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Free complexes defining maximal quasi-Buchsbaum graded modules over polynomial rings

By

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Introduction

In the study of homogeneous ideals defining two codimensional locally Cohen-Macaulay subschemes of a projective space, any knowledge of maximal generalized Cohen-Macaulay graded modules over a polynomial ring R (i.e. graded modules M of the same Krull dimension as that of the ground ring R such that $l_R(H^i_{\mathfrak{m}}(M)) < \infty$ for all $i < \dim(R)$, $\mathfrak{m}$ denoting the irrelevent maximal ideal of R) is very useful in two respects. First, for a homogeneous ideal $\mathfrak{a}$ of height two in a polynomial ring $R = k[x_1, \dots, x_r]$ having the property $l_R(H^i_{\mathfrak{m}}(R/\mathfrak{a})) < \infty$ for $i < r-2$, there is an exact sequence

$$0 \to N \xrightarrow{\tau} M \to \mathfrak{a} \to 0$$

with a maximal generalized Cohen-Macaulay graded R-module M and a graded free R-module N such that $H^{r-1}_{\mathfrak{m}}(M) = 0$, so that once the structure of M, such as its syzygies, is fully understood, the problem can be reduced to the analysis of the linear mapping τ. Second, denoting by t_0 the minimum of all t such that $\mathfrak{a}_t \neq 0$, let M' be the module over $R' = k[x_2, \dots, x_r]$ defined by the exact sequence

$$0 \to M' \to \bigoplus_{l=0}^{t_0-1} R'(-l) \xrightarrow{(1, x_1^1, \dots, x_1^{t_0-1})} R/\mathfrak{a} \to 0,$$

where the linear forms $x_1, \dots, x_r$ are chosen sufficiently generally. Then $H^i_{\mathfrak{m}'}(M') \cong H^{i-1}_{\mathfrak{m}}(R/\mathfrak{a})$ for each $i < r-1$ as an R'-module, so M' is a maximal generalized Cohen-Macaulay R'-module satisfying $e_{\mathfrak{m}'}(M') = t_0$, which bears a lot of information on the generators of $\mathfrak{a}$. In fact, applying Goto's structure theorem for maximal Buchsbaum modules over regular local rings (see [G2, (3.1)], [EG, Theorem 3.2]) to the above M, M', we could give a complete classification of homogeneous prime ideals that define arithmetically Buchsbaum subvarieties of codimension two in projective spaces (see [A1, § 7]).

Keeping that in mind, we will investigate the structure of maximal quasi-Buchsbaum graded modules over polynomial rings (i.e. graded modules M over R with $\mathfrak{m}H^i_{\mathfrak{m}}(M) = 0$ for all $i < \dim(R)$), especially in a simple case where $\iota(M) :=$

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$\#\{i | H_{\mathfrak{m}}^i(M) \neq 0,\ i < \dim(R)\} \leq 2$ as the first step. Our main results, described in section four, are obtained in the following manner. Given a finitely generated graded module M having no free direct summand over a noetherian graded k-algebra R and its minimal free resolution

$$\cdots \to L'_2 \to L'_1 \xrightarrow{\partial_1^{L'}} L'_0 \to M \to 0\,,$$

let $L.$ denote the minimal complex such that $H_i(L.) \cong \mathrm{Ext}_R^{b-i}(M, R)$ for $0 \leq i < b := \dim(R) - \mathrm{depth}_{\mathfrak{m}}(M)$, which is obtained by connecting the dual of $L'.$ and the complex $L''.$ giving a minimal free resolution of the kernel of the dual of $\partial_1^{L'}$ (see (4.2)). Then $L.$ is the minimal part of the mapping cone of a certain system of successive chain maps (see (1.6)), which forms the basis of our approach. When R is a polynomial ring and M is a maximal quasi-Buchsbaum graded R-module with $\imath(M) \leq 2$, only one chain map is involved, and besides, it can be handled by the linear algebra over k (see (2.1), (2.5)). Thanks to C. M. Ringel's theorem (see (3.3)) and an explicit formula for the chain maps from a direct sum of Koszul complexes to another provided in (2.6), we thus get detailed results for the case $\imath(M) \leq 2$, along with some observations in the case $\imath(M) = 3$. Note that the same method applies also to generalized Cohen-Macaulay R-modules satisfying $\imath(M) = 1$, $\mathfrak{m}^2 H_{\mathfrak{m}}^i(M) = 0$ for $i < \dim(R)$.

One may naturally feel inclined to give a necessary and sufficient condition for those successive chain maps mentioned above to correspond to a maximal Buchsbaum module, in order to generalize Goto's structure theorem. It is possible indeed at least over Gorenstein rings, which will be discussed in a forthcoming paper.

The idea of making use of mapping cones and its first applications (2.5), (4.6), (4.10) were reported for the most part at the conference held in Kyoto in autumn, 1990 (see [A2]). After that meeting, a joint work [CHP] was sent to me, which contains another treatment of maximal quasi-Buchsbaum modules over regular local rings. Also, Y. Yoshino argued about the same subject, laying stress on equivalence of categories, mainly over Gorenstein local rings in [Y]. He pointed out, among other things, that Schenzel's characterization of Buchsbaum modules in terms of dualizing complexes (see [SV, Chapter II, Theorem 4.1]) is false, unless the ground ring is regular. I am grateful to him for stimulating discussions and for providing me with some elementary knowledge of representations of quivers, especially Ringel's results.

§ 1. Mapping cones

Throughout this paper R denotes either a local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, or a graded ring $\bigoplus_{t \geq 0} R_t$ with $\mathfrak{m} = \bigoplus_{t \geq 1} R_t$ which is generated over a field $k = R_0$ by a finite number of homogeneous elements as a k-algebra. In the second case, R-modules are always assumed to be graded and $\mathrm{Hom}_R(M, N) = \bigoplus_{t \in \mathbf{Z}} \mathrm{Hom}_R(M, N)_t$ for all pairs of R-modules M, N, where $\mathrm{Hom}_R(M, N)_t$ denotes the additive group consisting of homomorphisms of degree t. Moreover when we consider a homomorphism $f: M \to N$, it is assumed that $f \in \mathrm{Hom}_R(M, N)_0$, unless otherwise specified. Let $L_\bullet = (L_i)_{i \in \mathbf{Z}}$ be a complex of R-modules with differentials $\partial_i^L: L_i \to L_{i-1}$ ($i \in \mathbf{Z}$). We say that $L_\bullet$ is minimal (resp. split exact) when $\mathrm{Im}(\partial_i^L) \subset \mathfrak{m}L_{i-1}$ (resp. $H_i(L_\bullet) = 0$, $L_i \cong \mathrm{Im}(\partial_i^L) \oplus \mathrm{Ker}(\partial_i^L)$) for all $i \in \mathbf{Z}$. Let $L^{*\bullet}$, and for an integer n, let $L[-n]_\bullet$, $L[\geq n]_\bullet$ denote the complexes obtained by setting $L^{*i} = \mathrm{Hom}_R(L_i, R)$, $\partial_{L^*}^i = (\partial_{i+1}^L)^*: L^{*i} \to L^{*i+1}$ (dual map), $L[-n]_i = L_{i-n}$, $\partial_i^{L[-n]} = \partial_{i-n}^L$ and

$$L[\geq n]_i = \begin{cases} L_i \ (i \geq n) \\ 0 \ (i < n) \end{cases}, \qquad \partial_i^{L[\geq n]} = \begin{cases} \partial_i^L \ (i > n) \\ 0 \ (i \leq n) \end{cases}$$

respectively. We set

$$\alpha(L_\bullet) = \sup\{i \mid H_i(L_\bullet) \neq 0\}, \qquad \beta(L_\bullet) = \inf\{i \mid H^i(L^{*\bullet}) \neq 0\},$$

where $\alpha(L_\bullet) = -\infty$ (resp. $\beta(L_\bullet) = \infty$) if and only if $L_\bullet$ (resp. $L^{*\bullet}$) is exact. The mapping cone of a chain map $\mu_\bullet: L_\bullet \to L'_\bullet$ will be denoted by $\mathrm{con}(\mu_\bullet)_\bullet$. Recall that

$$\mathrm{con}(\mu_\bullet)_i = L'_{i+1} \oplus L_i, \qquad \partial_i^{\mathrm{con}(\mu_\bullet)} = \begin{bmatrix} -\partial_{i+1}^{L'} & \mu_i \\ 0 & \partial_i^L \end{bmatrix}$$

for all $i \in \mathbf{Z}$. Further, we denote the set of chain maps $\mu_\bullet$ as above by $\mathrm{Hom}_R(L_\bullet, L'_\bullet)$, the set of sequences $\nu_\bullet = (\nu_i)_{i \in \mathbf{Z}}$ with $\nu_i \in \mathrm{Hom}_R(L_i, L'_i)$ by $\mathrm{hom}_R(L_\bullet, L'_\bullet)$ and the set of chain automorphisms (resp. chain endomorphisms) of $L_\bullet$ by $\mathrm{Aut}_R(L_\bullet)$ (resp. $\mathrm{End}_R(L_\bullet)$). There is a natural identification of the sets $\mathrm{hom}_R(L_\bullet, L'_\bullet)$ and $\mathrm{hom}_R(L[-n]_\bullet, L'[-n]_\bullet)$ obtained by shifting subscripts from i to $i+n$, under which the element of $\mathrm{hom}_R(L[-n]_\bullet, L'[-n]_\bullet)$ corresponding to $\nu_\bullet \in \mathrm{hom}_R(L_\bullet, L'_\bullet)$ will be denoted by $\nu[-n]_\bullet$. Let $\mathscr{C}(R)$, $\mathscr{C}m(R)$, $\mathscr{C}x(R)$, $\mathscr{C}y(R)$, $\mathscr{C}_0(R)$, $\mathscr{C}xm(R)$, $\mathscr{C}m_0(R)$ be sets of complexes defined by

$$\mathscr{C}(R) = \{L_\bullet \mid \textit{each } L_i \textit{ is a finitely generated free } R\textit{-module}\},$$

$$\mathscr{C}m(R) = \{L_\bullet \in \mathscr{C}(R) \mid L_\bullet \textit{ is minimal}\},$$

$$\mathscr{C}x(R) = \{L_\bullet \in \mathscr{C}(R) \mid \mathrm{Im}(\partial_i^L) \subset \mathfrak{m}L_{i-1} \textit{ for all } i \leq 0, \alpha(L_\bullet) < \beta(L_\bullet)\},$$

$$\mathscr{C}y(R) = \{L_\bullet \in \mathscr{C}(R) \mid \mathrm{Im}(\partial_i^L) \subset \mathfrak{m}L_{i-1} \textit{ for all } i \leq 0\},$$

$$\mathscr{C}_0(R) = \{L_\bullet \in \mathscr{C}(R) \mid L_i = 0 \textit{ for all } i < 0, H_i(L_\bullet) = 0 \textit{ for all } i > 0\},$$

$$\mathscr{C}xm(R) = \mathscr{C}x(R) \cap \mathscr{C}m(R), \qquad \mathscr{C}m_0(R) = \mathscr{C}_0(R) \cap \mathscr{C}m(R).$$

The $\mathscr{C}m_0(R)$ is nothing but the set of the minimal free resolutions of finitely generated R-modules. Under this identification we will denote the element of $\mathscr{C}m_0(R)$ which gives the minimal free resolution of a finitely generated R-module E by $\mathrm{res}(E)_\bullet$.

(1.1) Lemma. *For each complex $L_\bullet \in \mathscr{C}x(R)$ (resp. $\mathscr{C}y(R)$), there are a minimal $P_\bullet \in \mathscr{C}xm(R)$ (resp. $\mathscr{C}m(R)$) and a split exact $Q_\bullet \in \mathscr{C}_0(R)$ such that $L_\bullet \cong P_\bullet \oplus Q_\bullet$. Moreover these $P_\bullet$, $Q_\bullet$ are determined uniquely by this condition up to isomorphism.*

Proof. For each $i \in \mathbf{Z}$, let L'_i be a free direct summand of L_i such that

$$\mathrm{rank}_R(L'_i) = l_R(\partial_i^L(L'_i) + \mathfrak{m}L_{i-1}/\mathfrak{m}L_{i-1}) = l_R(\mathrm{Im}(\partial_i^L) + \mathfrak{m}L_{i-1}/\mathfrak{m}L_{i-1})$$

and put $L''_{i-1} = \partial_i^L(L'_i)$. Observe that $\partial_i^L|_{L'_i}: L'_i \to L''_{i-1}$ is an isomorphism. The relation $\partial_i^L(L''_i) = 0$ implies that $(L'_i + \mathfrak{m}L_i/\mathfrak{m}L_i) \cap (L''_i + \mathfrak{m}L_i/\mathfrak{m}L_i) = 0$ in the vector space $L_i/\mathfrak{m}L_i = L_i \otimes R/\mathfrak{m}$, therefore

$$L_i = L'''_i \oplus L'_i \oplus L''_i$$

with a suitable free R-submodule L'''_i of L_i. Let us replace L'''_i with another free submodule $P_i \subset L_i$ for each i, constructing P_i in the following manner. First, put $P_i = L'''_i$ for $i \leq 0$. Since $L'_i \oplus L''_i = 0$, $P_i = L'''_i = L_i$ for $i < 0$, we have

$$(1.1.\mathrm{i}) \qquad \begin{cases} \mathrm{rank}_R(P_i) = \mathrm{rank}_R(L'''_i)\,, \\ P_i + L'_i = L'''_i + L'_i\,, \\ L_i = P_i \oplus L'_i \oplus L''_i\,, \\ \partial_i^L(P_i) \subset \mathfrak{m}P_{i-1} \end{cases}$$

for $i \leq 0$. Suppose there are P_i satisfying (1.1.i) ($i \leq l$) for some $l \geq 0$. Then the equality $\partial_l^L(\partial_{l+1}^L(L'''_{l+1})) = 0$ and (1.1.l), (1.1.$l-1$) imply

$$\partial_{l+1}^L(L'''_{l+1}) \subset P_l + L''_l = P_l + \partial_{l+1}^L(L'_{l+1})\,,$$

so that modifying free bases of L'''_{l+1}, we obtain a free submodule P_{l+1} which satisfies (1.1.$l+1$). Thus we get a minimal complex $P_\bullet \in \mathscr{C}\mathscr{m}(R)$ ($\partial_i^P = \partial_i^L|_{P_i}$) and a split exact complex $Q_\bullet \in \mathscr{C}_0(R)$ ($\partial_i^Q = \partial_i^L|_{Q_i}$, $Q_i = L'_i \oplus L''_i$) such that $L_\bullet = P_\bullet \oplus Q_\bullet$. It is clear that $P_\bullet \in \mathscr{C}\mathscr{x}\mathscr{m}(R)$ if $L_\bullet \in \mathscr{C}\mathscr{x}(R)$. Let $P'_\bullet \in \mathscr{C}\mathscr{m}(R)$, $Q'_\bullet \in \mathscr{C}_0(R)$ be another pair of minimal and split exact complexes satisfying $L_\bullet \cong P'_\bullet \oplus Q'_\bullet$ and let $\pi_\bullet: L_\bullet \to P'_\bullet$ be the natural projection. Then $\varphi_\bullet := \pi_\bullet|_{P_\bullet}: P_\bullet \to P'_\bullet$ is a chain map such that

$$\varphi_i \otimes R/\mathfrak{m}: P_i \otimes R/\mathfrak{m} \tilde{\to} H_i(L_\bullet \otimes R/\mathfrak{m}) \tilde{\to} P'_i \otimes R/\mathfrak{m}$$

is an isomorphism for all $i \in \mathbf{Z}$, since both $P_\bullet$ and $P'_\bullet$ are minimal and free. Hence $P_\bullet \cong P'_\bullet$, in particular $\mathrm{rank}_R(P_i) = \mathrm{rank}_R(P'_i)$ and $\mathrm{rank}_R(Q_i) = \mathrm{rank}_R(Q'_i)$ for all $i \in \mathbf{Z}$. As a result, $Q'_i = 0$ for $i < 0$ and $Q_\bullet \cong Q'_\bullet$.

(1.2) Definition. With the notation of (1.1), we put $\min(L_\bullet)_\bullet = P_\bullet$, $\mathrm{se}(L_\bullet)_\bullet = Q_\bullet$.

(1.3) Lemma. *Let $L_\bullet$, $L'_\bullet \in \mathscr{C}\mathscr{y}(R)$ satisfy $l_R(L_i \otimes R/\mathfrak{m}) = l_R(L'_i \otimes R/\mathfrak{m})$ for all $i \in \mathbf{Z}$. Then $L_\bullet \cong L'_\bullet$ if and only if $\min(L_\bullet)_\bullet \cong \min(L'_\bullet)_\bullet$.*

Proof. Almost the same as the last part of the proof of (1.1).

In the argument below, let a and b denote nonnegative integers with $a < b$.

