J. Math. Kyoto Univ. (JMKYAZ)
33-1 (1993) 195-203

Some prime ideals in the extensions of Noetherian rings

By

Junro SATO

Introduction

Let R and S be commutative rings with unity and let $f: R \to S$ be a homomorphism with $f(1) = 1$. Mcquillan, in his paper [2], introduces the notions of S-prime and S-primary ideals. In fact, he proved that if f is a flat homomorphism, then every prime ideal of S is S-prime, and that if R is integrally closed integral domain, S is integral over R and no non-zero element of R is a divisor of zero in S, then every prime ideal of R is S-primary. The aim of this paper is to investigate more closely S-prime and S-primary ideals. Further, we introduce the notion of S-quasi-primary, and determine its structure. We are mainly interested in the case where R and S are Noetherian. If $f: R \to S$ is a homomorphism of rings and if I is an ideal of S, then $f^{-1}(I)$ is denoted by $I \cap R$.

In the first section, we consider S-prime ideals. In fact, we shall prove that $\mathfrak{a}$ is S-prime if and only if $\mathfrak{a}$ is a prime ideal of R and $\mathfrak{a}S \cap R = \mathfrak{a}$, provided that $\mathfrak{a}S \neq S$.

In the second section, we discuss S-primary ideals.

In the final section, we introduce and study S-quasi-primary ideals. The main result is that $\mathfrak{a}$ is an S-quasi-primary if and only if $\mathfrak{a}S \cap R$ is an R-primary ideal of R such that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{a}S \cap R} = p$, and that $\mathrm{Ass}_R(S/\mathfrak{a}S) = \{p\}$.

In this article R and S are assumed to be commutative rings and to have unity unless otherwise specified, and $f: R \to S$ is a homomorphism with $f(1) = 1$, and that our general references for unexplained technical terms are [1] and [3].

§ 1. S-prime ideals

First, we recall the following definition.

Definition 1.1. Let $f: R \to S$ be a homomorphism and let q be an ideal of R. We say that q is S-prime if, $a \in R$, $\alpha \in S$ and $f(a)\alpha \in qS$, implies $a \in q$, or $\alpha \in qS$.

Remark 1.2. Let q be an ideal of R such that $qS = S$. Then q is S-prime.

Now, we show our key lemma.

Communicated by Prof. K. Ueno, August 5, 1991

Lemma 1.3. *Let $f: R \to S$ be a homomorphism, and let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}S \neq S$. If $\mathfrak{a}$ is S-prime, then $\mathfrak{a}S \cap R$ is a prime ideal of R.*

Proof. Let a, $b \in R$ be such that $ab \in \mathfrak{a}S \cap R$ and $a \notin \mathfrak{a}S \cap R$. Then, since $f(a)b \in \mathfrak{a}S$ and $a \notin \mathfrak{a}$, we have $f(b) \in \mathfrak{a}S$ by definition. Thus $b \in \mathfrak{a}S \cap R$, and $\mathfrak{a}S \cap R$ is a prime ideal of R. Q.E.D.

As a consequence:

Theorem 1.4. *Let $f: R \to S$ be a homomorphism, and let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}S \neq S$. Assume that $\mathfrak{a}$ is S-prime. Then $\mathfrak{a}S \cap R = \mathfrak{a}$. There $\mathfrak{a}$ is a prime ideal of R.*

Proof. First we shall show $\mathfrak{a}S \cap R = \mathfrak{a}$. One inclusion is obvious. For the converse inclusion, let $r \in \mathfrak{a}S \cap R$. Since $1 \in S$, $f(r) = f(r) \cdot 1 \in \mathfrak{a}S$. Since $\mathfrak{a}$ is S-prime, we have $r \in \mathfrak{a}$ or $1 \in \mathfrak{a}S$. By our assumption $\mathfrak{a}S \neq S$, and we have $r \in \mathfrak{a}$, and hence $\mathfrak{a}S \cap R = \mathfrak{a}$, and $\mathfrak{a}$ is prime by Lemma 1.3. Q.E.D.

