

## Charge transfer model and (2-cluster) $\rightarrow$ (2-cluster) three-body scattering

By

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### §1. Introduction

We consider a three-body system consisting of two heavy particles (particles 1, 2) with the masses  $M_1, M_2$  and a light particle (particle 3) with  $m$ . We set  $\mu = (M_1, M_2)$  and write  $\mu \gg 1$  ( $\mu \rightarrow \infty$ ) for  $M_1, M_2 \gg 1$  ( $M_1, M_2 \rightarrow \infty$ ). Let  $r_j \in \mathbf{R}^N$  ( $j = 1, 2, 3$ ),  $N \geq 2$ , be the position of particle  $j$ , and let  $V_{jk}$  be the pair potential between particle  $j$  and particle  $k$ . Then the three-body Hamiltonian is

$$(1.1) \quad \tilde{H}^\mu = - \sum_{j=1}^2 (2M_j)^{-1} \Delta_{r_j} - (2m)^{-1} \Delta_{r_3} + V \quad \text{in } L^2(\mathbf{R}^{3N}),$$

$$V = V(r_1, r_2, r_3) = V_{23}(r_3 - r_2) + V_{13}(r_3 - r_1) + V_{12}(r_2 - r_1).$$

We assume the following throughout this paper:

(V)  $V_{ij}(x)$  ( $1 \leq i < j \leq 3$ ) is a smooth real-valued function on  $\mathbf{R}^N$ , and there exists  $\varepsilon_0 > N + (3/2)$  such that

$$|\partial_x^\gamma V_{ij}(x)| \leq C_\gamma (1 + |x|)^{-\varepsilon_0}$$

for all multi-indices  $\gamma$ .

Our main results are Theorems 1.1 and 1.3, which will be stated at the end of this section. For the proof of Theorem 1.1, we assume further

(V)'  $V_{ij}(x)$  ( $1 \leq i < j \leq 3$ ) satisfies (V) with

$$\varepsilon_0 > [(N - 1)/2] + N + (3/2). \quad ([ \ ] \text{ is Gauss' symbol.})$$

As usual, we remove the kinetic energy of the center of mass from  $\tilde{H}^\mu$  to get an operator  $H^\mu$  in  $L^2(\mathbf{R}^{2N})$ . A 2-cluster decomposition of the set  $\{1, 2, 3\}$  is a partition of  $\{1, 2, 3\}$  into two nonempty subsets, and in particular we use only the following 2-cluster decompositions:

$$(1.2) \quad a_1 := \{1, (2, 3)\}, \quad a_2 := \{2, (1, 3)\},$$

and we define  $\mathbf{A} := \{a_1, a_2\}$ .

For each  $a \in \mathbf{A}$ , the Jacobi coordinates  $\{x_a, y_a\}$  are defined by

$$(1.3) \quad \begin{aligned} x_a &:= r_3 - r_2, & y_a &:= r_1 - \frac{M_2 r_2 + m r_3}{M_2 + m} & \text{for } a = a_1, \\ x_a &:= r_3 - r_1, & y_a &:= \frac{M_1 r_1 + m r_3}{M_1 + m} - r_2 & \text{for } a = a_2. \end{aligned}$$

Let  $m_a = m_a^\mu$  and  $n_a = n_a^\mu$  ( $a \in \mathbf{A}$ ) be the reduced masses defined by

$$(1.4) \quad \frac{1}{m_a} = \frac{1}{M_i} + \frac{1}{m}, \quad \frac{1}{n_a} = \frac{1}{M_j} + \frac{1}{M_i + m} \quad \text{for } a = \{j, (i, 3)\}.$$

Then  $H^\mu$  is expressed as follows:

$$(1.5) \quad H^\mu = -\frac{1}{2m_a} \Delta_{x_a} - \frac{1}{2n_a} \Delta_{y_a} + V \quad \text{in } L^2(\mathbf{R}^{2N}).$$

$\{x_{a_1}, y_{a_1}\}$  and  $\{x_{a_2}, y_{a_2}\}$  are related as follows:

$$(1.6) \quad \begin{aligned} x_{a_1} &= \frac{m_{a_2}}{m} x_{a_2} + y_{a_2}, & y_{a_1} &= -\frac{m_{a_2}}{n_{a_1}} x_{a_2} + \frac{m_{a_1}}{m} y_{a_2}, \\ x_{a_2} &= \frac{m_{a_1}}{m} x_{a_1} - y_{a_1}, & y_{a_2} &= \frac{m_{a_1}}{n_{a_2}} x_{a_1} + \frac{m_{a_2}}{m} y_{a_1}. \end{aligned}$$

Under assumption (V),  $H^\mu$  is self-adjoint in  $\mathcal{H} := L^2(\mathbf{R}^{2N})$  with domain  $D(H^\mu) = H^2(\mathbf{R}^{2N})$ , the Sobolev space of order 2. For  $a = \{i, (j, 3)\} \in \mathbf{A}$  the 2-body Schrödinger operator  $h_a^\mu$  is defined by

$$(1.7) \quad h_a^\mu := -(2m_a)^{-1} \Delta_{x_a} + V_{j3}(x_a),$$

which is self-adjoint in  $L^2(\mathbf{R}_{x_a}^N)$  with domain  $D(h_a^\mu) = H^2(\mathbf{R}^N)$ . Since  $m_a \rightarrow m$  as  $\mu \rightarrow \infty$ ,  $h_a^\mu$  converges to a self-adjoint operator

$$(1.8) \quad h_a^\infty := -(2m)^{-1} \Delta_{x_a} + V_{j3}(x_a)$$

in the norm resolvent sense as  $\mu \rightarrow \infty$ . Furthermore we note that  $-z\Delta_{x_a} + V_{j3}(x_a)$ ,  $z \in \mathbf{C} \setminus \{0\}$ , is an analytic family of type (A) ([K], VII.2, [R-S] IV, XII.2). Let  $k(a)$  be the number of negative eigenvalues (counting multiplicity) of  $h_a^\infty$ . Under assumption (V), it is known that  $k(a)$  is finite ([R-S] IV, XIII.3). We set

$$\mathbf{CH} := \{\alpha = (a, k); a \in \mathbf{A}, 1 \leq k \leq k(a), k \in \mathbf{N}\},$$

where  $\mathbf{N} := \{1, 2, \dots\}$ , and write  $D(\alpha) = a$  for “channel”  $\alpha = (a, k) \in \mathbf{CH}$ . For  $\alpha = (a, k) \in \mathbf{CH}$  we denote by  $\lambda_\alpha^\infty$  ( $< 0$ ) the  $k$ -th negative eigenvalue of  $h_a^\infty$  and by  $\phi_\alpha^\infty$  the eigenfunction of  $h_a^\infty$  with eigenvalue  $\lambda_\alpha^\infty$  such that  $\{\phi_\alpha^\infty\}$  ( $\alpha \in \mathbf{CH}, D(\alpha) = a$ ) is an orthonormal system for each  $a \in \mathbf{A}$ . If  $\mu \gg 1$ , we can find negative eigenvalues  $\lambda_\alpha^\mu$  of  $h_{D(\alpha)}^\mu$  and associated normalized eigenfunctions  $\phi_\alpha^\mu$  ( $h_{D(\alpha)}^\mu \phi_\alpha^\mu = \lambda_\alpha^\mu \phi_\alpha^\mu$ ) for every  $\alpha \in \mathbf{CH}$  such that (i)  $\lambda_\alpha^\mu \rightarrow \lambda_\alpha^\infty$  as  $\mu \rightarrow \infty$ , (ii)  $\phi_\alpha^\mu \rightarrow \phi_\alpha^\infty$  in  $L^2(\mathbf{R}^N)$  as  $\mu \rightarrow \infty$  and (iii)  $\{\phi_\alpha^\mu\}$  ( $\alpha \in \mathbf{CH}, D(\alpha) = a$ ) is an orthonormal system for each  $a \in \mathbf{A}$ . (See [K], II.1, 4.)

For each  $\mu \gg 1$  and each  $\alpha \in \mathbf{CH}$ ,  $D(\alpha) = a$ , we define the channel embedding  $P_\alpha^\mu \in \mathbf{B}(L^2(\mathbf{R}_{y_a}^N), \mathcal{H})$  and the channel Hamiltonian  $T_\alpha^\mu$  by

$$(1.9) \quad (P_\alpha^\mu f)(x_a, y_a) = \phi_\alpha^\mu(x_a) f(y_a), \quad T_\alpha^\mu := -(2n_a)^{-1} \Delta_{y_a} + \lambda_\alpha^\mu,$$

respectively. Here we denote by  $\mathbf{B}(X, Y)$  the space of all bounded linear operators from a Banach space  $X$  to a Banach space  $Y$ .

Under assumption (V), the channel wave operators

$$(1.10) \quad W_{\alpha_\pm}^\mu := s - \lim_{t \rightarrow \pm\infty} e^{itH^\mu} P_\alpha^\mu e^{-itT_\alpha^\mu}$$

exist in  $\mathbf{B}(L^2(\mathbf{R}_{y_a}^N), \mathcal{H})$  (see [R-S] III, Theorem XI.35). For  $\alpha, \beta \in \mathbf{CH}$ , the scattering operator for scattering  $\alpha \rightarrow \beta$  is defined by

$$(1.11) \quad S_{\beta\alpha}^\mu := W_{\beta+}^{\mu*} W_{\alpha-}^\mu : L^2(\mathbf{R}_{y_{D(\alpha)}}^N) \rightarrow L^2(\mathbf{R}_{y_{D(\beta)}}^N).$$

Here  $A^*$  denotes the adjoint of the operator  $A$ .

For each  $\alpha \in \mathbf{CH}$  we give the spectral representation of  $T_\alpha^\mu$ . We define maps  $Z_\alpha^\mu(\lambda)$ ,  $\lambda > \lambda_\alpha^\mu$ , from  $\mathcal{S}(\mathbf{R}^N)$  (the Schwartz space of rapidly decreasing functions) to  $\Sigma := L^2(S^{N-1})$  ( $S^{N-1}$  is the unit sphere in  $\mathbf{R}^N$ ), by

$$(1.12) \quad (Z_\alpha^\mu(\lambda)f)(\omega) = (2\pi)^{-N/2} n_a^{1/2} (2n_a(\lambda - \lambda_\alpha^\mu))^{(N-2)/4} \\ \times \int e^{-i(2n_a(\lambda - \lambda_\alpha^\mu))^{1/2} \omega \cdot y_a} f(y_a) dy_a, \quad a = D(\alpha),$$

where  $\omega \in S^{N-1}$ . It is known that  $Z_\alpha^\mu(\lambda)$  can be extended to bounded operators from  $L_s^2(\mathbf{R}_{y_a}^N)$  to  $\Sigma$  for  $s > 1/2$ , where  $L_s^2(\mathbf{R}_{y_a}^N) := L^2(\mathbf{R}_{y_a}^N; \langle y \rangle^{2s} dy)$ ,  $\langle y \rangle := (1 + |y|^2)^{1/2}$  (cf. [G-M], Proposition 2.1). We define a map  $Z_\alpha^\mu$  from  $L_s^2(\mathbf{R}_{y_a}^N)$  to  $L^2((\lambda_\alpha^\mu, \infty); \Sigma)$  by

$$(1.13) \quad (Z_\alpha^\mu f)(\lambda, *) = (Z_\alpha^\mu(\lambda)f)(*), \quad \text{for } \lambda > \lambda_\alpha^\mu.$$

Then  $Z_\alpha^\mu$  can be extended to a unitary operator from  $L^2(\mathbf{R}_{y_a}^N)$  to  $L^2((\lambda_\alpha^\mu, \infty); \Sigma)$  and gives the spectral representation of  $T_\alpha^\mu$ , that is,

$$(1.14) \quad (Z_\alpha^\mu T_\alpha^\mu f)(\lambda, *) = \lambda (Z_\alpha^\mu(\lambda)f)(*), \quad \text{for a.e. } \lambda > \lambda_\alpha^\mu,$$

for  $f \in D(T_\alpha^\mu)$ . We can see that  $Z_\beta^\mu S_{\beta\alpha}^\mu Z_\alpha^{\mu*}$  is decomposable by a family of operators  $\{S_{\beta\alpha}^\mu(\lambda)\}$  ([A-J-S], 15-3):

$$(1.15) \quad (Z_\beta^\mu S_{\beta\alpha}^\mu Z_\alpha^{\mu*} h)(\lambda) = S_{\beta\alpha}^\mu(\lambda) h(\lambda) \text{ in } \Sigma \quad \text{for a.e. } \lambda \in (\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu,$$

for  $h \in L^2((\lambda_{\beta\alpha}^\mu, \infty); \Sigma)$ , where  $\lambda_{\beta\alpha}^\mu := \max(\lambda_\alpha^\mu, \lambda_\beta^\mu)$  and  $A^\mu = \{\text{the thresholds of } H^\mu\} \cup \sigma_p(H^\mu)$  ( $\sigma_p(H^\mu)$  denotes the set of all eigenvalues of  $H^\mu$ ). We will show that  $S_{\beta\alpha}^\mu(\lambda)$  is a  $\mathbf{B}(\Sigma)$ -valued norm continuous function in  $(\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu$  ( $\mathbf{B}(\Sigma) = \mathbf{B}(\Sigma, \Sigma)$ ). Furthermore, in Sect. 2 we will show that

$$(1.16) \quad T_{\beta\alpha}^\mu(\lambda) := S_{\beta\alpha}^\mu(\lambda) - \delta_{\beta\alpha}$$

has an integral kernel  $T_{\beta\alpha}^\mu(\lambda, \omega, \omega')$ , which is continuous for  $(\lambda, \omega, \omega') \in ((\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu) \times S^{N-1} \times S^{N-1}$ . Here  $\delta_{\beta\alpha} = 1$  (resp. 0) if  $\alpha = \beta$  (resp.  $\alpha \neq \beta$ ). In particular, the total scattering cross section for scattering  $\alpha \rightarrow \beta$  at relative energy  $\lambda$  and relative initial direction  $\omega$  (see [A-J-S], p. 627).

$$(1.17) \quad \sigma_{\beta\alpha}^\mu(\lambda; \omega) := (2\pi)^{N-1} (2n_{D(\alpha)}(\lambda - \lambda_{\beta\alpha}^\mu))^{(1-N)/2} \int_{S^{N-1}} |T_{\beta\alpha}^\mu(\lambda, \omega', \omega)|^2 d\omega'$$

is finite for all  $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu$  and  $\omega \in S^{N-1}$ .

We next consider the following time-dependent Schrödinger equation for the charge transfer or impact parameter model:

$$(1.18) \quad \begin{aligned} i\partial_t \psi(t) &= h_{\xi, \eta}(t) \psi(t) \quad \text{in } L^2(\mathbf{R}^N), \\ h_{\xi, \eta}(t) &= [-(2m)^{-1} \Delta_x + V_{23}(x) + V_{13}(x - \xi t - \eta)] \psi(t), \\ \xi &\in \mathbf{R}^N \setminus \{0\}, \quad \eta \in \Pi_\xi := \{\eta \in \mathbf{R}^N; \xi \cdot \eta = 0\}. \end{aligned}$$

(See [Y], [Ha], [G], [W].) The equation describes the motion of the light particle (particle 3) under the influence of interaction potential  $V_{13}$  and  $V_{23}$  due to two heavy particles 1 and 2; particle 2 is assumed to stay at the origin and particle 1 is assumed to move classically on the straight line  $\xi t + \eta$ .

Under assumption (V), (1.18) has a unique propagator

$$(1.19) \quad U(t, s) = U(\xi, \eta; t, s), \quad s, t \in \mathbf{R},$$

such that

(U-i)  $U(t, s)$  is a unitary operator on  $L^2(\mathbf{R}^N)$  and jointly strongly continuous in  $s$  and  $t$ .

(U-ii)  $U(t, r)U(r, s) = U(t, s)$  for  $r, s, t \in \mathbf{R}$ .

(U-iii) If  $f \in H^2(\mathbf{R}^N)$ , then  $U(t, s)f \in H^2(\mathbf{R}^N)$  for  $s, t \in \mathbf{R}$ , and  $U(t, s)f$  (which is strongly differentiable in  $s$  and  $t$ , respectively) satisfies

$$i\partial_t U(t, s)f = h_{\xi, \eta}(t)U(t, s)f, \quad i\partial_s U(t, s)f = -U(t, s)h_{\xi, \eta}(s)f$$

(see e.g. [R-S]II, Theorem X.71).

The purpose of this paper is to relate the scattering theory for equation (1.18) to that for the three-body system (1.5). We restrict ourselves to the (2-cluster)  $\rightarrow$  (2-cluster) scattering such that the initial and final channels belong to **CH**.

For  $\alpha \in \mathbf{CH}$  we define a function  $\psi_\alpha^\infty(x, t) = \psi_\alpha^\infty(\xi, \eta; x, t)$  by

$$(1.20) \quad \begin{aligned} \psi_\alpha^\infty(x, t) &:= e^{-i\lambda_\alpha^\infty t} \phi_\alpha^\infty(x) \quad \text{for } D(\alpha) = a_1, \\ \psi_\alpha^\infty(x, t) &:= e^{i\text{im } \xi \cdot x - i((m/2)|\xi|^2 + \lambda_\alpha^\infty)t} \phi_\alpha^\infty(x - \xi t - \eta) \quad \text{for } D(\alpha) = a_2. \end{aligned}$$

It is easy to see that  $\psi_\alpha^\infty(x, t)$  satisfies

$$(1.21) \quad \begin{aligned} i\partial_t \psi_\alpha^\infty(t) &= h_{a_1}^\infty \psi_\alpha^\infty(t) \quad \text{for } D(\alpha) = a_1, \\ i\partial_t \psi_\alpha^\infty(t) &= [-(2m)^{-1} \Delta_x + V_{13}(x - \xi t - \eta)] \psi_\alpha^\infty(t) \quad \text{for } D(\alpha) = a_2. \end{aligned}$$

Furthermore the strong limits

$$(1.22) \quad \Omega_\alpha^\pm = \Omega_\alpha^\pm(\xi, \eta) := s - \lim_{t \rightarrow \pm\infty} U(\xi, \eta; 0, t)\psi_\alpha^\infty(t)$$

exist in  $L^2(\mathbf{R}^N)$  for each  $\alpha \in \mathbf{CH}$  and

$$(1.23) \quad (\Omega_\alpha^\pm, \Omega_\beta^\pm) = \delta_{\alpha\beta}$$

holds for  $\alpha, \beta \in \mathbf{CH}$  ([Y], p. 155), where  $(\cdot, \cdot)$  denotes the inner product in  $L^2(\mathbf{R}^N)$ . Let  $\alpha, \beta \in \mathbf{CH}$  be, for example, such that  $D(\alpha) = a_1, D(\beta) = a_2$ . Then the quantity  $|(\Omega_\alpha^-, \Omega_\beta^+)|^2$  is the transition probability that particle 3 forming a bound state  $\phi_\alpha^\infty$  with particle 2 in the remote past will be captured by particle 1 (moving along the orbit  $\xi t + \eta$ ) in  $\phi_\beta^\infty$  in the far future.

