cousin complexes $C(\mathscr{D}(A), M)$, $C(\mathscr{H}(A), M)$ and $C(\mathscr{H}(M), M)$ and to explore whether a prime ideal $\mathfrak{q}$ of A with $\operatorname{ht} \mathfrak{q} + \dim A/\mathfrak{q} < \dim A$ could have any significance for any of these Cousin complexes. In fact, we shall show, as an example illustrating our results, that such a $\mathfrak{q}$ has no significance for any of the three complexes, and that the three complexes are isomorphic.

We now revert to the situation where A denotes a commutative Noetherian ring (with non-zero identity), and M denotes an arbitrary A-module. Also, $\mathscr{F} = (F_i)_{i \in \mathbf{N}_0}$ will always denote a filtration of $\operatorname{Spec}(A)$ which admits M, and the Cousin complex $C(\mathscr{F}, M)$ will always be denoted by

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \to \cdots \to M^n \xrightarrow{d^n} M^{n+1} \to \cdots .$$

1. Technical results about Cousin complexes

In this section, we collect together some results which are fundamental for working with Cousin complexes. Although some of them can be approached by means of straightforward modifications of arguments already in the literature, we do take the opportunity to point out that the work in [2, (3.1), (3.2) and (3.3)] often provides a short proof that a complex is isomorphic to a Cousin complex. We thus begin with a brief reminder about complexes of Cousin type.

1.1. Definition. (See [2, (3.1)].) A complex $C^{\bullet} = (C^i)_{i \geqslant -2}$ of A-modules and A-homomorphisms is said to be *of Cousin type for M with respect to $\mathscr{F}$* if it has the form

$$0 \xrightarrow{d_{C^\cdot}^{-2}} M \xrightarrow{d_{C^\cdot}^{-1}} C^0 \xrightarrow{d_{C^\cdot}^{0}} C^1 \to \cdots \to C^n \xrightarrow{d_{C^\cdot}^{n}} C^{n+1} \to \cdots$$

and satisfies the following for each $n \in \mathbf{N}_0$:

(i) $\operatorname{Supp}(C^n) \subseteq F_n$;

(ii) $\operatorname{Supp}(\operatorname{Coker} d_{C^\cdot}^{n-2}) \subseteq F_n$;

(iii) $\operatorname{Supp}(H^{n-1}(C^\cdot)) \subseteq F_{n+1}$; and

(iv) the natural A-homomorphism $\xi(C^n): C^n \to \bigoplus_{\mathfrak{p} \in \partial F_n}(C^n)_\mathfrak{p}$ for which, for $y \in C^n$ and $\mathfrak{p} \in \partial F_n$, the component of $\xi(C^n)(y)$ in the summand $(C^n)_\mathfrak{p}$ is $y/1$ (it follows from condition (i) and [3, (2.2) and (2.3)] that there is such an A-homomorphism) is an isomorphism.

1.2. Remarks. (i) By [2, (3.2)], the Cousin complex $C(\mathscr{F}, M)$ is a complex of Cousin type for M with respect to $\mathscr{F}$.

(ii) Let $C^\cdot = (C^i)_{i \geqslant -2}$ and $Y^\cdot = (Y^i)_{i \geqslant -2}$ be complexes of Cousin type for M with respect to $\mathscr{F}$. By [2, (3.3)], there is exactly one morphism of complexes

$$\Phi = (\phi^i)_{i \geqslant -2}: C^\cdot \to Y^\cdot$$

which is such that $\phi^{-1}: M \to M$ is the identity mapping; moreover, this morphism is an isomorphism.

(iii) Let L be an A-module such that, for some $n \in \mathbf{N}_0$, we have $\operatorname{Supp}(L) \subseteq F_n$; let $L' = \bigoplus_{\mathfrak{p} \in \partial F_n} L_\mathfrak{p}$.

Note that, for $\mathfrak{p} \in \partial F_n$, we have $\operatorname{Supp}_{A_\mathfrak{p}} L_\mathfrak{p} \subseteq \{\mathfrak{p}A_\mathfrak{p}\}$, so that each element of $L_\mathfrak{p}$ (considered as A-module) is annihilated by some power of $\mathfrak{p}$, and that, by [2, (1.2)(ii)] (applied to L and the filtration $(F_{i+n})_{i \in \mathbf{N}_0}$!), the natural A-homomorphism $\xi(L'): L' \to \bigoplus_{\mathfrak{p} \in \partial F_n} L'_\mathfrak{p}$ is an isomorphism.

The following example of the use of 1.2(ii) establishes a result which we shall need later.

1.3. Example. Let S be a multiplicatively closed subset of A. As in [8, 1.1], we denote by $S^{-1}\mathscr{F}$ the filtration $(G_i)_{i \in \mathbf{N}_0}$ of $\operatorname{spec}(S^{-1}A)$, where $G_i = \{S^{-1}\mathfrak{p} : \mathfrak{p} \in F_i$ and $\mathfrak{p} \cap S = \varnothing\}$.