As another characterization of being S-prime, we have the following:

Proposition 1.5. *Let $f: R \to S$ be a homomorphism, and let $p \in \mathrm{Spec}(R)$. Then p is S-prime if and only if $pS = S$ or $pS_p \cap S = pS$.*

Proof. ($\Rightarrow$): Assume that $pS \neq S$. We have only to show that $pS_p \cap S = pS$. One inclusion is clear. For the converse inclusion, let $\alpha \in pS_p \cap S$. Then there exists an element $a \in R \backslash p$ such that $f(a)\alpha \in pS$. Since p is S-prime, we have $\alpha \in pS$.
($\Leftarrow$): We may assume that $pS \neq S$. Suppose that $f(a)\alpha \in pS$ with $a \in R$ and $\alpha \in S$. If $a \in R \backslash p$, then $\alpha \in pS_p \cap S = pS$, and hence p is S-prime. Q.E.D.

By a localization, S-prime ideals behave nicely. In fact, the following proposition holds:

Proposition 1.6. *Let $f: R \to S$ be a homomorphism, and let $p \subseteq q$ be in $\mathrm{Spec}(R)$. If p is S-prime, then pR_q is S_q-prime.*

Proof. Assume that $f_q(a)\ \alpha \in pS_q$ with $a \in R_q$ and $\alpha \in S_q$, where $f_q: R_q \to S_q$ is defined by $f_q(b/c) = f(b)/c$ for all $b \in R$ and $c \in R \backslash q$. Then there exists an element $x \in R \backslash q$ such that $xa \in R$, $f(x)\alpha \in S$ and $f(xa)f(x)\alpha \in pS$. Since p is S-prime, we have $xa \in p$ or $f(x)\alpha \in pS$, and hence $a \in pR_q$ or $\alpha \in pS_q$. Thus pR_q is S_q-prime. Q.E.D.

The following proposition is well applicable to the case of extensions of Notherian rings.

Proposition 1.7. *Let $f: R \to S$ be a homomorphism of Noetherian rings, and let $p \in \mathrm{Spec}(R)$ be such that $pS \neq S$. Suppose that there exist only a finite number of prime ideals $P_1, \ldots, P_n$ of S lying over p. If $P_1, \ldots, P_n$ are only the prime divisors of pS, then p is S-prime.*

Proof. Let $pS = Q_1 \cap \cdots \cap Q_n$, $\sqrt{Q_i} = P_i$ $(1 \leqq i \leqq n)$ be an irredundant primary decomposition of the ideal pS of S. Suppose that $f(a)\alpha \in pS$ with $a \in R$ and $\alpha \in S$. If $a \notin p$, then $f(a) \notin P_1, \ldots, P_n$, and hence $\alpha \in Q_1 \cap \cdots \cap Q_n = pS$. Thus p is S-prime. Q.E.D.

In [2], the following proposition is proved:

Proposition 1.8 (cf. [2, Proposition 2]). *Let $f: R \to S$ be a homomorphism and suppose that S is flat as an R-module. Then every prime ideal of R is S-prime.*

As an application of this proposition, we have:

Proposition 1.9. *Let $f: R \to S$ be a homomorphism of Noetherian rings. Suppose that f is flat and integral. Then, for any $p \in \operatorname{Spec}(R)$, pS has no embedded prime divisors.*

Proof. Let $pS = Q_1 \cap \cdots \cap Q_n \cap T_1 \cap \cdots \cap T_m$ be an irredundant primary decomposition with embedded primary components $T_1, \ldots, T_m$. Since, by assumption, f is integral, we see that $\sqrt{T_1} \cap \cdots \cap \sqrt{T_m} \cap R \supsetneqq p$. Hence there exist elements $a \in R$ and $\alpha \in S$ such that $a \in \sqrt{T_1} \cap \cdots \cap \sqrt{T_m} \cap (R \backslash p)$ and $\alpha \in (Q_1 \cap \cdots \cap Q_n) \backslash pS$. Thus $a^k \in T_1 \cap \cdots \cap T_m \cap (R \backslash p)$ and $f(a^k)\alpha \in pS$ for some k. This implies that p is not S-prime. However, f is flat, and p is S-prime by Proposition 1.7. This is a contradiction. Q.E.D.

In terms of the prime divisors of pS, we give the following criterion to be S-prime.