Now we state the main results. For  $\xi \in \mathbf{R} \setminus \{0\}$  and  $\eta \in \Pi_\xi$ , we define

$$(1.24) \quad S_{\beta\alpha}^\infty(\xi, \eta) := e^{-i \int_{-\infty}^{\infty} V_{12}(-\xi t - \eta) dt} (\Omega_\alpha^-(\xi, \eta), \Omega_\beta^+(\xi, \eta)).$$

**Theorem 1.1.** *Let  $\alpha, \beta \in \mathbf{CH}$ , and assume (V). Then for  $f \in C(S^{N-1})$ , the continuous functions on  $S^{N-1}$ , and  $v_0 > 0$ , we have*

$$(1.25) \quad \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} (S_{\beta\alpha}^\mu((1/2)n_{D(\alpha)}v^2 + \lambda_\alpha^\mu)f)(\omega) = S_{\beta\alpha}^\infty(v_0\omega, 0)f(\omega)$$

uniformly on  $S^{N-1}$ .

Since  $\|S_{\beta\alpha}^\mu(\lambda)\|_{\mathbf{B}(\Sigma)} \leq 1$  for  $\lambda > 0$  and  $C(S^{N-1})$  is dense in  $\Sigma$ , we have

**Corollary 1.2.** *Let  $\alpha, \beta \in \mathbf{CH}$ , and assume (V) and fix  $v_0 > 0$ . Then*

$$(1.26) \quad s - \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} (S_{\beta\alpha}^\mu((1/2)n_{D(\alpha)}v^2 + \lambda_\alpha^\mu)f)(*) = S_{\beta\alpha}^\infty(v_0*, 0)f(*) \quad \text{in } \Sigma$$

for any  $f \in \Sigma$ .

**Theorem 1.3.** *Let  $\alpha, \beta \in \mathbf{CH}$  and assume (V) and fix  $v_0 > 0$ . Then*

$$(1.27) \quad \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} \sigma_{\beta\alpha}^\mu((1/2)n_{D(\alpha)}v^2 + \lambda_\alpha^\mu; \omega) = \int_{\Pi_\omega} |S_{\beta\alpha}^\infty(v_0\omega, \eta) - \delta_{\beta\alpha}|^2 d\eta$$

uniformly for  $\omega \in S^{N-1}$ , where  $d\eta$  is the Lebesgue measure on  $\Pi_\omega$ .

Scattering theory for the charge transfer model has first been studied by Yajima [Y]. He has proved asymptotic completeness for equation (1.18). His idea is to reduce the scattering theory for the time-dependent Hamiltonian to that for a time-independent Hamiltonian following Howland [Ho] and then to use the stationary method for three-body problem (cf. [G-M]). Hagedorn [Ha] has obtained similar results by a time-dependent approach. Recently Wüller [W] and Graf [G] have extended Yajima's results by using geometric methods of Enss [E].

Now we explain the organization of this paper. In the preliminary Sect. 2 we shall give the exact form of the scattering matrix (Theorem 2.3) for scattering

$\alpha \rightarrow \beta$ , though Theorem 2.3 will be proved in Sect. 8. We shall need certain uniform estimates for a family of self-adjoint operators which can be obtained by extending multiple commutator methods of Jensen, Mourre and Perry [J-M-P] (see also [J]), that had been originated from Mourre's work [M] (see also [P-S-S], [F-H], [T], [A-B-G], [Yaf]). These resolvent estimates will be given in Sect. 3 by an abstract setting. In Sect. 4 we shall give a stationary expression for  $(\Omega_\alpha^-(\xi, \eta), \Omega_\beta^+(\xi, \eta))$ . Our main theorem will be proved in Sect. 5. Lemma 5.4 contains essential estimates in our proof. The proof of this lemma will be given in Sect. 6 by using the abstract commutator estimates in Sect. 3. Certain lemmas of Sect. 5 will be proved in Sect. 7.

## §2. Preliminaries

For  $k, s \in \mathbf{R}$  the weighted Sobolev space  $H_s^k(\mathbf{R}^d)$  is defined by

$$(2.1) \quad H_s^k(\mathbf{R}^d) := \{f \in \mathcal{S}'(\mathbf{R}^d); \|f\|_{k,s} := \|\langle \xi \rangle^s (1 - \Delta)^{k/2} f\| < +\infty\},$$

where  $\mathcal{S}'$  denotes the tempered distributions,  $\Delta$  the  $d$ -dimensional Laplacian and  $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ ,  $\xi \in \mathbf{R}^d$ . Note that  $\|f\|_{k,s}$  is equivalent to  $\|(1 - \Delta)^{k/2} \langle \xi \rangle^s f\|$  and  $\sum_{|\gamma| \leq k} \|\langle \xi \rangle^s D_\xi^\gamma f\|$  if  $k \in \mathbf{N} \cup \{0\}$ , where  $|\gamma| = \gamma_1 + \cdots + \gamma_d$ ,  $D_\xi^\gamma := D_{\xi_1}^{\gamma_1} \cdots D_{\xi_d}^{\gamma_d}$ ,  $D_{\xi_j} := -i(\partial/\partial \xi_j)$  for multi-indices  $\gamma = (\gamma_1, \dots, \gamma_d)$ . We write  $H^k(\mathbf{R}^d) := H_0^k(\mathbf{R}^d)$  and  $L_s^2(\mathbf{R}^d) := H_s^0(\mathbf{R}^d)$ . Note that the Fourier transform on  $\mathcal{S}'(\mathbf{R}^d)$  maps  $H_s^k(\mathbf{R}^d)$  onto  $H_k^s(\mathbf{R}^d)$  boundedly for all  $k, s \in \mathbf{R}$ .

**Lemma 2.1.** *Let  $\alpha \in \mathbf{CH}$  ( $a = D(\alpha)$ ), and assume (V). Then  $\phi_\alpha^\mu, \phi_\alpha^\infty \in \mathcal{S}(\mathbf{R}^N)$  and  $\phi_\alpha^\mu \rightarrow \phi_\alpha^\infty$  in  $\mathcal{S}(\mathbf{R}^N)$  as  $\mu \rightarrow \infty$ .*

*Proof.* Let  $a = \{i, (j, 3)\}$ . By  $(-\Delta + 1)\phi_\alpha^\mu = (2m_a(\lambda_\alpha^\mu - V_{j3}) + 1)\phi_\alpha^\mu$ , we have for any  $k \in \mathbf{N}$

$$(2.2) \quad \phi_\alpha^\mu = [(-\Delta + 1)^{-1}(2m_a(\lambda_\alpha^\mu - V_{j3}) + 1)]^k \phi_\alpha^\mu.$$

Similarly we have for any  $k \in \mathbf{N}$

$$(2.3) \quad \phi_\alpha^\infty = [(-\Delta + 1)^{-1}(2m(\lambda_\alpha^\infty - V_{j3}) + 1)]^k \phi_\alpha^\infty.$$

Since for any  $\ell \in \mathbf{N} \cup \{0\}$  and  $f \in H^\ell(\mathbf{R}^N)$

$$s - \lim_{\mu \rightarrow \infty} (-\Delta + 1)^{-1}(2m_a(\lambda_\alpha^\mu - V_{j3}) + 1)f = (-\Delta + 1)^{-1}(2m(\lambda_\alpha^\infty - V_{j3}) + 1)f$$

in  $H^{\ell+2}(\mathbf{R}^N)$ , we see that

$$(2.4) \quad \phi_\alpha^\mu \rightarrow \phi_\alpha^\infty \quad \text{strongly in } H^k(\mathbf{R}^N)$$

as  $\mu \rightarrow \infty$  for any  $k \in \mathbf{N}$ . The following estimates are easily verified:

$$(2.5) \quad \sup_{\mu \gg 1} \{ \|[(-\Delta + 1)^{-1}(2m_a(\lambda_\alpha^\mu - V_{j3}) + 1)]^k\|_{\mathbf{B}(L_s^2, H_s^{2k})} + \|[(-\Delta + 1)^{-1}(2m(\lambda_\alpha^\infty - V_{j3}) + 1)]^k\|_{\mathbf{B}(L_s^2, H_s^{2k})} \} < \infty$$

for any  $k \in \mathbf{N}$  and  $s \geq 0$ . Here  $\sup_{\mu \gg 1} \{\cdots\} := \sup_{M_1, M_2 > M_0} \{\cdots\}$  for some large  $M_0$ . We claim that for each  $k \in \mathbf{N}$  and  $s \geq 0$ :

$$(2.6) \quad \sup_{\mu \gg 1} \{ \|\phi_\alpha^\mu\|_{2k,s} + \|\phi_\alpha^\infty\|_{2k,s} \} < \infty .$$

Indeed, (2.6) for  $k = 0$  follows from [Ag], p. 52, and (2.6) for  $k \geq 1$  from (2.2), (2.3) and (2.5). Thus by the Schwarz inequality, (2.4) and (2.6), we obtain

$$\begin{aligned} \|\langle x \rangle^s (-\Delta + 1)^k (\phi_\alpha^\mu - \phi_\alpha^\infty)\|^2 &\leq \|(-\Delta + 1)^k (\phi_\alpha^\mu - \phi_\alpha^\infty)\| \|\langle x \rangle^{2s} (-\Delta + 1)^k (\phi_\alpha^\mu - \phi_\alpha^\infty)\| \\ &\leq \|\phi_\alpha^\mu - \phi_\alpha^\infty\|_{2k,0} (\|\phi_\alpha^\mu\|_{2k,2s} + \|\phi_\alpha^\infty\|_{2k,2s}) \rightarrow 0 \end{aligned}$$

as  $\mu \rightarrow \infty$ . Since  $k$  and  $s$  are arbitrary, this and (2.6) imply the desired results.

The following limiting absorption principle is important for a representation of  $S_{\beta\alpha}^\mu(\lambda)$ .

**Lemma 2.2** ([M], [P-S-S]). *Assume (V) and fix  $\mu \gg 1$ . Let  $J$  be any compact interval in  $\mathbf{R} \setminus \mathcal{A}^\mu$  and fix  $s > 1/2$ . Then the norm limits*

$$(2.7) \quad (H^\mu - \lambda \pm i0)^{-1} := \lim_{\varepsilon \downarrow 0} (H^\mu - \lambda \pm i\varepsilon)^{-1}$$

exist in  $\mathbf{B}(L_s^2(\mathbf{R}^{2N}), L_{-s}^2(\mathbf{R}^{2N}))$  uniformly for  $\lambda \in J$ , and  $\mathbf{B}(L_s^2(\mathbf{R}^{2N}), L_{-s}^2(\mathbf{R}^{2N}))$ -valued functions  $(H^\mu - \lambda \pm i0)^{-1}$  are Hölder continuous in  $\lambda \in J$ .

**Remark.** Resolvent estimates for three-body Schrödinger operators have been studied by Mourre [M] for more general class of potentials including long range potentials, and Mourre's results have been extended by Perry, Sigal and Simon [P-S-S] (see also [F-H]) to many-body Schrödinger operators. Recently these results have been developed by Tamura [T] and Amrein, Berthier and Georgescu [A-B-G].

For  $a = \{i, (j, 3)\} \in \mathbf{A}$ , we define the intercluster potential  $I_a$  by  $I_a := V - V_{j3}(r_3 - r_j)$ . For  $\alpha \in \mathbf{CH}$  ( $D(\alpha) = a$ ),  $P_\alpha^{\mu*}$  is given by

$$(2.8) \quad (P_\alpha^{\mu*} f)(y_a) = \int \overline{\phi_\alpha^\mu(x_a)} f(x_a, y_a) dx_a \quad (\text{see (1.9)}).$$

Thus, by Lemma 2.1,  $P_\alpha^\mu$  and  $P_\alpha^{\mu*}$  can be regarded as operators in

$$\mathbf{B}(L_t^2(\mathbf{R}_{y_a}^N), L_s^2(\mathbf{R}_{x_a}^N) \otimes L_t^2(\mathbf{R}_{y_a}^N)), \quad \mathbf{B}(L_{-s}^2(\mathbf{R}_{x_a}^N) \otimes L_t^2(\mathbf{R}_{y_a}^N), L_t^2(\mathbf{R}_{y_a}^N)),$$

respectively, for any  $s, t \in \mathbf{R}$ . Thus, by (V), we can see that

$$(2.9) \quad \begin{aligned} P_\beta^{\mu*} I_a P_\alpha^\mu &\in \mathbf{B}(L_{-s}^2(\mathbf{R}_{y_a}^N), L_s^2(\mathbf{R}_{y_b}^N)), \\ I_a P_\alpha^\mu &\in \mathbf{B}(L_{-s}^2(\mathbf{R}_{y_a}^N), L_{\varepsilon_0-s}^2(\mathbf{R}^{2N})), \\ P_\beta^{\mu*} I_b &\in \mathbf{B}(L_{s-\varepsilon_0}^2(\mathbf{R}^{2N}), L_s^2(\mathbf{R}_{y_b}^N)) \end{aligned}$$

for any  $\alpha, \beta \in \mathbf{CH}$  ( $a = D(\alpha)$ ,  $b = D(\beta)$ ) and  $s$  with  $1/2 < s \leq \varepsilon_0/2$ . Furthermore, we note that  $Z_\alpha^\mu(\lambda)^* \in \mathbf{B}(\Sigma, L_{-s}^2(\mathbf{R}_y^N))$ ,  $s > 1/2$  (see (1.12)). Now we give an expression of  $S_{\beta\alpha}^\mu(\lambda)$  for each  $\mu \gg 1$ .

**Theorem 2.3.** *Let  $\alpha, \beta \in \mathbf{CH}$ , and assume (V). Then*

$$(2.10) \quad S_{\beta\alpha}^\mu(\lambda) = \delta_{\beta\alpha} + 2\pi i Z_\beta^\mu(\lambda) P_\beta^{\mu*} [-I_{D(\alpha)} + I_{D(\beta)}(H^\mu - \lambda - i0)^{-1} I_{D(\alpha)}] P_\alpha^\mu Z_\alpha^\mu(\lambda)^*$$

for a.e.  $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu$ . Furthermore, the R.H.S. of (2.10) is a  $\mathbf{B}(\Sigma)$ -valued norm continuous function of  $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu$ .

**Remark.**  $S_{\beta\alpha}^\mu(\lambda)$  is well defined for all  $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu$  by (2.10).

*Proof.* We only prove the second half of the statment. The formula (2.10) will be shown in Sect. 8. Let  $1/2 < s \leq \varepsilon_0/2$ . Then, it follows from (2.9) and Lemma 2.2 that

$$(2.11) \quad P_\beta^{\mu*} [-I_a + I_b(H^\mu - \lambda - i0)^{-1} I_a] P_\alpha^\mu \in \mathbf{B}(L_{-s}^2(\mathbf{R}_y^N), L_s^2(\mathbf{R}_y^N))$$

( $a = D(\alpha)$ ,  $b = D(\beta)$ ). Furthermore  $Z_\gamma^\mu(\lambda)$ ,  $\gamma \in \mathbf{CH}$ , is a  $\mathbf{B}(L_s^2(\mathbf{R}^N), \Sigma)$ -valued norm continuous function in  $\lambda$  (cf. [G-M], Proposition (2.1)), which together with (2.11) and Lemma 2.2 implies the second half of the theorem.

**Proposition 2.4.** *Let  $\alpha, \beta \in \mathbf{CH}$ , and assume (V). Then  $T_{\beta\alpha}^\mu(\lambda)$  (see (1.16)) has an integral kernel  $T_{\beta\alpha}^\mu(\lambda, \omega, \omega')$  given by*

$$(2.12) \quad T_{\beta\alpha}^\mu(\lambda, \omega, \omega') = i(2\pi)^{-N+1} (n_a n_b)^{1/2} (4n_a n_b (\lambda - \lambda_\alpha^\mu) (\lambda - \lambda_\beta^\mu))^{(N-2)/4} \\ \times ([-I_a + I_b(H^\mu - \lambda - i0)^{-1} I_a] \phi_\alpha^\mu e^{i(2n_a(\lambda - \lambda_\alpha^\mu))^{1/2} \omega \cdot y_a}, \\ \phi_\beta^\mu e^{i(2n_b(\lambda - \lambda_\beta^\mu))^{1/2} \omega \cdot y_b})_{L^2(\mathbf{R}^{2N})} \quad (a = D(\alpha), b = D(\beta)).$$

Furthermore,  $T_{\beta\alpha}^\mu(\lambda, \omega, \omega')$  is continuous in  $(\lambda, \omega, \omega') \in ((\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu) \times S^{N-1} \times S^{N-1}$ , and so  $T_{\beta\alpha}^\mu(\lambda) \in \mathbf{B}(C(S^{N-1}))$  and is also a Hilbert-Schmidt operator on  $\Sigma$ .