Now $S^{-1}\mathscr{F}$ admits $S^{-1}M$, and we need to know that $S^{-1}(C(\mathscr{F}, M))$ is isomorphic, as a complex of $S^{-1}A$-modules and $S^{-1}A$-homomorphisms, to

$$C(S^{-1}\mathscr{F}, S^{-1}M),$$

the Cousin complex for $S^{-1}M$ with respect to $S^{-1}\mathscr{F}$.

One can approach this by straightforward modifications of the arguments given in [3, Section 3], as is suggested in [8, p. 475]. However, another approach is by means of 1.2(ii), as we now show. It is clear from 1.2(i) that $S^{-1}(C(\mathscr{F}, M))$ satisfies conditions (i), (ii) and (iii) of 1.1. To check condition (iv), let $n \in \mathbf{N}_0$, and note that, for $\mathfrak{p} \in \partial F_n$ with $\mathfrak{p} \cap S = \varnothing$, the A-module $(\operatorname{Coker} d^{n-2})_\mathfrak{p}$ has a natural structure as an $S^{-1}A$-module, and, on use of the comments in 1.2(iii), we can see that there are $S^{-1}A$-isomorphisms

$$S^{-1}M^n \xrightarrow{\cong} \bigoplus_{\substack{\mathfrak{p} \in \partial F_n \\ \mathfrak{p} \cap S = \emptyset}} (\operatorname{Coker} d^{n-2})_{\mathfrak{p}} \xrightarrow{\cong} \bigoplus_{\substack{\mathfrak{p} \in \partial F_n \\ \mathfrak{p} \cap S = \emptyset}} (S^{-1}(\operatorname{Coker} d^{n-2}))_{S^{-1}\mathfrak{p}}$$

It now follows from 1.2(iii) applied to the $S^{-1}A$-module $S^{-1}(\operatorname{Coker} d^{n-2})$ that $S^{-1}(C(\mathscr{F}, M))$ satisfies condition (iv) of 1.1.

We can now apply 1.2(i), (ii) to see that there is a unique isomorphism of complexes of $S^{-1}A$-modules and $S^{-1}A$-homomorphisms

$$\Psi = (\psi^i)_{i \geqslant -2}: S^{-1}(C(\mathscr{F}, M)) \to C(S^{-1}\mathscr{F}, S^{-1}M)$$

which is such that $\psi^{-1}: S^{-1}M \to S^{-1}M$ is the identity mapping.

1.4. Remark. Let $n \in \mathbf{N}_0$ and $\mathfrak{q} \in \partial F_n$. It follows from the last two paragraphs of 1.3 (used with the particular choice $S = A \backslash \mathfrak{q}$) that the direct summand $(\operatorname{Coker} d^{n-2})_{\mathfrak{q}}$ of M^n is $A_{\mathfrak{q}}$-isomorphic to the n-th term in the Cousin complex $C(\mathscr{F}_{\mathfrak{q}}, M_{\mathfrak{q}})$.

2. Comparison of Cousin complexes

This section contains our main results which relate $C(\mathscr{H}(M), M)$ and $C(\mathscr{F}, M)$.

2.1. Notation. In addition to the notation introduced at the end of the Introduction, the following will be in force for the whole of this section.

We shall write the basic Cousin complex for M, that is $C(\mathscr{H}(M), M)$ where the M-height filtration $\mathscr{H}(M) = (H_i)_{i \in \mathbf{N}_0}$ of $\operatorname{Spec}(A)$ is given by

$$H_i = \{\mathfrak{p} \in \operatorname{Supp}(M): \operatorname{ht}_M \mathfrak{p} \geqslant i\} \qquad \text{for all } i \in \mathbf{N}_0\,,$$

as

$$0 \xrightarrow{b^{-2}} M \xrightarrow{b^{-1}} B^0 \xrightarrow{b^0} B^1 \to \cdots \to B^n \xrightarrow{b^n} B^{n+1} \to \cdots.$$

We shall also let $\mathscr{G} = (G_i)_{i \in \mathbf{N}_0}$ denote a second filtration of $\operatorname{Spec}(A)$ which admits M, and the Cousin complex $C(\mathscr{G}, M)$ will be denoted by

$$0 \xrightarrow{e^{-2}} M \xrightarrow{e^{-1}} N^0 \xrightarrow{e^0} N^1 \to \cdots \to N^n \xrightarrow{e^n} N^{n+1} \to \cdots.$$

2.2. Definition. For each $\mathfrak{p} \in \operatorname{Spec}(A)$, we define the *$\mathscr{F}$-height of* $\mathfrak{p}$, denoted by $\operatorname{ht}_{\mathscr{F}} \mathfrak{p}$, as follows.

If $\mathfrak{p} \notin F_0$, then we set $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = -1$. If $\mathfrak{p} \in \bigcap_{i \in \mathbf{N}_0} F_i$, then we set $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = \infty$. If neither of these conditions is satisfied, then the set $\{i \in \mathbf{N}_0: \mathfrak{p} \in F_i\}$ has a greatest member, n say, and we set $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = n$.

Note that, for $\mathfrak{p} \in \operatorname{Supp}(M)$ we have $\operatorname{ht}_{\mathscr{H}(M)} \mathfrak{p} = \operatorname{ht}_M \mathfrak{p}$.