Theorem 1.10. *Let $f: R \to S$ be a homomorphism of Noetherian rings, and let $p \in \operatorname{Spec}(R)$. Then p is S-prime if and only if either $pS = S$ or $pS \neq S$ and $P \cap R = p$ for any prime divisor P of the ideal pS.*

Proof. $(\Rightarrow)$: Assume that $pS \neq S$. Let $pS = Q_1 \cap \cdots \cap Q_n \cap T_1 \cap \cdots \cap T_m$ be an irredundant primary decomposition of the ideal pS where $\sqrt{Q_i} \cap R = p$ $(1 \leqq i \leqq n)$ and $\sqrt{T_j} \cap R \supsetneqq p$ $(1 \leqq j \leqq m)$. Then there exist elements $a \in R$ and $\alpha \in S$ such that $a \in T_1 \cap \cdots \cap T_m \cap (R \backslash p)$, $\alpha \in (Q_1 \cap \cdots \cap Q_n) \backslash pS$ and $f(a)\alpha \in pS$, and pS is not S-prime.

$(\Leftarrow)$: We may assume that $pS \neq S$. Then, by assumption, we have an irredundant primary decomposition $pS = Q_1 \cap \cdots \cap Q_n$, and $\sqrt{Q_i} \cap R = p$ $(1 \leqq i \leqq n)$. Suppose that $f(a)\alpha \in pS$ with $a \in R$ and $\alpha \in S$, then $f(a)\alpha \in Q_1 \cap \cdots \cap Q_n$. If $a \notin p$, then $\alpha \in Q_1 \cap \cdots \cap Q_n = pS$, and hence p is S-prime. Q.E.D.

Combining this result with Proposition 1.8, we have the following well-known result.

Corollary 1.11 (cf. [1], (9.B), Theorem 12.). *Let $f: R \to S$ be a flat homomorphism of Noetherian rings and $p \in \operatorname{Spec}(R)$ such that $pS \neq S$. Then, for all prime divisors P of the ideal pS of S, we have $P \cap R = p$.*

We shall show an example if a prime ideal p of R which is not S-prime even if S is integral over R.

Example 1.12. Let $\mathbf{Q}$ be a rational number field and consider $R = \mathbf{Q}[x^2, y^2] \subset A = \mathbf{Q}[x, y]$. Let $S = \{a \in A | a(1, 1) = a(-1, 1) = a(-1, -1) = a(1, -1)\} \supset R$ and let $f: R \hookrightarrow S$ be the natural inclusion map. Note that f is an integral morphism. Put $m = (x - 1, y - 1)A \cap S = (x^2 - 1, y^2 - 1)A$. Then $\mathrm{ht}(m) = 2$. Let $\mathfrak{c}(A/S)$ is a conductor ideal of A over S. Then we can show that $\mathfrak{c}(A/S) = (x^2 - 1, y^2 - 1)A$. Indeed, $\mathfrak{c}(A/S) \supseteq (x^2 - 1, y^2 - 1)A$ is obvious. Take $a(x, y) \in$ Then we can write $a(x, y) = a_0 + a_1 x + a_2 y + a_3 xy + b(x, y)$, $a_i \in \mathbf{Q}$ $(1 \leqq i \leqq n)$, $b(x, y) \in (x^2 - 1, y^2 - 1)A$. Since $xa(x, y) \in S$, we can see easily that $a_0 = a_1 = a_2 = a_3 = 0$. Thus $m = \mathfrak{c}(A/S)$. Therefore we see that depth $S_m = 1$. (cf. [6], Proposition 1.9, Proposition 1.10 and Proposition 1.13) Let $x^2 R = p \in \mathrm{Spec}(R)$. Then it follows that m is a prime divisor of $pS = x^2 S$ and $m \cap R = (x^2 - 1, y^2 - 1)R$ $(\neq p)$. Hence p is not S-prime by Theorem 1.5. Further, this example shows that p is not S-prime, though $p \in \mathrm{Spec}(R)$, $pS \neq S$ and satisfies the condition $pS \cap R = p$.

The following is a corollary to Theorem 1.10.

Corollary 1.13. *Let $f: R \to S$ be a homomorphism of Noetherian rings and let m be a maximal ideal of R. Then m is S-prime.*

Proof. By Remark 1.2, we may assume that $mS \neq S$. Let $mS = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of the ideal mS of S, and let $p_i = \sqrt{Q_i} \cap R$ $(1 \leqq i \leqq n)$. Then, since $p_i \neq R$ and $p_i \subseteq m$, and we have $p_i = m$ for all i. Hence, by Theorem 1.10, m is S-prime. Q.E.D.