*Proof.* Fix a real  $s$  with  $N/2 < s \leq \varepsilon_0/2$ . Since the map  $\mathbf{R}^N \ni \xi \rightarrow \exp(i\xi \cdot *) \in L_{-s}^2(\mathbf{R}^N)$  is strongly continuous, the continuity of  $T_{\beta\alpha}^\mu(\lambda, \omega, \omega')$  in  $(\lambda, \omega, \omega')$  follows in the same way as Theorem 2.3. To finish the proof, we have only to show that  $T_{\beta\alpha}^\mu(\lambda)$  is an integral operator with kernel  $T_{\beta\alpha}^\mu(\lambda, \omega, \omega')$ . Let  $\gamma = \alpha, \beta$ . For each  $h_\gamma \in C(S^{N-1})$ ,

$$(2.13) \quad (Z_\gamma^{\mu*}(\lambda) h_\gamma)(y) = (2\pi)^{-N/2} n_c^{1/2} (2n_c (\lambda - \lambda_\gamma^\mu))^{(N-2)/4} \\ \times \int e^{i(2n_c(\lambda - \lambda_\gamma^\mu))^{1/2} \omega \cdot y} h_\gamma(\omega) d\omega \quad (c = D(\gamma)),$$

which follows from (1.12). Therefore we have

$$(T_{\beta\alpha}^\mu(\lambda) h_\alpha, h_\beta) = \int_{S^{N-1}} \int_{S^{N-1}} T_{\beta\alpha}^\mu(\lambda, \omega, \omega') h_\alpha(\omega') \overline{h_\beta(\omega)} d\omega' d\omega,$$

where  $T_{\beta\alpha}^\mu(\lambda, \omega, \omega')$  is the R.H.S. of (2.12). Since  $C(S^{N-1})$  is dense in  $\Sigma$ , this implies that  $T_{\beta\alpha}^\mu(\lambda)$  is an integral operator with kernel  $T_{\beta\alpha}^\mu(\lambda, \omega, \omega')$ . This completes the proof.



§3. Abstract resolvent estimates

This section is devoted to extending the abstract results developed in [J-M-P].

Let  $H$  be a self-adjoint operator in a Hilbert space  $\mathbf{H}$  whose inner product and norm will be denoted by  $(f, g)$  and  $\|f\|$ . Then we define the scale of spaces  $\mathbf{H}_{+2}$  and  $\mathbf{H}_{-2}$  associated to the self-adjoint operator  $H$  as follows.  $\mathbf{H}_{+2}$  is the domain  $D(H)$  with the graph norm  $\|f\|_{+2} = \|(H + i)f\|$  and  $\mathbf{H}_{-2}$  is the dual of  $\mathbf{H}_{+2}$  obtained via the inner product in  $\mathbf{H}$ .

Let  $H, A$  be self-adjoint operators in  $\mathbf{H}$ ,  $I$  a compact interval in  $\mathbf{R}$ , and  $d \in \mathbf{N}$ .

**Assumption 3.1.**

(H-i)  $D(A) \cap D(H)$  is a core for  $H$ .

(H-ii)  $e^{i\theta A}$  leaves  $D(H)$  invariant, and for each  $f \in D(H)$

$$\sup_{|\theta| < 1} \|He^{i\theta A}f\| < \infty .$$

(H-iii) Let  $H^{(0)} = H$ . There are self-adjoint operators  $iH^{(1)}, \dots, i^d H^{(d)}$  satisfying the following:

$$D(i^j H^{(j)}) \supset D(H) \quad (j = 1, \dots, d),$$

the form  $i[i^{j-1}H^{(j-1)}, A]$ , defined on  $D(H) \cap D(A)$  is bounded from below and closable, and the self-adjoint operator associated with its closure is  $i^j H^{(j)}$  ( $j = 1, \dots, d$ ).

$$(([B, C]u, v) := (Cu, B^*v) - (Bu, C^*v).)$$

(H-iv) The form  $[H^{(d)}, A]$  defined on  $D(A) \cap D(H)$  extends to a bounded operator from  $\mathbf{H}_{+2}$  to  $\mathbf{H}_{-2}$ , which is denoted by  $[H^{(d)}, A]_0$ .

(H-v) There exist  $C_0 > 0$  and  $\phi \in C_0^\infty(\mathbf{R})$  supported in a sufficiently small neighborhood of  $I$  and satisfying  $0 \leq \phi \leq 1$ ,  $\phi \equiv 1$  on  $I$ , such that

$$(3.1) \quad \phi(H)iH^{(1)}\phi(H) \geq C_0\phi(H)^2 .$$

Let  $W$  be a bounded operator on  $\mathbf{H}$ , and  $A$  a self-adjoint operator in  $\mathbf{H}$ .

**Assumption 3.2.** Let  $W^{(0)} = W$ . There are bounded operators  $W^{(1)}, \dots, W^{(d)}$  on  $\mathbf{H}$  satisfying the following properties:

The form  $[W^{(j-1)}, A]$ , defined on  $D(A)$ , extends to the bounded operator  $W^{(j)}$  ( $j = 1, \dots, d$ ).

**Theorem 3.3.** Let  $H, A$  be self-adjoint operators in  $\mathbf{H}$ ,  $I$  a compact interval in  $\mathbf{R}$ , and  $d \in \mathbf{N}$ . Furthermore, if  $d \geq 2$ , let  $W_1, \dots, W_{d-1}$  be bounded operators on  $\mathbf{H}$ . Assume assumptions 3.1 and 3.2 with  $W = W_1, \dots, W_{d-1}$ . Fix a real  $s > d - 1/2$ , and set

$$I_\pm = \{z \in \mathbf{C}; \operatorname{Re} z \in I, 0 < \pm \operatorname{Im} z < 1\} .$$

Define

$$D(z) = \langle A \rangle^{-s} R(z) W_1 R(z) \dots W_{d-1} R(z) \langle A \rangle^{-s} \quad \text{for } d \geq 2 ,$$

$$D(z) = \langle A \rangle^{-s} R(z) \langle A \rangle^{-s} \quad \text{for } d = 1 ,$$

for  $z \in \mathbf{C} \setminus \mathbf{R}$ , where  $R(z) = (H - z)^{-1}$ ,  $\langle A \rangle = (1 + A^2)^{1/2}$ . Then there exists a constant  $K$ , such that:

$$(3.2) \quad \begin{aligned} & \text{(i)} \quad \sup_{z \in I_{\pm}} \|D(z)\| \leq K. \\ & \text{(ii)} \quad \|D(z) - D(z')\| \leq K|z - z'|^{\delta_1} \text{ for } z, z' \in I_{\pm}, \text{ where} \\ & \delta_1 = \delta_1(s, d) = \frac{1}{1 + \frac{sd}{s - d + 1/2}}. \end{aligned}$$

(iii) For  $\lambda \in I$  the norm limits

$$D(\lambda \pm i0) := \lim_{\delta \downarrow 0} D(\lambda \pm i\delta)$$

exist in  $\mathbf{B}(\mathbf{H})$ , and  $D(\lambda \pm i0)$  are Hölder continuous with exponent  $\delta_1$  in  $\lambda \in I$  in the operator norm.

Moreover, if  $H, A, (W_1, \dots, W_{d-1})$  if  $d \geq 2$  depend on a parameter  $v$  such that  $\phi, C_0$  can be taken independently of  $v$ , and that

$$(3.3) \quad \begin{aligned} & \|H^{(j)}R(i)\| (j = 1, \dots, d), \quad \|R(i)[H^{(d)}, A]_0 R(i)\|, \\ & (\|W_k^{(j)}\| (j = 0, \dots, d; k = 1, \dots, d-1) \text{ if } d \geq 2) \end{aligned}$$

remain bounded in  $v$ , then  $K$  can be taken independently of  $v$ .

Theorem 3.3 gives an extension of Theorem 2.2 of [J-M-P], in which all  $W_k$  are the identity operator. Note that assumption 3.1 implies the non-existence of the point spectrum of  $H$  in  $I$  ([M]).

Furthermore under assumption 3.1 the absence of the singular continuous spectrum in  $I$  can be proved ([M], [P-S-S]).

We prove Theorem 3.3 by the commutator method of [J-M-P]. The following Lemma 3.4 plays an important role in our proof.

For small  $|\varepsilon| > 0$  the operator

$$(3.4) \quad Q_d(\varepsilon) = \sum_{j=1}^d \frac{\varepsilon^j}{j!} H^{(j)}$$

is  $H$ -bounded with  $H$ -bound  $< 1$ , since each  $H^{(j)}$  is  $H$ -bounded by (H-iii) and the closed graph theorem. Thus the operator  $H + Q_d(\varepsilon)$  is a closed operator with  $D(H + Q_d(\varepsilon)) = D(H)$ , and furthermore the resolvents of this operator have the following properties.

**Lemma 3.4** ([J-M-P], Lemma 3.1). *Let  $H, A$  be self-adjoint operators in  $\mathbf{H}$ ,  $I$  a compact interval in  $\mathbf{R}$ . Assume assumption 3.1. Then there exists a positive constant  $\varepsilon_1$  such that for  $0 < \pm\varepsilon < \varepsilon_1$ ,  $z \in I_{\pm}$ , the following results hold:*

- (i) *There exists a bounded inverse  $G_z(\varepsilon)$  of  $H + Q_d(\varepsilon) - z$ .*
- (ii) *The following estimates hold for  $G_z(\varepsilon)$ :*

$$(3.5) \quad \|G_z(\varepsilon)\| \leq C \cdot |\varepsilon|^{-1},$$

$$(3.6) \quad \|(H + i)G_z(\varepsilon)\| \leq C \cdot |\varepsilon|^{-1}, \quad \|G_z(\varepsilon)(H + i)\| \leq C \cdot |\varepsilon|^{-1},$$

$$(3.7) \quad \begin{aligned} \|(H + i)G_z(\varepsilon)\langle A \rangle^{-1}\| &\leq C \cdot |\varepsilon|^{-1/2}, \\ \|\langle A \rangle^{-1}G_z(\varepsilon)(H + i)\| &\leq C \cdot |\varepsilon|^{-1/2}. \end{aligned}$$

(iii) The form  $[A, G_z(\varepsilon)]$ , defined on  $D(A)$ , extends to a bounded operator on  $\mathbf{H}$ , which is denoted by  $[A, G_z(\varepsilon)]_0$ . Furthermore  $G_z(\varepsilon)$  maps  $D(A)$  into  $D(A) \cap D(H)$ .

(iv) For each  $z \in I_+$  (resp.  $I_-$ ),

$$(3.8) \quad \begin{aligned} G_z(\varepsilon) &\in C^1((0, \varepsilon_1); \mathbf{B}(\mathbf{H})) \text{ (resp. } C^1((-\varepsilon_1, 0); \mathbf{B}(\mathbf{H})), \quad \text{and} \\ \frac{d}{d\varepsilon} G_z(\varepsilon) &= [G_z(\varepsilon), A]_0 + \frac{\varepsilon^d}{d!} G_z(\varepsilon) [H^{(d)}, A]_0 G_z(\varepsilon). \end{aligned}$$

Moreover, if  $H, A$  depend on a parameter  $v$  such that  $\phi, C_0$  can be taken independently of  $v$ , and that

$$(3.9) \quad \|H^{(j)}R(i)\| (j = 1, \dots, d), \quad \|R(i)[H^{(d)}, A]_0 R(i)\|,$$

remain bounded in  $v$ , then  $C$  can be taken independently of  $v$ .

See [J-M-P] for the proof of the first half of the lemma. The last half can be shown by carefully checking the estimates carried out in [J-M-P] (see also [M], [P-S-S]).

Moreover we need the following elementary lemma.

**Lemma 3.5.** Fix an integer  $k \geq 0$  and let  $f_k(\varepsilon) = |\log \varepsilon|$  for  $k = 0$ ,  $f_k(\varepsilon) = \varepsilon^{-k}$  for  $k \in \mathbf{N}$ . Assume that a  $\mathbf{B}(\mathbf{H})$ -valued  $C^1$ -function  $X(\varepsilon)$ ,  $\varepsilon \in (0, \varepsilon_1)$  ( $\varepsilon_1 > 0$ ), satisfies the following inequalities:

$$(3.10) \quad \left\| \frac{d}{d\varepsilon} X(\varepsilon) \right\| \leq C_1 \cdot (\|X(\varepsilon)\|^p \cdot \varepsilon^{-q} + f_k(\varepsilon) + 1),$$

$$(3.11) \quad \|X(\varepsilon)\| \leq C_2 \cdot \varepsilon^{-r},$$

where  $p, q, r, C_1, C_2$  are constants satisfying  $0 \leq p, q < 1, r \geq 0, C_1 > 0, C_2 > 0$ . Then the following estimates hold:

$$(3.12) \quad \|X(\varepsilon)\| \leq C \cdot \varepsilon^{-k+1} \quad \text{when } k \geq 2,$$

$$(3.13) \quad \|X(\varepsilon)\| \leq C \cdot |\log \varepsilon| \quad \text{when } k = 1,$$

$$(3.14) \quad \|X(\varepsilon)\| \leq C \quad \text{when } k = 0,$$

where  $C = C(C_1, C_2, \varepsilon_1, p, q, r) > 0$ . Furthermore, when  $k = 0$ , the norm limit  $X(0) := \lim_{\varepsilon \downarrow 0} X(\varepsilon)$  exists in  $\mathbf{B}(\mathbf{H})$ .

*Proof.* Putting (3.11) in (3.10), we have

$$(3.15) \quad \left\| \frac{d}{d\varepsilon} X(\varepsilon) \right\| \leq C \cdot (\varepsilon^{-pr-q} + f_k(\varepsilon) + 1).$$

We first consider for  $k \geq 1$ . By integrating with respect to  $\varepsilon$ , we have (3.12), (3.13) when  $pr + q \leq k$ . When  $pr + q > k$ , we get  $\|X(\varepsilon)\| \leq C \cdot \varepsilon^{-r_1}$ , where  $r_1 = pr + q - 1$ . (Note that  $r - r_1 = (1 - p)r + (1 - q) > 1 - q > 0$ .) Putting this into (3.10), we have the inequality replaced  $r$  by  $r_1$  in (3.15). If  $pr_1 + q \leq k$ , we obtain (3.12), (3.13). If  $pr_1 + q > k$ , we get  $\|X(\varepsilon)\| \leq C \cdot \varepsilon^{-r_2}$ , where  $r_2 = pr_1 + q - 1$ . Continuing this process, we can find some  $r_n (= pr_{n-1} + q - 1)$  with  $pr_n + q \leq k$ ,  $pr_{n-1} + q > k$ , since  $r_j - r_{j-1} \geq 1 - q > 0$ . Thus we obtain (3.12), (3.13). When  $k = 0$ , (3.14) is obtained similarly and the existence of  $X(0)$  follows from the integrability of the R.H.S. of (3.10).

*Proof of Theorem 3.3.* We have only to prove this theorem for  $d \geq 2$  because the theorem for  $d = 1$  has been proved in [P-S-S]. Moreover we give only the proof for  $z \in I_+$ .

(i) For multi-indices of nonnegative integers  $\alpha = (\alpha_1, \dots, \alpha_{d-1})$ ,  $\beta = (\beta_1, \dots, \beta_{d-1})$  we write  $|\alpha| = \alpha_1 + \dots + \alpha_{d-1}$ , and  $\alpha \leq \beta$  if and only if  $\alpha_j \leq \beta_j$  for all  $j$ . Let  $\Gamma_\alpha$  be a family of all multi-indices  $\beta$  with  $\alpha \leq \beta$ ,  $|\beta| = |\alpha| + 1$ . Namely  $\beta \in \Gamma_\alpha$  implies that  $\alpha_j = \beta_j - 1$  for some  $j$  and  $\beta_i = \alpha_i$  for  $i \neq j$ . We set

$$F_z^\alpha(\varepsilon) := \langle A \rangle^{-s} G_z(\varepsilon) W_1^{(\alpha_1)} G_z(\varepsilon) W_2^{(\alpha_2)} G_z(\varepsilon) \dots W_{d-1}^{(\alpha_{d-1})} G_z(\varepsilon) \langle A \rangle^{-s}$$

for  $z \in I_+$ ,  $\varepsilon > 0$ ,  $\alpha = (\alpha_1, \dots, \alpha_{d-1})$  with  $|\alpha| \leq d$ .

By Lemma 3.4 (iv), we have for  $|\alpha| \leq d - 1$ ,

$$\begin{aligned} (3.16) \quad \frac{d}{d\varepsilon} F_z^\alpha(\varepsilon) &= \langle A \rangle^{-s} \left( \frac{d}{d\varepsilon} G_z(\varepsilon) \right) W_1^{(\alpha_1)} G_z(\varepsilon) \dots W_{d-1}^{(\alpha_{d-1})} G_z(\varepsilon) \langle A \rangle^{-s} \\ &\quad + \dots + \langle A \rangle^{-s} G_z(\varepsilon) W_1^{(\alpha_1)} G_z(\varepsilon) \dots W_{d-1}^{(\alpha_{d-1})} \left( \frac{d}{d\varepsilon} G_z(\varepsilon) \right) \langle A \rangle^{-s} \\ &= \langle A \rangle^{-s} \{ [G_z(\varepsilon), A]_0 W_1^{(\alpha_1)} G_z(\varepsilon) \dots W_{d-1}^{(\alpha_{d-1})} G_z(\varepsilon) \\ &\quad + \dots + G_z(\varepsilon) W_1^{(\alpha_1)} G_z(\varepsilon) \dots W_{d-1}^{(\alpha_{d-1})} [G_z(\varepsilon), A]_0 \} \langle A \rangle^{-s} \\ &\quad + \frac{\varepsilon^d}{d!} \{ \langle A \rangle^{-s} G_z(\varepsilon) [H^{(d)}, A]_0 G_z(\varepsilon) W_1^{(\alpha_1)} G_z(\varepsilon) \dots W_{d-1}^{(\alpha_{d-1})} G_z(\varepsilon) \langle A \rangle^{-s} \\ &\quad + \dots + \langle A \rangle^{-s} G_z(\varepsilon) W_1^{(\alpha_1)} G_z(\varepsilon) \dots W_{d-1}^{(\alpha_{d-1})} G_z(\varepsilon) \\ &\quad \times [H^{(d)}, A]_0 G_z(\varepsilon) \langle A \rangle^{-s} \} \\ &= I_1(\varepsilon) + I_2(\varepsilon). \end{aligned}$$

First we estimate  $I_2(\varepsilon)$ . Since  $s > 1$  and  $(H + i)^{-1} [H^{(d)}, A]_0 (H + i)^{-1}$ ,  $W_j^{(\alpha_j)}$  ( $j = 1, \dots, d - 1$ ) are bounded, by assumption 3.1 (iv) and assumption 3.2, we have

$$(3.17) \quad \|I_2(\varepsilon)\| \leq C \cdot \varepsilon^d \cdot \varepsilon^{-1/2} \cdot \varepsilon^{-d+1} \cdot \varepsilon^{-1/2} \leq C$$

by Lemma 3.4 (ii).