Our first lemma in this section establishes some simple but useful properties of the $\mathscr{F}$-height of a prime ideal in $\operatorname{Supp}(M)$.

2.3. Lemma. *Let* $\mathfrak{p}, \mathfrak{q} \in \operatorname{Supp}(M)$ *with* $\mathfrak{p} \subset \mathfrak{q}$. (*The symbol* '$\subset$' *is reserved to denote strict inclusion.*)

(i) *If* $\operatorname{ht}_{\mathscr{F}} \mathfrak{p}$ *is finite, then* $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} < \operatorname{ht}_{\mathscr{F}} \mathfrak{q}$.

(ii) *If* $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = \infty$, *then* $\operatorname{ht}_{\mathscr{F}} \mathfrak{q} = \infty$.

(iii) *In any event*, $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{q}$.

(iv) *For every* $\mathfrak{p} \in \operatorname{Supp}(M)$, *we have* $\operatorname{ht}_M \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p}$.

Proof. (i) Let $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = h$. Suppose that $\operatorname{ht}_{\mathscr{F}} \mathfrak{q} = k \leqslant h$, and look for a contradiction. Then $\mathfrak{q} \in F_k \backslash F_{k+1}$ and $\mathfrak{p} \in F_h \subseteq F_k$. Thus $\mathfrak{q}$ is not a minimal member of F_k, and this is a contradiction.

(ii) Suppose that $\operatorname{ht}_{\mathscr{F}} \mathfrak{q} = k$, finite, and look for a contradiction. Then $\mathfrak{q} \in F_k \backslash F_{k+1}$ and $\mathfrak{p} \in F_k$. Thus $\mathfrak{q}$ is not a minimal member of F_k, and this is a contradiction.

(iii) This is immediate from (i) and (ii).

(iv) Let $\operatorname{ht}_M \mathfrak{p} = n$, so that there is a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of prime ideals in $\operatorname{Supp}(M)$. If $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = \infty$, there is nothing to prove. Otherwise, it follows from (i) and (ii) that $\operatorname{ht}_{\mathscr{F}} \mathfrak{p}_i$ is finite for all $i = 0, \ldots, n$ and

$$\operatorname{ht}_{\mathscr{F}} \mathfrak{p}_0 < \operatorname{ht}_{\mathscr{F}} \mathfrak{p}_1 < \cdots < \operatorname{ht}_{\mathscr{F}} \mathfrak{p}_n = \operatorname{ht}_{\mathscr{F}} \mathfrak{p} .$$

Hence $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} \geqslant n$.

2.4. Lemma. *Suppose, temporarily, that* A *is local with maximal ideal* $\mathfrak{m}$, *and that* $\mathfrak{m} \in \partial F_n$. *Then* $\dim H^{i-1}(C(\mathscr{F}, M)) \leqslant n - i - 1$ *for all* $i = 0, \ldots, n - 1$. (*We interpret the dimension of the zero* A*-module as* -1.)

Proof. Let $i \in \mathbf{N}_0$ with $0 \leqslant i \leqslant n - 1$. We may assume that $H^{i-1}(C(\mathscr{F}, M)) \neq 0$. Let

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{m}$$

be a chain of prime ideals in $\operatorname{Supp}(H^{i-1}(C(\mathscr{F}, M)))$. By 2.3,

$$\operatorname{ht}_{\mathscr{F}} \mathfrak{p}_0 < \operatorname{ht}_{\mathscr{F}} \mathfrak{p}_1 < \cdots < \operatorname{ht}_{\mathscr{F}} \mathfrak{p}_t = \operatorname{ht}_{\mathscr{F}} \mathfrak{m} = n .$$

But, by [8, 1.4(i)], we have $\mathfrak{p}_0 \in F_{i+1}$, and so $i + 1 \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p}_0$. Hence $n \geqslant t + i + 1$, and the result follows from this.

Our next result is a generalization of part of the Theorem of [6], and of [8, 1.8].

2.5. Proposition. *Suppose, temporarily, that* A *is local with maximal ideal* $\mathfrak{m}$, *and that* $\mathfrak{m} \in \partial F_n$. *Then* M^n, *the* n*-th term in the Cousin complex* $C(\mathscr{F}, M)$, *is isomorphic to* $H^n_{\mathfrak{m}}(M)$, *the* n*-th local cohomology module of* M *with respect to* $\mathfrak{m}$.

Proof. Now that Lemma 2.4 has been proved, this can be achieved by straightforward modification of the argument given in the proof of the Theorem in [6], and so will be left to the reader.

2.6. Corollary. *Let* $n \in \mathbf{N}_0$ *and* $\mathfrak{q} \in \partial F_n$. *Then the direct summand* $(\operatorname{Coker} d^{n-2})_{\mathfrak{q}}$ *of* M^n *is* $A_{\mathfrak{q}}$*-isomorphic to* $H^n_{\mathfrak{q}A_{\mathfrak{q}}}(M_{\mathfrak{q}})$, *the n-th local cohomology module of* $M_{\mathfrak{q}}$ *with respect to the maximal ideal of the local ring* $A_{\mathfrak{q}}$.