(1.4) Lemma. *Given complexes $L_\bullet \in \mathscr{C}\mathscr{y}(R)$, $G_\bullet \in \mathscr{C}_0(R)$ satisfying $\alpha(L_\bullet) < a$ and a chain map $\mu_\bullet: L_\bullet \to G[-a-1]_\bullet$, we have $H_i(\mathrm{con}(\mu_\bullet)_\bullet) \cong H_i(L_\bullet)$ for $i < a$, $H_a(\mathrm{con}(\mu_\bullet)_\bullet) \cong H_0(G_\bullet)$ and $H_i(\mathrm{con}(\mu_\bullet)_\bullet) = 0$ for $i > a$. In particular $\alpha(\mathrm{con}(\mu_\bullet)_\bullet) \leq a$,* *with equality if and only if* $H_0(G.) \neq 0$. *Moreover if* $b \leq \beta(L.)$ *and* $b - a \leq \beta(G.)$, *then* $H^i(\mathrm{con}(\mu.)^{*\cdot}) = 0$ *for* $i < b$, *i.e.* $b \leq \beta(\mathrm{con}(\mu.).)$, $\mathrm{con}(\mu.). \in \mathscr{C}x(R)$.

Proof. Easily verified by the long exact sequences arising from the short exact sequence

$$0 \to G[-a]. \to \mathrm{con}(\mu.). \to L. \to 0 \tag{1.4.1}$$

and its dual.

For a complex $L. \in \mathscr{C}(R)$ and an integer n, we define a complex $\sigma_n(L.).$ by

$$\sigma_n(L.)_i = \begin{cases} L_i \ (i < n) \\ P_{i-n} \ (i \geq n) \end{cases}, \qquad \partial_i^{\sigma_n(L.)} = \begin{cases} \partial_i^L \ (i < n) \\ \varepsilon^P \ (i = n) \\ \partial_{i-n}^P \ (i > n) \end{cases},$$

where $P. = \mathrm{res}(\mathrm{Im}(\partial_n^L)).$ and ε^P is the map $P_0 \to P_0/\mathrm{Im}(\partial_1^P) \cong \mathrm{Im}(\partial_n^L) \subset L_{n-1}$. Note that $\sigma_n(L.). \in \mathscr{C}m(R)$ if $L. \in \mathscr{C}m(R)$ and that $\alpha(\sigma_n(L.).) \leq \alpha(L.)$ with equality if and only if $\alpha(L.) < n$.

(1.5) Proposition. *Let* $L. \in \mathscr{C}xm(R)$, $G. \in \mathscr{C}m_0(R)$, *and suppose* $\alpha(L.) \leq a < b \leq \beta(L.)$, $G. = \mathrm{res}(H_a(L.)).$, $\mathrm{Ext}_R^i(H_a(L.), R) = 0$ *for all* i $(0 \leq i < b - a)$. *Then* $\sigma_a(L.). \in \mathscr{C}xm(R)$, $\alpha(\sigma_a(L.).) < a < b \leq \beta(\sigma_a(L.).)$ *and there is a chain map* $\mu.: \sigma_a(L.). \to G[-a-1].$ *such that* $L. \cong \min(\mathrm{con}(\mu.).).$.

Proof. Put $G. = \mathrm{res}(H_a(L.)).$, $P. = \mathrm{res}(\mathrm{Im}(\partial_a^L)).$. Since

$$0 \to H_a(L.) \to L_a/\mathrm{Im}(\partial_{a+1}^L) \overset{\partial_a^L}{\to} \mathrm{Im}(\partial_a^L) \to 0 \tag{1.5.1}$$

is exact, there is a chain map $\mu'.: P. \to G[-1].$ such that $E := L_a/\mathrm{Im}(\partial_{a+1}^L) \cong \mathrm{Coker}(\partial_1^{\mathrm{con}(\mu.)})$. Moreover $(L[\geq a])[a]. \cong \min(\mathrm{con}(\mu'.).).$ since both $(L[\geq a])[a].$ and $\min(\mathrm{con}(\mu'.).).$ give minimal free resolutions of E. Let $\mu.: \sigma_a(L.). \to G[-a-1].$ be the chain map obtained by setting $\mu_i = \mu'_{i-a}$ for $i \geq a$, $\mu_i = 0$ for $i < a$. Then $\mathrm{con}(\mu.)_i = \mathrm{con}(\mu'.)_{i-a}$ for $i \geq a$, $\mathrm{con}(\mu.)_i = L_i$ for $i < a$, $\partial_a^{\mathrm{con}(\mu.)}$ coincides with the composite map $\mathrm{con}(\mu.)_a = G_0 \oplus P_0 \to E \to \mathrm{Im}(\partial_a^L) \hookrightarrow L_{a-1}$ and $\mathrm{Im}(\partial_a^{\mathrm{con}(\mu.)}) \subset \mathfrak{m}L_{a-1}$, therefore $L. \cong \min(\mathrm{con}(\mu.).).$. Clearly $\alpha(\sigma_a(L.).) < a < b$, while $b \leq \beta(\sigma_a(L.).)$ follows from the long exact sequences arising from the dual of (1.4.1) with $L.$ replaced by $\sigma_a(L.).$, since $\beta(L.) = \beta(\mathrm{con}(\mu.).)$ and $b - a \leq \beta(G.)$ by hypothesis.

(1.6) Corollary. *For a complex* $L. \in \mathscr{C}xm(R)$, *suppose* $\mathrm{Ext}_R^i(H_j(L.), R) = 0$ *for all* i, j $(0 \leq i < b - j,\ 0 \leq j \leq a)$, $\alpha(L.) \leq a < b \leq \beta(L.)$. *Put* $G^{(j)}. = \mathrm{res}(H_j(L.)). \in \mathscr{C}m_0(R)$ *for each* j $(0 \leq j \leq a)$. *Then* $\sigma_0(L.). \in \mathscr{C}xm(R)$, $\alpha(\sigma_0(L.).) < 0 < b \leq \beta(\sigma_0(L.).)$ *and there are inductively defined chain maps* $\mu^{(0)}.: \sigma_0(L.). \to G^{(0)}[-1].$, $\mu^{(j)}.: \mathrm{con}(\mu^{(j-1)}.). \to G^{(j)}[-j-1].$ $(1 \leq j \leq a)$ *such that* $L. \cong \min(\mathrm{con}(\mu^{(a)}.).).$. *Moreover* $\sigma_0(L.). = 0$ *if* $L_i = 0$ *for all* $i < 0$, *and* $H_i(\sigma_0(L).) = 0$, $H^i(\sigma_0(L.)^{*\cdot}) = 0$ *for all* $i \in \mathbf{Z}$ *if* R *is Gorenstein and* $H_i(L.) = 0$ *for all* $i < 0$.

Proof. The case $a=0$ follows from (1.5). Suppose that $a>0$ and that our assertion is true for $a-1$. Put $L'_\bullet = \sigma_a(L_\bullet)_\bullet$. By (1.5) there is a chain map $\mu_\bullet: L'_\bullet \to G^{(a)}[-a-1]_\bullet$ such that $L_\bullet \cong \min(\mathrm{con}(\mu_\bullet)_\bullet)_\bullet$, $\alpha(L'_\bullet) \le a-1$, $b \le \beta(L'_\bullet)$. Since $H_j(L'_\bullet) \cong H_j(L_\bullet)$, $\mathrm{Ext}^i_R(H_j(L'_\bullet), R) \cong \mathrm{Ext}^i_R(H_j(L_\bullet), R) = 0$ for all i, j $(0 \le i < b-j$, $0 \le j \le a-1)$ by hypothesis and (1.4), there are chain maps $\mu^{(0)}: \sigma_0(L'_\bullet)_\bullet \to G^{(0)}_\bullet$, $\mu^{(j)}_\bullet: \mathrm{con}(\mu^{(j-1)}_\bullet)_\bullet \to G^{(j)}[-j-1]_\bullet$ $(1 \le j \le a-1)$ such that $L'_\bullet = \min(\mathrm{con}(\mu^{(a-1)}_\bullet)_\bullet)_\bullet$, $b \le \beta(\sigma_0(L'_\bullet)_\bullet)$ by the induction hypothesis. Let $\mu^{(a)}_\bullet: \mathrm{con}(\mu^{(a-1)}_\bullet)_\bullet \to G[-a-1]_\bullet$ be the chain map extending $\mu_\bullet$ by $\mu^{(a)}_\bullet|_{\mathrm{se}(\mathrm{con}(\mu^{(a-1)}_\bullet)_\bullet)_\bullet} = 0$. Then $\mathrm{con}(\mu^{(a)}_\bullet)_\bullet = \mathrm{con}(\mu_\bullet)_\bullet \oplus \mathrm{se}(\mathrm{con}(\mu^{(a-1)}_\bullet)_\bullet)_\bullet$, so that $L_\bullet \cong \min(\mathrm{con}(\mu_\bullet)_\bullet)_\bullet \cong \min(\mathrm{con}(\mu^{(a)}_\bullet)_\bullet)_\bullet$. Since $\sigma_0(L'_\bullet)_\bullet = \sigma_0(L_\bullet)_\bullet$, the inequality $\alpha(\sigma_0(L_\bullet)_\bullet) < 0 < b \le \beta(\sigma_0(L_\bullet)_\bullet)$ follows. It is clear that $\sigma_0(L_\bullet)_\bullet = 0$ if $L_\bullet \in \mathscr{C}m_0(R)$. When R is Gorenstein, its injective dimention is finite, therefore $\beta(\sigma_0(L_\bullet)_\bullet) = \infty$ if $\alpha(\sigma_0(L_\bullet)_\bullet) = -\infty$, which implies the last part.

(1.7) Remark. As is seen by the proofs, the existence of the chain maps $\mu_\bullet$, $\mu^{(j)}_\bullet$ $(0 \le j \le a)$ with the property $L_\bullet \cong \min(\mathrm{con}(\mu_\bullet)_\bullet)_\bullet$, $L_\bullet \cong \min(\mathrm{con}(\mu^{(a)}_\bullet)_\bullet)_\bullet$ stated in (1.5), (1.6) is a consequence of the hypothesis $\alpha(L_\bullet) \le a$ only and other conditions such as the inequality $\alpha(L_\bullet) < \beta(L_\bullet)$ appearing in the definition of $\mathscr{C}x(R)$ have nothing to do with it.

(1.8) Lemma. *Let $L_\bullet$, $L'_\bullet \in \mathscr{C}m(R)$, $G_\bullet$, $G'_\bullet \in \mathscr{C}m_0(R)$ be complexes with $a > \max(\alpha(L_\bullet), \alpha(L'_\bullet))$ and let $\mu_\bullet: L_\bullet \to G[-a-1]_\bullet$, $\mu'_\bullet: L'_\bullet \to G'[-a-1]_\bullet$ be chain maps.*

(1) *If* $\min(\mathrm{con}(\mu_\bullet)_\bullet)_\bullet \cong \min(\mathrm{con}(\mu'_\bullet)_\bullet)_\bullet$, *then* $L_\bullet \cong L'_\bullet$ *and* $G_\bullet \cong G'_\bullet$.

(2) *We have* $\mathrm{con}(\mu_\bullet \oplus \mu'_\bullet)_\bullet \cong \mathrm{con}(\mu_\bullet)_\bullet \oplus \mathrm{con}(\mu'_\bullet)_\bullet$ *for the chain map* $\mu_\bullet \oplus \mu'_\bullet$: $L_\bullet \oplus L'_\bullet \to G[-a-1]_\bullet \oplus G'[-a-1]_\bullet$.

Proof. (1) Since $L_i = \min(\mathrm{con}(\mu_\bullet)_\bullet)_i$ for all $i < a$, $\mathrm{Im}(\partial_a^{\mathrm{con}(\mu_\bullet)}) = \mathrm{Im}(\partial_a^L)$, $(L[\ge a])[a]_\bullet = \mathrm{res}(\mathrm{Im}(\partial_a^L))_\bullet$ and the same holds for $L'_\bullet$, one finds that the hypothesis $\min(\mathrm{con}(\mu_\bullet)_\bullet)_\bullet \cong \min(\mathrm{con}(\mu'_\bullet)_\bullet)_\bullet$ implies $L_\bullet \cong L'_\bullet$. To prove $G_\bullet \cong G'_\bullet$, it is enough to observe that $G_\bullet$ (resp. $G'_\bullet$) gives a minimal free resolution of $H_a(\min(\mathrm{con}(\mu_\bullet)_\bullet)_\bullet)$ (resp. $H_a(\min(\mathrm{con}(\mu'_\bullet)_\bullet)_\bullet)$) by (1.4).

(2) Obvious.

(1.9) Proposition. *Given chain maps $\mu_\bullet$, $\mu'_\bullet: L_\bullet \to G[-a-1]_\bullet$ with $L_\bullet \in \mathscr{C}y(R)$, $G_\bullet \in \mathscr{C}_0(R)$, $a > \alpha(L_\bullet)$, a necessary and sufficient condition for* $\min(\mathrm{con}(\mu_\bullet)_\bullet)_\bullet \cong \min(\mathrm{con}(\mu'_\bullet)_\bullet)_\bullet$ *is that there are chain automorphisms $\varphi_\bullet \in \mathrm{Aut}_R(L_\bullet)$, $\psi_\bullet \in \mathrm{Aut}_R(G_\bullet)$ such that $\mu'_\bullet\varphi_\bullet \simeq \psi[-a-1]_\bullet\mu_\bullet$ (chain homotopic).*

Proof. If there are chain automorphisms $\varphi_\bullet$ and $\psi_\bullet$ satisfying the condition, we have

$$\mu'_i\varphi_i - \psi_{i-a-1}\mu_i = \partial^G_{i-a}\nu_i + \nu_{i-1}\partial^L_i \qquad \text{for all } i \in \mathbf{Z} \tag{1.9.1}$$

with suitable $\nu_i \in \mathrm{Hom}_R(L_i, G_{i-a})$ $(i \in \mathbf{Z})$. The maps

$$\lambda_i := \begin{bmatrix} \psi_{i-a} & \nu_i \\ 0 & \varphi_i \end{bmatrix} \qquad (i \in \mathbf{Z})$$

therefore give a chain isomorphism $\lambda_{\bullet}\colon \mathrm{con}(\mu_{\bullet})_{\bullet} \to \mathrm{con}(\mu'_{\bullet})_{\bullet}$. Hence $\min(\mathrm{con}(\mu_{\bullet})_{\bullet})_{\bullet} \cong \min(\mathrm{con}(\mu'_{\bullet})_{\bullet})_{\bullet}$. Conversely, suppose $\min(\mathrm{con}(\mu_{\bullet})_{\bullet})_{\bullet} \cong \min(\mathrm{con}(\mu'_{\bullet})_{\bullet})_{\bullet}$. Then there is a chain isomorphism $\lambda_{\bullet}\colon \mathrm{con}(\mu_{\bullet})_{\bullet} \to \mathrm{con}(\mu'_{\bullet})_{\bullet}$ by (1.3). Put $\partial_{\bullet} = \partial_{\bullet}^{\mathrm{con}(\mu_{\bullet})}$, $\partial'_{\bullet} = \partial_{\bullet}^{\mathrm{con}(\mu'_{\bullet})}$, $M = \mathrm{Coker}(\partial_{a+1})$, $M' = \mathrm{Coker}(\partial'_{a+1})$, $E = H_0(G_{\bullet})$, $N = \mathrm{Im}(\partial_a^L)$, $\eta^N = \lambda_{a-1}|_N$, and let $\eta\colon M \to M'$ (resp. $\delta\colon M \to N$, $\delta'\colon M' \to N$) denote the isomorphism (resp. homomorphisms) induced by λ_a (resp. ∂_a, ∂'_a). We have a commutative diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & G_1 \oplus L_{a+1} & \xrightarrow{\partial_{a+1}} & G_0 \oplus L_a & \xrightarrow{\partial_a} & N \subset L_{a-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \lambda_{a+1}} & & \downarrow{\scriptstyle \lambda_a} & & \downarrow{\scriptstyle \lambda_{a-1}} & & \\
\cdots & \longrightarrow & G_1 \oplus L_{a+1} & \xrightarrow{\partial'_{a+1}} & G_0 \oplus L_a & \xrightarrow{\partial'_a} & N \subset L_{a-1} & \longrightarrow & \cdots
\end{array} \tag{1.9.2}
$$

Since $H_i(L_{\bullet}) = 0$ for all $i \geq a$ by hypothesis, one finds $\mathrm{Ker}(\delta) \cong E \cong \mathrm{Ker}(\delta')$, moreover $\eta^N \delta = \delta' \eta$ by (1.9.2). There exists therefore an automorphism $\eta^E\colon E \to E$ which makes the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & M & \xrightarrow{\delta} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta^E} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta^N} & & \\
0 & \longrightarrow & E & \longrightarrow & M' & \xrightarrow{\delta'} & N & \longrightarrow & 0
\end{array} \tag{1.9.3}
$$

commutative. Now observe that $G_{\bullet}$, $L[\geq a]_{\bullet}$ give free resolutions of E, N respectively and that M (resp. M') is the module defined by the element of $\mathrm{Ext}_R^1(N, E)$ represented by $\varepsilon^G \mu_{a+1}$ (resp. $\varepsilon^G \mu'_{a+1}$), where $\varepsilon^G\colon G_0 \to E$ is the natural surjection. Let $\psi_{\bullet}$, $\varphi'_{\bullet}$ be chain automorphisms of $G_{\bullet}$, $L[\geq a]_{\bullet}$ compatible with η^E, η^N respectively and let $\varphi_{\bullet}$ be the chain automorphism of $L_{\bullet}$ defined by $\varphi_i = \varphi'_{i-a}$ $(i \geq a)$, $\varphi_i = \lambda_i^{\mathrm{con}(\mu_{\bullet})}$ $(i < a)$. The diagram (1.9.3) implies that $\eta^E \varepsilon^G \mu_{a+1} = \varepsilon^G \psi_0 \mu_{a+1}$ and $\varepsilon^G \mu'_{a+1} \varphi'_1 = \varepsilon^G \mu'_{a+1} \varphi_{a+1}$ represent the same element of $\mathrm{Ext}_R^1(N, E)$, in other words there are $v_i \in \mathrm{Hom}_R(L_i, G_{i-a})$ $(i = a, a+1)$ such that

$$\mu'_{a+1} \varphi_{a+1} - \psi_0 \mu_{a+1} = \partial_1^G v_{a+1} + v_a \partial_{a+1}^L \,.$$

Since $L_{\bullet} \in \mathscr{C}(R)$, this can be completed to the form (1.9.1), therefore $\mu'_{\bullet} \varphi_{\bullet} \simeq \psi[-a-1]_{\bullet} \mu_{\bullet}$ as desired.