Next we estimate  $I_1(\varepsilon)$ . Noting that  $G_z(\varepsilon)$  maps  $D(A)$  into  $D(A)$  and  $W_j^{(\alpha_j)}$  maps  $D(A)$  into  $D(A)$ , as follows from assumption 3.2 and Lemma 3.4 (iii), we

have, by elementary computation,

$$I_1(\varepsilon) = [F_z^\alpha(\varepsilon), A] - \sum_{\beta \in \Gamma_\alpha} F_z^\beta(\varepsilon).$$

Since  $\|\langle A \rangle^s F_z^\alpha(\varepsilon)\|$ ,  $\|F_z^\alpha(\varepsilon) \langle A \rangle^s\| \leq C \cdot \varepsilon^{-d+(1/2)}$  by Lemma 3.4 (ii) and

$$\begin{aligned} \|F_z^\alpha(\varepsilon) \langle A \rangle\| &\leq \|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \|F_z^\alpha(\varepsilon) \langle A \rangle^s\|^{1/s}, \\ \|\langle A \rangle F_z^\alpha(\varepsilon)\| &\leq \|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \|\langle A \rangle^s F_z^\alpha(\varepsilon)\|^{1/s} \end{aligned}$$

by interpolation, we have

$$\begin{aligned} \|[F_z^\alpha(\varepsilon), A]\| &\leq \|F_z^\alpha(\varepsilon) \langle A \rangle\| + \|\langle A \rangle F_z^\alpha(\varepsilon)\| \\ &\leq C \cdot \|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s}. \end{aligned}$$

Thus we get

$$(3.18) \quad \|I_1(\varepsilon)\| \leq C \left( \|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s} + \sum_{\beta \in \Gamma_\alpha} \|F_z^\beta(\varepsilon)\| \right).$$

Therefore  $F_z^\alpha(\varepsilon)$  satisfies

$$(3.19) \quad \left\| \frac{d}{d\varepsilon} F_z^\alpha(\varepsilon) \right\| \leq C \left( \|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s} + \sum_{\beta \in \Gamma_\alpha} \|F_z^\beta(\varepsilon)\| + 1 \right)$$

for all multi-indices  $\alpha$  with  $|\alpha| \leq d-1$ .

Furthermore, it follows from Lemma 3.4 (ii) that

$$(3.20) \quad \|F_z^\gamma(\varepsilon)\| \leq C \varepsilon^{-d+1}$$

for all multi-indices  $\gamma$  with  $|\gamma| \leq d$ .

Let  $|\alpha| = d-1$ . Then we have by (3.19) and (3.20)

$$\left\| \frac{d}{d\varepsilon} F_z^\alpha(\varepsilon) \right\| \leq C (\|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s} + \varepsilon^{-d+1} + 1).$$

Applying Lemma 3.5 to this, we have

$$(3.21) \quad \|F_z^\alpha(\varepsilon)\| \leq C \varepsilon^{-d+2}.$$

Next let  $|\alpha| = d-2$ . Then  $|\beta| = d-1$  for  $\beta \in \Gamma_\alpha$ . Thus we obtain by (3.19)

$$\left\| \frac{d}{d\varepsilon} F_z^\alpha(\varepsilon) \right\| \leq C (\|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s} + \varepsilon^{-d+2} + 1).$$

Applying Lemma 3.5, we have

$$\|F_z^\alpha(\varepsilon)\| \leq C \varepsilon^{-d+3}.$$

Continuing, we have for  $|\alpha| = 0$

$$(3.22) \quad \left\| \frac{d}{d\varepsilon} F_z^\alpha(\varepsilon) \right\| \leq C (\|F_z^\alpha(\varepsilon)\|^{1-(1/s)} \varepsilon^{(-d+(1/2))/s} + |\log \varepsilon| + 1).$$

Thus we have the following estimate, by Lemma 3.5,

$$(3.23) \quad \sup_{z \in I_+, 0 < \varepsilon < 1} \|F_z(\varepsilon)\| \leq K < \infty,$$

where  $F_z(\varepsilon) := F_z^\alpha(\varepsilon)$  for  $|\alpha| = 0$ .

Since  $\lim_{\varepsilon \downarrow 0} \|Q_d(\varepsilon)R(z)\| = 0$  for each  $z \in \mathbf{C} \setminus \mathbf{R}$ ,  $1 + Q_d(\varepsilon)R(z)$  has a bounded inverse, and so

$$G_z(\varepsilon) = R(z)(1 + Q_d(\varepsilon)R(z))^{-1}$$

holds for each  $z \in \mathbf{C} \setminus \mathbf{R}$  when  $\varepsilon > 0$  is small. Therefore we get

$$\lim_{\varepsilon \downarrow 0} G_z(\varepsilon) = R(z)$$

for each  $z \in \mathbf{C} \setminus \mathbf{R}$ , and so we have by (3.23)

$$\sup_{z \in I_+} \|D(z)\| \leq K.$$

(ii) For simplicity we write  $n = (d - 1/2)/s$ . By (3.22), (3.23) we obtain

$$\left\| \frac{d}{d\varepsilon} F_z(\varepsilon) \right\| \leq C(\varepsilon^{-n} + 1).$$

Integrating this we have

$$(3.24) \quad \|F_z(\varepsilon) - F_z(0)\| \leq C \cdot \varepsilon^{1-n}.$$

On the other hand  $G_z(\varepsilon)$  is differentiable in  $z \in I_+$  for each  $\varepsilon > 0$  by Lemma 3.4. We have the following estimate by Lemma 3.4 (ii):

$$\begin{aligned} \left\| \frac{d}{dz} F_z(\varepsilon) \right\| &\leq \|\langle A \rangle^{-s} G_z(\varepsilon)^2 W_1 \dots G_z(\varepsilon) \langle A \rangle^{-s}\| \\ &\quad + \dots + \|\langle A \rangle^{-s} G_z(\varepsilon) W_1 \dots W_{d-1} G_z(\varepsilon)^2 \langle A \rangle^{-s}\| \\ &\leq C \cdot \varepsilon^{-d}, \end{aligned}$$

which implies

$$(3.25) \quad \|F_z(\varepsilon) - F_{z'}(\varepsilon)\| \leq C \cdot \varepsilon^{-d} |z - z'|$$

for  $z, z' \in I_+$ ,  $\varepsilon > 0$ . Let  $\varepsilon = |z - z'|^{\delta_2}$ ,  $\delta_2 = (1 - n)^{-1} \delta_1$  (see (3.2) for  $\delta_1$ ). Then by (3.24), (3.25) we have

$$\begin{aligned} \|F_z(0) - F_{z'}(0)\| &\leq \|F_z(0) - F_z(\varepsilon)\| + \|F_z(\varepsilon) - F_{z'}(\varepsilon)\| + \|F_{z'}(\varepsilon) - F_{z'}(0)\| \\ &\leq C \cdot |z - z'|^{\delta_1}. \end{aligned}$$

Thus we have proved (ii). (iii) follows from (ii).

The proof of the last half can be obtained if one takes into consideration the last part of Lemma 3.4 and the proof carried out above.

**§4. The quantity  $(\Omega_\alpha^-(\xi, \eta), \Omega_\beta^+(\xi, \eta))$** 

We fix  $\xi \in \mathbf{R}^N \setminus \{0\}$  and  $\eta \in \Pi_\xi$  and assume (V) throughout this section. We define an operator in  $L^2(\mathbf{R}^{N+1})$  ( $\mathbf{R}^{N+1} = \mathbf{R}_x^N \times \mathbf{R}_t$ ):

$$(4.1) \quad K_{\xi, \eta} := -\frac{1}{2m} \Delta_x - i\partial_t + V_{23}(x) + V_{13}(x - \xi t - \eta) + V_{12}(-\xi t - \eta).$$

Since  $-\frac{1}{2m} \Delta_x - i\partial_t$  is a self-adjoint operator with domain  $D_0 := \{u \in L^2(\mathbf{R}^{N+1}); -\frac{1}{2m} \Delta_x u - i\partial_t u \in L^2(\mathbf{R}^{N+1})\}$  and has a core  $\mathcal{S}(\mathbf{R}^{N+1})$ ,  $K_{\xi, \eta}$  is a self-adjoint operator with domain  $D_0$  and with core  $\mathcal{S}(\mathbf{R}^{N+1})$  by (V).

**Lemma 4.1.** *Let  $J$  be a compact interval and  $s > 1/2$ . Then the norm limits*

$$(4.2) \quad (K_{\xi, \eta} - \lambda \pm i0)^{-1} := \lim_{\varepsilon \downarrow 0} (K_{\xi, \eta} - \lambda \pm i\varepsilon)^{-1}$$

exist in  $\mathbf{B}(L^2(\mathbf{R}^N) \otimes L_s^2(\mathbf{R}), L^2(\mathbf{R}^N) \otimes L_{-s}^2(\mathbf{R}))$  uniformly for  $\lambda \in J$ .

*Proof.* We shall apply Theorem 3.3. We set  $\mathbf{H} = L^2(\mathbf{R}^{N+1})$ ,  $d = 1$ ,  $H = K_{\xi, \eta}$ ,  $A = t \times$ , and  $I = J$ . Then assumption 3.1, (H-i) is satisfied because  $\mathcal{S}(\mathbf{R}^{N+1})$  is a common core for  $H$  and  $A$ . (H-ii) is obvious. Since  $i[H, A] = 1$ , (H-iii)  $\sim$  (H-v) follow. The conclusion follows from Theorem 3.3.

It is easy to verify that

$$(4.3) \quad U_1(t, s) := e^{-i \int_s^t V_{12}(-\xi \tau - \eta) d\tau} U(\xi, \eta; t, s), \quad (t, s \in \mathbf{R})$$

is the unique propagator of the following equation:

$$(4.4) \quad i\partial_t \psi(t) = [h_{\xi, \eta}(t) + V_{12}(-\xi t - \eta)] \psi(t),$$

where  $U(\xi, \eta; t, s)$  and  $h_{\xi, \eta}(t)$  have been defined in (1.19), (1.18), respectively. We identify  $L^2(\mathbf{R}^{N+1})$  with  $L^2(\mathbf{R}; L^2(\mathbf{R}_x^N))$  and introduce a family of unitary operators  $\tilde{U}(\tau)$  ( $\tau \in \mathbf{R}$ ) on  $L^2(\mathbf{R}^{N+1})$ :

$$(4.5) \quad (\tilde{U}(\tau)f)(x, t) = (U_1(t, t - \tau)f(*, t - \tau))(x)$$

for  $f \in L^2(\mathbf{R}^{N+1})$ .

This family is a strongly continuous unitary group on  $L^2(\mathbf{R}^{N+1})$ , and  $\tilde{U}(\tau)f$  is strongly differentiable in  $\tau$  for each  $f \in \mathcal{S}(\mathbf{R}^{N+1})$  and

$$\frac{d}{d\tau} \tilde{U}(\tau)f|_{\tau=0} = -iK_{\xi, \eta}f.$$

Thus we have  $\tilde{U}(\tau) = e^{-itK_{\xi, \eta}}$ , and so

$$(4.6) \quad (K_{\xi, \eta} - i\varepsilon)^{-1} = i \int_0^\infty e^{-\varepsilon\tau} \tilde{U}(\tau) d\tau \quad \text{for } \varepsilon > 0$$

(cf. [Ho] and [Y] for the above discussion).

For  $a \in \mathbf{A}$  we define  $W_a(x, t) = W_a(\xi, \eta; x, t)$  by

$$(4.7) \quad \begin{aligned} W_{a_1}(x, t) &:= V_{13}(x - \xi t - \eta) + V_{12}(-\xi t - \eta), \\ W_{a_2}(x, t) &:= V_{23}(x) + V_{12}(-\xi t - \eta). \end{aligned}$$

**Lemma 4.2.** *Let  $\alpha, \beta \in \mathbf{CH}$  with  $a = D(\alpha)$ ,  $b = D(\beta)$ , and assume (V). Then*

$$(4.8) \quad \begin{aligned} e^{-i \int_{-\infty}^{\infty} V_{12}(-\xi\tau - \eta) d\tau} (\Omega_{\alpha}^{-}(\xi, \eta), \Omega_{\beta}^{+}(\xi, \eta))_{L^2(\mathbf{R}^N)} - \delta_{\alpha\beta} \\ = -i(W_a \psi_{\alpha}^{\infty}, \psi_{\beta}^{\infty})_{L^2(\mathbf{R}^{N+1})} + i((K_{\xi, \eta} - i0)^{-1} W_a \psi_{\alpha}^{\infty}, W_b \psi_{\beta}^{\infty})_{L^2(\mathbf{R}^{N+1})}, \end{aligned}$$

where  $\psi_{\gamma}^{\infty} = \psi_{\gamma}^{\infty}(x, t)$  ( $\gamma = \alpha, \beta$ ; see (1.20)) and  $W_c = W_c(x, t)$  ( $c \in \mathbf{A}$ ).

**Remark.** By (V) and Lemma 2.1, it is easy to see that  $W_{D(\gamma)} \psi_{\gamma}^{\infty} \in L_s^2(\mathbf{R}^{N+1})$  for some  $s > 1/2$ . Therefore the second term in the R.H.S. of (4.8) is well-defined by Lemma 4.1.

*Proof.* By  $V_{12}(-\xi t - \eta) \in L^1(R_t)$  and (4.3), we have

$$(4.9) \quad \begin{aligned} \Omega_{1, \gamma}^{\pm} &:= e^{-i \int_{\pm\infty}^0 V_{12}(-\xi\tau - \eta) d\tau} \Omega_{\gamma}^{\pm}(\xi, \eta) \quad (\text{see (1.22)}) \\ &= s - \lim_{t \rightarrow \pm\infty} U_1(0, t) \psi_{\gamma}^{\infty}(*, t) \quad \text{in } L^2(\mathbf{R}^N) \end{aligned}$$

and  $(\Omega_{1, \alpha}^+, \Omega_{1, \beta}^+) = \delta_{\beta\alpha}$  (see (1.23)). Thus the L.H.S. of (4.8) equals  $(\Omega_{1, \alpha}^- - \Omega_{1, \alpha}^+, \Omega_{1, \beta}^+)$ . Since for  $\gamma \in \mathbf{CH}$ ,  $\psi_{\gamma}^{\infty}(*, t) \in H^2(\mathbf{R}^N)$  for each  $t \in \mathbf{R}$ ,  $U_1(0, t) \psi_{\gamma}^{\infty}(*, t)$  is continuously differentiable with respect to  $t$  in  $L^2(\mathbf{R}^N)$  and satisfies

$$(4.10) \quad \partial_t U_1(0, t) \psi_{\gamma}^{\infty}(*, t) = i U_1(0, t) W_{D(\gamma)} \psi_{\gamma}^{\infty}(*, t).$$

Therefore we obtain

$$(4.11) \quad \begin{aligned} (\Omega_{1, \alpha}^- - \Omega_{1, \alpha}^+, \Omega_{1, \beta}^+) &= - \lim_{\tau \rightarrow \infty} (U_1(0, \tau) \psi_{\alpha}^{\infty}(*, \tau) - U_1(0, -\tau) \psi_{\alpha}^{\infty}(*, -\tau), \Omega_{1, \beta}^+) \\ &= - \lim_{\tau \rightarrow \infty} \int_{-\tau}^{\tau} \frac{d}{dt} (U_1(0, t) \psi_{\alpha}^{\infty}(*, t), \Omega_{1, \beta}^+) dt \\ &= -i \int_{-\infty}^{\infty} (U_1(0, t) W_a(*, t) \psi_{\alpha}^{\infty}(*, t), \Omega_{1, \beta}^+) dt. \end{aligned}$$

Here we note that the integral converges absolutely because the following estimates follow from (V) and Lemma 2.1:

$$(4.12) \quad \|W_{D(\gamma)}(*, t) \psi_{\gamma}^{\infty}(*, t)\|_{L^2(\mathbf{R}^N)} \leq \text{const} \cdot (1 + |t|)^{-\epsilon_0}, \quad \gamma \in \mathbf{CH}.$$

By using (4.10) for  $\gamma = \alpha$ , we have in the same way as (4.11)

$$(4.13) \quad \begin{aligned} (U_1(0, t) W_a(*, t) \psi_{\alpha}^{\infty}(*, t), \Omega_{1, \beta}^+) &= (U_1(0, t) W_a(*, t) \psi_{\alpha}^{\infty}(*, t), U_1(0, t) \psi_{\beta}^{\infty}(*, t)) \\ &\quad + \lim_{\tau \rightarrow \infty} (U_1(0, t) W_a(*, t) \psi_{\alpha}^{\infty}(*, t), [U_1(0, \tau) \psi_{\beta}^{\infty}(*, \tau) - U_1(0, t) \psi_{\beta}^{\infty}(*, t)]) \\ &= (W_a(*, t) \psi_{\alpha}^{\infty}(*, t), \psi_{\beta}^{\infty}(*, t)) \end{aligned}$$



$$\begin{aligned}
 & -i \int_t^\infty (U_1(0, t)W_a(*, t)\psi_\alpha^\infty(*, t), U_1(0, s)W_b(*, s)\psi_\beta^\infty(*, s))ds \\
 &= (W_a(*, t)\psi_\alpha^\infty(*, t), \psi_\beta^\infty(*, t)) \\
 & -i \int_t^\infty (U_1(s, t)W_a(*, t)\psi_\alpha^\infty(*, t), W_b(*, s)\psi_\beta^\infty(*, s))ds \quad (\text{by (U-ii)}),
 \end{aligned}$$

where the integral converges absolutely by (4.12) for  $\gamma = \beta$ . Thus by (4.11) and (4.13) we obtain

$$\begin{aligned}
 (\Omega_{1,\alpha}^- - \Omega_{1,\alpha}^+, \Omega_{1,\beta}^+) &= -i(W_a\psi_\alpha^\infty, \psi_\beta^\infty)_{L^2(\mathbb{R}^{N+1})} \\
 & - \int_{-\infty}^\infty dt \int_t^\infty (U_1(s, t)W_a(*, t)\psi_\alpha^\infty(*, t), W_b(*, s)\psi_\beta^\infty(*, s))ds.
 \end{aligned}$$

The double integral absolutely converges, since the inner integral is  $O(|t|^{-2\epsilon_0+1})$  by (4.12), and this is calculated as

$$\begin{aligned}
 & \int_{-\infty}^\infty ds \int_{-\infty}^s (U_1(s, t)W_a(*, t)\psi_\alpha^\infty(*, t), W_b(*, s)\psi_\beta^\infty(*, s))dt \\
 &= \int_{-\infty}^\infty ds \int_0^\infty (U_1(s, s-t)W_a(*, s-t)\psi_\alpha^\infty(*, s-t), W_b(*, s)\psi_\beta^\infty(*, s))dt \\
 &= \int_0^\infty (\tilde{U}(t)W_a\psi_\alpha^\infty, W_b\psi_\beta^\infty)_{L^2(\mathbb{R}^{N-1})} dt \quad (\text{see (4.5)}) \\
 &= \lim_{\varepsilon \downarrow 0} \int_0^\infty e^{-\varepsilon t} (\tilde{U}(t)W_a\psi_\alpha^\infty, W_b\psi_\beta^\infty)_{L^2(\mathbb{R}^{N+1})} \\
 &= -i((K_{\xi,\eta} - i0)^{-1}W_a\psi_\alpha^\infty, W_b\psi_\beta^\infty)_{L^2(\mathbb{R}^{N+1})},
 \end{aligned}$$

where we have used (4.6) and Lemma 4.1 in the last step. This completes the proof.