In [4], we introduce the notion of a "surper-primitive element". Now, we study the relationship between S-prime ideals and surper-primitive elements. For a moment, we consider birational-extensions of Noetherian domains. For a an element $\alpha \in K$, we set $I_\alpha = \{a \in R | a \cdot \alpha \in R\}$. I_α is an ideal of R and is called the denominator ideal of α in R.

Definition 1.14. Let R be a Noetherian domain with quotient field K. Then $\alpha \in K$ is called a super-primitive element over R if $I_\alpha + \alpha I_\alpha \nsubseteq p$ for for any $p \in Dp_1(R)$.

When α is a surper-primitive element, we say that $R[\alpha]$ is a surper-primitive extension of R.

We have a characterization of flatness using surper-primitive extension and S-prime ideals of depth one. In fact, the following proposition holds:

Proposition 1.15. *Let $\{\alpha_i | 1 \leqq i \leqq n\}$ be a set of super-primitive elements over a Noetherian domain R. Let $f: R \hookrightarrow S = R[\alpha_1, \ldots, \alpha_n]$ be the natural inclusion map. Then the following conditions are equivalent.*

(1) *For any $p \in Dp_1(R)$, p is S-prime.*

(2) *S is flat over R.*

(3) $pS_p \cap S = pS$ for any $p \in \mathrm{Spec}(R)$.

(4) $\mathrm{Ass}_R(S/pS) = \{p\}$ or $pS = S$ for any $p \in \mathrm{Spec}(R)$.

Proof. (2) $\Rightarrow$ (3): Since $R \to S \to S_q$ is flat, $pS_p \cap S = (pR_p \cap R) \otimes_R S = p \otimes_R S = pS$, as desired. (3) $\Rightarrow$ (4): Assume that $pS \neq S$. For any $q \in \mathrm{Ass}_R(S/pS)$, by definition, we have $q = (pS : \alpha)$ for some $\alpha \in S \backslash pS$, and hence $\alpha q S \subseteq pS$. Note that $p \subseteq q$. Now we assume that $p \subsetneqq q$, then $qR_p = R_p$, and $\alpha \in pS_p \cap S = pS$, a contradiction. Therefore $p = q$, as desired. (4) $\Rightarrow$ (1): By Remark 1.2, we may assume that $pS \neq S$. Let $f(a)\alpha \in pS$ where $a \in R$ and $\alpha \in S \backslash pS$. We shall show that $a \in p$. Now $a \cdot \bar{\alpha} = \bar{0}$ in S/pS. Thus a is a zero divisor in S/pS. Since $\mathrm{Ass}_R(S/pS) = \{p\}$, this implies $a \in p$, as desired.
(1) $\Rightarrow$ (2): If S is not flat over R, then we have $(I_{\alpha_1} \cap \cdots \cap I_{\alpha_n})S \neq S$ by [5, Proposition 1]. Now, let $I_{\alpha_1} \cap \cdots \cap I_{\alpha_n} = q_1 \cap \cdots \cap q_m$, $\sqrt{q_i} = p_i$ $(1 \leqq i \leqq m)$ be an irredundant primary decomposition of the ideal $I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$ of R. Note that $p_i \in Dp_1(R)$ by [6, Proposition, 1.10]. If $p_i S = S$ for all i, then $(I_{\alpha_1} \cap \cdots \cap I_{\alpha_n})S = S$. Hence $p_i S \neq S$ for some i. Since $p_i \supseteq I_{\alpha_j}$ for some j, we have $p_i S \supseteq I_{\alpha_j} + \alpha_j I_{\alpha_j}$. Put $\mathfrak{a}_j = I_{\alpha_j} + \alpha_j I_{\alpha_j}$ for all j. Thus we get $p_i S \cap R \supseteq \mathfrak{a}_j$. Since p_i is S-prime, $p_i = p_i S \cap R$ by Theorem 1.4. Thus $p_i \supseteq \mathfrak{a}_j$. But α_i is super-primitive, and $p_i \not\supseteq \mathfrak{a}_j$ by definition. This is a contradiction. Q.E.D.