Proof. This is immediate from 1.4 and 2.5.

2.7. Corollary. *If* $\mathfrak{p}$ *is a prime ideal of* A *which is of significance for* $C(\mathscr{F}, M)$, *then* $\mathfrak{p} \in \operatorname{Supp}(M)$ *and* $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = \operatorname{ht}_M \mathfrak{p}$.

Proof. By [8, 1.5], $\mathfrak{p} \in \operatorname{Supp}(M)$. Let $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} = n$, necessarily finite. By 2.6,

$$H^n_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cong (\operatorname{Coker} d^{n-2})_{\mathfrak{p}} \neq 0 .$$

Hence, by [5, 6.1], $n \leqslant \dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{ht}_M \mathfrak{p}$. On the other hand, $\operatorname{ht}_M \mathfrak{p} \leqslant n$ by 2.3(iv), and so the proof is complete.

2.8. Remark. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. It follows from 2.7 and 2.6 that, for $\mathfrak{p}$ to be of significance for $C(\mathscr{F}, M)$, several conditions are necessary. It is worthwhile for us to stress the details.

(i) If $\mathfrak{p} \in \operatorname{Spec}(A) \backslash F_0$, then $\mathfrak{p}$ is of no significance for $C(\mathscr{F}, M)$.

(ii) If $\mathfrak{p} \in F_0 \backslash \operatorname{Supp}(M)$, then $\mathfrak{p}$ is of no significance for $C(\mathscr{F}, M)$.

(iii) If $\mathfrak{p} \in \operatorname{Supp}(M)$, then $\operatorname{ht}_M \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p}$, and $\mathfrak{p}$ only has significance for $C(\mathscr{F}, M)$ if $\operatorname{ht}_M \mathfrak{p} = \operatorname{ht}_{\mathscr{F}} \mathfrak{p}$; in fact, when this condition is satisfied, $\mathfrak{p}$ has significance for $C(\mathscr{F}, M)$ if and only if $H^{\operatorname{ht}_M \mathfrak{p}}_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$.

(iv) In particular, if $\mathfrak{p} \in \bigcap_{n \in \mathbf{N}_0} F_n$, then $\mathfrak{p}$ is of no significance for $C(\mathscr{F}, M)$.

2.9. Remark. If follows from 2.8 that, for each $n \in \mathbf{N}_0$,

$$M^n = \bigoplus_{\mathfrak{p} \in \partial F_n \cap \partial H_n} (\operatorname{Coker} d^{n-2})_{\mathfrak{p}} .$$

2.10. Theorem. (*We use the notation of* 2.1.) *Suppose that* $F_i \subseteq G_i$ *for all* $i \in \mathbf{N}_0$, *so that, on use of* 2.3(iv), $\operatorname{ht}_M \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{G}} \mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M)$.

For each $n \in \mathbf{N}_0$, *let*

$$S^n = \bigoplus_{\mathfrak{p} \in \partial F_n \cap \partial H_n \backslash \partial G_n} (\operatorname{Coker} d^{n-2})_{\mathfrak{p}} .$$

Then $d^n(S^n) \subseteq S^{n+1}$ *for all* $n \in \mathbf{N}_0$, *and so, if* u^n *denotes the restriction of* d^n *to* S^n (*for each* $n \geqslant -2$) (*interpret* $S^{-2} = S^{-1} = 0$), *then*

$$0 \xrightarrow{u^{-2}} 0 \xrightarrow{u^{-1}} S^0 \xrightarrow{u^0} S^1 \to \cdots \to S^n \xrightarrow{u^n} S^{n+1} \to \cdots$$

is a subcomplex of $C(\mathscr{F}, M)$; *we denote this subcomplex by* $S(\mathscr{F}, \mathscr{G}, M)$.

The quotient complex $C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)$ *is isomorphic to the Cousin complex* $C(\mathscr{G}, M)$.

Proof. Let $n \in \mathbf{N}_0$ and $\mathfrak{p} \in \partial F_n \cap \partial H_n \backslash \partial G_n$, and suppose that $0 \neq x \in (\operatorname{Coker} d^{n-2})_{\mathfrak{p}}$. Let $\mathfrak{q} \in \partial F_{n+1}$ be such that the component $(x + \operatorname{Im} d^{n-1})/1$ of $d^n(x)$ in the direct summand $(\operatorname{Coker} d^{n-1})_{\mathfrak{q}}$ of M^{n+1} is non-zero.

By 2.8(iii), $\mathfrak{q} \in \partial F_{n+1} \cap \partial H_{n+1}$. By 1.2(iii), x is annihilated by some power of $\mathfrak{p}$; hence, since multiplication by an element of $A \backslash \mathfrak{q}$ provides an automorphism of the A-module $(\text{Coker } d^{n-1})_{\mathfrak{q}}$, we must have $\mathfrak{p} \subset \mathfrak{q}$. This means that $\mathfrak{q} \notin \partial G_{n+1}$, for otherwise, since $\mathfrak{p} \in G_{n+1}$, we should have a contradiction to the fact that each member of ∂G_{n+1} is a minimal member of G_{n+1} with respect to inclusion. Hence $\mathfrak{q} \in \partial F_{n+1} \cap \partial H_{n+1} \backslash \partial G_{n+1}$ and it follows that $d^n(S^n) \subseteq S^{n+1}$, as claimed.