## §5. Proofs of the main theorems

In this section we will prove Theorems 1.1, and 1.3. To do so we prepare some lemmas and propositions. Throughout this section, we assume (V) and  $\mu \gg 1$ , and fix  $v_0 > 0$  and a 2-body initial channel  $\alpha \in \mathbf{CH}$  with  $D(\alpha) = a$  and a 2-body final channel  $\beta \in \mathbf{CH}$  with  $D(\beta) = b$ ,

**5.1.** The purpose of this subsection is to rewrite  $T_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega, \omega')$  in a form convenient for later purposes. We first note that when  $\mu \rightarrow \infty$ ,

$$(5.1) \quad m_a, m_b \rightarrow m; \quad n_a, n_b \rightarrow \infty; \quad n_a/n_b \rightarrow 1; \quad \lambda_\gamma^\mu \rightarrow \lambda_\gamma^\infty$$

for  $\gamma = \alpha, \beta$ . Thus for any  $v > 0$ , there exists a unique  $v' = v'(v, \mu) > 0$  such that

$$(5.2) \quad \frac{1}{2}n_a v^2 + \lambda_\alpha^\mu = \frac{1}{2}n_b v'^2 + \lambda_\beta^\mu,$$

since  $\mu \gg 1$ . Clearly  $v' \rightarrow v_0$  as  $\mu \rightarrow \infty$ ,  $v \rightarrow v_0$ . Throughout this section we assume  $|v - v_0| \ll 1$  in addition to  $\mu \gg 1$ . Therefore, using the coordinates  $(x_a, y_a)$ , we have for  $\lambda = (1/2)n_a v^2 + \lambda_\alpha^\mu$

(5.3)

$$T_{\beta\alpha}^\mu(\lambda; \omega, \omega') = C_1(v, \mu) \int e^{in_a v \omega' \cdot y_a} dy_a \int \phi_\alpha^\mu(x_a) \times \overline{([-I_a + I_a(H^\mu - \lambda + i0)^{-1}I_b] \phi_\beta^\mu(x_b(\cdot, \cdot)) e^{in_b v' \omega \cdot y_b(\cdot, \cdot)})(x_a, y_a)} dx_a$$

(see (2.12) for  $T_{\beta\alpha}^\mu(\lambda; \omega, \omega')$  and (1.6) for  $x_b = x_b(x_a, y_a)$ ,  $y_b = y_b(x_a, y_a)$ ), where

(5.4)

$$C_1(v, \mu) = i(2\pi)^{-N+1} (vv')^{(N-2)/2} (n_a n_b)^{(N-1)/2}.$$

For each  $\omega \in S^{N-1}$ ,  $v$  and  $\mu$  we define a self-adjoint operator  $H^\mu(\omega, v)$  in  $L^2(\mathbf{R}^{2N})$  by

(5.5)

$$H^\mu(\omega, v) := e^{-in_a v \omega \cdot y_a} \left( H^\mu - \frac{1}{2} n_a v^2 \right) e^{in_a v \omega \cdot y_a} \\ = H^\mu - iv\omega \cdot \nabla_{y_a}$$

with domain  $D(H^\mu(\omega, v)) = H^2(\mathbf{R}^{2N})$ , and denote the resolvent of  $H^\mu(\omega, v)$  by

(5.6)

$$R^\mu(\omega, v; z) = (H^\mu(\omega, v) - z)^{-1}.$$

By Lemma 2.2 and (5.5) we have the norm limits

$$R^\mu(\omega, v; \lambda_\alpha^\mu \pm i0) := \lim_{\varepsilon \downarrow 0} R^\mu(\omega, v; \lambda_\alpha^\mu \pm i\varepsilon)$$

in  $\mathbf{B}(L_s^2(\mathbf{R}^{2N}), L_{-s}^2(\mathbf{R}^{2N}))$ ,  $s > 1/2$ , and

$$R^\mu(\omega, v; \lambda_\alpha^\mu \pm i0) = e^{-in_a v \omega \cdot y_a} (H^\mu - (1/2)n_a v^2 + \lambda_\alpha^\mu \mp i0)^{-1} e^{in_a v \omega \cdot y_a}.$$

Thus  $T_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega, \omega')$  can be written as follows:

(5.7)

$$T_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega, \omega') = C_1(v, \mu) \int e^{in_a v (\omega' - \omega) \cdot y_a} dy_a \int \phi_\alpha^\mu(x_a) \times \overline{([-I_a + I_a R^\mu(\omega, v; \lambda_\alpha^\mu - i0)I_b] \phi_\beta^\mu(x_b(\cdot, \cdot)) \chi^\mu(\omega, v))}(x_a, y_a) dx_a,$$

where

(5.8)

$$\chi^\mu(\omega, v) = \chi^\mu(\omega, v; x_a, y_a) := e^{in_b v' \omega \cdot y_b - in_a v \omega \cdot y_a}, \\ y_b = y_b(x_a, y_a).$$

We define

(5.9)

$$E^\mu(\omega, v; x_a, y_a) := (2\pi)^{N/2} (n_a v)^{1-N} C_1(v, \mu) \phi_\alpha^\mu(x_a) \times \overline{([-I_a + I_a R^\mu(\omega, v; \lambda_\alpha^\mu - i0)I_b] \phi_\beta^\mu(x_b(\cdot, \cdot)) \chi^\mu(\omega, v))}(x_a, y_a), \\ G^\mu(\omega, v; y_a) := \int E^\mu(\omega, v; x_a, y_a) dx_a,$$

and we denote by  $\check{v}$  the inverse Fourier transform i.e.

$$(5.10) \quad \check{g}(\xi) = (2\pi)^{-N/2} \int e^{i\xi \cdot y_a} g(y_a) \mathrm{d}y_a .$$

Then we obtain

$$(5.11) \quad T_{\beta a}^\mu((1/2)n_a v^2 + \lambda_a^\mu; \omega, \omega') = (n_a v)^{N-1} \check{G}^\mu(\omega, v; n_a v(\omega' - \omega)) .$$

for each  $\omega, \omega' \in S^{N-1}$ ,  $v > 0$  sufficiently near  $v_0$  and  $\mu \gg 1$ .

**5.2.** This subsection is devoted to showing the existence of the limit of  $G^\mu(\omega, v; y_a)$  as  $\mu \rightarrow \infty$  and  $v \rightarrow v_0$  in an appropriate topology. We write  $x = x_a$ ,  $y = y_a$  for simplicity.  $x_b, y_b$  are linear combinations of  $x, y$  with  $\mu$ -dependent coefficients (see (1.6)). There are four cases of pairs of the initial and final 2-cluster decompositions  $a$  and  $b$ : Case  $j$ - $k$  means the pair  $(a, b)$  with  $a = a_j$ ,  $b = a_k$  ( $1 \leq j, k \leq 2$ ). We want to express  $V, I_a, I_b$  and  $\phi_\beta^\mu$  as functions of  $x, y$  with parameter  $\mu$

(5.12)

$$\text{Case 1-1: } V = V^\mu(x, y) = V_{23}(x) + V_{13} \left( \frac{m_a}{m} x - y \right) + V_{12} \left( -\frac{m_a}{M_2} x - y \right),$$

$$I_a = I_b = I_a^\mu(x, y) = I_b^\mu(x, y) = V_{13} \left( \frac{m_a}{m} x - y \right) + V_{12} \left( -\frac{m_a}{M_2} x - y \right),$$

$$\phi_\beta^\mu(x_b) = \Phi_\beta^\mu(x, y) = \phi_\beta^\mu(x),$$

Case 1-2:  $V = V^\mu$  and  $I_a = I_a^\mu$  are the same as Case 1-1,

$$I_b = I_b^\mu(x, y) = V_{23}(x) + V_{12} \left( -\frac{m_a}{M_2} x - y \right),$$

$$\phi_\beta^\mu(x_b) = \Phi_\beta^\mu(x, y) = \phi_\beta^\mu \left( \frac{m_a}{m} x - y \right),$$

$$\text{Case 2-1: } V = V^\mu(x, y) = V_{23} \left( \frac{m_a}{m} x + y \right) + V_{13}(x) + V_{12} \left( \frac{m_a}{M_1} x - y \right),$$

$$I_a = I_a^\mu(x, y) = V_{23} \left( \frac{m_a}{m} x + y \right) + V_{12} \left( \frac{m_a}{M_1} x - y \right),$$

$$I_b = I_b^\mu(x, y) = V_{13}(x) + V_{12} \left( \frac{m_a}{M_1} x - y \right),$$

$$\phi_\beta^\mu(x_b) = \Phi_\beta^\mu(x, y) = \phi_\beta^\mu \left( \frac{m_a}{m} x + y \right),$$

Case 2-2:  $V = V^\mu$  and  $I_a = I_a^\mu$  are the same as Case 2-1,

$$I_b^\mu(x, y) = I_a^\mu(x, y),$$

$$\phi_\beta^\mu(x_b) = \Phi_\beta^\mu(x, y) = \phi_\beta^\mu(x) .$$

We see that when  $\mu \rightarrow \infty$ ,  $V^\mu$ ,  $I_a^\mu$ ,  $I_b^\mu$  and  $\Phi_\beta^\mu$  have limits  $V^\infty$ ,  $I_a^\infty$ ,  $I_b^\infty$  and  $\Phi_\beta^\infty$  pointwise on  $\mathbf{R}^{2N}$  (see (1.4)). These limits have the following forms:

(5.13)

Case 1-1:  $V^\infty(x, y) = V_{23}(x) + V_{13}(x - y) + V_{12}(-y)$ ,

$$I_a^\infty(x, y) = I_b^\infty(x, y) = V_{13}(x - y) + V_{12}(-y),$$

$$\Phi_\beta^\infty(x, y) = \phi_\beta^\infty(x),$$

Case 1-2:  $V^\infty$  and  $I_a^\infty$  are the same as Case 1-1,

$$I_b^\infty(x, y) = V_{23}(x) + V_{12}(-y),$$

$$\Phi_\beta^\infty(x, y) = \phi_\beta^\infty(x - y),$$

Case 2-1:  $V^\infty(x, y) = V_{23}(x + y) + V_{13}(x) + V_{12}(-y)$ ,

$$I_a^\infty(x, y) = V_{23}(x + y) + V_{12}(-y), \quad I_b^\infty(x, y) = V_{13}(x) + V_{12}(-y),$$

$$\Phi_\beta^\infty(x, y) = \phi_\beta^\infty(x + y)$$

Case 2-2:  $V^\infty$  and  $I_a^\infty$  are the same as Case 2-1,

$$I_b^\infty(x, y) = I_a^\infty(x, y),$$

$$\Phi_\beta^\infty(x, y) = \phi_\beta^\infty(x).$$

Now we investigate  $\chi^\mu(\omega, v)$  as  $\mu \rightarrow \infty$ ,  $v \rightarrow v_0$ .  $\chi^\mu(\omega, v)$  are given as follows:

$$(5.14) \quad \begin{aligned} \chi^\mu(\omega, v; x, y) &= e^{in_a(v'(v, \mu) - v)\omega \cdot y} \quad (\text{Case 1-1, 2-2}), \\ &= e^{im_a v'(v, \mu)\omega \cdot x + i((m_b n_b v'(v, \mu)/m) - n_a v)\omega \cdot y} \quad (\text{Case 1-2}), \\ &= e^{-im_a v'(v, \mu)\omega \cdot x + i((m_b n_b v'(v, \mu)/m) - n_a v)\omega \cdot y} \quad (\text{Case 2-1}), \end{aligned}$$

(see (5.2) for  $v' = v'(v, \mu)$ ).

**Lemma 5.1.** Define  $\chi(\omega) = \chi(\omega; x, y)$ ,  $\omega \in S^{N-1}$ ,  $b y$

$$(5.15) \quad \begin{aligned} \chi(\omega; x, y) &= e^{i(\lambda_\alpha^\infty - \lambda_\beta^\infty)v_0^{-1}\omega \cdot y} \quad (\text{Case 1-1, 2-2}), \\ &= e^{imv_0\omega \cdot x - i((m/2)v_0^2 + \lambda_\beta^\infty - \lambda_\alpha^\infty)v_0^{-1}\omega \cdot y} \quad (\text{Case 1-2}), \\ &= e^{-imv_0\omega \cdot x - i((m/2)v_0^2 + \lambda_\beta^\infty - \lambda_\alpha^\infty)v_0^{-1}\omega \cdot y} \quad (\text{Case 2-1}). \end{aligned}$$

Then we have for any  $\delta > 0$  and any multi-index  $\gamma$ ,

$$(5.16) \quad \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} \sup_{\omega \in S^{N-1}} \|\langle x; y \rangle^{-\delta} D^\gamma [\chi^\mu(\omega, v) - \chi(\omega)]\|_{L^\infty(\mathbf{R}^{2N})} = 0,$$

where  $D = (\partial_x, \partial_y)$  and  $\langle x; y \rangle := (1 + |x|^2 + |y|^2)^{1/2}$ .

*Proof.* We only prove this in Case 2-1, because the others can be proved similarly. We first note that  $m_a v'(v, \mu) \rightarrow mv_0$  as  $\mu \rightarrow \infty$ ,  $v \rightarrow v_0$ . By (5.2) we

have

$$\frac{m_b n_b v'}{m} - n_a v = \frac{\left(\frac{m_b^2 n_b}{m^2} - n_a\right) n_a v^2 + \frac{2m_b^2 n_b}{m^2} (\lambda_\alpha^\mu - \lambda_\beta^\mu)}{n_b \left[\frac{m_b v'}{m} + \frac{n_a v}{n_b}\right]}.$$

Since  $\frac{m_b^2 n_b}{m^2} - n_a = -\frac{M_2 m}{M_2 + m}$ , we obtain

$$\frac{m_b n_b v'}{m} - n_a v \rightarrow -\left(\frac{1}{2} m v_0^2 + \lambda_\beta^\infty - \lambda_\alpha^\infty\right) \frac{1}{v_0} \quad \text{as } \mu \rightarrow \infty, \quad v \rightarrow v_0.$$

Therefore, in view of (5.14), (5.15), it follows that

$$\lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} \sup_{\omega \in S^{N-1}} \|D^\gamma [\chi^\mu(\omega, v) - \chi(\omega)]\|_{L^\infty(K)} = 0$$

for any compact set  $K$  in  $\mathbf{R}^{2N}$  and multi-index  $\gamma$ . The lemma follows from this together with the estimates

$$|D^\gamma \chi^\mu(\omega, v; x, y)| + |D^\gamma \chi(\omega; x, y)| \leq C_\gamma \quad \text{on } \mathbf{R}^{2N},$$

where  $C_\gamma$  is independent of  $\mu \gg 1$ ,  $\omega \in S^{N-1}$  and  $v$  with  $|v - v_0| \ll 1$ .

We note that there exists a constant  $C > 0$  independent of  $\mu \gg 1$  such that

$$(5.17) \quad C^{-1} \langle x; y \rangle \leq \rho^\mu(x, y) \leq C \langle x; y \rangle \quad \text{on } \mathbf{R}^{2N}$$

for  $\rho^\mu(x, y) = \langle \frac{m_a}{m} x \pm y; x \rangle$ ,  $\langle \pm \frac{m_a}{M_j} x - y; x \rangle$  ( $j = 1, 2$ ),

$$\langle \frac{m_a}{m} x - y; -\frac{m_a}{M_2} x - y \rangle, \quad \langle \frac{m_a}{m} x + y; \frac{m_a}{M_1} x - y \rangle, \quad \langle x; x \pm y \rangle.$$

**Lemma 5.2.** *Let  $k \in \mathbf{N} \cup \{0\}$  and let  $s$  be a real with  $s < \varepsilon_0 - N$ , and assume (V). Then*

$$(5.18) \quad s - \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} I_b^\mu \Phi_\beta^\mu \chi^\mu(\omega, v) = I_b^\infty \Phi_\beta^\infty \chi(\omega) \quad \text{in } H_s^k(\mathbf{R}^{2N})$$

uniformly for  $\omega \in S^{N-1}$ .