Corollary 1.16. *Let R be a Noetherian normal domain with quotient field K, and let $\alpha_1, \ldots, \alpha_n \in K$. Then $S = R[\alpha_1, \ldots, \alpha_n]$ is flat over R if and only if every prime ideal $p \in \mathrm{Ht}_1(R)$ is S-prime.*

Proof. Since R is normal, α_i $(1 \leqq i \leqq n)$ is a super-primitive element over R by [4, 1.13 Theorem]. Hence the assertion follows easily from Proposition 1.15. Q.E.D.

Summarizing the above results, we have the main theorem of this section.

Theorem 1.17. *Let $f: R \to S$ be a homomorphism of Noetherian rings, and let $\mathfrak{a}$ be an ideal of R. Then $\mathfrak{a}$ is S-prime if and only if $\mathfrak{a} \in \mathrm{Spec}(R)$ and $\mathrm{Ass}_R(S/\mathfrak{a}S) = \{\mathfrak{a}\}$ or $\mathfrak{a}S = S$.*

§ 2. *S*-primary ideals

In this section, we start with the following definition.

Definition 2.1. Let $f: R \to S$ be a homomorphism and let q be an ideal of R. We say that q is S-primary if, $f(a)\alpha \in qS$ with $a \in R$, $\alpha \in S$, implies $a \in q$, or $\alpha \in \sqrt{qS}$.

If q is S-primary and if $qS \neq S$, then we can show that q is a primary ideal of R. Hence the notion of "S-primary" is similar to the ordinary primary ideal of R.

Theorem 2.2. *Let $f: R \to S$ be a homomorphism and let q be an ideal of R. If q is S-primary, then $qS \cap R = q$ and q is a primary ideal of R or $qS = S$.*

Proof. Suppose that $qS \neq S$. We note that $qS \cap R = q$. Indeed $qS \cap R \supseteq q$ obviously. To see the other inclusion, let $r \in qS \cap R$. Since $f(r) = f(r) \cdot 1 \in qS$ and q is S-primary, we have $r \in q$ or $1 \in \sqrt{qS}$. Since $qS \neq S$, we get $r \in q$. Therefore $qS \cap R = q$. It remains only to show that q is a primary ideal of R. Assume that $ab \in q$ with $a \in R \backslash q$ and $b \in R$, then $af(b) \in qS$. Since q is S-primary, it follows that $f(b) \in \sqrt{qS}$, and hence $b \in \sqrt{qS} \cap R = \sqrt{qS \cap R} = \sqrt{q}$. Thus q is a primary ideal of R. Q.E.D.

Proposition 2.3. *Let $f: R \rightarrow S$ be a homomorphism of Noetherian rings and let q be an ideal of R. Let $qS = Q_1 \cap \cdots \cap Q_n \cap T_1 \cap \cdots \cap T_m$ be an irredundant primary decomposition of the ideal qS of S, where T_j $(1 \leqq j \leqq n)$ are the embedded primary components of the expression of qS. If $Q_i \cap R = q$ for all i, then q is S-primary.*

Proof. Assume that $f(a)\alpha \in qS$ with $a \in R$ and $\alpha \in S \backslash \sqrt{qS}$. Then, since $\sqrt{qS} = \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_n}$, we have $\alpha \notin \sqrt{Q_i}$ for some i, and $a \in Q_i \cap R = q$. It follows that q is S-primary. Q.E.D.

If the ideal qS have no embedded primary components, then the converse of proposition holds:

Proposition 2.4. *Let $f: R \rightarrow S$ be a homomorphism of Noetherian rings, and let q be an ideal of R. Assume that the ideal qS has no embedded primary component. $qS = Q_1 \cap \cdots \cap Q_n$ be an irredundant primary decomposition of qS. If q is S-primary, then $Q_i \cap R = q$ for all i.*

Proof. We may assume that $Q_1 \cap \cdots \cap Q_n$ is the shortest primary decomposition. Suppose that $Q_i \cap R \supsetneqq q$ for some i. We may assume that $i = 1$. Then there exists an element $a \in (Q_1 \cap R) \backslash q$. Since $\sqrt{Q_i}$ $(1 \leqq i \leqq n)$ are minimal prime divisors, there is an element $\alpha \in Q_2 \cap \cdots \cap Q_n$ not contained in $\sqrt{Q_1}$. Since $\sqrt{qS} = \sqrt{Q_1} \cap \cdots \cap \sqrt{Q_n}$, $\alpha \notin \sqrt{qS}$. Note that $f(a)\alpha \in Q_1 \cap \cdots \cap Q_n = qS$. This contradicts that q is S-primary. Q.E.D.