Let us write the quotient complex $C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)$ as

$$0 \xrightarrow{v^{-2}} M \xrightarrow{v^{-1}} Q^0 \xrightarrow{v^0} Q^1 \to \cdots \to Q^n \xrightarrow{v^n} Q^{n+1} \to \cdots .$$

Our strategy is to show that this complex is of Cousin type for M with respect to $\mathscr{G}$ and then appeal to 1.2(ii). First note that, for each $n \in \mathbf{N}_0$,

$$Q^n \cong \bigoplus_{\mathfrak{p} \in \partial G_n \cap \partial F_n \cap \partial H_n} (\text{Coker } d^{n-2})_{\mathfrak{p}} = \bigoplus_{\mathfrak{p} \in \partial G_n \cap \partial H_n} (\text{Coker } d^{n-2})_{\mathfrak{p}}$$

$$\cong \bigoplus_{\mathfrak{p} \in \partial G_n \cap \partial H_n} H^n_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cong \bigoplus_{\mathfrak{p} \in \partial G_n \cap \partial H_n} (\text{Coker } e^{n-2})_{\mathfrak{p}} \cong \bigoplus_{\mathfrak{p} \in \partial G_n} (\text{Coker } e^{n-2})_{\mathfrak{p}}$$

$$= N^n,$$

in view of 2.6, 2.8(iii) and 2.9. It is thus immediate from 1.2 that $\text{Supp}\,(Q^n) \subseteq G_n$ and the natural A-homomorphism

$$\xi(Q^n) \colon Q^n \to \bigoplus_{\mathfrak{p} \in \partial G_n} Q^n_{\mathfrak{p}}$$

is an isomorphism. Thus $C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)$ satisfies conditions (i) and (iv) of 1.1 (for $\mathscr{G}$).

Let us now turn attention to condition 1.1(ii). Of course, $\text{Supp}\,(M) \subseteq G_0$, so let $n \in \mathbf{N}$ (we use $\mathbf{N}$ to denote the set of positive integers). Then

$$\text{Supp}\,(\text{Coker } v^{n-2}) \subseteq \text{Supp}\,(\text{Coker } d^{n-2}) \cap \text{Supp}\,(Q^{n-1}) \subseteq F_n \cap G_{n-1} .$$

Suppose that $\mathfrak{p} \in \partial G_{n-1} \cap \text{Supp}\,(\text{Coker } v^{n-2})$ and look for a contradiction. Then $\text{ht}_{\mathscr{F}}\, \mathfrak{p} \leqslant \text{ht}_{\mathscr{G}}\, \mathfrak{p} = n-1$, so that $\mathfrak{p} \in \text{Supp}\,(\text{Coker } d^{n-2}) \backslash F_n$, a contradiction. Hence $\text{Supp}\,(\text{Coker } v^{n-2}) \subseteq G_n$, and $C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)$ satisfies condition (ii) of 1.1 for $\mathscr{G}$. It remains for us to check condition (iii).

Let $n \in \mathbf{N}_0$. First of all, on use of the results of the immediately preceding paragraph,

$$\text{Supp}\,(H^{n-1}(C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M))) \subseteq \text{Supp}\,(\text{Coker } v^{n-2})$$
$$\subseteq \text{Supp}\,(\text{Coker } d^{n-2}) \cap G_n .$$

Suppose that $\mathfrak{p} \in \partial G_n \ \text{Supp}(H^{n-1}(C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)))$, and look for a contradiction. Then $\text{ht}_{\mathscr{F}}\, \mathfrak{p} \leqslant \text{ht}_{\mathscr{G}}\, \mathfrak{p} = n$, so that $\mathfrak{p} \in \text{Supp}\,(\text{Coker } d^{n-2}) \backslash F_{n+1}$. Since $C(\mathscr{F}, M)$ is of Cousin type for M with respect to $\mathscr{F}$, it follows that

$$(H^{n-1}(C(\mathscr{F}, M)))_{\mathfrak{p}} = (H^n(C(\mathscr{F}, M)))_{\mathfrak{p}} = 0 .$$

It therefore follows from the long exact sequence of cohomology modules which results from the exact sequence of complexes

$$0 \to S(\mathscr{F}, \mathscr{G}, M) \to C(\mathscr{F}, M) \to C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M) \to 0$$

that $(H^{n-1}(C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)))_{\mathfrak{p}} \cong (H^n(S(\mathscr{F}, \mathscr{G}, M)))_{\mathfrak{p}}$. However, it follows from 1.2(iii) that $(S^n)_{\mathfrak{p}} = 0$, because

$$S^n = \bigoplus_{\mathfrak{p}' \in \partial F_n \cap \partial H_n \setminus \partial G_n} (\operatorname{Coker} d^{n-2})_{\mathfrak{p}'} .$$

Hence $(H^{n-1}(C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)))_{\mathfrak{p}} \cong (H^n(S(\mathscr{F}, \mathscr{G}, M)))_{\mathfrak{p}} = 0$, and we have a contradiction. Thus $\operatorname{Supp}(H^{n-1}(C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M))) \subseteq G_{n+1}$. We have therefore proved that the complex $C(\mathscr{F}, M)/S(\mathscr{F}, \mathscr{G}, M)$ is of Cousin type for M with respect to $\mathscr{G}$, and so an application of 1.2(ii) completes the proof.