Given complexes $L_{\bullet}$, $L'_{\bullet} \in \mathscr{C}(R)$, a chain map $\mu_{\bullet}\colon L_{\bullet} \to L'_{\bullet}$ and homomorphisms $\varphi_{\bullet} \in \mathrm{hom}_R(L_{\bullet}, L_{\bullet})$, $\psi_{\bullet} \in \mathrm{hom}_R(L'_{\bullet}, L'_{\bullet})$, $v_{\bullet} \in \mathrm{hom}_R(L_{\bullet}, L'[1]_{\bullet})$, we define $\theta(\psi_{\bullet}, \varphi_{\bullet}, v_{\bullet})_{\bullet} \in \mathrm{hom}_R(\mathrm{con}(\mu_{\bullet})_{\bullet}, \mathrm{con}(\mu_{\bullet})_{\bullet})$ by

$$\theta(\psi_{\bullet}, \varphi_{\bullet}, v_{\bullet})_i = \begin{bmatrix} \psi_{i+1} & v_i \\ 0 & \varphi_i \end{bmatrix} \quad \text{for all } i \in \mathbf{Z}\,.$$

Also, we put $^{-}v_{\bullet} = ((-1)^i v_i)_{i \in \mathbf{Z}} \in \mathrm{hom}_R(L_{\bullet}, L'[1]_{\bullet})$.

(1.10) Lemma. *Let $L_{\bullet}$, $L'_{\bullet}$, $\mu_{\bullet}$ be as above and put $L''_{\bullet} = \mathrm{con}(\mu_{\bullet})_{\bullet}$.*

(1) *For* $\varphi_.$, $\varphi'_. \in \mathrm{Aut}_R(L_.)$, $\psi_.$, $\psi'_. \in \mathrm{Aut}_R(L'_.)$, $\nu_. \in \mathrm{hom}_R(L_., L'[1]_.)$ *satisfying the condition* $\varphi_. \simeq \varphi'_.$, $\psi_. \simeq \psi'_.$, $\theta(\psi_., \varphi_., \nu_.)_. \in \mathrm{Aut}_R(L''_.)$, *there is a homomorphism* $\nu'_. \in \mathrm{hom}_R(L_., L'[1]_.)$ *such that* $\theta(\psi'_., \varphi'_., \nu'_.)_.$ *is a chain automorphism which is chain homotopic to* $\theta(\psi_., \varphi_., \nu_.)_.$.

(2) *Let* $\varphi_. \in \mathrm{Aut}_R(L_.)$, $\psi_. \in \mathrm{Aut}_R(L'_.)$, $\nu_.$, $\nu'_. \in \mathrm{hom}_R(L_., L'[1]_.)$ *and suppose* $\theta(\psi_., \varphi_., \nu_.)_. \in \mathrm{Aut}_R(L''_.)$. *Then* $\theta(\psi_., \varphi_., \nu'_.)_. \in \mathrm{Aut}_R(L''_.)$ *if and only if* $^-\nu_. - {}^-\nu'_.$ *is a chain map. Moreover, when this is the case,* $\theta(\psi_., \varphi_., \nu_.)_. \simeq \theta(\psi_., \varphi_., \nu'_.)_.$ *if* $^-\nu_. - {}^-\nu'_. \simeq 0$.

Proof. (1) By hypothesis there are $\zeta_. \in \mathrm{hom}_R(L_., L[1]_.)$, $\zeta'_. \in \mathrm{hom}_R(L'_., L'[1]_.)$ such that

$$\varphi'_i - \varphi_i = \partial^L_{i+1}\zeta_i + \zeta_{i-1}\partial^L_i , \qquad \psi'_i - \psi_i = \partial^{L'}_{i+1}\zeta'_i + \zeta'_{i-1}\partial^{L'}_i$$

for all $i \in \mathbf{Z}$. Put

$$\nu'_i = \nu_i + \mu_{i+1}\zeta_i - \zeta'_i\mu_i \qquad (i \in \mathbf{Z}) .$$

Then we have

$$\theta(\psi'_., \varphi'_., \nu'_.)_i - \theta(\psi_., \varphi_., \nu_.)_i = \partial^{\mathrm{con}(\mu_.)}_{i+1}\begin{bmatrix} -\zeta'_{i+1} & 0 \\ 0 & \zeta_i \end{bmatrix} + \begin{bmatrix} -\zeta'_i & 0 \\ 0 & \zeta_{i-1} \end{bmatrix}\partial^{\mathrm{con}(\mu_.)}_i$$

for all i by direct computations.

(2) Clear by definition.

The following lemma follows from the proof of (1.9).

(1.11) Lemma. *Given* $L_. \in \mathscr{C}_{\mathscr{Y}}(R)$ *with* $a > \alpha(L_.)$, $G_. \in \mathscr{C}_0(R)$ *and a chain map* $\mu_.: L_. \to G[-a-1]_.$, *put* $L''_. = \mathrm{con}(\mu_.)_.$. *For each* $\lambda_. \in \mathrm{Aut}_R(L''_.)$, *there are* $\varphi_. \in \mathrm{Aut}_R(L_.)$, $\psi_. \in \mathrm{Aut}_R(G_.)$, $\nu_. \in \mathrm{hom}_R(L_., G[-a]_.)$ *such that* $\theta(\psi[-a-1]_., \varphi_., \nu_.)_.$ *is a chain automorphism of* $L''_.$ *which is chain homotopic to* $\lambda_.$.

(1.12) Lemma. *Let* $L_. \in \mathscr{C}_{\mathscr{Y}}(R)$, $L'_. \in \mathscr{C}(R)$, $P_. = \min(L_.)_.$ *and let* $\mu_.$, $\mu'_.: L_. \to L'_.$ *be chain maps. Then* $\mu_. \simeq \mu'_.$ *if and only if* $\mu_.|_{P_.} \simeq \mu'_.|_{P_.}$.

Proof. Put $Q_. = \mathrm{se}(L_.)_.$. Since every chain map from a split exact complex to an arbitrary complex is chain homotopic to zero, we have $\mu_.|_{Q_.} \simeq 0$, $\mu'_.|_{Q_.} \simeq 0$. Hence the conclusion.

§ 2. Complexes whose homology modules are vector spaces

From now on $K_.$ denotes the complex giving the minimal free resolution of the residue field $k = R/\mathfrak{m}$, i.e. $K_. = \mathrm{res}(k)_.$. Let p be a nonnegative integer and $G_. = K_.^p$. In the graded case, let p_n ($n \in \mathbf{Z}$) be nonnegative integers such that $p_n = 0$ for all but a finite number of n and put $G_. = \bigoplus_n K_.(n)^{p_n}$. Note that $G_. \cong K_. \otimes_R G_0$, where $G_0 = R^p$ in the first case and $G_0 = \bigoplus_n R(n)^{p_n}$ in the second.

(2.1) Lemma. *Let* $L_.$ *be a complex in* $\mathscr{C}_{\mathscr{Y}}(R)$ *and* c *an integer.*

(1) *For chain maps* $\mu_\cdot$, $\mu'_\cdot: L_\cdot \to G[-c]_\cdot$, *the following conditions are equivalent.*

(i) $\mu_\cdot \simeq \mu'_\cdot$,

(ii) $\mu_c|_{(\partial_c^L)^{-1}(\mathfrak{m}L_{c-1})} \equiv \mu'_c|_{(\partial_c^L)^{-1}(\mathfrak{m}L_{c-1})}$ (mod $\mathfrak{m}$),

(iii) $\mu_i|_{(\partial_i^L)^{-1}(\mathfrak{m}L_{i-1})} \equiv \mu'_i|_{(\partial_i^L)^{-1}(\mathfrak{m}L_{i-1})}$ (mod $\mathfrak{m}$) *for all* $i \in \mathbf{Z}$.

(2) *For a chain map* $\mu_\cdot: L_\cdot \to G[-c]_\cdot$, *we have* $\mu_c\partial_{c+1}^L \equiv 0$ (mod $\mathfrak{m}$). *Conversely, given a linear map* $h \in \mathrm{Hom}_R(L_c, G_0)$ *satisfying* $h\partial_{c+1}^L \equiv 0$ (mod $\mathfrak{m}$), *there is a chain map* $\mu_\cdot: L_\cdot \to G[-c]_\cdot$ *such that* $\mu_c = h$.

(3) *For every* $g \in \mathrm{Aut}_R(G_0)$, *we have* $id_{K_\cdot} \otimes g \in \mathrm{Aut}_R(G_\cdot)$. *Moreover for each* $\lambda_\cdot \in \mathrm{Aut}_R(G_\cdot)$, *there is an automorphism* $g \in \mathrm{Aut}_R(G_0)$ *such that* $\lambda_\cdot \simeq id_{K_\cdot} \otimes g$.

Proof. (1) Since $\mu_\cdot \simeq \mu'_\cdot$ if and only if

$$\begin{cases}\mu_i - \mu'_i = \partial_{i+1-c}^G v_i + v_{i-1}\partial_i^L & \text{for all } i \in \mathbf{Z}\\ \text{with } \quad v_\cdot \in \mathrm{hom}_R(L_\cdot, G[-c+1]_\cdot),\end{cases}$$

the implications (i)⇒(iii)⇒(ii) are clear. Suppose (ii) holds and put $P_\cdot = \min(L_\cdot)_\cdot$. Then $\mathrm{Im}(\mu_c|_{P_c} - \mu'_c|_{P_c}) \subset \mathfrak{m}G_0 = \mathrm{Im}(\partial_1^G)$, so there is a map $v_c \in \mathrm{Hom}_R(P_c, G_1)$ fulfilling $\mu_c|_{P_c} - \mu'_c|_{P_c} = \partial_1^G v_c$. This can be completed to a chain homotopy $\mu_\cdot|_{P_\cdot} - \mu'_\cdot|_{P_\cdot} \simeq 0$, therefore $\mu_\cdot \simeq \mu'_\cdot$ by (1.12).

(2) It is clear that $\mu_c\partial_{c+1}^L = \partial_1^G\mu_{c+1} \equiv 0$ (mod $\mathfrak{m}$). Conversely, let $h \in \mathrm{Hom}_R(L_c, G_0)$ satisfy $h\partial_{c+1}^L \equiv 0$ (mod $\mathfrak{m}$). Since $\mathrm{Im}(h\partial_{c+1}^L) \subset \mathfrak{m}G_0 = \mathrm{Im}(\partial_1^G)$, there is a linear map $\mu_{c+1} \in \mathrm{Hom}_R(L_{c+1}, G_1)$ satisfying $\partial_1^G\mu_{c+1} = h\partial_{c+1}^L$. Hence the existence of $\mu_\cdot$ as stated.

(3) The first part is obvious. The second follows from (1).

(2.2) Remark. By (2) of (2.1), we can associate with each $h \in \mathrm{Hom}_R(L_c, G_0)$ satisfying $h\partial_{c+1}^L \equiv 0$ (mod $\mathfrak{m}$) complexes $\mathrm{con}(\mu_\cdot)_\cdot$, $\min(\mathrm{con}(\mu_\cdot)_\cdot)_\cdot$, taking a chain map $\mu_\cdot: L_\cdot \to G[-c]_\cdot$ such that $\mu_c = h$. The property (1) of (2.1) implies that they are determined uniquely by h up to isomorphism.

(2.3) Lemma. *Assume* R *is graded. For a complex* $L_\cdot \in \mathscr{C}m(R)$ *and an integer* c, *suppose that* $L_\cdot \cong \bigoplus_n L^{(n)}_\cdot$ *with* $L^{(n)}_\cdot \in \mathscr{C}m(R)$ $(n \in \mathbf{Z})$, $L^{(n)}_\cdot = 0$ *for all but a finite number of* n *and that each* $L_c^{(n)}$ *is isomorphic to the direct sum of a finite number (possibly may be zero) of copies of* $R(n)$. *Then for every chain map* $\mu_\cdot: L_\cdot \to G[-c]_\cdot$, *we have* $\mu_\cdot \simeq \bigoplus_n \pi^{(n)}_\cdot\mu_\cdot|_{L^{(n)}_\cdot}$, *where* $\pi^{(n)}_\cdot: G[-c]_\cdot \to K[-c]_\cdot(n)^{p_n}$ *denotes the natural projection for all* $n \in \mathbf{Z}$.

Proof. Put $v^{n,n'}_\cdot = \pi^{(n)}_\cdot\mu_\cdot|_{L^{(n')}_\cdot}$. We see $v_c^{n,n'} \equiv 0$ (mod $\mathfrak{m}$) for $n \neq n'$, therefore $\mu_\cdot \simeq \bigoplus_n v^{n,n}_\cdot$ by (1) of (2.1).

Let q be a nonnegative integer and $F_\cdot = K_\cdot^q$. In the graded case let q_m $(m \in \mathbf{Z})$ be nonnegative integers that are zero for all but a finite number of m and set $F_\cdot = \bigoplus_m K_\cdot(m)^{q_m}$. As in the case of $G_\cdot$, we have $F_\cdot = K_\cdot \otimes_R F_0$ with $F_0 = R^q$ or $F_0 = \bigoplus_m R(m)^{q_m}$. Fix a positive integer a and denote by $\{v_1, \ldots, v_s\}$ a free basis of K_{a+1}, where we assume that each v_l is a homogeneous element of K_{a+1} in the graded case.

(2.4) Definition. A minimal complex $L. \in \mathscr{C}m(R)$ is called decomposable if there are minimal complexes $L'., L''. \in \mathscr{C}m(R)$ different from zero such that $L. \cong L'. \oplus L''.$. When $L.$ is not decomposable, we call it indecomposable.

(2.5) Proposition. *Given chain maps $\mu., \mu'.: F. \to G[-a-1].$, put $h = \mu_{a+1}$, $h' = \mu'_{a+1}$ and let h_l, h'_l $(1 \le l \le s)$ be the elements of $\mathrm{Hom}_R(F_0, G_0)$ $(\mathrm{Hom}_R(F_0, G_0)_{\deg(v_l)}$ in the graded case) such that $h = \sum_{l=1}^{s} v_l^* \otimes h_l$, $h' = \sum_{l=1}^{s} v_l^* \otimes h'_l$.*

(1) *We have $\min(\mathrm{con}(\mu.).). \cong \min(\mathrm{con}(\mu'.).).$ if and only if there are $g \in \mathrm{Aut}_R(G_0)$, $f \in \mathrm{Aut}_R(F_0)$ such that*

$$gh_l f \equiv h'_l \pmod{\mathfrak{m}} \quad \text{for all } l \ (1 \le l \le s). \tag{2.5.1}$$

(2) *Suppose R is graded and that $\deg(v_l) = d$ for all l $(1 \le l \le s)$. Then*

$$\min(\mathrm{con}(\mu.).). \cong \bigoplus_m \min(\mathrm{con}(\mu(m, m-d).).).,$$

where $\mu(m, n).$ denotes the composite map $K.(m)^{q_m} \hookrightarrow F. \xrightarrow{\mu.} G[-a-1]. \to K[-a-1].(n)^{p_n}$ for each pair $(m, n) \in \mathbf{Z}^2$.

(3) *Suppose R is a local ring. The minimal complex $\min(\mathrm{con}(\mu.).).$ is decomposable if and only if there are $g \in \mathrm{Aut}_R(G_0)$, $f \in \mathrm{Aut}_R(F_0)$, integers $p^{(j)}$, $q^{(j)}$ $(j = 1, 2, 0 \le p^{(j)} \le p,\ 0 \le q^{(j)} \le q,\ (p^{(j)}, q^{(j)}) \ne (0, 0), (p, q))$ such that*

$$\begin{cases} p^{(1)} + p^{(2)} = p, \quad q^{(1)} + q^{(2)} = q, \\ gh_l f \equiv \begin{bmatrix} h_l^{(1)} & 0 \\ 0 & h_l^{(2)} \end{bmatrix} \pmod{\mathfrak{m}} \\ \text{with } \ h_l^{(j)} \in \mathrm{Hom}_R(R^{q^{(j)}}, R^{p^{(j)}}) \ \text{ for all } l \ (1 \le l \le s), \end{cases} \tag{2.5.2}$$

where the linear maps are identified with matrices.