*Proof.* We set  $g^\mu := I_b^\mu \Phi_\beta^\mu$ ,  $g^\infty := I_b^\infty \Phi_\beta^\infty$ . Then, from (V), Lemma 2.1 and (5.17), the estimate

$$(5.19) \quad |D^\gamma g^\mu(x, y)| + |D^\gamma g^\infty(x, y)| \leq C_\gamma \langle x; y \rangle^{-\varepsilon_0}$$

follows for any  $\gamma$ , where  $C_\gamma$  is independent of  $\mu \gg 1$ . (5.19) yields  $g^\mu \chi^\mu(\omega, v)$ ,  $g^\infty \chi(\omega) \in H_s^k(\mathbf{R}^{2N})$  because of  $s < \varepsilon_0 - N$ . Fix  $\delta > 0$  with  $\delta + s < \varepsilon_0 - N$ . Then

we get

$$(5.20) \quad \|g^\mu \chi^\mu(\omega, v) - g^\infty \chi(\omega)\|_{k,s} \leq \|g^\mu [\chi^\mu(\omega, v) - \chi(\omega)]\|_{k,s} + \|[g^\mu - g^\infty] \chi(\omega)\|_{k,s} \\ \leq C \left( \sum_{|\gamma| \leq k} \|\langle x; y \rangle^{-\delta} D^\gamma [\chi^\mu(\omega, v) - \chi(\omega)]\|_{L^\infty} \|g^\mu\|_{k,s+\delta} \right. \\ \left. + \sum_{|\gamma| \leq k} \|\langle x; y \rangle^{-\delta} D^\gamma \chi(\omega)\|_{L^\infty} \|g^\mu - g^\infty\|_{k,s+\delta} \right),$$

where  $C$  is independent of  $\mu \gg 1$ ,  $\omega \in S^{N-1}$  and  $v$  near  $v_0$ .  $\|g^\mu\|_{k,s+\delta}$  are uniformly bounded for  $\mu \gg 1$  by (5.19). So by Lemma 5.1 the first term in the R.H.S. of (5.20) goes to zero uniformly for  $\omega \in S^{N-1}$  as  $\mu \rightarrow \infty$ ,  $v \rightarrow v_0$ . On the other hand, by (V) and Lemma 2.1 we have

$$D^\gamma g^\mu \rightarrow D^\gamma g^\infty \quad \text{pointwise on } \mathbf{R}^{2N}$$

as  $\mu \rightarrow \infty$  for any  $\gamma$ . Thus, using (5.19) and the Lebesgue dominated convergence theorem, we get  $\lim_{\mu \rightarrow \infty} \|g^\mu - g^\infty\|_{k,s+\delta} = 0$ . Therefore the second term in the R.H.S. of (5.20) goes to zero uniformly for  $\omega \in S^{N-1}$  as  $\mu \rightarrow \infty$ .

**Lemma 5.3.** *Let  $k \in \mathbf{N} \cup \{0\}$  and  $s \geq 0$ . Then the map  $T$  defined by*

$$Tf(y) := \int \langle x \rangle^{-N} f(x, y) dx \quad \text{for } f \in H_s^k(\mathbf{R}^{2N})$$

is a bounded operator from  $H_s^k(\mathbf{R}^{2N})$  to  $H_s^k(\mathbf{R}^N)$ . Furthermore

$$D_y^\gamma Tf = TD_y^\gamma f$$

for any  $\gamma$  with  $|\gamma| \leq k$  and any  $f \in H_s^k(\mathbf{R}^{2N})$ .

*Proof.* Apply the Schwarz inequality.

For each  $\omega \in S^{N-1}$ ,  $-(2m)^{-1} \Delta_x - iv_0 \omega \cdot \nabla_y$  is a self-adjoint operator in  $L^2(\mathbf{R}^{2N})$  with domain

$$D_\omega = \{f \in L^2(\mathbf{R}^{2N}); -(2m)^{-1} \Delta_x f - iv_0 \omega \cdot \nabla_y f \in L^2(\mathbf{R}^{2N})\},$$

and  $\mathcal{S}(\mathbf{R}^{2N})$  is a core of this self-adjoint operator. Therefore

$$(5.21) \quad H(\omega) := -(2m)^{-1} \Delta_x - iv_0 \omega \cdot \nabla_y + V^\infty$$

is a self-adjoint operator with domain  $D_\omega$ , and  $\mathcal{S}(\mathbf{R}^{2N})$  is a core of  $H(\omega)$  since  $V^\infty$  is a bounded real-valued function. We denote the resolvent of  $H(\omega)$  by

$$(5.22) \quad R(\omega; z) = (H(\omega) - z)^{-1}.$$

The next lemma will be proved in Sect. 6 by using the abstract resolvent estimates of Sect. 3.

**Lemma 5.4.** *Assume (V). Let  $J$  be any compact interval in  $\mathbf{R}$ , fix  $k \in \mathbf{N} \cup \{0\}$  and  $s \in \mathbf{R}$  with  $0 \leq k \leq \varepsilon_0 - 2$  and  $k + 1/2 < s$ , and set  $J_\pm := \{z \in \mathbf{C}; \operatorname{Re} z \in J, 0 < \pm \operatorname{Im} z < 1\}$  and  $\mathbf{B}_{k,s} := \mathbf{B}(H_s^k(\mathbf{R}^{2N}), H_{-s}^k(\mathbf{R}^{2N}))$ . Then:*

(i) There exist a large  $M > 0$  and a small  $\delta_0 > 0$  such that

$$\sup_{\substack{M_1, M_2 > M, |v-v_0| < \delta_0 \\ \omega \in S^{N-1}, z \in J_{\pm}}} \{ \|R^\mu(\omega, v; z)\|_{\mathbf{B}_{k,s}} + \|R(\omega; z)\|_{\mathbf{B}_{k,s}} \} < \infty,$$

where  $\mu = (M_1, M_2)$ .

(ii) The norm limits

$$R^\mu(\omega, v; \lambda \pm i0) := \lim_{\varepsilon \downarrow 0} R^\mu(\omega, v; \lambda \pm i\varepsilon), \quad R(\omega; \lambda \pm i0) := \lim_{\varepsilon \downarrow 0} R(\omega; \lambda \pm i\varepsilon)$$

exist in  $\mathbf{B}_{k,s}$  uniformly for  $\lambda \in J$ ,  $\omega \in S^{N-1}$ ,  $\mu = (M_1, M_2)$  and  $v$  with  $M_1, M_2 > M$ ,  $|v - v_0| < \delta_0$ .

(iii) For each  $f \in H_s^k(\mathbf{R}^{2N})$  and  $\lambda \in J$ ,

$$s - \lim_{\substack{\mu \rightarrow \infty, v \rightarrow v_0 \\ J \ni \lambda' \rightarrow \lambda}} R^\mu(\omega, v; \lambda' \pm i0)f = R(\omega; \lambda \pm i0)f$$

in  $H_{-s}^k(\mathbf{R}^{2N})$  uniformly for  $\omega \in S^{N-1}$ .

(iv) Let  $f \in H_s^k(\mathbf{R}^{2N})$  and  $\lambda \in J$ . Then  $R(\omega; \lambda \pm i0)f$  is an  $H_{-s}^k(\mathbf{R}^{2N})$ -valued strongly continuous function of  $\omega \in S^{N-1}$ .

We define

(5.23)

$$E(\omega; x, y) := i(2\pi)^{(2-N)/2} v_0^{-1} \phi_\alpha^\infty(x) \times \overline{[-I_a^\infty + I_a^\infty R(\omega; \lambda_\alpha^\infty - i0)I_b^\infty] \Phi_\beta^\infty \chi(\omega)}(x, y).$$

$$G(\omega; y) := \int E(\omega; x, y) dx.$$

**Lemma 5.5.** (i) Assume (V). Then

$$(5.24) \quad \sup_{\substack{\mu \gg 1, |v-v_0| \ll 1 \\ \omega \in S^{N-1}}} \{ \|G^\mu(\omega, v)\|_{1,1} + \|G(\omega)\|_{1,1} \} < \infty,$$

$$(5.25) \quad \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} \sup_{\omega \in S^{N-1}} \|G^\mu(\omega, v) - G(\omega)\|_{1,1} = 0,$$

where  $\| \cdot \|_{1,1}$  is the  $H_1^1(\mathbf{R}^N)$ -norm and  $G^\mu(\omega, v) = G^\mu(\omega, v; y)$ , etc. (see (5.9)), and

$$\sup_{\substack{\mu \gg 1, |v-v_0| \ll 1 \\ \omega \in S^{N-1}}} \{ \cdots \} := \sup_{\substack{M_1, M_2 > M, |v-v_0| < \delta \\ \omega \in S^{N-1}}} \{ \cdots \}$$

for some  $M > 0$  and  $\delta > 0$ .

(ii) If we replace (V) by (VY) and  $\| \cdot \|_{1,1}$  by  $\| \cdot \|_{[(N-1)/2+1, (N+1)/2]}$ , then (i) still holds ( $[ \ ]$  is Gauss' symbol).

*Proof.* (i) Here we denote by  $\| \cdot \|_{k,s}$  the  $H_s^k(\mathbf{R}^{2N})$ -norm. By (5.9), (5.23) and Lemma 5.3 we can see that it suffices to prove

$$(5.26) \quad \sup_{\substack{\mu \gg 1, |v-v_0| \ll 1 \\ \omega \in S^{N-1}}} \{ \|\langle x \rangle^N E^\mu(\omega, v)\|_{1,1} + \|\langle x \rangle^N E(\omega)\|_{1,1} \} < \infty,$$

$$(5.27) \quad \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} \sup_{\omega \in S^{N-1}} \|\langle x \rangle^N [E^\mu(\omega, v) - E(\omega)]\|_{1,1} = 0.$$

We first note that

$$(5.28) \quad (vv')^{(N-2)/2} (n_a n_b)^{(N-1)/2} (vn_a)^{1-N} \rightarrow v_0^{-1} \quad \text{as } \mu \rightarrow \infty, \quad v \rightarrow v_0.$$

By Lemmas 2.1 and 5.2, we have

$$(5.29) \quad \sup_{\substack{\mu \gg 1, |v-v_0| \ll 1 \\ \omega \in S^{N-1}}} \{ \|\langle x \rangle^N \phi_\alpha^\mu I_a^\mu \Phi_\beta^\mu \chi^\mu(\omega, v)\|_{1,1} + \|\langle x \rangle^N \phi_\alpha^\infty I_a^\infty \Phi_\beta^\infty \chi(\omega)\|_{1,1} \} < \infty,$$

$$\lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} \sup_{\omega \in S^{N-1}} \|\langle x \rangle^N [\phi_\alpha^\mu I_a^\mu \Phi_\beta^\mu \chi^\mu(\omega, v) - \phi_\alpha^\infty I_a^\infty \Phi_\beta^\infty \chi(\omega)]\|_{1,1} = 0.$$

Set

$$(5.30) \quad f^\mu(\omega, v) := R^\mu(\omega, v; \lambda_\alpha^\mu - i0) I_b^\mu \Phi_\beta^\mu \chi^\mu(\omega, v),$$

$$(5.31) \quad f(\omega) := R(\omega; \lambda_\alpha^\infty - i0) I_b^\infty \Phi_\beta^\infty \chi(\omega).$$

Then (5.26) and (5.27) can be reduced to the following estimates:

$$(5.32) \quad \sup_{\substack{\mu \gg 1, |v-v_0| \ll 1 \\ \omega \in S^{N-1}}} \{ \|\langle x \rangle^N \phi_\alpha^\mu I_a^\mu \overline{f^\mu(\omega, v)}\|_{1,1} + \|\langle x \rangle^N \phi_\alpha^\infty I_a^\infty \overline{f(\omega)}\|_{1,1} \} < \infty,$$

$$(5.33) \quad \lim_{\substack{\mu \rightarrow \infty \\ v \rightarrow v_0}} \sup_{\omega \in S^{N-1}} \|\langle x \rangle^N [\phi_\alpha^\mu I_a^\mu \overline{f^\mu(\omega, v)} - \phi_\alpha^\infty I_a^\infty \overline{f(\omega)}]\|_{1,1} = 0.$$

Fix  $s$  with  $3/2 < s < \varepsilon_0 - N$ . Then, by Lemmas 5.2 and 5.4, we have

$$(5.34) \quad \sup_{\substack{\mu \gg 1, |v-v_0| \ll 1 \\ \omega \in S^{N-1}}} \{ \|f^\mu(\omega, v)\|_{1,-s} + \|f(\omega)\|_{1,-s} \} < \infty.$$

By (5.34) and the following estimates

$$(5.35) \quad \|\langle x; y \rangle^{1+s} D^\gamma (\langle x \rangle^N \phi_\alpha^\infty I_a^\infty)\|_{L^\infty} < \infty, \quad \sup_{\mu \gg 1} \|\langle x; y \rangle^{1+s} D^\gamma (\langle x \rangle^N \phi_\alpha^\mu I_a^\mu)\|_{L^\infty} < \infty$$

for  $|\gamma| \leq 1$ , we obtain (5.32).

Let  $\theta, \omega \in S^{N-1}$ . We have

$$(5.36) \quad \|f^\mu(\omega, v) - f(\omega)\|_{1,-s} \leq \|R^\mu(\omega, v; \lambda_\alpha^\mu - i0)\|_{\mathbf{B}_{1,s}} \|g^\mu \chi^\mu(\omega, v) - g^\infty \chi(\omega)\|_{1,s}$$

$$+ \|[R^\mu(\omega, v; \lambda_\alpha^\mu - i0) - R(\omega; \lambda_\alpha^\infty - i0)]g^\infty \chi(\theta)\|_{1,-s}$$

$$+ \|R^\mu(\omega, v; \lambda_\alpha^\mu - i0) - R(\omega; \lambda_\alpha^\infty - i0)\|_{\mathbf{B}_{1,s}} \|g^\infty (\chi(\omega) - \chi(\theta))\|_{1,s},$$

where  $\mathbf{B}_{k,s}$  is as in Lemma 5.4, and  $g^\mu, g^\infty$  are the same as in the proof of Lemma 5.2. By Lemmas 5.2 and 5.4 (iii), the first two terms go to zero uniformly for  $\omega \in S^{N-1}$  as  $\mu \rightarrow \infty, v \rightarrow v_0$  for fixed  $\theta$ . The operator norm in the third term is uniformly bounded for  $\omega \in S^{N-1}, \mu \gg 1$  and  $v$  with  $|v - v_0| \ll 1$  (Lemma 5.4 (i)) and  $g^\infty \chi(\omega)$  is an  $H_s^1$ -valued strongly continuous function of  $\omega$ . Thus, we see that for any  $\varepsilon > 0$  and  $\theta \in S^{N-1}$  there exist  $M > 0, \delta > 0$  and a neighborhood of  $\theta, U(\theta)$ , such that

$$\|f^\mu(\omega, v) - f(\omega)\|_{1,-s} < \varepsilon \quad (\mu = (M_1, M_2))$$



if  $\omega \in U(\theta)$ ,  $M_1, M_2 > M$ ,  $|v - v_0| < \delta$ . Therefore, by using the finite covering argument, we see that

$$(5.37) \quad \lim_{\mu \rightarrow \infty, v \rightarrow v_0} \sup_{\omega \in S^{N-1}} \|f^\mu(\omega, v) - f(\omega)\|_{1, -s} = 0.$$

To prove (5.33) we write

$$(5.38) \quad \begin{aligned} \langle x \rangle^N [\phi_\alpha^\mu I_\alpha^\mu \overline{f^\mu(\omega, v)} - \phi_\alpha^\infty I_\alpha^\infty \overline{f(\omega)}] &= \langle x \rangle^N \phi_\alpha^\mu I_\alpha^\mu [\overline{f^\mu(\omega, v)} - \overline{f(\omega)}] \\ &\quad + \langle x \rangle^N [\phi_\alpha^\mu I_\alpha^\mu - \phi_\alpha^\infty I_\alpha^\infty] \overline{f(\omega)} \end{aligned}$$

By (5.35) and (5.37), the first term tends to zero in  $H_1^1(\mathbf{R}^{2N})$  uniformly for  $\omega$  as  $\mu \rightarrow \infty$  and  $v \rightarrow v_0$ . We next prove that the second term of (5.38) tends to zero uniformly for  $\omega$  as  $\mu \rightarrow \infty$ . First we claim that  $f(\omega)$  is  $H_{-s}^1(\mathbf{R}^{2N})$ -valued strongly continuous in  $\omega$ . To see this we write for  $\theta, \omega \in S^{N-1}$

$$(5.39) \quad \begin{aligned} f(\omega) - f(\theta) &= (R(\omega; \lambda_\alpha^\infty - i0) - R(\theta; \lambda_\alpha^\infty - i0)) I_b^\infty \Phi_\beta^\infty \chi(\theta) \\ &\quad + R(\omega; \lambda_\alpha^\infty - i0) I_b^\infty \Phi_\beta^\infty (\chi(\omega) - \chi(\theta)). \end{aligned}$$

The first term goes to zero in  $H_{-s}^1(\mathbf{R}^{2N})$  as  $\omega \rightarrow \theta$  by Lemma 5.4 (iv). Similarly for the second term by Lemmas 5.1 and 5.4(i). This proves the continuity of  $f(\omega)$ . Now we will prove

$$(5.40) \quad \lim_{\mu \rightarrow \infty} \sup_{\omega \in S^{N-1}} \|F^\mu f(\omega)\|_{1,1} = 0, \quad F^\mu := \langle x \rangle^N [\phi_\alpha^\mu I_\alpha^\mu - \phi_\alpha^\infty I_\alpha^\infty].$$

We have for  $\omega, \theta \in S^{N-1}$

$$\|F^\mu f(\omega)\|_{1,1} \leq \|F^\mu f(\theta)\|_{1,1} + C \sum_{|\gamma| \leq 1} \|\langle x; y \rangle^{1+s} D^\gamma F^\mu\|_{L^\infty} \|f(\omega) - f(\theta)\|_{1, -s}.$$

By Lemma 2.1, (V) and  $f(\theta) \in H_{-s}^1(\mathbf{R}^{2N})$ , the first term tends to zero in  $H_1^1(\mathbf{R}^{2N})$  as  $\mu \rightarrow \infty$  for fixed  $\theta \in S^{N-1}$ . In view of the continuity of  $f(\omega)$  and the boundedness  $\sum_{|\gamma| \leq 1} \|\langle x; y \rangle^{1+s} D^\gamma F^\mu\|_{L^\infty}$  (see (5.35)), the argument similar to that in the proof of (5.37) yields (5.40), and hence (5.33).