We end this section with the following result.

Proposition 2.5. *Let $f: R \rightarrow S$ be a homomorphism and let q be an ideal of R. If q is S-primary, then $q: x$ is S-primary.*

Proof. We may assume that $x \notin q$. Now, suppose that $a \in R$, $\alpha \in S$, $f(a) \cdot \alpha \in (q : x)S$ and that $\alpha \notin \sqrt{(q : x)S}$. Then, since $\sqrt{qS} \subseteq \sqrt{(q : x)S}$, $\alpha \notin \sqrt{qS}$, and hence $f(ax)\alpha = f(a)\alpha\ f(x) \in qS$. Since q is S-primary, we have $ax \in q$, and $a \in q : x$. Therefore $q : x$ is S-primary. Q.E.D.

§ 3. S-quasi-primary ideals

In this section, we introduce the notion of "S-quasi-primary" and investigate its several properties.

Definition 3.1. Let $f: R \to S$ be a homomorphism and let q be an ideal of R. We say that q is S-quasi-primary if, $f(a)\alpha \in qS$ with $a \in R$, $\alpha \in S$, implies $a \in \sqrt{q}$ or $\alpha \in qS$.

Remark 3.2. For a prime ideal q of R, we see that q is S-quasi-primary if and only if q is S-prime.

Let q be an ideal of R, and let $\tilde{q} = qS \cap R$. The following example shows that for an S-quasi-primary ideal q of R it does not holds $q = \tilde{q}$.

Example 3.3. With a field k and an indeterminate t, we consider $R = k[t^2, t^3] \subset S = k[t]$, $q = t^2R$ and let $f: R \to S$ be the natural inclusion map. Then it follows that q is S-quasi-primary. Indeed, suppose that $a(t) \in R$, $\alpha \in S$, $a(t)\alpha \in qS = t^2k[t]$ and that $a(t) \notin \sqrt{q} = (t^2, t^3)R$. Then we have $a(0) \neq 0$, and hence $\alpha \in t^2k[t] = qS$. Therefore q is S-quasi-primary. Further, we can see easily that $t^3 \notin q$ and $t^3 = t^3 \cdot 1 \in (t^2, t^3)R = qS \cap R = \tilde{q}$. Hence we have $q \neq \tilde{q}$.

Next, for an S-quasi-primary ideal q of R, we consider relationship between q and $\tilde{q}$.

Proposition 3.4. *Let $f: R \to S$ be a homomorphism.*

For an ideal of R, we write $\tilde{q} = qS \cap R$. If q is S-quasi-primary and $qS \neq S$, then

(1) *$\tilde{q}$ is a primary ideal of R.*

(2) $\sqrt{q} = \sqrt{\tilde{q}}$.

Proof. (1) Let $ab \in \tilde{q}$ where $a \in R$ and $b \notin \tilde{q}$. Note that $af(b) \in qS$ and $f(b) \notin qS$. Since q is S-quasi-primary, we conclude $a \in \sqrt{q} \subseteq \sqrt{\tilde{q}}$. Thus $\tilde{q}$ is a primary ideal of R.

(2) The inclusion $\sqrt{q} \subseteq \sqrt{\tilde{q}}$ is clear. If $x \in \sqrt{\tilde{q}}$, then $x^n \in \tilde{q}$ for some n. Since $f(x^n) \cdot 1 \in qS$ and q is S-quasi-primary, it follows that $x^n \in \sqrt{q}$ or $1 \in qS$. Since $qS \neq S$, we have $x^n \in \sqrt{q}$, and $x \in \sqrt{q}$. Therefore $\sqrt{q} = \sqrt{\tilde{q}}$, as desired.

Q.E.D.

Consequently, we see that if q is S-quasi-primary, then $\sqrt{q} = p$ is a prime ideal of R. Furthermore, for the ideals q and $\tilde{q}$ of R, we have the following proposition.