(4) *Suppose R is graded, $\deg(v_l) = d$ for all l $(1 \le l \le s)$, $F. = K.(d)^q$ and $G. = K.^p$. Then the above (3) holds, with "$\equiv \pmod{\mathfrak{m}}$", $\mathrm{Aut}_R(G_0)$, $\mathrm{Aut}_R(F_0)$ and $\mathrm{Hom}_R(R^t, R^{t'})$ being replaced by "$=$", $GL(p, k)$, $GL(q, k)$ and $\mathrm{Hom}_k(k^t, k^{t'})$ respectively.*

Proof. Put $P. = \min(\mathrm{con}(\mu.).).$, $P'. = \min(\mathrm{con}(\mu'.).).$.

(1) By (1.9), we have $P. \cong P'.$ if and only if there are chain automorphisms $\varphi. \in \mathrm{Aut}_R(F.)$, $\psi. \in \mathrm{Aut}_R(G.)$ satisfying $\mu'.\varphi. \simeq \psi[-a-1].\mu.$. On the other hand, for such $\varphi., \psi.$ there are $f \in \mathrm{Aut}_R(F_0)$, $g \in \mathrm{Aut}_R(G_0)$ with the property $\varphi. \simeq id_{K.} \otimes f^{-1}$, $\psi. \simeq id_{K.} \otimes g$ by (3) of (2.1). Therefore $P. \cong P'.$ if and only if $\mu'.(id_{K.} \otimes f^{-1}) \simeq (id_{K[-a-1].} \otimes g)\mu.$ for some $f \in \mathrm{Aut}_R(F_0)$, $g \in \mathrm{Aut}_R(G_0)$. Moreover this condition is equivalent to

$$\mu'_{a+1}(id_{K_{a+1}} \otimes f^{-1}) \equiv (id_{K_0} \otimes g)\mu_{a+1} \pmod{\mathfrak{m}}, \tag{2.5.3}$$

by (1) of (2.1). Since $\mu'_{a+1}(id_{K_{a+1}} \otimes f^{-1}) = \sum_{l=1}^{s} v_l^* \otimes h'_l f^{-1}$, $(id_{K_0} \otimes g)\mu_{a+1} = (id_R \otimes g)\mu_{a+1} = \sum_{l=1}^{s} v_l^* \otimes gh_l$, one sees that (2.5.1) and (2.5.3) are the same.

(2) Since $K_{a+1}(m)^{q_m} \cong R(m-d)^{sq_m}$, our assertion is an immediate consequence of (2.3) and (2) of (1.8).

(3) Suppose $P_{\bullet} \cong L^{(1)}_{\bullet} \oplus L^{(2)}_{\bullet}$ with minimal complexes $L^{(j)}_{\bullet}$ $(j = 1, 2)$. Then each $L^{(j)}_{\bullet}$ satisfies $L^{(j)}_i = 0$ for all $i < 0$, $H_i(L^{(j)}_{\bullet}) = 0$ for $i \neq 0$, a and $\mathfrak{m}H_i(L^{(j)}_{\bullet}) = 0$ for $i = 0$, a, since $P_{\bullet}$ has the same property. There are therefore chain maps $\mu^{(j)}_{\bullet}\colon K_{\bullet}^{q^{(j)}} \to K[-a-1]_{\bullet}^{p^{(j)}}$ $(j = 1, 2)$ for suitable nonnegative integers $p^{(j)}$, $q^{(j)}$ such that $L^{(j)}_{\bullet} = \min(\mathrm{con}(\mu^{(j)}_{\bullet})_{\bullet})_{\bullet}$ by (1.6), (1.7). Thus

$$P_{\bullet} \cong \min(\mathrm{con}(\mu^{(1)}_{\bullet} \oplus \mu^{(2)}_{\bullet})_{\bullet})_{\bullet} \tag{2.5.4}$$

by (2) of (1.8). We see $p^{(1)} + p^{(2)} = p$, $q^{(1)} + q^{(2)} = q$ by (1) of (1.8), so that the "only if" part of our assertion follows from (1). Conversely, if (2.5.2) holds, there are chain maps $\mu^{(j)}_{\bullet}$ $(j = 1, 2)$ as above defined by the condition $\mu^{(j)}_{a+1} = \sum_{l=1}^{s} v_l^* \otimes h_l^{(j)}$ by (2) of (2.1), for which we have (2.5.4) by (1). This together with (2) of (1.8) shows the "if" part.

(4) As stated at the beginning of section one, all homomorphisms are homogeneous of degree zero in the case R is graded.

When R is a regular local ring (resp. a polynomial ring over k), the complex $K_{\bullet}$ is the Koszul complex of R with respect to a regular system of parameters (resp. indeterminates over k), so the chain map $\mu_{\bullet}$ can be expressed by an explicit formula.

(2.6) Lemma. *Assume R is a regular local ring of dimension r with $\mathfrak{m} = (x_1, \ldots, x_r)$ or a polynomial ring $k[x_1, \ldots, x_r]$ with $\deg(x_j) = 1$ $(1 \leq j \leq r)$. Let $\{u_1, \ldots, u_r\}$ be a free basis of K_1 such that $\partial_1^K(u_j) = x_j$ $(1 \leq j \leq r)$, $\Gamma_l = \{(j_1, \ldots, j_l) \mid 1 \leq j_1 < \cdots < j_l \leq r\}$ for $0 \leq l \leq r$ and $u_I = u_{j_1} \wedge \cdots \wedge u_{j_l}$ for $I = (j_1, \ldots, j_l) \in \Gamma_l$, where $\Gamma_0 = \{\varnothing\}$, $u_{\varnothing} = 1$. Given an R-linear map $h = \sum_{I \in \Gamma_c} u_I^* \otimes h_I \in \mathrm{Hom}_R(F_c, G_0)$ with $1 \leq c \leq r$, $h_I \in \mathrm{Hom}_R(F_0, G_0)$ $(\mathrm{Hom}_R(F_0, G_0)_c$ in the case $R = k[x_1, \ldots, x_r])$, we set*

$$\mu_i = 0 \qquad \text{for } i < c\,,$$

$$\mu_c = h\,,$$

$$\mu_{c+l} = \sum_{I \in \Gamma_{c+l}} \sum_{\substack{J \in \Gamma_l \\ J \subset I}} \mathrm{sgn}\begin{pmatrix} I \\ J,\ I \backslash J \end{pmatrix} u_I^* \otimes u_J \otimes h_{I \backslash J} \qquad \text{for } l \geq 1\,.$$

Then $\mu_{\bullet}\colon F_{\bullet} \to G[-c]_{\bullet}$ is a chain map satisfying $\mu_c = h$.

Proof. Note that $\partial_{\bullet}^F = \partial_{\bullet}^K \otimes id_{F_0}$, $\partial_{\bullet}^G = \partial_{\bullet}^K \otimes id_{G_0}$, $\partial_i^{G[-c]} = \partial_{i-c}^G$. Clearly $\partial_i^{G[-c]}\mu_i = 0 = \mu_{i-1}\partial_i^F$ for $i \leq c$. Suppose $l \geq 0$. First

$$(\partial_{l+1}^K \otimes id_{G_0})\mu_{c+l+1} = \sum_{I \in \Gamma_{c+l+1}} \sum_{\substack{J \in \Gamma_{l+1} \\ J \subset I}} \mathrm{sgn}\begin{pmatrix} I \\ J,\ I \backslash J \end{pmatrix} u_I^* \otimes (\partial_{l+1}^K(u_J)) \otimes h_{I \backslash J}\,.$$

On the other hand,

$$\begin{aligned} &\mu_{c+l}(\partial_{c+l+1}^K \otimes id_{F_0}) \\ &= \mu_{c+l}\left(\sum_{\substack{j \in I \\ I \in \Gamma_{c+l+1}}} \mathrm{sgn}\begin{pmatrix} I \\ j,\ I \backslash j \end{pmatrix} u_I^* \otimes (x_j u_{I \backslash j}) \otimes id_{F_0} \right) \end{aligned}$$

$$= \sum_{\substack{j \in I \\ I \in \Gamma_{c+l+1}}} \sum_{\substack{J \in \Gamma_l \\ J \subset I \setminus j}} \operatorname{sgn}\begin{pmatrix} I \\ j, I\setminus j \end{pmatrix} \operatorname{sgn}\begin{pmatrix} I\setminus j \\ J, I\setminus(j, J) \end{pmatrix} u_I^* \otimes (x_j u_J) \otimes h_{I\setminus(j,J)}$$

$$= \sum_{\substack{j \in I \\ I \in \Gamma_{c+l-1}}} \sum_{\substack{J \in \Gamma_l \\ J \subset I \setminus j}} \operatorname{sgn}\begin{pmatrix} I \\ (j, J), I\setminus(j, J) \end{pmatrix} u_I^* \otimes (x_j u_J) \otimes h_{I\setminus(j,J)}$$

$$= \sum_{I \in \Gamma_{c+l+1}} \sum_{\substack{J \in \Gamma_{l+1} \\ J \subset I}} \sum_{j \in J} \operatorname{sgn}\begin{pmatrix} I \\ J, I\setminus J \end{pmatrix} u_I^* \otimes \left(\operatorname{sgn}\begin{pmatrix} J \\ j, J\setminus j \end{pmatrix} (x_j u_{J\setminus j})\right) \otimes h_{I\setminus J}$$

$$= \sum_{I \in \Gamma_{c+l+1}} \sum_{\substack{J \in \Gamma_{l+1} \\ J \subset I}} \operatorname{sgn}\begin{pmatrix} I \\ J, I\setminus J \end{pmatrix} u_I^* \otimes (\partial_{l+1}^K(u_J)) \otimes h_{I\setminus J},$$

therefore $\partial_i^{G[-c]}\mu_i = \mu_{i-1}\partial_i^F$ for $i > c$ also.

For the rest of this section we confine ourselves to the case R is graded. We will denote the set $\{t | H_i(L.)_t \neq 0\}$ by $T_i(L.)$ $(i \in \mathbf{Z})$ for each $L. \in \mathscr{C}(R)$. Given integers $i_1, \ldots, i_l$, $t_1, \ldots, t_l$ with $i_1 < \cdots < i_l$, $l \geq 1$, let $C_{t_1,\ldots,t_l}^{i_1,\ldots,i_l}$ denote the set of minimal complexes $L. \in \mathscr{C}m(R)$ satisfying the condition

$$\begin{cases} H_i(L.) = 0 & \text{for } i \neq i_1, \ldots, i_l, \\ \mathfrak{m}H_i(L.) = 0 & \text{for } i = i_1, \ldots, i_l, \\ T_{i_j}(L.) \subset \{t_j\} & \text{for all } j\ (1 \leq j \leq l). \end{cases}$$

(2.7) Corollary. *Assume that R is a polynomial ring $k[x_1, \ldots, x_r]$ with $\deg(x_j) = 1$ $(1 \leq j \leq r, r \geq 2)$ and that $0 < a \leq r - 1$. Let $L. \in \mathscr{C}m(R)$ be an indecomposable minimal complex different from zero satisfying $L_i = 0$ for $i < 0$, $H_i(L.) = 0$ for $i \neq 0$, a, $\mathfrak{m}H_i(L.) = 0$ for $i = 0$, a. Then $L. \in C_{-m}^0$, $L_r \cong R(m - r)$ for some $m \in \mathbf{Z}$ or $L. \in C_{-m,-m+a+1}^{0,a}$, $L_r \cong R(m - r - 1)^{\binom{r}{r-a}p}$ for some $m \in \mathbf{Z}$, $p \geq 1$.*

Proof. If $H_a(L.) = 0$ (resp. $H_0(L.) = 0$), the complex $L.$ coincides with $K.(m)$ (resp. $K[-a].(m - a - 1)$) for some $m \in \mathbf{Z}$, since $L.$ is indecomposable by hypothesis. Hence our assertion in that simple case. Suppose that neither of $H_0(L.)$ and $H_a(L.)$ is zero. By (1.6) and (2) of (2.5), there are integers m, p, q $(p \geq 1$, $q \geq 1)$ and a chain map $\mu.: K.(m)^q \to K[-a-1].(m - a - 1)^p$ such that $L. \cong \min(\operatorname{con}(\mu.).)$., therefore $L. \in C_{-m,-m+a+1}^{0,a}$. Let $h_I \in \operatorname{Hom}_R(R(m)^q, R(m-a-1)^p)_{a+1} = \operatorname{Hom}_k(k^q, k^p)$ $(I \in \Gamma_{a+1})$ be the linear maps determined by the condition $\mu_{a+1} = \sum_{I \in \Gamma_{a+1}} u_I^* \otimes h_I$. Then

$$\mu_r = \sum_{J \in \Gamma_{r-a-1}} \operatorname{sgn}\begin{pmatrix} (1, \ldots, r) \\ J, (1, \ldots, r)\setminus J \end{pmatrix} (u_1 \wedge \cdots \wedge u_r)^* \otimes u_J \otimes h_{(1,\ldots,r)\setminus J}$$

by (2.6). Since $\{h_{(1,\ldots,r)\setminus J} | J \in \Gamma_{r-a-1}\} = \{h_I | I \in \Gamma_{a+1}\}$, the indecomposability of $L.$ implies that $\operatorname{rank}_k(\mu_r \otimes k) = q$ by (4) of (2.5). In other words,

$$\operatorname{rank}_k(\partial_r^{\operatorname{con}(\mu.)} \otimes k|_{K_r(m)^q}) = q,$$

so that $\mathrm{se}(\mathrm{con}(\mu.).)_r \cong R(m-r)^q$, $L_r \cong \min(\mathrm{con}(\mu.).)_r \cong K[-a]_r(m-a-1)^p \cong R(m-r-1)^{\binom{r}{r-a}p}$ as desired.

(2.8) Lemma. *Let R be as in (2.6), $F.$ be as above and $L. \in \mathscr{C}m(R)$. For each linear map $\tilde{h}: F_r \to L_r$, there exists a chain map $\mu.: F. \to L.$ such that $\mu_r = \tilde{h}$. In particular, there is a chain automorphism $\lambda. \in \mathrm{Aut}_R(F. \oplus L.)$ such that*

$$\lambda_r = \begin{bmatrix} id_{F_r} & 0 \\ \tilde{h} & id_{L_r} \end{bmatrix}.$$

Proof. Passing to the duals, set $F'_i = F^{*r-i}$, $L'_i = L^{*r-i}$ for $i \in \mathbf{Z}$ and let $F'.$, $L'.$ be the complexes in $\mathscr{C}m(R)$ whose differentials are induced naturally from those of $F.$, $L.$ respectively. Since $F'.$ has the same structure as $G.$, there is a chain map $\mu'.: L'. \to F'.$ such that $\mu'_0 = \tilde{h}^*$ by (2) of (2.1). It is enough to put $\mu_i = \mu^*_{r-i}: F_i \to L_i$ for all $i \in \mathbf{Z}$.