(ii) Fix  $s_1$  with  $[(N-1)/2] + 1 + (1/2) < s_1 < \varepsilon_0 - N$ . If we replace  $s$ ,  $H_1^1(\mathbf{R}^{2N})$  and  $H_{-s}^1(\mathbf{R}^{2N})$  by  $s_1$ ,  $H_L^M(\mathbf{R}^{2N})$  and  $H_{-s_1}^M(\mathbf{R}^{2N})$ , respectively ( $M = [(N-1)/2] + 1, L = (N+1)/2$ ) in the proof of (i), we obtain the desired results in the same way as above.

**5.3.** We fix  $\omega \in S^{N-1}$  in this subsection. We first introduce a family of operators  $\{L_\omega(\eta)\}$  ( $\eta \in \Pi_\omega$ ) in  $L^2(\mathbf{R}^{N+1})$ ,  $\mathbf{R}^{N+1} = \mathbf{R}_x^N \times \mathbf{R}_t^1$ :

$$(5.41) \quad \begin{aligned} L_\omega(\eta) &:= -(2m)^{-1} \Delta_x - iv_0 \partial_t + V^\infty(x, t\omega + \eta), \\ D(L_\omega(\eta)) &:= \{f \in L^2; -(2m)^{-1} \Delta_x f - iv_0 \partial_t f \in L^2\}. \end{aligned}$$

By (V),  $L_\omega(\eta)$  is self-adjoint for each  $\eta \in \Pi_\omega$ , and  $\mathcal{S}(\mathbf{R}^{N+1})$  is a core of  $L_\omega(\eta)$ . Furthermore the norm limits

$$(5.42) \quad (L_\omega(\eta) - \lambda \pm i0)^{-1} := \lim_{\varepsilon \downarrow 0} (L_\omega(\eta) - \lambda \pm i\varepsilon)^{-1}$$

exist in  $\mathbf{B}(L^2(\mathbf{R}^N) \otimes L_s^2(\mathbf{R}), L^2(\mathbf{R}^N) \otimes L_{-s}^2(\mathbf{R}))$  uniformly for  $\lambda$  in any compact set in  $\mathbf{R}$ , where  $s > 1/2$  (see Lemma 4.1).

By the correspondence  $f(x, y) \rightarrow f(x, t\omega + \eta)$  ( $t \in \mathbf{R}, \eta \in \Pi_\omega$ ) we have  $L^2(\mathbf{R}^{2N}) = \int_{\Pi_\omega}^{\oplus} L^2(\mathbf{R}^{N+1})d\eta$  (see [R-S]IV, XIII.16 for constant fiber direct integrals). Then, by (5.5) and (5.41), we have

$$(5.43) \quad H(\omega) = \int_{\Pi_\omega}^{\oplus} L_\omega(\eta)d\eta, \quad (H(\omega) - z)^{-1} = \int_{\Pi_\omega}^{\oplus} (L_\omega(\eta) - z)^{-1}d\eta$$

for  $z \in \mathbf{C} \setminus \mathbf{R}$ . Thus, for each  $f \in L^2(\mathbf{R}^N \times \mathbf{R}^N)$  and  $z \in \mathbf{C} \setminus \mathbf{R}$ , we have

$$(5.44) \quad (R(\omega; z)f)(*, *\omega + \eta) = (L_\omega(\eta) - z)^{-1}f(*, *\omega + \eta)$$

in  $L^2(\mathbf{R}_x^N \times \mathbf{R}_t)$  for a.e.  $\eta \in \Pi_\omega$ . Here we note that  $(L_\omega(\eta) - z)^{-1}$  operates on the variable  $x, t$  (the first  $*$  stands for  $x$  and the second  $*$  for  $t$ ). The limiting absorption principle for  $H(\omega)$  (Lemma 5.4) and  $L_\omega(\eta)$  ((5.42)) together with (5.44) yields the following lemma.

**Lemma 5.6.** *Assume (V), and fix  $\lambda \in \mathbf{R}$  and  $f \in L_s^2(\mathbf{R}^{2N})$  ( $s > 1/2$ ). Then*

$$(5.45) \quad (R(\omega; \lambda - i0)f)(*, *\omega + \eta) = (L_\omega(\eta) - \lambda + i0)^{-1}f(*, *\omega + \eta)$$

in  $L_{-s}^2(\mathbf{R}^{N+1})$  for a.e.  $\eta \in \Pi_\omega$ .

*Proof.* Set

$$B_\varepsilon(\eta) := \|\langle *, *\omega + \eta \rangle^{-s} [(R(\omega; \lambda - i\varepsilon)f)(*, *\omega + \eta) - (R(\omega; \lambda - i0)f)(*, *\omega + \eta)]\|_{L^2(\mathbf{R}^{N+1})}^2$$

for each  $\varepsilon > 0$  and  $\eta \in \Pi_\omega$ . Then, by Fubini's theorem,  $B_\varepsilon(\eta)$  is well defined for a.e.  $\eta \in \Pi_\omega$  and

$$\|B_\varepsilon(\eta)\|_{L^1(\Pi_\omega)} = \|[R(\omega; \lambda - i\varepsilon) - R(\omega; \lambda - i0)]f\|_{L^2(\mathbf{R}^{2N})}^2,$$

Since the R.H.S. goes to zero as  $\varepsilon \downarrow 0$  by Lemma 5.4, we can choose a sequence  $\varepsilon_1 > \varepsilon_2 > \dots \rightarrow 0$  and a null subset  $e_0$  of  $\Pi_\omega$  such that as  $j \rightarrow \infty$ ,  $B_{\varepsilon_j}(\eta) \rightarrow 0$  for every  $\eta \in \Pi_\omega \setminus e_0$ . This implies that

$$(5.46) \quad \|(R(\omega; \lambda - i\varepsilon_j)f)(*, *\omega + \eta) - (R(\omega; \lambda - i0)f)(*, *\omega + \eta)\|_{L^2(\mathbf{R}^{N+1})}^2 \rightarrow 0$$

as  $j \rightarrow \infty$  for every  $\eta \in \Pi_\omega \setminus e_0$ . On the other hand, there exists a null subset  $e_1$  of  $\Pi_\omega$  such that for every  $\eta \in \Pi_\omega \setminus e_1$  and all  $j$ ,

$$(5.47) \quad \begin{aligned} f(*, *\omega + \eta) &\in L_s^2(\mathbf{R}^{N+1}), \\ (R(\omega; \lambda - i\varepsilon_j)f)(*, *\omega + \eta) &= (L_\omega(\eta) - \lambda + i\varepsilon_j)^{-1}f(*, *\omega + \eta) \end{aligned}$$

in  $L_{-s}^2(\mathbf{R}^{N+1})$ . Thus, by (5.42) and (5.47), we have

$$(5.48) \quad s - \lim_{j \rightarrow \infty} (R(\omega; \lambda - i\varepsilon_j)f)(*, *\omega + \eta) = (L_\omega(\eta) - \lambda + i0)^{-1}f(*, *\omega + \eta)$$

in  $L_{-s}^2(\mathbf{R}^{N+1})$  for every  $\eta \in \Pi_\omega \setminus e_1$ . By (5.46) and (5.48) we get the desired result.

In Lemma 5.8 we shall prove that (5.45) holds for all  $\eta \in \Pi_\omega$  under the stronger assumption (V)' if we require a regularity of  $f$ . To this end we review the trace theorem.

**Trace Theorem** (see e.g. the proof of Theorem IX.38 of [R-S]II). *Let  $p, q \in \mathbf{N}$ , and let  $\sigma$  be a real with  $\sigma > p/2$ . Then there exists a constant  $C$  such that*

$$\|f(z, *)\|_{L^2(\mathbf{R}^q)} \leq C \|f\|_{H^\sigma(\mathbf{R}^p \times \mathbf{R}^q)}$$

for all  $f \in \mathcal{S}(\mathbf{R}^p \times \mathbf{R}^q)$  and  $z \in \mathbf{R}^p$ . In particular, the trace  $T_z f := f(z, *) \in L^2(\mathbf{R}^q)$  is well-defined for all  $f \in H^\sigma(\mathbf{R}^p \times \mathbf{R}^q)$  and  $z \in \mathbf{R}^p$ . Furthermore  $T_z$  is a  $\mathbf{B}(H^\sigma(\mathbf{R}^p \times \mathbf{R}^q), L^2(\mathbf{R}^q))$ -valued norm continuous function of  $z \in \mathbf{R}^p$ .

For each  $\eta \in \Pi_\omega$  we define a map  $\gamma_\eta$  from  $\mathcal{S}(\mathbf{R}^{2N})$  to  $\mathcal{S}(\mathbf{R}^{N+1})$  by  $(\gamma_\eta f)(x, t) := f(x, t\omega + \eta)$ .  $\gamma_\eta f$  is the restriction of  $f$  on a plane of codimension  $N - 1$  in  $\mathbf{R}^{2N}$ . The trace theorem guarantees that  $\gamma_\eta$  can be uniquely extended to a bounded operator from  $H^k(\mathbf{R}^{2N})$  to  $L^2(\mathbf{R}^{N+1})$  for any  $k > (N - 1)/2$ , and that  $\gamma_\eta$  is a  $\mathbf{B}(H^k(\mathbf{R}^{2N}), L^2(\mathbf{R}^{N+1}))$ -valued norm continuous function of  $\eta \in \Pi_\omega$ . Furthermore we have

**Lemma 5.7.** *Fix  $k > (N - 1)/2$  and  $s \in \mathbf{R}$ . Then  $\gamma_\eta$  can be uniquely extended to a bounded operator from  $H_s^k(\mathbf{R}^{2N})$  to  $L_s^2(\mathbf{R}^{N+1})$ , and  $\gamma_\eta$  is a  $\mathbf{B}(H_s^k(\mathbf{R}^{2N}), L_s^2(\mathbf{R}^{N+1}))$ -valued norm continuous function of  $\eta \in \Pi_\omega$ .*

*Proof.* The relation  $\gamma_\eta \langle x; y \rangle^{-s} = \langle x; t\omega + \eta \rangle^{-s} \gamma_\eta$  and the trace theorem yield  $\gamma_\eta \in \mathbf{B}(H_s^k(\mathbf{R}^{2N}), L_s^2(\mathbf{R}^{N+1}))$ . The continuity is easily verified.

**Lemma 5.8.** *Assume (V)', and fix  $\lambda \in \mathbf{R}$ ,  $s > [(N - 1)/2] + (3/2)$  and  $f \in H_s^{[(N-1)/2]+1}(\mathbf{R}^{2N})$ . Then*

$$(5.49) \quad \gamma_\eta R(\omega; \lambda - i0)f = (L_\omega(\eta) - \lambda + i0)^{-1} \gamma_\eta f$$

in  $L_{-s}^2(\mathbf{R}^{N+1})$  for all  $\eta \in \Pi_\omega$ .

*Proof.*  $R(\omega; \lambda - i\varepsilon)$  maps  $H_s^{[(N-1)/2]+1}(\mathbf{R}^{2N})$  into  $H_{-s}^{[(N-1)/2]+1}(\mathbf{R}^{2N})$  for each  $\varepsilon > 0$ , and we have

$$(5.50) \quad \gamma_\eta R(\omega; \lambda - i\varepsilon)f = (L_\omega(\eta) - \lambda + i\varepsilon)^{-1} \gamma_\eta f$$

in  $L_{-s}^2(\mathbf{R}^{N+1})$  for a.e.  $\eta \in \Pi_\omega$  by (5.44) and Lemma 5.7. On the other hand, it is easy to see that  $(L_\omega(\eta) - \lambda + i\varepsilon)^{-1}$  is strongly continuous in  $\eta \in \Pi_\omega$  as a  $\mathbf{B}(L^2(\mathbf{R}^{N+1}))$ -valued function. Thus both sides of (5.50) are strongly continuous in  $\eta \in \Pi_\omega$  in  $L_{-s}^2(\mathbf{R}^{N+1})$  by Lemma 5.7, and so (5.50) holds for all  $\eta \in \Pi_\omega$ . We fix  $\eta \in \Pi_\omega$ , and let  $\varepsilon \downarrow 0$  in both sides. Then Lemmas 5.4 and 5.7 yield

$$\gamma_\eta R(\omega; \lambda - i\varepsilon)f \rightarrow \gamma_\eta R(\omega; \lambda - i0)f \quad \text{in } L_{-s}^2(\mathbf{R}^{N+1}).$$

Since  $\gamma_\eta f \in L_s^2(\mathbf{R}^{N+1})$ , we get

$$(L_\omega(\eta) - \lambda + i\varepsilon)^{-1} \gamma_\eta f \rightarrow (L_\omega(\eta) - \lambda + i0)^{-1} \gamma_\eta f \quad \text{in } L_{-s}^2(\mathbf{R}^{N+1})$$

by (5.42). This completes the proof.

**Lemma 5.9.** *We assume (V). Then we have*

$$(5.51) \quad (2\pi)^{(N-2)/2} \int G(\omega; t\omega + \eta) dt = S_{\beta\alpha}^{\infty}(v_0\omega, \eta) - \delta_{\alpha\beta}$$

for a.e.  $\eta \in \Pi_{\omega}$ . (See (1.24) for  $S_{\beta\alpha}^{\infty}(v_0\omega, \eta)$ .)

*Proof.* By (5.23), we have

$$(5.52) \quad (2\pi)^{(N-2)/2} \int G(\omega; t\omega + \eta) dt = (2\pi)^{(N-2)/2} \iint E(\omega; x, t\omega + \eta) dx dt$$

for a.e.  $\eta \in \Pi_{\omega}$ . Using Lemma 5.6 and (5.23), in the R.H.S. of (5.52), we get

$$(5.53) \quad (2\pi)^{(N-2)/2} \iint E(\omega; x, t\omega + \eta) dx dt \\ = iv_0^{-1} \{ -(\phi_{\alpha}^{\infty}(x)I_a^{\infty}(x, t\omega + \eta), \Phi_{\beta}^{\infty}(x, t\omega + \eta)\chi(\omega; x, t\omega + \eta)) \\ + (\phi_{\alpha}^{\infty}(x)I_a^{\infty}(x, t\omega + \eta), ((L_{\omega}(\eta) - \lambda_{\alpha}^{\infty} + i0)^{-1}I_b^{\infty}(*, *\omega + \eta)\Phi_{\beta}^{\infty}(*, *\omega + \eta) \\ \times \chi(\omega; *, *\omega + \eta))(x, t)) \}$$

for a.e.  $\eta \in \Pi_{\omega}$ . By scaling  $t \rightarrow v_0 t$ ,  $L_{\omega}(\eta)$  turns into

$$\tilde{L}_{\omega}(\eta) := -(2m)^{-1}\Delta_x - i\partial_t + V_{23}(x) + V_{13}(x - v_0 t\omega - \eta) + V_{12}(-v_0 t\omega - \eta) \\ \text{(for Case 1-1, 1-2)} \\ := -(2m)^{-1}\Delta_x - i\partial_t + V_{23}(x + v_0 t\omega + \eta) + V_{13}(x) + V_{12}(-v_0 t\omega - \eta) \\ \text{(for Case 2-1, 2-2).}$$

Thus this scaling yields

$$(5.54) \quad \text{the R.H.S. of (5.53)} \\ = -i(\phi_{\alpha}^{\infty}(x)I_a^{\infty}(x, v_0 t\omega + \eta), \Phi_{\beta}^{\infty}(x, v_0 t\omega + \eta)\chi(\omega; x, v_0 t\omega + \eta)) \\ + i(((\tilde{L}_{\omega}(\eta) - \lambda_{\alpha}^{\infty} - i0)^{-1}I_a^{\infty}(*, v_0 *\omega + \eta)\phi_{\alpha}^{\infty}(*))(x, t), I_b^{\infty}(x, v_0 t\omega + \eta) \\ \times \Phi_{\beta}^{\infty}(x, v_0 t\omega + \eta)\chi(\omega; x, v_0 t\omega + \eta)).$$

To finish our proof, in view of (5.53), (5.52), (4.8) and (1.24), we have only to show that (5.54) is equal to the R.H.S. of (4.8) for  $\xi = v_0\omega$  and each  $\eta \in \Pi_{\omega}$ .