With the aid of 1.2(ii) and 2.9, we can now immediately deduce the following corollary of 2.10.

2.11. Corollary. (*We use the notation of* 2.1.) *Suppose that* $F_i \subseteq G_i$ *for all* $i \in \mathbf{N}_0$. *There is a morphism of complexes*

$$\Theta = (\theta^i)_{i \geqslant -2}: C(\mathscr{F}, M) \to C(\mathscr{G}, M)$$

which has the following properties:

(i) $\theta^{-1}: M \to M$ *is the identity mapping*;

(ii) *for each* $n \in \mathbf{N}_0$, *the map* $\theta^n: M^n \to N^n$ *is an epimorphism whose kernel is a direct summand of* M^n;

(iii) *for each* $n \in \mathbf{N}_0$ *and each* $\mathfrak{p} \in \partial G_n \cap \partial H_n$ *we have* ($\mathfrak{p} \in \partial F_n \cap \partial H_n$ *too, and*) *the restriction of* θ^n *to the direct summand* $(\operatorname{Coker} d^{n-2})_{\mathfrak{p}}$ *of* M^n *gives an isomorphism* $(\operatorname{Coker} d^{n-2})_{\mathfrak{p}} \xrightarrow{\cong} (\operatorname{Coker} e^{n-2})_{\mathfrak{p}}$;

(iv) *for each* $n \in \mathbf{N}_0$ *and each* $\mathfrak{p} \in \partial F_n \cap \partial H_n \setminus \partial G_n$, *the restriction of* θ^n *to the direct summand* $(\operatorname{Coker} d^{n-2})_{\mathfrak{p}}$ *of* M^n *is zero.*

2.12. Remark. Theorem 2.10 and Corollary 2.11 can be applied to the basic Cousin complex $C(\mathscr{H}(M), M)$ for M and $C(\mathscr{F}, M)$, because, by 2.3(iv), $H_i \subseteq F_i$ for all $i \in \mathbf{N}_0$.

Thus it follows from 2.10 that the modules

$$S^n = \bigoplus_{\mathfrak{p} \in \partial H_n \setminus \partial F_n} (\operatorname{Coker} b^{n-2})_{\mathfrak{p}}$$

form the terms of a subcomplex of $C(\mathscr{H}(M), M)$, and the corresponding quotient complex is isomorphic to $C(\mathscr{F}, M)$. Furthermore, 2.11 tells us that there is a morphism of complexes $\Omega = (\omega^i)_{i \geqslant -2}: C(\mathscr{H}(M), M) \to C(\mathscr{F}, M)$ such that $\omega^{-1}: M \to M$ is the identity mapping, and for each $n \in \mathbf{N}_0$ and each $\mathfrak{p} \in \partial H_n$, the restriction of ω^n to the direct summand $(\operatorname{Coker} b^{n-2})_{\mathfrak{p}}$ of B^n is zero if $\mathfrak{p} \notin \partial F_n$, but gives an isomorphism of $(\operatorname{Coker} b^{n-2})_{\mathfrak{p}}$ onto $(\operatorname{Coker} d^{n-2})_{\mathfrak{p}}$ if $\mathfrak{p} \in \partial F_n$.

3. Applications to exactness of Cousin complexes

We begin this section with reminders of the definitions of the small support and the depth of the A-module M (recall that M is not assumed to be finitely generated).

3.1. Definitions. Recall from [1, Section 1] that, when A is local with maximal ideal $\mathfrak{m}$, the *depth of* M, denoted by depth M or $\operatorname{depth}_A M$, is defined to be the smallest integer i such that $\operatorname{Ext}^i_A(A/\mathfrak{m}, M) \neq 0$ if any such integer exists, and ∞ otherwise. For non-zero finitely generated M, this depth is, of course, finite, but for a non-zero M which is not finitely generated, it is possible for this depth to be ∞.

By [1, (2.1)], $H^i_{\mathfrak{m}}(M) = 0$ for all $i < \operatorname{depth} M$, while $H^i_{\mathfrak{m}}(M) \neq 0$ for $i = \operatorname{depth} M$ if depth M is finite. Thus, by [5, 6.1], if depth M is finite, then depth $M \leqslant \dim M$.

We now revert to the situation where A denotes a general commutative Noetherian ring, not necessarily local. The *small support*, or *little support*, *of* M, denoted by $\operatorname{supp}(M)$ or $\operatorname{supp}_A(M)$, is defined by

$$\operatorname{supp}(M) = \{\mathfrak{p} \in \operatorname{Spec}(A) : \operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \text{ is finite}\}.$$

It is clear that $\operatorname{supp}(M) \subseteq \operatorname{Supp}(M)$; if M is finitely generated, then these two sets are equal, but in general this need not be the case.