(2.9) Proposition. *Let R be as in (2.7) with $r \geq 3$ and a be an integer satisfying $0 < a < r-1$. For every $L. \in \mathscr{C}m(R)$ having the property $L_i = 0$ for $i<0$, $H_i(L.) = 0$ for $i \neq 0, a, r-1$, $\mathfrak{m}H_i(L.) = 0$ for $i = 0, a, r-1$, there are $L'^{(m)}. \in C^{0,r-1}_{-m,-m+r}$, $L''^{(m)}. \in C^{0,a,r-1}_{-m,-m+a+1,-m+r+1}$ $(m \in \mathbf{Z})$, which are zero for all but a finite number of m, such that*

$$L. \cong \left(\bigoplus_m L'^{(m)}.\right) \oplus \left(\bigoplus_m L''^{(m)}.\right).$$

Proof. By (1.5), there is a chain map $\mu.: \sigma_{r-1}(L.). \to G[-r].$ for suitable p_n $(n \in \mathbf{Z})$ such that $L. = \min(\mathrm{con}(\mu.).).$. Since $\sigma_{r-1}(L.)_i = 0$ for $i < 0$, $H_i(\sigma_{r-1}(L.).) = 0$ for $i \neq 0, a$, $\mathfrak{m}H_i(\sigma_{r-1}(L.).) = 0$ for $i = 0, a$, it follows from (2.7) that

$$\sigma_{r-1}(L.). \cong \left(\bigoplus_m P'^{(m)}.\right) \oplus \left(\bigoplus_m P''^{(m)}.\right)$$

for some minimal complexes $P'^{(m)}. \in C^0_{-m}$, $P''^{(m)}. \in C^{0,a}_{-m,-m+a+1}(m \in \mathbf{Z})$, which are zero for all but a finite number of m, satisfying $P'^{(m)}_r \cong R(m-r)^{l'_m}$, $P''^{(m)}_r \cong R(m-r-1)^{\binom{r}{r-a}l''_m}$ $(l'_m, l''_m \geq 0)$. Put $P^{(n)}. = P'^{(n+r)}. \oplus P''^{(n+r+1)}.$ and let $\pi^{(n)}.$ be the projection as in (2.3) with $c = r$ $(n \in \mathbf{Z})$. We have $\mu. \simeq \bigoplus_n \nu^{(n)}.$ with $\nu^{(n)}. := \pi^{(n)}.\mu.|_{P^{(n)}.}$ by that lemma, so that

$$L. \cong \bigoplus_n \min(\mathrm{con}(\nu^{(n)}.).). \tag{2.9.1}$$

Fix n and let $P. = P^{(n)}.$, $P'. = P'^{(n+r)}.$, $P''. = P''^{(n+r+1)}.$, $\nu. = \nu^{(n)}.$, $p = p_n$. Obviously there is an automorphism $g \in \mathrm{Aut}_R(R(n)^p) = GL(p, k)$ such that

$$g\nu_r|_{P''_r} = \begin{bmatrix} 0 \\ h'' \end{bmatrix}$$

with a surjective $h'' \in \mathrm{Hom}_R(P''_r, R(n)^{p''})$ for some p'' $(1 \leq p'' \leq p)$. Write

$$g\nu_r|_{P'_r} = \begin{bmatrix} h' \\ h''' \end{bmatrix}$$

with $h' \in \mathrm{Hom}_R(P'_r, R(n)^{p'})$, $h''' \in \mathrm{Hom}_R(P'_r, R(n)^{p''})$ $(p' = p - p'')$ and let $\tilde{h} \in \mathrm{Hom}_R(P'_r, P''_r)$ be a linear map such that $h''' + h''\tilde{h} = 0$. Then (2.8) guarantees the existence of a chain automorphism $\lambda. \in \mathrm{Aut}_R(P.)$ such that $\lambda_r = \begin{bmatrix} id_{P'_r} & 0 \\ \tilde{h} & id_{P''_r} \end{bmatrix}$, for which we have $gv_r\lambda_r = h' \oplus h'': P'_r \oplus P''_r \to R(n)^{p'} \oplus R(n)^{p''}$. Let $v'.: P'. \to K[-r].(n)^{p'}$, $v''.: P''. \to K[-r].(n)^{p''}$ be the chain maps satisfying $v'_r = h'$, $v''_r = h''$. Since $(id_{K[-r].} \otimes g)v.\lambda. \simeq v'. \oplus v''.$ by (1) of (2.1), one finds that

$$\text{(2.9.2)} \quad \min(\mathrm{con}(v^{(n)}.).). = \min(\mathrm{con}(v.).). \cong \min(\mathrm{con}(v'.).). \oplus \min(\mathrm{con}(v''.).).$$

by (1.9) and moreover that

$$\text{(2.9.3)} \quad \begin{cases} \min(\mathrm{con}(v'.).). \in C^{0,r-1}_{-(n+r),-(n+r)+r}, \\ \min(\mathrm{con}(v''.).). \in C^{0,a,r-1}_{-(n+r+1),-(n+r+1)+a+1,-(n+r+1)+r+1}. \end{cases}$$

This holds for all $n \in \mathbf{Z}$. Our assertion therefore follows from (2.9.1)–(2.9.3).

The above (2.7) and (2.9) are special results. Unlike the cases treated there, in general it cannot be expected that every indecomposable minimal complex $L. \in \mathscr{Cm}(R)$ satisfying $L_i = 0$ $(i < 0)$, $H_i(L.) = 0$ $(i \neq i_1, \ldots, i_l)$, $\mathfrak{m}H_i(L.) = 0$ $(i = i_1, \ldots, i_l)$ is contained in $C^{i_1,\ldots,i_l}_{t_1,\ldots,t_l}$ for suitable $t_1, \ldots, t_l$, even though R is a polynomial ring (see (2.12) below). The following proposition gives a general formulation. For a set $T \subset \mathbf{Z}$ and an integer n, let $T+n$ denote the set $\{t+n | t \in T\}$.

(2.10) Proposition. *Assume that R is graded and that K_i is the direct sum of a finite number of copies of $R(-i)$ for every $i \geq 0$, namely, the residue field k has a linear free resolution over R. Let $L. \in \mathscr{Cm}(R)$ be a minimal complex satisfying $a := \alpha(L.) < \infty$, $L_i = 0$ for $i < 0$, and $\mathfrak{m}H_i(L.) = 0$ for $0 \leq i \leq a$. If there are subsets $T_i^{(j)} \subset T_i(L.)$ $(j = 1, 2, 0 \leq i \leq a)$ such that*

$$\text{(2.10.1)} \quad T_i^{(1)} \cup T_i^{(2)} = T_i(L.), \quad T_i^{(1)} \cap T_i^{(2)} = \varnothing \quad \text{for all } i,$$

$$\text{(2.10.2)} \quad \begin{cases} (T_i^{(1)} + (i' - i + 1)) \cap T_{i'}^{(2)} = \varnothing, \quad (T_i^{(2)} + (i' - i + 1)) \cap T_{i'}^{(1)} = \varnothing \\ \text{for all pairs } i, i' \text{ with } 0 \leq i < i' \leq a, \end{cases}$$

then $L. \cong L^{(1)}. \oplus L^{(2)}.$ with minimal complexes $L^{(j)}. \in \mathscr{Cm}(R)$ $(j = 1, 2)$ fulfilling

$$\text{(2.10.3)} \quad T_i(L^{(j)}.) = T_i^{(j)} \quad (j = 1, 2, 0 \leq i \leq a).$$

Moreover

$$\text{(2.10.4)} \quad L_{i'}^{(j)} \cong \bigoplus_{i=0}^{a} \bigoplus_{t \in T_i^{(j)}} R(-t - i' + i)^{l^{(j)}_{i,i',t}} \quad (j = 1, 2)$$

for all $i' \geq a$, where $l^{(j)}_{i,i',t}$ are suitable nonnegative integers.

Proof. In the case $a = 0$, one has $L. \cong G.$ with suitable p_n $(n \in \mathbf{Z})$, which is nothing but our assertion. Suppose $a > 0$ and that our assertion is true for smaller values of a. Let $\mu.: \sigma_a(L.). \to G[-a-1].$ be the chain map satisfying $L. \cong \min(\mathrm{con}\,\mu.).$ (see (1.5), (1.7)). Observe first that $G. \cong G^{(1)}. \oplus G^{(2)}.$ with

$$\text{(2.10.5)} \quad G^{(j)}. = \bigoplus_{-n \in T_a^{(j)}} K.(n)^{p_n},$$

since $H_0(G.) \cong H_a(L.)$ by (1.4) and $T_a^{(1)} \cup T_a^{(2)} = T_a(L.)$, $T_a^{(1)} \cap T_a^{(2)} = \varnothing$ by (2.10.1). Besides, since $\alpha(\sigma_a(L.).) < a$, $T_i(\sigma_a(L.).) = T_i(L.)$ for $0 \le i < a$ by (1.4), there are $P^{(j)}. \in \mathscr{Cm}(R)$ $(j = 1, 2)$ with

$$T_i(P^{(j)}.) = T_i^{(j)} \qquad (j = 1, 2, 0 \le i < a) \tag{2.10.6}$$

such that $\sigma_a(L.). \cong P^{(1)}. \oplus P^{(2)}.$ by the induction hypothesis. Here we may assume

$$P_{i'}^{(j)} \cong \bigoplus_{i=0}^{a-1} \bigoplus_{t \in T_i^{(j)}} R(-t - i' + i)^{\tilde{l}_{i,i',t}^{(j)}} \qquad (j = 1, 2, i' \ge a - 1) \tag{2.10.7}$$

for suitable $\tilde{l}_{i,i',t}^{(j)} \ge 0$. Denote the projection from $G[-a-1].$ to $G^{(j)}[-a-1].$ by $\varpi^{(j)}.$ for each $j = 1, 2$. The conditions (2.10.2), (2.10.5), (2.10.7) imply that $\varpi_{a+1}^{(j)} \mu_{a+1}|_{P_{a+1}^{(j')}} \equiv 0 \pmod{\mathfrak{m}}$ for $j \ne j'$, therefore $L. \cong L^{(1)}. \oplus L^{(2)}.$ with

$$L^{(j)}. = \min(\mathrm{con}(\varpi^{(j)}.\mu.|_{P^{(j)}.}).). \qquad (j = 1, 2) \tag{2.10.8}$$

by (1) of (2.1), (2) of (1.8). The properties (2.10.3) and (2.10.4) follow from (2.10.5)–(2.10.8) immediately.

(2.11) Lemma. *Suppose R is graded and let $\mu.: F. \to G[-c].$ be a chain map such that μ_i is a matrix with entries in k for every $i \in \mathbf{Z}$, where $F.$, $G.$ are the minimal complexes defined before and c an integer. Given nonsingular matrices $f \in \mathrm{Aut}_R(F_0)$, $g \in \mathrm{Aut}_R(G_0)$ with entries in k and a homomorphism $\nu. \in$* $\mathrm{hom}_R(F., G[-c+1].)$, *the map $\theta(\psi[-c]., \varphi., \nu.).$ with $\varphi. = id_{K.} \otimes f$, $\psi. = id_{K.} \otimes g$ is a chain automorphism of $L. := \mathrm{con}(\mu.).$ if and only if $\mu_c \varphi_c = \psi_0 \mu_c$ and $^-\nu. \in$* $\mathrm{Hom}_R(F., G[-c+1].)$.

Proof. Note first that the four conditions $\theta(\psi[-c]., \varphi., 0). \in \mathrm{Aut}_R(L.)$, $\mu.\varphi. = \psi[-c].\mu.$, $\mu.\varphi. \simeq \psi[-c].\mu.$ and $\mu_c \varphi_c = \psi_0 \mu_c$ are equivalent by hypothesis and (1) of (2.1). Moreover, $\mu.\varphi. \simeq \psi[-c].\mu.$ if $\theta(\psi[-c]., \varphi., \nu.). \in \mathrm{Aut}_R(L.)$. Our assertion therefore follows from (2) of (1.10).

(2.12) Example. Assume $R = k[x_1, \dots, x_r]$ with $r \ge 4$, $\deg(x_j) = 1$ $(1 \le j \le r)$. Let $h \in \mathrm{Hom}_R(K_2, K_0(-2))_0$ be an R-linear map and $\zeta'.: K. \to K[-2].(-2)$ the chain map defined by the formula in (2.6) with $c = 2$, $\zeta_2' = h$. Let further d be a positive integer, $F. = \bigoplus_{m=1}^d K.(m)$, $\zeta. = \underbrace{\zeta'. \oplus \cdots \oplus \zeta'.}_{d \text{ times}}: F. \to F[-2].(-2)$, $L'. =$ $\mathrm{con}(\zeta'.).$, $L. = \mathrm{con}(\zeta.). \cong \bigoplus_{m=1}^d L'.(m)$. Take another chain map $\mu.: L. \to G[-3].$ with $G. = \bigoplus_{m=1}^{d+1} K.(m-4)$ which satisfies, for reasons of degree,

$$\mu_3|_{L_3'(m)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ w^{(m)} & 0 \\ 0 & w'^{(m)} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \qquad \text{for each } m \ (1 \le m \le d),$$

where the upper $m-1$ and the lower $d-m$ rows are zero, $w^{(m)} \in \mathrm{Hom}_R(K[-1]_3(m-2), K_0(m-4))$, $w^{(m)}\zeta_4' \equiv 0 \pmod{\mathfrak{m}}$ (see (2) of (2.1)), $w'^{(m)} \in \mathrm{Hom}_R(K_3(m), K_0(m-3))$ and

$$L_3'(m) = K[-1]_3(m-2) \oplus K_3(m) \cong R(m-4)^{\binom{r}{2}} \oplus R(m-3)^{\binom{r}{3}}.$$

Put $P. = \min(\mathrm{con}(\mu.).).$. We have

$$T_0(P.) = \{t \mid -d \le t \le -1\}, \qquad T_1(P.) = \{t \mid 2-d \le t \le 1\},$$

$$T_2(P.) = \{t \mid 3-d \le t \le 3\} \quad \text{and} \quad H_i(P.) = 0 \qquad \text{for } i \neq 0, 1, 2.$$

(1) If $h = (u_1 \wedge u_2)^* \otimes 1$ and $w^{(m)} = (u_1 \wedge u_2)^* \otimes 1$, $w'^{(m)} = (u_1 \wedge u_3 \wedge u_4)^* \otimes 1$ for all m $(1 \le m \le d)$, then $P.$ is indecomposable.

(2) If $r = 4$ and h is generic, then

$$P. \cong \left(\bigoplus_{m=1}^{d} P^{(m)}. \right) \oplus K[-2].(d-3), \qquad 0 \neq P^{(m)}. \in C^{0,1,2}_{-m,-m+2,-m+4}.$$

Proof. Let us prove (2) first. Write $h = \sum_{1 \le i < j \le 4} (u_i \wedge u_j)^* \otimes h_{ij}$ with $h_{ij} \in k$. Then, with respect to the bases $u_2 \wedge u_3 \wedge u_4$, $-u_1 \wedge u_3 \wedge u_4$, $u_1 \wedge u_2 \wedge u_4$, $-u_1 \wedge u_2 \wedge u_3$ of K_3 and u_1, u_2, u_3, u_4 of $K[-2]_3(-2)$, the expression of ζ_3' is the alternating matrix

$$\begin{bmatrix} 0 & -h_{34} & h_{24} & -h_{23} \\ h_{34} & 0 & -h_{14} & h_{13} \\ -h_{24} & h_{14} & 0 & -h_{12} \\ h_{23} & -h_{13} & h_{12} & 0 \end{bmatrix}$$

by (2.6). For a generic h, the map ζ_3' is therefore an isomorphism, so $\min(L'.)_3$ must be the direct sum of some copies of $R(-4)$. Our assertion follows from this and (2.3).

Next we prove (1). Suppose $P.$ is decomposable, say

$$P. \cong Q^{(1)}. \oplus Q^{(2)}., \qquad 0 \neq Q^{(j)}. \in \mathscr{C}\mathscr{m}(R) \qquad (j = 1, 2).$$

Since $\min(L.). \cong \sigma_2(P.). \cong \sigma_2(Q^{(1)}.). \oplus \sigma_2(Q^{(2)}.).$ by (1) of (1.8), it is not hard to verify using (1.9) that

$$\sigma_2(Q^{(j)}.). = \bigoplus_{-m \in T_0(Q^{(j)}.)} \min(L'.).(m) \qquad (j = 1, 2).$$

Here we may assume without any loss of generality that $d' := -\inf(T_0(Q^{(1)}.))$ satisfies $1 \le d' < d$, on account of the conditions $Q^{(j)}. \neq 0$ $(j = 1, 2)$ and $\dim_k(H_0(L.)_t) \le 1$ for all $t \in \mathbf{Z}$. Put

$$G^{(j)}. = \bigoplus_{-(m-4) \in T_2(Q^{(j)}.)} K.(m-4)$$

and let

$$\mu^{(j)}.: \bigoplus_{-m \in T_0(Q^{(j)}.)} L'.(m) \to G^{(j)}[-3].$$

be a chain map satisfying $\min(\operatorname{con}(\mu^{(j)}.).).\cong Q^{(j)}.$ $(j=1,2)$. Further, let $\pi^{(m-4)}.$: $G[-3].\to K.(m-4)$ be the projection. By the construction of $P.$, $\dim_k(H_2(P.)_t)\le 1$ for all $t\in\mathbf{Z}$, so $T_2(Q^{(1)}.)\cap T_2(Q^{(2)}.) = \varnothing$, $T_2(Q^{(1)}.)\cup T_2(Q^{(2)}.) = T_2(P.)$ and $G^{(1)}[-3].\oplus G^{(2)}[-3].\cong G[-3].$. In addition, $-(d'-3)\in T_2(P.)$. It follows therefore that

$$(2.12.1)\qquad \begin{cases}\pi_3^{(d'-3)}(\mu_3^{(1)}\oplus\mu_3^{(2)})|_{L_3'(d'+1)}=0 & \text{if } -(d'-3)\in T_2(Q^{(1)}.)\,,\\ \pi_3^{(d'-3)}(\mu_3^{(1)}\oplus\mu_3^{(2)})|_{L_3'(d')}=0 & \text{if } -(d'-3)\notin T_2(Q^{(1)}.)\,.\end{cases}$$

To apply (1.9) to the present case, we have to know the structures of $\operatorname{Aut}_R(L.)$, $\operatorname{Aut}_R(G.)$.