Proposition 3.5. *Let $f: R \to S$ be a homomorphism and let q be an ideal of R. Put $\tilde{q} = qS \cap R$. If q S-quasi-primary, then $\tilde{q}$ is S-quasi-primary.*

Proof. Suppose that $a \in R$, $\alpha \in S$, $f(a) \cdot \alpha \in \tilde{q}S$ and that $\alpha \notin \tilde{q}S$. Note that $qS \subseteq \tilde{q}S = (qS \cap R)S \subseteq qS$, and hence $qS = \tilde{q}S$. Thus we have $f(a) \cdot \alpha \in \tilde{q}S$ and $\alpha \notin qS$. Since q is S-quasi-primary, we see $a \in \sqrt{q} = \sqrt{\tilde{q}}$. Therefore $\tilde{q}$ is S-quasi-primary.

Q.E.D.

Using Proposition 3.4, we have the following:

Theorem 3.6. *Let $f: R \to S$ be a homomorphism of Noetherian rings, and let q be an ideal of R such that $qS \neq S$. Let $qS = Q_1 \cap \cdots \cap Q_n$, $\sqrt{Q_i} = P_i$ $(1 \leqq i \leqq n)$*

be an irredundant primary decomposition of the ideal qS. *Put* $\tilde{q}=qS\cap R$. *Then* q *is S-quasi-primary if and only if* $\tilde{q}$ *is a primary ideal of* R, $\sqrt{q}=\sqrt{\tilde{q}}=p$ *and* $P_i\cap R=p$ *for all* i.

Proof. ($\Rightarrow$): By Proposition 3.4 $\tilde{q}$ is a primary ideal of R and $\sqrt{q}=\sqrt{\tilde{q}}=p$. Suppose that $P_i\cap R\supsetneqq p$ for some i. We may assume that $i=1$. Thus there exists an element $a\in P_1\cap R\backslash p$. We may assume that $a\in Q_1$. Further, we can take an element $\alpha\in Q_2\cap\cdots\cap Q_n$ such that $\alpha\notin Q_1$. Then $f(a)\alpha\in Q_1\cap\cdots\cap Q_n=qS$. Since $a\notin p=\sqrt{q}=\sqrt{\tilde{q}}$ and $\alpha\in qS$, we see that q is not S-quasi-primary.

($\Leftarrow$) Assume that $f(a)\alpha\in qS$, $a\notin p=\sqrt{q}=\sqrt{\tilde{q}}$ and that $\alpha\in S$. Then $f(a)\notin P_i$ for all i. It follows that $\alpha\in Q_1\cap\cdots\cap Q_n=qS$. Therefore q is S-quasi-primary. Q.E.D.

Now, we have the following structure theorem of the S-quasi-primary ideals, which is similar to Theorem 1.10.

Theorem 3.7. *Let* $f\colon R\to S$ *be a homomorphism of Noetherian rings, and let* q *be an ideal of* R *such that* $qS\neq S$. *Then* q *is S-quasi-primary if and only if* $\tilde{q}=qS\cap R$ *is a primary ideal of* R *such that* $\sqrt{q}=\sqrt{\tilde{q}}=p\in\operatorname{Spec}(R)$, *and* $\operatorname{Ass}_R(S/qS)=\{p\}$.

Proof. ($\Rightarrow$): We have only to prove that $\operatorname{Ass}_R(S/qS)=\{p\}$. Let $qS=Q_1\cap\cdots\cap Q_n$ $\sqrt{Q_i}=P_i$ $(1\leqq i\leqq n)$ be an irredundant primary decomposition of the ideal qS. Take $p'\in\operatorname{Ass}_R(S/qS)$. Then, by definition of Ass, there exists an element $x\in S\backslash qS$ such that $p'=(qS:x)$, and $xp'S\subseteq qS$. Hence $f(x)p'S\subseteq Q_j$ for all j. Since $x\notin Q_i$ for some i, $p'S\subseteq P_i$, and $p'\subseteq P_i\cap R=p$ by Theorem 3.6. On the other hand, since $q\subseteq p'$, $p=\sqrt{q}=\sqrt{\tilde{q}}\subseteq p'$. Therefore $p'=p$, as desired.

($\Leftarrow$): By Theorem 3.6, we have only to show that $P_i\cap R=p$ for all i, and this is clear because $\{p\}=\operatorname{Ass}_R(S/qS)=\operatorname{Ass}_S(S/qS)\cap R$ by [1, (9.A), Proposition]. Q.E.D.

In the rest of this section, we discuss relationship among the S-prime, S-primary and S-quasi-primary ideals.