3.2. Remark. Note that, by 3.1 and 2.3(iv), for $\mathfrak{p} \in \operatorname{supp}(M)$,

$$\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{ht}_M \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p}.$$

Note also that if $\mathfrak{p}$ is a prime ideal of significance for $C(\mathscr{F}, M)$ then, by 2.8, we must have $\mathfrak{p} \in \operatorname{supp}(M)$.

3.3. Remark. Suppose that $n \in \mathbb{N}$ and that the Cousin complex $C(\mathscr{F}, M)$ is exact at $M = M^{-1}, M^0, \ldots, M^{n-2}$. Let L be a finitely generated A-module such that $(0 : L)$ is not contained in any prime ideal $\mathfrak{p}$ which is of significance for $C(\mathscr{F}, M)$ and satisfies $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} \leqslant n - 1$. Then an argument entirely similar to the 'partially exact Cousin complex argument' of [3, (4.6)] will show that

$$\operatorname{Ext}^i_A(L, M) \begin{cases} = 0 & \text{for } i < n; \\ \cong \operatorname{Hom}_A(L, \operatorname{Coker} d^{n-2}) & \text{for } i = n. \end{cases}$$

3.4. Proposition. *Let* $n \in \mathbb{N}$. *Then the Cousin complex* $C(\mathscr{F}, M)$ *is exact at* $M = M^{-1}, M^0, \ldots, M^{n-2}$ *if and only if, for every* $\mathfrak{p} \in \operatorname{supp}(M)$, *we have* $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geqslant \min\{n, \operatorname{ht}_{\mathscr{F}} \mathfrak{p}\}$.

Proof. $(\Rightarrow)$ Let $\mathfrak{p} \in \operatorname{supp}(M)$, so that, $\mathfrak{p} \in \operatorname{Supp}(M)$ by 3.1 and $\operatorname{ht}_{\mathscr{F}} \mathfrak{p}$ is non-negative: let this height be r. Then, by 2.3(iii), $\mathfrak{p}$ is not contained in any prime ideal $\mathfrak{q}$ which is of significance for $C(\mathscr{F}, M)$ and satisfies $\operatorname{ht}_{\mathscr{F}} \mathfrak{q} < r$. Hence, by 3.3, $\operatorname{Ext}^i_A(A/\mathfrak{p}, M) = 0$ for all $i < \min\{n, r\}$, so that $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geqslant \min\{n, r\}$.

($\Leftarrow$) Assume that $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geqslant \min\{n, \operatorname{ht}_{\mathscr{F}} \mathfrak{p}\}$ for every $\mathfrak{p} \in \operatorname{supp}(M)$. Suppose that $t \in \mathbf{N}$ is such that $C(\mathscr{F}, M)$ is exact at $M = M^{-1}, M^0, \ldots, M^{t-2}$ but not exact at M^{t-1}; we suppose that $t < n$, and look for a contradiction.

Let $\mathfrak{p}$ be a minimal member of $\operatorname{Supp}(H^{t-1}(C(\mathscr{F}, M)))$, so that

$$\mathfrak{p} \in \operatorname{Ass}(H^{t-1}(C(\mathscr{F}, M))).$$

By [8, 1.4], $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} \geqslant t+1$. By 3.3, $\operatorname{Ext}^t_A(A/\mathfrak{p}, M) \cong \operatorname{Hom}_A(A/\mathfrak{p}, \operatorname{Coker} d^{t-2})$, and so $\operatorname{Ext}^t_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, M_\mathfrak{p}) \cong \operatorname{Hom}_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}, (\operatorname{Coker} d^{t-2})_\mathfrak{p})$, and this is non-zero because

$$\mathfrak{p}A_\mathfrak{p} \in \operatorname{Ass}_{A_\mathfrak{p}}((H^{t-1}(C(\mathscr{F}, M)))_\mathfrak{p}) \subseteq \operatorname{Ass}_{A_\mathfrak{p}}((\operatorname{Coker} d^{t-2})_\mathfrak{p}).$$

Hence $\mathfrak{p} \in \operatorname{supp}(M)$ and $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \leqslant t$. However, $\operatorname{ht}_{\mathscr{F}} \mathfrak{p} \geqslant t+1$, and so $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} < \min\{n, \operatorname{ht}_{\mathscr{F}} \mathfrak{p}\}$. This contradiction completes the proof.

3.5. Corollary. *The following conditions are equivalent*:
(i) $C(\mathscr{F}, M)$ *is exact*;
(ii) $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \geqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M)$;
(iii) $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} = \operatorname{ht}_{\mathscr{F}} \mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{supp}(M)$.

Proof. This is now immediate from 3.2 and 3.4.

We are now ready to state and prove our main result in this section. It shows that there can be essentially at most one exact Cousin complex for M, as it shows that if $C(\mathscr{F}, M)$ is exact, then the basic Cousin complex for M, $C(\mathscr{H}(M), M)$ is exact and isomorphic to $C(\mathscr{F}, M)$.