CLAIM 1. For each $\lambda.\in\operatorname{Aut}_R(L.)$, there are $f_m\in k$ $(1\le m\le d)$ different from zero, chain maps $v^{(l)}.$: $K.(l-1)\to K[-1].(l-2)$ $(2\le l\le d)$ and $v.\in\hom_R(F., F[-1].(-2))$ such that $\lambda.\simeq\theta(\varphi[-2]., \varphi., v.).$ with

$$\varphi.=id_K.\otimes\begin{bmatrix} f_1 & & 0\\ & \ddots & \\ 0 & & f_d\end{bmatrix}\in\operatorname{Aut}_R(F.)\,,$$

$$(2.12.2)\qquad {}^-v.=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ v^{(2)}. & 0 & 0 & \cdots & 0\\ 0 & v^{(3)}. & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & v^{(d)}. & 0\end{bmatrix}.$$

Proof of Claim 1. By (1.10), (1.11), (2.1) and (2.11) there are maps $\varphi^{(1)}.=id_K.\otimes f^{(1)}\in\operatorname{Aut}_R(F.)$, $\varphi^{(2)}.=id_K.\otimes f^{(2)}\in\operatorname{Aut}_R(F.(-2))=\operatorname{Aut}_R(F.)$, $v'.\in\hom_R(F., F[-1].(-2))$ satisfying $\lambda.\simeq\theta(\varphi^{(2)}[-2]., \varphi^{(1)}., v'.).$, $\zeta_2\varphi_2^{(1)}=\varphi_0^{(2)}\zeta_2$, ${}^-v'.\in\operatorname{Hom}_R(F., F[-1].(-2))$, where

$$f^{(j)}=\begin{bmatrix} f_1^{(j)} & & 0\\ & \ddots & \\ 0 & & f_d^{(j)}\end{bmatrix}\in\operatorname{Aut}_R(F_0)$$

with $0\ne f_m^{(j)}\in k$ $(j=1,2,\ 1\le m\le d)$. One sees $\zeta_2'f_m^{(1)}=f_m^{(2)}\zeta_2'$ for all m $(1\le m\le d)$, therefore $f_m^{(1)}=f_m^{(2)}$ $(1\le m\le d)$, namely, $\varphi^{(1)}.$ and $\varphi^{(2)}.$ coincide with one and the same chain map, which we will denote by $\varphi.$. On the other hand, for reasons of degree, there are chain maps $v^{(1)}.$ $(2\le l\le d)$ and $v.\in\hom_R(F., F[-1].(-2))$ defined by (2.12.2) such that ${}^-v'.\simeq{}^-v.$ by (1) of (2.1). It follows from (2) of (1.10) that $\theta(\varphi[-2]., \varphi., v.).$ is a chain automorphism which is chain homotopic to $\lambda.$.

CLAIM 2. For each $\lambda'.\in\operatorname{Aut}_R(G.)$, there are $g_l\in k$ $(1\le l\le d+1)$ different from zero satisfying

$$\lambda'.\simeq\varphi'.=id_K.\otimes\begin{bmatrix} g_1 & & 0\\ & \ddots & \\ 0 & & g_{d+1}\end{bmatrix}.$$

Now let us return to the proof of (1) of (2.12). Let $\varphi_\bullet$, $v_\bullet$ be as in Claim 1 and $\varphi'_\bullet$ be as in Claim 2. Then

(2.12.3) $\pi_3^{(d'-3)}\varphi'[-3]_3\mu_3\theta(\varphi[-2]_\bullet, \varphi_\bullet, v_\bullet)_3|_{L'_3(d'+1)} = (g_{d'+1}f_{d'+1}w^{(d'+1)}, 0)$

(2.12.4)
$$\pi_3^{(d'-3)}\varphi'[-3]_3\mu_3\theta(\varphi[-2]_\bullet, \varphi_\bullet, v_\bullet)_3|_{L'_3(d')} = (0, g_{d'+1}(f_{d'}w'^{(d')} - w^{(d'+1)}v_3^{(d'+1)})).$$

When $h=(u_1 \wedge u_2)^* \otimes 1, w^{(m)}=(u_1 \wedge u_2)^* \otimes 1, w'^{(m)}=(u_1 \wedge u_3 \wedge u_4)^* \otimes 1$ $(1 \le m \le d)$, we find

$$E' := (\zeta'_3)^{-1}(\mathfrak{m}K[-2]_3(-2))$$
$$= \Big(\bigoplus_{I \in \Gamma} \mathfrak{m}\cdot u_I\Big) \oplus \Big(\bigoplus_{I \in \Gamma_3\setminus\Gamma} R\cdot u_I\Big) \subset K_3$$

with $\Gamma=\{(1, 2, j) | 3 \le j \le r\}$ by (2.6), so that $w^{(d'+1)}v_3^{(d'+1)}|_{E'(d')} \equiv 0 \pmod{\mathfrak{m}}$ again by (2.6) applied to $v^{(d'+1)}_\bullet$. Therefore

(2.12.5) $$g_{d'+1}(f_{d'}w'^{(d')} - w^{(d'+1)}v_3^{(d'+1)})|_{E'(d')} \not\equiv 0 \pmod{\mathfrak{m}}.$$

Moreover

(2.12.6) $$g_{d'+1}f_{d'+1}w^{(d'+1)}|_{K[-1]_3(d'-1)} \not\equiv 0 \pmod{\mathfrak{m}}.$$

Since $(\partial_3^L)^{-1}(\mathfrak{m}L_2) = \bigoplus_{m=1}^d (\partial_3^{L'})^{-1}(\mathfrak{m}L'_2)(m)$, $(\partial_3^{L'})^{-1}(\mathfrak{m}L'_2)(m) = K[-1]_3(m-2) \oplus E'(m) \subset L'_3(m)$, the relations (2.12.1), (2.12.3)–(2.12.6) lead to a contradiction by (1) of (2.1), if $\varphi'[-3]_\bullet\mu_\bullet\theta(\varphi[-2]_\bullet, \varphi_\bullet, v_\bullet)_\bullet \simeq \mu^{(1)}_\bullet \oplus \mu^{(2)}_\bullet$. Thus $P_\bullet$ cannot be decomposable by (1.9).

§ 3. Classification of matrices

In order to characterize indecomposable minimal complexes $L_\bullet \in \mathscr{C}m(R)$ having the property $L_i = 0$ $(i < 0)$, $H_i(L_\bullet) = 0$ $(i \ne 0, a)$, $\mathfrak{m}H_i(L_\bullet) = 0$ $(i = 0, a)$, it is important to analyze the condition stated in (3) of (2.5). In his paper [R], C. M. Ringel dealt with this problem many years ago in a more general setting (see [Ka, pp. 82, 83] also). Here, for the convenience of the reader, we will explain a part of his results necessary for our study.

Let k be a field of arbitrary characteristic, $\mathbf{Z}_0$ the set of nonnegative integers and s a positive integer. For each pair $(p, q) \in \mathbf{Z}_0^2$, let mat$(p, q)$ denote the set of $p \times q$-matrices with entries in k and mat(p, q, s) the product of s copies of mat(p, q) equipped with the natural Zariski topology, where we understand that mat(p, q) consists of the single element representing the unique k-linear map from 0 to k^q or k^p to 0 in the case $p = 0$ or $q = 0$. Two elements $h = (h_l)_{1 \le l \le s}$, $h' = (h'_l)_{1 \le l \le s}$ of mat(p, q, s) are, by definition, equivalent if there are nonsingular matrices $g \in GL(p, k)$, $f \in GL(q, k)$ such that $gh_lf = h'_l$ for all l, in which case we denote $h \sim h'$. It is clear that "$\sim$" is an equivalence relation. Put Mat$(p, q, s) =$ mat$(p, q, s)/\sim$ and for each $h \in$ mat(p, q, s), let $\langle h\rangle \in$ Mat(p, q, s) denote the equivalence class represented by h. We say that $\langle h\rangle$ is decomposable and denote $\langle h\rangle = \langle h'\rangle \oplus \langle h''\rangle$ if there are $h' = (h'_l)_{1\le l\le s} \in \mathrm{mat}(p', q', s)$, $h'' = (h''_l)_{1\le l\le s} \in \mathrm{mat}(p-p', q-q', s)$ for some $(p', q') \in \mathbf{Z}_0^2$ with $0 \le p' \le p$, $0 \le q' \le q$, $(p', q') \ne (0, 0)$, (p, q) satisfying $h \sim \left(\begin{bmatrix} h'_l & 0 \\ 0 & h''_l \end{bmatrix}\right)_{1\le l\le s}$. When $\langle h\rangle$ is not decomposable, we call it indecomposable. Let $\{\gamma_t\}_{t\ge 0}$ denote the sequence of integers defined by the condition $\gamma_0 = 0$, $\gamma_1 = 1$, $\gamma_{t+2} = s\gamma_{t+1} - \gamma_t$ $(t \ge 0)$ and put

$$\Lambda_t = \{(y\gamma_{t+2} + z\gamma_{t+1}, y\gamma_{t+1} + z\gamma_t) | y, z \in \mathbf{Z}_0, z > 0, y + z \ge 2\}$$

for $t \ge 0$ and $\Lambda = \bigcup_{t\ge 0} \Lambda_t$. Observe that $(\gamma_{t+2}, \gamma_{t+1}) = (\gamma_{t+1}, \gamma_t)\chi$, $\Lambda_{t+1} = \Lambda_t \chi$ for all $t \ge 0$, where

$$\chi = \begin{bmatrix} s & 1 \\ -1 & 0 \end{bmatrix}, \quad \chi^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & s \end{bmatrix}.$$

(3.1) Remark (cf. [R], [Ka, p. 82]). Put $\rho(p, q) = p^2 + q^2 - spq$. We have $\{(p, q) \in \mathbf{Z}_0^2 | \rho(p, q) = 0\} = \{(0, 0)\}$ for $s \ge 3$ and $= \{(p, p) | p \in \mathbf{Z}_0\}$ for $s = 2$. Using the fact that $\rho((p, q)\chi) = \rho(p, q)$ for all $(p, q) \in \mathbf{Z}^2$, one can prove

$$\{(p, q) \in \mathbf{Z}_0^2 | \rho(p, q) > 0\} = \{(p, q) | (p, q) \text{ or } (q, p) \in \Lambda\}$$
$$\cup \{(\gamma_{t+1}, \gamma_t) | t \ge 0\} \cup \{(\gamma_t, \gamma_{t+1}) | t \ge 0\}$$

for $s \ge 2$.

With each $h = (h_1, \ldots, h_s) \in \mathrm{mat}(p, q, s)$, regarding h as a $p \times sq$-matrix, we can associate an element of $\mathrm{mat}(sq - \mathrm{rank}(h), q, s)$, say $\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_s)$, such that the columns of the transposed matrix of $\tilde{h}$ form a free basis of $\mathrm{Ker}(h)$. If $h' = (h'_1, \ldots, h'_s) \sim h$ and $\tilde{h}' = (\tilde{h}'_1, \ldots, \tilde{h}'_s)$ is an element of $\mathrm{mat}(sq - \mathrm{rank}(h'), q, s)$ associated with h' in the same way as above, then $\mathrm{rank}(h') = \mathrm{rank}(h)$ and $\tilde{h}' \sim \tilde{h}$. This assignment therefore gives rise to a mapping Ψ: $\bigcup_{(p,q)\in \mathbf{Z}_0^2} \mathrm{Mat}(p, q, s) \to \bigcup_{(p,q)\in \mathbf{Z}_0^2} \mathrm{Mat}(p, q, s)$ (cf. [DlR, pp. 15, 16], [Ka, p. 82]). In general, $\Psi^2(\langle h\rangle) = \langle h\rangle$ for all $h \in \mathrm{mat}(p, q, s)$ satisfying $\mathrm{rank}(h) = p$. Note that $\langle h\rangle \in \mathrm{Mat}(p, q, s)$ is decomposable if $\mathrm{rank}(h) < p$, $(p, q) \ne (1, 0)$ and moreover that, in the case $\mathrm{rank}(h) = p$, $\langle h\rangle$ is decomposable if and only if so is $\Psi(\langle h\rangle)$. We define $\Theta_1(\langle(h_1, \ldots, h_s)\rangle) = \langle({}^t h_1, \ldots, {}^t h_s)\rangle$, $\Theta_2(\langle(h_1, \ldots, h_s)\rangle) = \langle(h_2, \ldots, h_s, -h_1)\rangle$.

(3.2) Lemma. *Assume* $s \ge 2$. *For each* $t \ge 0$, *let* $\omega_t = (\omega_{t,1}, \ldots, \omega_{t,s})$ *be an element of* $\mathrm{mat}(\gamma_{t+1}, \gamma_t, s)$ *such that*

$$\omega_{t,l} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1_{\gamma_t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (1 \le l \le s-1), \qquad \omega_{t,s} = \begin{bmatrix} \omega_{t-1,2} & \\ \omega_{t-1,3} & \\ \vdots & 0 \\ \omega_{t-1,s-1} & \\ \omega_{t-1,s} & \\ 0 & 1_{\gamma_t - \gamma_{t-1}} \end{bmatrix},$$

where $\omega_{0,l} = 0 \in \mathrm{mat}(1, 0)$ *for all* l $(1 \le l \le s)$, *the first* $(l-1)\gamma_t$ *rows of* $\omega_{t,l}$ *are zero for* $1 \le l \le s-1$ *and the matrices* $\omega_{t,s}$ $(t \ge 0)$ *are defined inductively on* t. *Put* $\Omega_t = \langle \omega_t \rangle \in \mathrm{Mat}(\gamma_{t+1}, \gamma_t, s)$. *Then* Ω_t *is indecomposable for every* $t \ge 0$.

Proof. Put $\omega'_t = (-\omega_{t,s}, \omega_{t,1}, \omega_{t,2}, \dots, \omega_{t,s-1})$ for $t \ge 1$. Since $\mathrm{rank}(\omega'_t) = \gamma_{t+1}$, $\Psi(\langle \omega'_t \rangle) = \Theta_1(\Omega_{t-1})$ and $\Theta_2(\langle \omega'_t \rangle) = \Omega_t$, the indecomposability of Ω_{t-1} implies that of Ω_t for $t \ge 1$. Besides, it is clear that Ω_0 and Ω_1 are indecomposable. Hence our assertion.

(3.3) Theorem (cf. [R, Theorem 3], [Ka, Theorem 4], [Kr], [Di]). *Let* s, p, q *be integers with* $s \ge 2$, $p \ge q \ge 0$ *and* Ω_t $(t \ge 0)$ *be as in the above lemma.*

(1) *For each* $t \ge 0$, *the set* $\mathrm{Mat}(\gamma_{t+1}, \gamma_t, s)$ *contains exactly one indecomposable element, namely,* Ω_t.

(2) *If* $(p, q) \in \Lambda$, *then* $\langle h \rangle$ *is decomposable for all* $h \in \mathrm{mat}(p, q, s)$.

(3) *For each pair* $(p, q) = (y\gamma_{t+2} + z\gamma_{t+1}, y\gamma_{t+1} + z\gamma_t)$ *with* $y \ge 0$, $z > 0$, $s \ge 2$, $t \ge 0$, *the set* $\{h \in \mathrm{mat}(p, q, s) | \langle h \rangle = \Omega_{t+1}{}^{\oplus y} \oplus \Omega_t{}^{\oplus z}\}$ *is a nonempty Zariski open subset of* $\mathrm{mat}(p, q, s)$.

(4) *Suppose* $s = 2$. *We have* $\gamma_t = t$ *for* $t \ge 0$. *If* $\langle h \rangle$ *is indecomposable for some* $h \in \mathrm{mat}(p, q, 2)$, *then* $(p, q) \notin \Lambda$, *that is,* $p = q$, $q + 1$.
Case 1. *Whe* $p = q$, *the pair*

$$\left[\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \begin{bmatrix} w & 1 & 0 & \cdots & 0 \\ 0 & w & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & w \end{bmatrix}\right] \in \mathrm{mat}(p, p, 2)$$

represents an indecomposable class for every $w \in k$. *If* k *is algebraically closed, each indecomposable element of* $\mathrm{Mat}(p, p, 2)$ *have a representative of this form.*
Case 2. *When* $p = q + 1$, *the* Ω_{p-1} *is the only indecomposable element of the set* $\mathrm{Mat}(p, p-1, 2)$.

(5) *For each positive integer* p, *the set* $\{h \in \mathrm{mat}(p, p, 2) | \langle h \rangle = \langle (1, 1) \rangle^{\oplus p}\}$ *is a nonempty Zariski open subset of* $\mathrm{mat}(p, p, 2)$ *if* k *is algebraically closed.*

(6) *Suppose* $s \ge 3$. *If* $\langle h \rangle$ *is indecomposable for some* $h \in \mathrm{mat}(p, q, s)$, *then* $(p, q) \notin \Lambda$. *Conversely, if* $(p, q) \notin \Lambda$ *and* $\# k = \infty$, *there exists a nonempty Zariski open subset* $U_{p,q} \subset \mathrm{mat}(p, q, s)$ *such that* $\langle h \rangle$ *is indecomposable for all* $h \in U_{p,q}$. *Moreover,* $\#\{\langle h \rangle | h \in U_{p,q}\} = \infty$ *when* $\rho(p, q) < 0$ *(see* (3.1)).

§ 4. Maximal quasi-Buchsbaum graded modules

Throughout this section r denotes $\dim(R)$.

(4.1) Definition. For a complex $L_\bullet \in \mathscr{C}x(R)$ and an integer b with $\alpha(L_\bullet) < b \le \beta(L_\bullet)$, we denote the R-module $\mathrm{Coker}((\partial_b^L)^*\colon L^{*b-1} \to L^{*b})$ by $\mathrm{md}(L_\bullet, b)$.