Proposition 3.8. *Let* $f\colon R\to S$ *be a homomorphism of Noetherian rings, and let* q *be an ideal of* R *such that* $qS\neq S$. *If* q *is S-primary and* qS *has no embedded prime divisor, then* q *is S-quasi-primary.*

Proof. By Theorem 2.2, $qS\cap R=q$ and q is a primary ideal of R. Hence we have only to show that $\operatorname{Ass}_R(S/qS)=\{p\}$ by Theorem 3.7. Let $qS=Q_1\cap\cdots\cap Q_n$ be an irredundant primary decomposition of the ideal qS. Now, by assumption, since qS has no embedded prime divisor, we have $Q_i\cap R=q$ for all i by Proposition 2.4. Thus $p=\sqrt{q}=\sqrt{Q_i\cap R}=\sqrt{Q_i}\cap R=P_i\cap R$. Therefore we conclude that $\{p\}=\operatorname{Ass}_S(S/qS)\cap R=\operatorname{Ass}_R(S/qS)$, as desired. Q.E.D.

The following result asserts that S-quasi-primary ideals are closely related to S-prime ideals.

Proposition 3.9. *Let $f: R \to S$ be a homomorphism and let q be an ideal of R. If q is S-quasi-primary, then $\sqrt{q} = p$ is S-prime.*

Proof. Suppose that $a \in R$, $\alpha \in S$, $f(a)\cdot\alpha \in pS$ and that $\alpha \notin pS$. Then, since $qS \subseteq \sqrt{qS} = pS$, we have $\alpha \notin qS$. Since q is S-quasi-primary, we have $a \in \sqrt{q} = p$. Therefore p is S-prime. Q.E.D.

Finally, we show an example of an S-primary ideal $q \in \operatorname{Spec}(R)$, which is not S-prime.

Example 3.10. Let $f: R \to S$ be a homomorphism and let $q \in \operatorname{Spec}(R)$. If the ideal qS of S has an irredundant primary decomposition $qS = Q \cap T$, $\sqrt{Q} \subsetneqq \sqrt{T}$, $Q \cap R = q$ and $\sqrt{T} \cap R \supsetneqq q$, then q is not S-prime, but S-primary. Indeed, suppose that $a \in R$, $\alpha \in S$, $a\cdot\alpha \in qS$ and that $\alpha \notin \sqrt{qS} = \sqrt{Q}$. Then we have $a \in Q \cap R = q$, and q is S-primary. Also, suppose that q is S-prime, then $\operatorname{Ass}_R(S/qS) = \{q\}$ by Theorem 1.17. However, since $\operatorname{Ass}_R(S/qS) = \{q, \sqrt{T} \cap R\}$, q is not S-prime. Also, note that since $q \in \operatorname{Spec}(R)$, q is not S-quasi-primary by Remark 3.2. Finally we construct an example satisfying these conditions. Let k be a field and let x, y are indeterminates. Let $R = k[x^2, y^2] \subset A = k[x, y]$, and let $S = \{a \in A \mid a(1, 1) = a(-1, 1)\}$. Put $m = (x - 1, y - 1)A \cap S = (x + 1, y - 1)A \cap S$, $P(x - 1)A \cap S$ and $q = (x^2 - 1)R$. Then it is easily seen $qS = Q \cap T$ for a primary ideal Q belonging to P and a primary ideal T belonging to m of S. Further, since $(x - 1)A \subsetneqq (x - 1, y - 1)A$ and $y^2 - 1 \in (x + 1, y - 1)A \cap R \backslash (x^2 - 1)R$, we see that $P \subsetneqq m$ and $m \cap R \supsetneqq q$. Also, $Q \cap R = (x^2 - 1)R = q$, as desired.

DEPARTMENT OF APPLIED MATHEMATICS
OKAYAMA UNIVERSITY OF SCIENCE

References

[1] H. Matsumura, Commutative Algebra, Benjamin, New York, 1970.
[2] D. L. Mcguillan, On prime ideals in ring extensions, Arch. Math., **33** (1979), 121–126.
[3] M. Nagata, Local ring, Interscience Tracts No. 13, John Wiley., New York (1961).
[4] S. Oda, J. Sato and K. Yoshida, High degree anti-integral extensions of Noetherian domains, to appear.
[5] J. Sato, Denominator ideals and the extensions of Noetherian domains, to appear.
[6] K. Yoshida, On birational-integral extension of rings and prime ideals of depth one, Japan J. Math., **8** (1982).