3.6. Theorem. *Here, we use the notation of* 2.1, *so that* $\mathscr{G} = (G)_{i \in \mathbf{N}_0}$ *denotes a second filtration of* Spec(A) *which admits* M. *We shall suppose that* $F_i \subseteq G_i$ *for all* $i \in \mathbf{N}_0$, *so that, by* 2.3(iv), $\operatorname{ht}_M \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{G}} \mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M)$ *and the results of* 2.10 *and* 2.11 *are available.*

If $C(\mathscr{G}, M)$ *is exact, then*

(i) $C(\mathscr{F}, M)$ *is exact*;

(ii) *the morphism of complexes* $\Theta = (\theta^i)_{i \geqslant -2}\colon C(\mathscr{F}, M) \to C(\mathscr{G}, M)$ *of* 2.11 *is an isomorphism*;

(iii) $\operatorname{supp}(M)$ *is equal to the set of prime ideals of A which are of significance for* $C(\mathscr{G}, M)$, *and also equal to the set of prime ideals of A which are of significance for* $C(\mathscr{F}, M)$;

(iv) *for* $\mathfrak{p} \in \operatorname{supp}(M)$ *with* $\operatorname{ht}_M \mathfrak{p} = n$, *we have* $\mathfrak{p} \in \partial F_n \cap \partial G_n$ *and the restriction of* θ^n *to the direct summand* $(\operatorname{Coker} d^{n-2})_\mathfrak{p}$ *of* M^n *gives an isomorphism* $(\operatorname{Coker} d^{n-2})_\mathfrak{p} \xrightarrow{\cong} (\operatorname{Coker} e^{n-2})_\mathfrak{p}$;

(v) *both the complexes* $C(\mathscr{G}, M)$ *and* $C(\mathscr{F}, M)$ *are isomorphic to the basic Cousin complex* $C(\mathscr{H}(M), M)$ *for* M, *so that all three complexes are exact.*

Note. Given an exact Cousin complex $C(\mathscr{G}, M)$, we can always use the basic Cousin complex $C(\mathscr{H}(M), M)$ in the rôle of $C(\mathscr{F}, M)$. Thus this theorem gives information about all exact Cousin complexes.

Proof. Let $\mathfrak{p} \in \operatorname{supp}(M)$. By 3.2 and 3.5,

$$\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} \leqslant \operatorname{ht}_M \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{F}} \mathfrak{p} \leqslant \operatorname{ht}_{\mathscr{G}} \mathfrak{p} = \operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} .$$

Therefore $\operatorname{depth}_{A_\mathfrak{p}} M_\mathfrak{p} = \operatorname{ht}_M \mathfrak{p} = \operatorname{ht}_{\mathscr{F}} \mathfrak{p} = \operatorname{ht}_{\mathscr{G}} \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{supp}(M)$, and so it follows from 3.5 that $C(\mathscr{F}, M)$ and $C(\mathscr{H}(M), M)$ are exact. All the remaining claims now follow easily from 2.11, 3.2, 2.8(iii) and the Note immediately preceding this proof.

3.7. Example. Suppose that A is local with maximal ideal $\mathfrak{m}$. In [8, 4.1], a Cousin complex characterization of balanced big Cohen-Macaulay A-modules was given: let $d := \dim A$, and let $\mathscr{D}(A) = (D_i)_{i \in \mathbf{N}_0}$ be the dimension filtration of $\operatorname{Spec}(A)$ given by $D_i = \{\mathfrak{p} \in \operatorname{Spec}(A)\colon \dim A/\mathfrak{p} \leqslant \dim A - i\}$ for all $i \in \mathbf{N}_0$; then M is a balanced big Cohen-Macaulay A-module if and only if $C(\mathscr{D}(A), M)$ is exact and $H^d_\mathfrak{m}(M) \neq 0$.

Let us assume that M is a balanced big Cohen-Macaulay A-module. Various characterizations of the small support of M are provided by [7, (3.2)] (where the small support was called the 'supersupport' of M).

Note that, for $i \in \mathbf{N}_0$, we have

$$\{\mathfrak{p} \in \operatorname{Spec}(A)\colon \operatorname{ht} \mathfrak{p} \geqslant i\} \subseteq \{\mathfrak{p} \in \operatorname{Spec}(A)\colon \dim A - \dim A/\mathfrak{p} \geqslant i\} .$$

It therefore follows from 3.6 and [8, 4.1] that all three Cousin complexes

$$C(\mathscr{D}(A), M)\,, \qquad C(\mathscr{H}(A), M)\,, \qquad C(\mathscr{H}(M), M)$$

are isomorphic and exact, and that only primes in $\operatorname{supp}(M)$ (which, by [7, (3.3)] or 3.6, must satisfy $\operatorname{ht}_M \mathfrak{p} = \operatorname{ht}_A \mathfrak{p} = \operatorname{ht}_{\mathscr{D}(A)} \mathfrak{p}$) have significance for these Cousin complexes.

Thus a prime ideal $\mathfrak{q}$ of A with $\operatorname{ht} \mathfrak{q} + \dim A/\mathfrak{q} < \dim A$ has no significance for any of these Cousin complexes: it was a desire to investigate this point, and compare these three Cousin complexes, which provided the impetus for this research.

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