(4.2) Lemma. (1) *Let* $L_\bullet$ *be a minimal complex in* $\mathscr{C}xm(R)$ *and* b *an integer satisfying* $\alpha(L_\bullet) < b \le \beta(L_\bullet)$. *Then, the* R*-module* $\mathrm{md}(L_\bullet, b)$ *has no free direct sum-*

mand, and besides, it has a minimal free resolution

$$\cdots \to L^{*i-1} \xrightarrow{(\partial_i^L)^*} L^{*i} \to \cdots \to L^{*b-1} \xrightarrow{(\partial_b^L)^*} L^{*b} \to \mathrm{md}(L_\bullet, b) \to 0 .$$

(2) *For each R-module M without any free direct summand, there is a minimal complex* $L_\bullet \in \mathscr{C}x\mathscr{m}(R)$ *and an integer b as in* (1) *such that* $M \cong \mathrm{md}(L_\bullet, b)$. *Moreover, we may choose b to be* $r - \mathrm{depth}_{\mathfrak{m}}(M)$.

(3) *Suppose R is Gorenstein and let* $L_\bullet$ *be a minimal complex in* $\mathscr{C}x\mathscr{m}(R)$ *which satisfies* $H_i(L_\bullet) = 0$ *for all* $i < 0$ *and* $H_0(L_\bullet) \neq 0$. *Then* $\beta(L_\bullet) \leq r$ *and* $\mathrm{depth}_{\mathfrak{m}}(\mathrm{md}(L_\bullet, b)) = r - b$ *for each b* ($\alpha(L_\bullet) < b \leq \beta(L_\bullet)$). *Moreover, the Krull dimension of* $\mathrm{md}(L_\bullet, b)$ *is r if* $l_R(H_i(L_\bullet)) < \infty$ *for all i* ($0 \leq i < b$), *except in the case* $b = r$, $H_i(L_\bullet) = 0$ *for* $i \neq 0$, $l_R(\mathrm{md}(L_\bullet, r)) < \infty$.

Proof. (1) For each $\tilde{L}_\bullet \in \mathscr{C}x(R)$ and $b > \alpha(\tilde{L}_\bullet)$, the R-module $\mathrm{md}(\tilde{L}_\bullet, b)$ have a free direct summand if and only if there is an automorphism $\eta \in \mathrm{Aut}_R(\tilde{L}^{*b})$ such that the matrix representing $\eta(\partial_b^{\tilde{L}})^*$ contains a row whose components are all zero. This being true if and only if $\mathrm{Im}(\partial_{b+1}^{\tilde{L}}) \not\subset \mathfrak{m}\tilde{L}_b$ since $\mathrm{Ker}(\partial_b^{\tilde{L}}) = \mathrm{Im}(\partial_{b+1}^{\tilde{L}})$, the minimality of $L_\bullet$ implies that $\mathrm{md}(L_\bullet, b)$ has no free direct summand. The last assertion is obvious.

(2) Put $b = r - \mathrm{depth}_{\mathfrak{m}}(M)$, $L'_\bullet = \mathrm{res}(M)_\bullet$, $L''_\bullet = \mathrm{res}(\mathrm{Ker}((\partial_1^{L'})^*))_\bullet$ and let ε denote the composite map $L''_0 \to \mathrm{Coker}(\partial_1^{L''}) \cong \mathrm{Ker}((\partial_1^{L'})^*) \hookrightarrow L'^{*0}$. We define a complex $L_\bullet$ by

$$L_i = \begin{cases} L'^{*b-i} & (i \leq b) \\ L''_{i-b-1} & (i > b) \end{cases}, \qquad \partial_i^L = \begin{cases} (\partial_{b-i+1}^{L'})^* & (i \leq b) \\ \varepsilon \quad (i = b+1) \\ \partial_{i-b-1}^{L''} & (i > b+1) \end{cases}.$$

Then it is clear by definition that $\mathrm{Im}(\partial_i^L) \subset \mathfrak{m}L_{i-1}$ for $i \neq b+1$, $\alpha(L_\bullet) < b \leq \beta(L_\bullet)$, $M \cong \mathrm{md}(L_\bullet, b)$. On the other hand, as remarked in the proof of (1), $\mathrm{Im}(\partial_{b+1}^L) \subset \mathfrak{m}L_b$ since M has no free direct summand. Hence $L_\bullet \in \mathscr{C}x\mathscr{m}(R)$.

(3) Since $\mathrm{Ext}_R^{b-i}(\mathrm{md}(L_\bullet, b), R) \cong H_i(L_\bullet)$ for $i < b \leq \beta(L_\bullet)$ and since the injective dimension of R is r, the first assertion follows immediately by local duality. The proof of the second half is left to the readers.

We need the following easy but fundamental lemma.

(4.3) Lemma. (1) *For minimal complexes* $L_\bullet$, $L'_\bullet \in \mathscr{C}x\mathscr{m}(R)$ *and integers b, b′ with* $\alpha(L_\bullet) < b \leq \beta(L_\bullet)$, $\alpha(L'_\bullet) < b' \leq \beta(L'_\bullet)$, *we have* $\mathrm{md}(L_\bullet, b) \cong \mathrm{md}(L'_\bullet, b')$ *if and only if* $L[b]_\bullet \cong L'[b']_\bullet$.

(2) *Let* $L_\bullet$, *b be as above. Then* $\mathrm{md}(L_\bullet, b)$ *is decomposable if and only if so is* $L_\bullet$.

(4.4) Remark. The correspondence between modules and complexes as above can further be developed. Restricting the consideration to modules of finite projective dimension, Y. Yoshino established an equivalence theorem for a stable category of R-modules and a subcategory of the derived category of complexes of R-modules (see [Y, § 1]).

In the argument below, we assume that R is Gorenstein or is a polynomial ring exclusively so that we can make use of the local duality in its simplest form. Given a maximal quasi-Buchsbaum R-module M of finite projective dimension having no free direct summand and a minimal complex $L_\bullet \in \mathscr{C}\mathit{xm}(R)$ such that $M = \mathrm{md}(L_\bullet, b)$ with $b = r - \mathrm{depth}_{\mathfrak{m}}(M)$, we have $L_i = 0$ for $i < 0$, $H^i_{\mathfrak{m}}(M) \cong \mathrm{Hom}_R(H_{b-r+i}(L_\bullet), H^r_{\mathfrak{m}}(R))$ for $0 \le i < r$ and $\mathfrak{m}H_i(L_\bullet) = 0$ for $i < b$. The results concerning $L_\bullet$ obtained in section two yield a number of consequences on the module M. When R is graded, for integers $i_1, \dots, i_l$, $t_1, \dots, t_l$ with $0 \le i_1 < \cdots < i_l < r$, $l \ge 1$, let $D^{i_1,\dots,i_l}_{t_1,\dots,t_l}$ denote the set of graded R-modules M satisfying the condition

$$\begin{cases} H^i_{\mathfrak{m}}(M) = 0 & \text{for } i \ne i_1, \dots, i_l, r\,, \\ \mathfrak{m}H^i_{\mathfrak{m}}(M) = 0 & \text{for } i = i_1, \dots, i_l\,, \\ T^{i_j}_{\mathfrak{m}}(M) \subset \{t_j\} & \text{for all } j\ (1 \le j \le l)\,, \end{cases}$$

where $T^i_{\mathfrak{m}}(M) = \{t \mid H^i_{\mathfrak{m}}(M)_t \ne 0\}$ for each $i \ge 0$. Note that $\dim(M) = r$ or 0 for all $M \in D^{i_1,\dots,i_l}_{t_1,\dots,t_l}$. We first state three results corresponding to (2.10), (2.7) and (2.9) in this order, omitting their proofs. For convenience sake, the residue field k will also be called maximal quasi-Buchsbaum from now on.

(4.5) Theorem. *Assume that R is a Gorenstein graded ring and that K_i is the direct sum of a finite number of copies of $R(-i)$ for all $i \ge 0$. Let M be a maximal quasi-Buchsbaum graded R-module of finite projective dimension which has no free direct summand. If there are subsets $T^{(j)i} \subset T^i_{\mathfrak{m}}(M)$ ($j = 1, 2, 0 \le i < r$) such that*

$$\begin{cases} T^{(1)i} \cup T^{(2)i} = T^i_{\mathfrak{m}}(M)\,, \quad T^{(1)i} \cap T^{(2)i} = \varnothing \quad \text{for all } i\ (0 \le i < r)\,, \\ (T^{(2)i'} + (i' - i + 1)) \cap T^{(1)i} = \varnothing\,, \quad (T^{(1)i'} + (i' - i + 1)) \cap T^{(2)i} = \varnothing \\ \text{for all pairs } i, i' \text{ with } 0 \le i < i' < r\,, \end{cases}$$

then $M \cong M^{(1)} \oplus M^{(2)}$ with maximal quasi-Buchsbaum graded R-modules $M^{(j)}$ ($j = 1, 2$) of finite projective dimension satisfying $T^i_{\mathfrak{m}}(M^{(j)}) = T^{(j)i}$ for all i, j ($j = 1, 2, 0 \le i < r$).

Put $\imath(M) = \#\{i \mid H^i_{\mathfrak{m}}(M) \ne 0, i < r\}$ for each R-module M.

(4.6) Corollary. *Let M be as in* (4.5) *and assume that $R = k[x_1, \dots, x_r]$ with* $\deg(x_j) = 1$ $(1 \le j \le r, r \ge 1)$. *If M is indecomposable and $\imath(M) \le 2$, then $M \cong$* $\mathrm{Coker}(\partial^K_{d+1})(-m) \in D^d_m$ *for some $m \in \mathbf{Z}$ (see* [G2]*) or $M \in D^{d,d+a}_{m,m-a-1}$ for some a, $m \in \mathbf{Z}$ with $0 < a < r - d$, where $d =$* $\mathrm{depth}_{\mathfrak{m}}(M)$.

(4.7) Proposition. *Let R be as in* (4.6) *with $r \ge 3$ and M be as in* (4.5). *If M is indecomposable and $\{i \mid H^i_{\mathfrak{m}}(M) \ne 0, i < r\} = \{0, a, r-1\}$ with $0 < a < r - 1$, then $M \in D^{0,a,r-1}_{m,m-a-1,m-r-1}$ for some $m \in \mathbf{Z}$.*

(4.8) Remark. The above results (4.6) and (4.7) imply that, if $R = k[x_1, \ldots, x_r]$ with $r = 1, 2, 3$, an indecomposable maximal quasi-Buchsbaum graded R-module having no free direct summand is contained in one of D^0_m, $D^{0,1}_{m,m-2}$, $D^{0,2}_{m,m-3}$, $D^{1,2}_{m,m-2}$, $D^{0,1,2}_{m,m-2,m-4}$ $(m \in \mathbf{Z})$. As is seen by (2.12), however, the sets $D^{i_1,\ldots,i_l}_{t_1,\ldots,t_l}$ $(1 \le l \le r-1, 0 \le i_1 < \cdots < i_l < r,\ t_1, \ldots, t_l \in \mathbf{Z})$ do not cover all indecomposable maximal quasi-Buchsbaum graded R-modules with no free direct summand, in the case $r \ge 4$.

More results can be deduced by the use of (2.5) and (3.3) when M is a maximal quasi-Buchsbaum R-module of finite projective dimension and $\imath(M) \le 2$. Given such an M, one sees easily that M is free if $\imath(M) = 0$ and that M is isomorphic to the direct sum of some copies of $\operatorname{Coker}((\partial^K_b)^*)$ $(\operatorname{Coker}((\partial^K_b)^*)(m)$ $(m \in \mathbf{Z})$ in the graded case) and of a free R-module if $\imath(M) = 1$, so we confine ourselves to the case $\imath(M) = 2$. Also, we only deal with the case R is a polynomial ring for the sake of simplicity, leaving a general formulation to the interested readers (cf. [Y, §6]).

(4.9) Lemma. *Let R be as in (4.7), i_1, i_2, a, t be integers with $0 \le i_1 < i_2 < r$, $a = i_2 - i_1 < r - 1$, $t \ge 0$ and $F_\bullet = K_\bullet(a+1+r)^{\gamma_t}$, $G_\bullet = K_\bullet(r)^{\gamma_{t+1}}$ (resp. $F_\bullet = K_\bullet(a+1+r)^{\gamma_{t+1}}$, $G_\bullet = K_\bullet(r)^{\gamma_t}$) be complexes, where $\{\gamma_t\}_{t\ge 0}$ denotes the sequence defined in the previous section with respect to $s = \binom{r}{a+1}$. Attaching subscripts to the elements of Γ_{a+1} so that $\{I_l \mid 1 \le l \le s\} = \Gamma_{a+1}$, we define a chain map $\mu_\bullet\colon F_\bullet \to G[-a-1]_\bullet$ by $\mu_{a+1} = \sum_{l=1}^{s} u^*_{I_l} \otimes \omega_{t,l}$ (resp. $\mu_{a+1} = \sum_{l=1}^{s} u^*_{I_l} \otimes {}^t\omega_{t,l}$) (see (2.2), (3.2)). Denote the R-module $\operatorname{md}(\min(\operatorname{con}(\mu_\bullet))_\bullet, r - i_1)$ by $A_{(i_1,i_2,t)}$ (resp. $B_{(i_1,i_2,t)}$). Then $A_{(i_1,i_2,t)}$ (resp. $B_{(i_1,i_2,t)}$) is the only indecomposable maximal quasi-Buchsbaum graded R-module in $D^{i_1,i_2}_{a+1,0}$ satisfying $l_R(H^{i_1}_{\mathfrak{m}}(A_{(i_1,i_2,t)})) = \gamma_t$, $l_R(H^{i_2}_{\mathfrak{m}}(A_{(i_1,i_2,t)})) = \gamma_{t+1}$ (resp. $l_R(H^{i_1}_{\mathfrak{m}}(B_{(i_1,i_2,t)})) = \gamma_{t+1}$, $l_R(H^{i_2}_{\mathfrak{m}}(B_{(i_1,i_2,t)})) = \gamma_t$).*

Proof. Notice that $s \ge 3$. Use (2.5), (3.2) and (4.3).

(4.10) Theorem. *Assume $R = k[x_1, \ldots, x_r]$ with $\deg(x_j) = 1$ $(1 \le j \le r)$, $r \ge 2$. Let M be a maximal quasi-Buchsbaum graded R-module which has no free direct summand such that $M \in D^{i_1,i_2}_{a+1,0}$, $q := l_R(H^{i_1}_{\mathfrak{m}}(M)) > 0$, $p := l_R(H^{i_2}_{\mathfrak{m}}(M)) > 0$ with $0 \le i_1 < i_2 < r$, $a = i_2 - i_1$. Put $s = \binom{r}{a+1}$, $F_\bullet = K_\bullet(a+1+r)^q$, $G_\bullet = K_\bullet(r)^p$. Let further $\{\gamma_t\}_{t\ge 0}$, Λ_t $(t \ge 0)$ and Λ be as in the preceding section if $a < r-1$.*

(1) *There is a chain map $\mu_\bullet\colon F_\bullet \to G[-a-1]_\bullet$ determined by the condition $\mu_{a+1} = \sum_{I \in \Gamma_{a+1}} u^*_I \otimes h_I$ for some $h = (h_I)_{I \in \Gamma_{a+1}} \in \operatorname{mat}(p, q, s)$, $h_I \in \operatorname{mat}(p, q)$ such that $M = \operatorname{md}(\min(\operatorname{con}(\mu_\bullet))_\bullet, r - i_1)$.*

(2) *Suppose $a < r-1$, $(p, q) = (\gamma_{t+1}, \gamma_t)$ (resp. $(q, p) = (\gamma_{t+1}, \gamma_t)$) for some $t \ge 1$. Then either M is decomposable or $M \cong A_{(i_1,i_2,t)}$ (resp. $M \cong B_{(i_1,i_2,t)}$). Moreover the latter case occurs as long as h is generic.*

(3) *Suppose that $a < r-1$ and that (p, q) (resp. (q, p)) coincides with $(y\gamma_{t+2} + z\gamma_{t+1}, y\gamma_{t+1} + z\gamma_t) \in \Lambda_t$ for some $t \ge 0$. Then M is decomposable. Moreover $M \cong A_{(i_1,i_2,t+1)}{}^{\oplus y} \oplus A_{(i_1,i_2,t)}{}^{\oplus z}$ (resp. $M \cong B_{(i_1,i_2,t+1)}{}^{\oplus y} \oplus B_{(i_1,i_2,t)}{}^{\oplus z}$) as long as h is generic.*

(4) *Suppose that* $a < r - 1$ *and that neither* (p, q) *nor* (q, p) *lies in* Λ. *If* $\#k = \infty$ *and* h *is generic, then* M *is indecomposable.*

(5) *Suppose* $a = r - 1$, *i.e.* $i_1 = 0$, $i_2 = r - 1$. *Then* M *is indecomposable if and only if* $p = q = 1$ *and* $h \sim (1) \in \mathrm{mat}(1, 1, 1)$.

Proof. (1) Note that $\mathrm{depth}_{\mathfrak{m}}(M) = i_1$, $\mathrm{Ext}_R^{r-i}(M, R)_t \cong H_{\mathfrak{m}}^i(M)_{-t-r}$ for $t \in \mathbf{Z}$, $i = i_1, i_2$ and $\mathrm{Ext}_R^{r-i}(M, R) = 0$ for $i \neq i_1, i_2, r$. Use (2) of (4.2), (1.6) and (2.1).

(2) Use (4.3), (4.9) and (3) of (3.3).

(3) Use (4.3), (4.9), (2.5), (2) of (3.3) and (3) of (3.3).

(4) Use (4.3), (2.5) and (6) of (3.3).

(5) Observe that $s = 1$, namely, $\mathrm{mat}(p, q, s) = \mathrm{mat}(p, q)$. In this case the classification is trivial.

(4.11) Proposition. *Let* R, i_1, i_2, a *be as in the above theorem and assume* $a < r - 1$, $\#k = \infty$. *For each pair of positive integers* p, q *satisfying* $\rho(p, q) < 0$, *there exist an infinite set* $X_{p,q}$ *and indecomposable maximal quasi-Buchsbaum graded* R*-modules* $M^{(j)}$ $(j \in X_{p,q})$ *having no free direct summand such that* $M^{(j)} \in D_{a+1,0}^{i_1,i_2}$, $l_R(H_{\mathfrak{m}}^{i_1}(M^{(j)})) = q$, $l_R(H_{\mathfrak{m}}^{i_2}(M^{(j)})) = p$ *for all* $j \in X_{p,q}$ *and* $M^{(j)} \ncong M^{(j')}$ *for* $j \neq j'$.

Proof. Use (2.2), (2.5), (4.3) and (6) of (3.3).

(4.12) Remark. (1) The elements of $D_{m,m-a-1}^{i_1,i_2}$ are obtained from those of $D_{a+1,0}^{i_1,i_2}$ by shift of grading, so that (4.6), (4.10) and (4.11), together with (2.6), describe fairly well the structure of a maximal quasi-Buchsbaum graded R-module M with $\imath(M) \leq 2$, when R is a polynomial ring.

(2) M. Cipu, J. Herzog and D. Popescu dealt with the same problem by an approach different from ours in [CHP, §2].

Finally, we state some partial results on the multiplicities and the numbers of minimal generators of the modules treated in the above theorem.

(4.13) Lemma. *Let the notation be as in* (4.10). *Then*

$$\mathrm{e}_{\mathfrak{m}}(M) = \binom{r-1}{i_1-1} q + \binom{r-1}{i_2-1} p - \mathrm{rank}_k(\mu_{r-i_1+1} \otimes k),$$

$$l_R(M/\mathfrak{m}M) = \binom{r}{i_1} q + \binom{r}{i_2} p - \mathrm{rank}_k(\mu_{r-i_1+1} \otimes k) - \mathrm{rank}_k(\mu_{r-i_1} \otimes k),$$

where we understand $\binom{n}{-1} = 0$ *for* $n \in \mathbf{Z}_0$.

Proof. Put $L_{\bullet} = \mathrm{con}(\mu_{\bullet})_{\bullet}$, $P_{\bullet} = \min(L_{\bullet})_{\bullet}$, $Q_{\bullet} = \mathrm{se}(L_{\bullet})_{\bullet}$, $b = r - i_1$. One sees easily $N := \mathrm{md}(L_{\bullet}, b) = \mathrm{md}(P_{\bullet}, b) \oplus \mathrm{md}(Q_{\bullet}, b)$, $M = \mathrm{md}(P_{\bullet}, b)$, $\mathrm{e}_{\mathfrak{m}}(N) = \binom{r-1}{i_1-1} q + \binom{r-1}{i_2-1} p$, $l_R(N/\mathfrak{m}N) = \binom{r}{i_1} q + \binom{r}{i_2} p - \mathrm{rank}_k(\partial_b^L \otimes k)$. Besides, $N' := \mathrm{md}(Q_{\bullet}, b)$ is a free R-module with $l_R(N'/\mathfrak{m}N') = \mathrm{rank}_k(\partial_{b+1}^L \otimes k)$ and $\mathrm{rank}_k(\partial_i^L \otimes k) = \mathrm{rank}_k(\mu_i \otimes k)$ for all $i \in \mathbf{Z}$. Hence follow the desired formulae.

(4.14) Lemma. *Let c, p, q, r $(1 \le c \le r)$ be positive integers, $R = k[x_1, \dots, x_r]$ a polynomial ring and $F_\bullet = K_\bullet(c+r)^q$, $G_\bullet = K_\bullet(r)^p$. Given $h = (h_I)_{I \in \Gamma_c} \in \mathrm{mat}(p, q, s)$ with $s = \binom{r}{c}$, let $\mu_\bullet\colon F_\bullet \to G[-c]_\bullet$ be the chain map as in (2.6), where $\mathrm{Hom}_R(F_c, G_0)$ and $\mathrm{mat}(p, q, s)$ are identified. Assume h is generic. Then*

(1) $\mathrm{rank}_k(\mu_c \otimes k) = \inf\left(p, \binom{r}{c} q\right)$,

(2) $\mathrm{rank}_k(\mu_r \otimes k) = \inf\left(\binom{r}{c} p, q\right)$,

(3) $\mathrm{rank}_k(\mu_{c+l} \otimes k) = \binom{r}{c+l} q$ *and μ_{c+l} is injective if $1 \le l \le r - c - 1$ and $p \ge \binom{r-l}{c} q$,*

(4) $\mathrm{rank}_k(\mu_{c+l} \otimes k) = \binom{r}{l} p$ *and μ_{c+l} is surjective if $1 \le l \le r - c - 1$ and $q \ge \binom{c+l}{c} p$.*

Proof. The assertions (1) and (2) are trivial.

(3) Suppose $r \ge 2$. Let $R' = k[x_1, \dots, x_{r-1}]$, $\Gamma'_l = \{(j_1, \dots, j_l) \mid 1 \le j_1 < \cdots < j_l \le r-1\}$ $(0 \le l \le r-1)$, $h' = (h_I)_{I \in \Gamma'_c}$, $K'_\bullet$ the Koszul complex of R' with respect to $x_1, \dots, x_{r-1}$, $F'_\bullet = K'_\bullet(c+r)^q$, $G'_\bullet = K'_\bullet(r)^p$ and $\mu'_\bullet\colon F'_\bullet \to G'[-c]_\bullet$ the chain map defined as in (2.6) such that $\mu'_c = h'$. By definition,

$$\mu_{c+l} = \mu^{(1)}_{c+l} + \mu^{(2)}_{c+l} + \mu^{(3)}_{c+l} \qquad \text{with}$$

$$\mu^{(1)}_{c+l} = \sum_{I \in \Gamma'_{c+l}} \sum_{\substack{J \in \Gamma'_l \\ J \subset I}} \mathrm{sgn}\begin{pmatrix} I \\ J,\ I \backslash J \end{pmatrix} u_I^* \otimes u_J \otimes h_{I \backslash J}\,,$$

$$\mu^{(2)}_{c+l} = \sum_{I \in \Gamma'_{c+l-1}} \sum_{\substack{J \in \Gamma'_{l-1} \\ J \subset I}} \mathrm{sgn}\begin{pmatrix} (I, r) \\ (J, r),\ I \backslash J \end{pmatrix} u^*_{(I,r)} \otimes u_{(J,r)} \otimes h_{I \backslash J}\,,$$

$$\mu^{(3)}_{c+l} = \sum_{I \in \Gamma'_{c+l-1}} \sum_{\substack{J \in \Gamma'_l \\ J \subset I}} \mathrm{sgn}\begin{pmatrix} (I, r) \\ J,\ (I \backslash J, r) \end{pmatrix} u^*_{(I,r)} \otimes u_J \otimes h_{(I \backslash J, r)}\,.$$

When regarded as matrices, the above linear maps satisfy $\mu^{(1)}_{c+l} = \mu'_{c+l}$, $\mu^{(2)}_{c+l} = (-1)^c \mu'_{c+l-1}$. In addition, $p \ge \binom{r-l}{c} q = \binom{(r-1)-(l-1)}{c} q > \binom{(r-1)-l}{c} q$, therefore, with the help of (1) and (2), we find by induction on r that μ_{c+l} is injective and $\mathrm{rank}_k(\mu_{c+l} \otimes k) = \binom{r}{c+l} q$.

(4) Similar to (3).

For positive integers c, r with $1 \leq c \leq r$, we define

$$S_{c,l,r} = \begin{cases} \mathbf{Z}_0^2 & \text{for } l \leq 0 \text{ or } l \geq r - c\,, \\ \left\{(p, q) \in \mathbf{Z}_0^2 \,\middle|\, p \geq \binom{r-l}{c} q \text{ or } q \geq \binom{c+l}{c} p\right\} & \text{for } 1 \leq l \leq r - c - 1\,. \end{cases}$$

One sees

$$\{(p, q) \in \mathbf{Z}_0^2 | \rho(p, q) > 0\} \subset \{(p, q) \in \mathbf{Z}_0^2 | p \geq (s-1)q \text{ or } q \geq (s-1)p\} \subset S_{c,l,r}$$

for every l, where $\rho(p, q)$ is the polynomial as in (3.1) with $s = \binom{r}{c}$.

(4.15) Proposition. *Let R, M and i_1, i_2, p, q be as in* (4.10). *Then*

$$-\inf\left(\binom{r}{i_2}p, \binom{r}{i_1-1}q\right) \leq e_{\mathfrak{m}}(M) - \binom{r-1}{i_1-1}q - \binom{r-1}{i_2-1}p \leq 0\,,$$

$$-\inf\left(\binom{r}{i_2}p, \binom{r}{i_1-1}q\right) - \inf\left(\binom{r}{i_2+1}p, \binom{r}{i_1}q\right)$$

$$\leq l_R(M/\mathfrak{m}M) - \binom{r}{i_1}q - \binom{r}{i_2}p \leq 0\,.$$

In the first inequality, the equality on the left can actually be attained if $(p, q) \in S_{i_2-i_1+1,r-i_2,r}$, and in the second, if $(p, q) \in S_{i_2-i_1+1,r-i_2,r} \cap S_{i_2-i_1+1,r-i_2-1,r}$. The equality on the right can always be attained in both inequalities.

Proof. Put $c = i_2 - i_1 + 1$ and let h, $\mu.$ be as in (4.10). Since $\operatorname{rank}_k(\mu_{c+l} \otimes k) \leq \inf\left(\binom{r}{l}p, \binom{r}{c+l}q\right)$, with equality if $(p, q) \in S_{c,l,r}$, $0 \leq l \leq r - c$ by (4.14) as long as h is generic, the formulae in (4.13) imply our assertion.

(4.16) Remark. (1) By (2) of (4.10), (4.13) and (4.14), one can deduce formulas for the multiplicies and the numbers of minimal generators of the modules defined in (4.9), since (γ_t, γ_{t+1}), $(\gamma_{t+1}, \gamma_t) \in S_{i_2-i_1+1,r-i_2,r} \cap S_{i_2-i_1+1,r-i_2-1,r}$ for $t \geq 0$ by (3.1).

(2) The Betti numbers of the modules treated in (4.10) can also be expressed by a formula similar to the second one described in (4.13) (see (1) of (4.1)).

(3) Examples show that the inequalities in (4.15) are not sharp in general, if (p, q) lies in the outside of the ranges mentioned there. Consider, for instance, the case $r = 5$, $i_1 = 2$, $i_2 = 4$, $p = q = 1$.

(4) The formulae in (4.13)–(4.15) also hold when R is a regular local ring of dimension r and M is a maximal quasi-Buchsbaum R-module with $\imath(M) = 2$.

§5. Remark on the case where three local cohomology modules are nonzero vector spaces

We end by giving a formulation of the classification problem of maximal quasi-Buchsbaum modules M with $\imath(M) = 3$ over a regular local ring R of dimension $r \geq 3$ which contains its residue field k, in terms of the linear algebra over k.

Let R, r be as above, p_1, p_2, p_3, a, a' be positive integers with $a < a' < r$, $G. = K.^{p_1}$, $G'. = K.^{p_2}$, $G''. = K.^{p_3}$ be complexes and $\mu^{(j)}.: G. \to G'[-a-1].$, $\mu'^{(j)}.: \mathrm{con}(\mu^{(j)}.). \to G''[-a'-1].$ $(j = 1, 2)$ be chain maps. Put $F^{(j)}. = \mathrm{con}(\mu^{(j)}.).$, $L^{(j)}. = \mathrm{con}(\mu'^{(j)}.).$ $(j = 1, 2)$. Suppose $L^{(1)}. \cong L^{(2)}.$. Then, there is a chain isomorphism $\lambda'.: F^{(2)}. \to F^{(1)}.$ by (1.8), so $\mathrm{con}(\mu'^{(1)}.\lambda'.). \cong L^{(1)}. \cong L^{(2)}.$. It follows therefore from (1.9) that

$$\mu'^{(1)}.\lambda'.\lambda''. \cong \psi''[-a'-1].\mu'^{(2)}. \tag{5.1}$$

for some chain automorphisms $\lambda''. \in \mathrm{Aut}_R(F^{(2)}.)$, $\psi''. \in \mathrm{Aut}_R(G''.)$, where

$$\lambda'.\lambda''. \simeq \lambda. := \begin{bmatrix} \psi'[-a]. & \nu. \\ 0 & \psi. \end{bmatrix} \tag{5.2}$$

with suitable $\psi'. \in \mathrm{Aut}_R(G'.)$, $\psi. \in \mathrm{Aut}_R(G.)$, $\nu. \in \hom_R(G., G'[-a].)$ by (1.11) or by the proof of (1.9). Besides, the formula in (2.6), together with (2.1) and (1) of (1.10), allows us to assume that ψ_i, ψ_i', ψ_i'' $(i \in \mathbf{Z})$, $\mu_i^{(j)}$ $(j = 1, 2, i \in \mathbf{Z})$, $\mu'^{(j)}_{a'+1}$ $(j = 1, 2)$ are all matrices with entries in k. In particular, as in (2.11), the fact that $\lambda.$ is a chain map implies

$$\mu^{(1)}.\psi. = \psi'[-a-1].\mu^{(2)}., \tag{5.3}$$

in other words $\begin{bmatrix} \psi'[-a]. & 0 \\ 0 & \psi. \end{bmatrix}$ is also a chain map, and hence so is $^{-}\nu.$ by (2) of (1.10). Again, we may assume that ν_i $(i \in \mathbf{Z})$ are matrices with components in k by (2.6). Put $f = \psi_0$, $f' = \psi_0'$, $f'' = \psi_0''$, $g = \nu_{a'+1}$, $g_i^{(j)} = \mu_{a'+i}^{(j)}$ $(i, j = 1, 2)$, $h^{(j)} = \mu_{a+1}^{(j)}$, $h_1'^{(j)} = \mu'^{(j)}_{a'+1}|_{G'_{a'-a+1}}$, $h_2'^{(j)} = \mu'^{(j)}_{a'+1}|_{G_{a'+1}}$. The conditions (5.1), (5.2) and (5.3) imply by (1) of (2.1) that

$$h^{(1)}(id_{K_{a+1}} \otimes f) = f'h^{(2)}, \tag{5.4.1}$$

$$h_1'^{(1)}(id_{K_{a'-a+1}} \otimes f') = f''h_1'^{(2)}, \tag{5.4.2}$$

$$(h_1'^{(1)}g + h_2'^{(1)}(id_{K_{a'+1}} \otimes f))|_{\mathrm{Ker}(g_1^{(2)})} = f''h_2'^{(2)}|_{\mathrm{Ker}(g_1^{(2)})}. \tag{5.4.3}$$

Moreover

$$h_1'^{(j)}g_2^{(j)} = 0 \qquad \text{for } j = 1, 2 \tag{5.5}$$

by (2) of (2.1). Conversely, the existence of $f \in GL(p, k)$, $f' \in GL(p', k)$, $f'' \in GL(p'', k)$ and $g = \nu_{a'+1}$ with $^{-}\nu. \in \mathrm{Hom}_R(G., G'[-a].)$ satisfying (5.4.1)–(5.4.3) implies $L^{(1)}. \cong L^{(2)}.$.

Thus the classification of maximal quasi-Buchsbaum R-modules M with the property $l_R(H_{\mathfrak{m}}^{i_1}(M)) = p_1 > 0$, $l_R(H_{\mathfrak{m}}^{i_2}(M)) = p_2 > 0$, $l_R(H_{\mathfrak{m}}^{i_3}(M)) = p_3 > 0$, $H_{\mathfrak{m}}^i(M) = 0$ $(i \neq i_1, i_2, i_3, r)$ for some i_1, i_2, i_3 with $0 \leq i_1 < i_2 < i_3 < r$ has been reduced to the classification of the elements (h, h_1', h_2') of

$$\mathrm{mat}(p_2, p_1, s_{21}) \times \mathrm{mat}(p_3, p_2, s_{32}) \times \mathrm{mat}(p_3, p_1, s_{31}),$$

which satisfy the condition corresponding to (5.5), under the equivalence relation $(h^{(1)}, h_1'^{(1)}, h_2'^{(1)}) \sim (h^{(2)}, h_1'^{(2)}, h_2'^{(2)})$ indicated by (5.4.1)–(5.4.3), where $s_{21} = \binom{r}{i_2 - i_1 + 1}$, $s_{32} = \binom{r}{i_3 - i_2 + 1}$, $s_{31} = \binom{r}{i_3 - i_1 + 1}$, $a = i_2 - i_1$, $a' = i_3 - i_1$.

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