*Case 1-1.* We can obtain the desired result by observing that

$$(5.55) \quad I_c^{\infty}(x, v_0 t\omega + \eta) = W_c(v_0\omega, \eta; x, t) \quad \text{for } c = a, b \text{ (see (4.7))},$$

$$(5.56) \quad \chi(\omega; x, v_0 t\omega + \eta) = e^{i(\lambda_{\alpha}^{\infty} - \lambda_{\beta}^{\infty})t}, \quad \Phi_{\beta}^{\infty}(x, v_0 t\omega + \eta) = \phi_{\beta}^{\infty}(x),$$

$$(5.57) \quad e^{-i\lambda_{\alpha}^{\infty}t}(\tilde{L}_{\omega}(\eta) - \lambda_{\alpha}^{\infty} - i0)^{-1} = (K_{v_0\omega, \eta} - i0)^{-1}e^{-i\lambda_{\alpha}^{\infty}t}.$$

Case 1-2. (5.55) and (5.57) hold also in this case. Instead of (5.56), we have only to note that

$$(5.58) \quad \begin{aligned} \chi(\omega; x, v_0 t \omega + \eta) &= e^{imv_0 \omega x - i((m/2)v_0^2 + \lambda_\beta^\infty - \lambda_\alpha^\infty)t}, \\ \Phi_\beta^\infty(x, v_0 t \omega + \eta) &= \phi_\beta^\infty(x - v_0 t \omega - \eta). \end{aligned}$$

Case 2-1. We use new variables  $(X, s) = (x + v_0 t \omega + \eta, t)$  in this case and the next case. In terms of the new variables we can write

$$(5.59) \quad I_c^\infty(x, v_0 t \omega + \eta) = W_c(v_0 \omega, \eta; X, s) \quad \text{for } c = a, b,$$

$$(5.60) \quad \chi(\omega; x, v_0 t \omega + \eta) = e^{-imv_0 \omega \cdot X + i((m/2)v_0^2 + \lambda_\alpha^\infty - \lambda_\beta^\infty)s},$$

$$(5.61) \quad \phi_\alpha^\infty(x) = \phi_\alpha^\infty(X - v_0 s \omega - \eta),$$

$$(5.62) \quad \Phi_\beta^\infty(x, v_0 t \omega + \eta) = \phi_\beta^\infty(X),$$

$$(5.63) \quad \begin{aligned} (\tilde{L}_\omega(\eta) - \lambda_\alpha^\infty - i0)^{-1} &= (-(2m)^{-1} \Delta_x - i\partial_s - iv_0 \omega \cdot \nabla_x + V_{23}(X) + V_{13}(X - v_0 s \omega - \eta) \\ &\quad + V_{12}(-v_0 s \omega - \eta) - \lambda_\alpha^\infty - i0)^{-1} \\ &= U(-(2m)^{-1} \Delta_x - i\partial_s + V_{23}(X) + V_{13}(X - v_0 s \omega - \eta) \\ &\quad + V_{12}(-v_0 s \omega - \eta) - i0)^{-1} U^*, \end{aligned}$$

where  $U = e^{-imv_0 \omega X + i((m/2)v_0^2 + \lambda_\alpha^\infty)s}$ , a unitary multiplication operator. Noting that the Jacobian for  $(x, t) \rightarrow (X, s)$  is one, we can compute the R.H.S. of (5.54) to obtain the desired result.

Case 2-2. We use the same variables as above. (5.59), (5.61), (5.63) hold also. Instead of (5.60), (5.62), we have only to note that

$$(5.64) \quad \chi(\omega; x, v_0 t \omega + \eta) = e^{i(\lambda_\alpha^\infty - \lambda_\beta^\infty)s}, \quad \Phi_\beta^\infty(x, v_0 t \omega + \eta) = \phi_\beta^\infty(X - v_0 s \omega - \eta).$$

We have shown that (5.54) equals the R.H.S. of (4.8) for  $\xi = v_0 \omega$  and  $\eta \in \Pi_\omega$ , and have finished the proof of Lemma 5.9.

Lemma 5.5 (ii) shows  $G(\omega) \in H_{(N+1)/2}^{((N-1)/2)+1}(\mathbf{R}^N)$  if we assume (V). Thus  $G(\omega; t \omega + \eta) \in L_{(N+1)/2}^2(\mathbf{R}_t) \subset L^1(\mathbf{R}_t)$  is well-defined for each  $\eta \in \Pi_\omega$ , in view of the trace theorem. At the end of this subsection we prove the following.

**Lemma 5.10.** *We assume (V). Then (5.51) holds for all  $\eta \in \Pi_\omega$ .*

*Proof.* By the trace theorem the L.H.S. of (5.51) is continuous in  $\eta \in \Pi_\omega$ , and by Lemma 5.8 and the smoothness of  $V_{ij}$ ,  $\phi_\alpha^\infty$ ,  $\phi_\beta^\infty$  the R.H.S. of (5.53) (= the R.H.S. of (5.51)) is continuous in  $\eta \in \Pi_\omega$ . Hence, (5.51) holds for all  $\eta \in \Pi_\omega$ .

**5.4.** We assume (V) and give the proof of Theorem 1.1 in this subsection. The following lemma will be proved in Sect. 7.

**Lemma 5.11.** *Let  $k > 1/2$ ,  $s > (N - 1)/2$ , and  $h \in C(S^{N-1})$ . Then for any  $\varepsilon > 0$  there exists  $R_0 = R_0(k, s, \varepsilon, h, N) > 0$  such that*

$$(5.65) \quad \left| R^{N-1} \int_{S^{N-1}} F(R(\omega' - \omega))h(\omega')d\omega' - h(\omega) \int_{\Pi_\omega} F(\eta)d\eta \right| \leq \varepsilon \|F\|_{k,s}$$

for all  $R \geq R_0$ ,  $\omega \in S^{N-1}$  and  $F \in H_s^k(\mathbf{R}^N)$ .

**Remark.**  $F(R(\star - \omega)) \in L^2(S^{N-1})$  and  $F(\eta) \in L_s^2(\Pi_\omega) \subset L^1(\Pi_\omega)$  are well-defined by the trace theorem.

*Proof of Theorem 1.1.* We fix  $f \in C(S^{N-1})$  and set  $M = [(N - 1)/2] + 1$ ,  $L = (N + 1)/2$ . Then by Lemma 5.5 we have  $\check{G}^\mu(\omega, v)$ ,  $\check{G}(\omega) \in H_M^L(\mathbf{R}^N)$ , where  $\check{G}^\mu(\omega, v; \xi)$ ,  $\check{G}(\omega; \xi)$  are the inverse Fourier transform of  $G^\mu(\omega, v; y)$ ,  $G(\omega; y)$ , respectively. Thus applying Lemma 5.11 with  $k = L$ ,  $s = M$ ,  $h = f$ ,  $F = \check{G}^\mu(\omega, v)$ , and using (5.11) (see (1.16)),  $\lim_{\mu \rightarrow \infty, v \rightarrow v_0} vn_a \rightarrow \infty$ , we obtain

$$(5.66) \quad \begin{aligned} & \left| (T_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu)f)(\omega) - \left( \int_{\Pi_\omega} \check{G}(\omega; \eta)d\eta \right) f(\omega) \right| \\ & \leq \left| (vn_a)^{N-1} \int_{S^{N-1}} \check{G}^\mu(\omega, v; vn_a(\omega' - \omega))f(\omega')d\omega' - \left( \int_{\Pi_\omega} \check{G}^\mu(\omega, v; \eta)d\eta \right) f(\omega) \right| \\ & \quad + \|\check{G}^\mu(\omega, v; \eta) - \check{G}(\omega; \eta)\|_{L^1(\Pi_\omega)} |f(\omega)| \\ & \leq \theta \|G^\mu(\omega, v)\|_{M,L} + C \|G^\mu(\omega, v) - G(\omega)\|_{M,L} \|f\|_{L^\infty(\Pi_\omega)}, \end{aligned}$$

where  $\theta = \theta(\mu, f, v)$ , which is independent of  $\omega \in S^{N-1}$ , satisfies  $\theta \rightarrow 0$  as  $\mu \rightarrow \infty$  and  $v \rightarrow v_0$ , and  $C$  is independent of  $\omega, v$  and  $\mu \gg 1$ . In the last step we have used the following estimate, which follows from the Schwarz inequality and Lemma 5.7,

$$(5.67) \quad \begin{aligned} \|\check{u}(\eta)\|_{L^1(\Pi_\omega)} & \leq \text{const.} \|\check{u}(\eta)\|_{L_M^2(\Pi_\omega)} \\ & \leq \text{const.} \|u\|_{H_M^L(\mathbf{R}^N)} \end{aligned}$$

for  $u \in H_L^M(\mathbf{R}^N)$ . Thus, by Lemma 5.5 and (5.66), we get

$$(5.68) \quad \lim_{\mu \rightarrow \infty, v \rightarrow v_0} (T_{\beta\alpha}^\mu((1/2)v^2 + \lambda_\alpha^\mu)f) = \left( \int_{\Pi_\omega} \check{G}(\omega; \eta)d\eta \right) f(\omega)$$

uniformly on  $S^{N-1}$ . Now direct calculation yields

$$(5.69) \quad \int_{\Pi_\omega} \check{u}(\eta)d\eta = (2\pi)^{(N-2)/2} \int_{\mathbf{R}} u(t\omega)dt$$

for each  $\omega \in S^{N-1}$  and  $u \in \mathcal{S}(\mathbf{R}^N)$ . By the Schwarz inequality and the trace theorem we have

$$(5.70) \quad \|u(\star\omega)\|_{L^1(\mathbf{R})} \leq \text{const.} \|u(\star\omega)\|_{L_L^2(\mathbf{R})} \leq \text{const.} \|u\|_{H_L^M(\mathbf{R}^N)}$$

for any  $u \in H_L^M(\mathbf{R}^N)$ , regarding  $u(\star\omega)$  as the trace of  $u$  to the one dimensional subspace  $\{t\omega; t \in \mathbf{R}\}$  in  $\mathbf{R}^N$ . This together with (5.67) implies that (5.69) holds for any  $u \in H_L^M(\mathbf{R}^N)$ . In particular we have

$$(5.71) \quad \int_{\Pi_\omega} \check{G}(\omega; \eta) d\eta = (2\pi)^{(N-2)/2} \int_{\mathbf{R}} G(\omega; t\omega) dt \\ = S_{\beta\alpha}^\infty(v_0\omega, 0) - \delta_{\alpha\beta}$$

for all  $\omega \in S^{N-1}$  by Lemma 5.9. Theorem 1.1 follows from (5.68) and (5.71) if we recall the definition of  $T_{\beta\alpha}^\mu(\lambda)$ .

**5.5.** We assume (V) and prove Theorem 1.3 in this subsection. The next lemma will be proved in Sect. 7.

**Lemma 5.12.** *Let  $0 < s < k - 1/2$ . Then for any  $\varepsilon > 0$  there exists  $R_0 = R_0(\varepsilon, k, s, N) \geq 1$  such that*

$$\left| R^{N-1} \int_{S^{N-1}} |F(R(\omega' - \omega))|^2 d\omega' - \int_{\Pi_\omega} |F(\eta)|^2 d\eta \right| \leq \varepsilon \|F\|_{k,s}^2$$

for all  $R \geq R_0$ ,  $\omega \in S^{N-1}$  and  $F \in H_s^k(\mathbf{R}^N)$ .

We define  $J$  by  $(Jf)(\xi) = \overline{f(\xi)}$  (the complex conjugation) and  $\tilde{P}_\gamma^\mu, \tilde{W}_{\gamma\pm}^\mu$  by  $(\tilde{P}_\gamma^\mu f)(x_c, y_c) := \overline{\phi_\gamma^\mu(x_c)} f(y_c)$ ,

$$(5.72) \quad \tilde{W}_{\gamma\pm}^\mu := s - \lim_{t \rightarrow \pm\infty} e^{itH^\mu} \tilde{P}_\gamma^\mu e^{-itT_\gamma^\mu}$$

for  $\gamma = \alpha, \beta$ ,  $c = D(\gamma)$ . Since

$$J\tilde{P}_\gamma^\mu J = P_\gamma^\mu, \quad e^{-itT_\gamma^\mu} = J e^{itT_\gamma^\mu} J, \quad e^{-itH^\mu} = J e^{itH^\mu} J \quad \text{for } \gamma = \alpha, \beta,$$

we have, by (1.10),  $W_{\gamma\pm}^\mu = J\tilde{W}_{\gamma\pm}^\mu J$  and so

$$(5.73) \quad S_{\beta\alpha}^\mu = J\tilde{S}_{\alpha\beta}^{\mu*} J,$$

where  $\tilde{S}_{\alpha\beta}^\mu := \tilde{W}_{\alpha+}^{\mu*} \tilde{W}_{\beta-}^\mu$ . We can see that  $Z_\alpha^\mu \tilde{S}_{\alpha\beta}^\mu Z_\beta^{\mu*}$  is decomposable in terms of a family of operators  $\{\tilde{S}_{\alpha\beta}^\mu(\lambda)\}$  ([A-J-S], 15-3):

$$(5.74) \quad Z_\alpha^\mu \tilde{S}_{\alpha\beta}^\mu Z_\beta^{\mu*} = \{\tilde{S}_{\alpha\beta}^\mu(\lambda)\} \quad (\text{see (1.15)}).$$

For  $\gamma = \alpha, \beta$  we note that  $JZ_\gamma^{\mu*}(\lambda) = Z_\gamma^{\mu*}(\lambda)\hat{J}$ , where  $\hat{J}$  is defined by  $(\hat{J}g)(\omega) = \overline{g(-\omega)}$  for  $g \in \Sigma = L^2(S^{N-1})$ . Then we have by (5.73), (5.74)

$$(5.75) \quad S_{\beta\alpha}^\mu(\lambda) = \hat{J}\tilde{S}_{\alpha\beta}^\mu(\lambda)*\hat{J} \quad \text{on } \Sigma, \quad \lambda \in (\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu,$$

where  $\lambda_{\beta\alpha}^\mu := \max(\lambda_\alpha^\mu, \lambda_\beta^\mu)$ ,  $A^\mu := \{\text{the thresholds of } H^\mu\} \cup \sigma_p(H^\mu)$ . Since both sides are norm continuous by Theorem 2.3, (5.75) holds for all  $\lambda \in (\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu$ . Let  $\tilde{T}_{\alpha\beta}^\mu(\lambda, \omega, \omega')$  be the integral kernel of  $\tilde{S}_{\alpha\beta}^\mu(\lambda) - \delta_{\alpha\beta}$ . Then, by (5.75), we get

$$(5.76) \quad T_{\beta\alpha}^\mu(\lambda, \omega', \omega) = \tilde{T}_{\alpha\beta}^\mu(\lambda, -\omega, -\omega') \quad (\text{cf. (1.16)}).$$

This equality holds for all  $(\lambda, \omega, \omega') \in ((\lambda_{\beta\alpha}^\mu, \infty) \setminus A^\mu) \times S^{N-1} \times S^{N-1}$ , because both sides are continuous in all variables (Proposition 2.4).

The arguments up to subsection 5.3 are valid even if we replace (the initial channel)  $\alpha$  and (the final channel)  $\beta$  by  $\beta$  and  $\alpha$ , respectively, and then relpace  $\phi_\gamma^\mu$  and  $\phi_\gamma^\infty$  by  $\bar{\phi}_\gamma^\mu$  and  $\bar{\phi}_\gamma^\infty$ , respectively for  $\gamma = \alpha, \beta$ . In the definition of  $G^\mu(\omega, v; y_a)$  and  $G(\omega; y)$  (see (5.9), (5.23)), we replace  $\alpha$  and  $\beta$  by  $\beta$  and  $\alpha$ , respectively (the initial speed  $v$  is replaced by  $v'$  (see (5.2)), and replace  $\phi_\gamma^\mu$  and  $\phi_\gamma^\infty$  by  $\bar{\phi}_\gamma^\mu$  and  $\bar{\phi}_\gamma^\infty$ , respectively. Denote the resulting function by  $p^\mu(\omega, v; y_b)$  and  $p^\infty(\omega; y)$ . Then we have

$$(5.77) \quad \begin{aligned} \tilde{T}_{\alpha\beta}^\mu(\lambda, \omega, \omega') &= (n_b v')^{N-1} \check{p}^\mu(\omega, v; n_b v'(\omega' - \omega)), \\ \lambda &= (1/2)n_a v^2 + \lambda_\alpha^\mu = (1/2)n_b v'^2 + \lambda_\beta^\mu \end{aligned}$$

in the same way as (5.11). Thus, in virtue of (5.76), the total cross section is represented as (see (1.17))

$$(5.78) \quad \begin{aligned} \sigma_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega) \\ = (2\pi)^{N-1} (n_a v)^{1-N} (n_b v')^{2N-2} \int_{S^{N-1}} |\check{p}^\mu(-\omega, v; n_b v'(\omega' + \omega))|^2 d\omega'. \end{aligned}$$

We have

$$(5.79) \quad \begin{aligned} &|\sigma_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega) - (2\pi)^{N-1} \|\check{p}^\infty(-\omega; *)\|_{L^2(\Pi_\omega)}^2| \\ &\leq |\sigma_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega) - (2\pi)^{N-1} \|\check{p}^\mu(-\omega, v; *)\|_{L^2(\Pi_\omega)}^2| \\ &\quad + (2\pi)^{N-1} (\|\check{p}^\infty(-\omega; *)\|_{L^2(\Pi_\omega)} + \|\check{p}^\mu(-\omega, v; *)\|_{L^2(\Pi_\omega)}) \\ &\quad \times \|\|\check{p}^\infty(-\omega; *)\|_{L^2(\Pi_\omega)} - \|\check{p}^\mu(-\omega, v; *)\|_{L^2(\Pi_\omega)}\|. \end{aligned}$$

Under assumption (V), Lemma 5.5(i) holds even if  $G^\mu(\omega, v; y_a)$  and  $G(\omega; y)$  are replaced by  $p^\mu(\omega, v; y_b)$  and  $p^\infty(\omega; y)$ , respectively. Thus, by using the trace theorem, Lemma 5.12 and (5.78), we obtain

$$(5.80) \quad \lim_{\mu \rightarrow \infty, v \rightarrow v_0} \sigma_{\beta\alpha}^\mu((1/2)n_a v^2 + \lambda_\alpha^\mu; \omega) = (2\pi)^{N-1} \|\check{p}^\infty(-\omega; *)\|_{L^2(\Pi_\omega)}^2$$

uniformly for  $\omega \in S^{N-1}$ . Since

$$\check{p}^\infty(-\omega; \xi) = (2\pi)^{-(N-1)/2} \int_{\Pi_\omega} e^{i\xi \cdot \eta} \left( (2\pi)^{-1/2} \int_{\mathbf{R}} p^\infty(-\omega; t\omega + \eta) dt \right) d\eta$$

for  $\xi \in \Pi_\omega$ , the R.H.S. of (5.80) equals

$$(2\pi)^{N-2} \int_{\Pi_\omega} \left| \int_{\mathbf{R}} p^\infty(-\omega; t\omega + \eta) dt \right|^2 d\eta$$

by Parseval's equality.

We define  $\check{\psi}_\gamma^\infty(\xi, \eta; x, t)$  by replacing  $\phi_\gamma^\infty$  by  $\bar{\phi}_\gamma^\infty$  in the definition of  $\psi_\gamma^\infty(\xi, \eta; x, t)$  (see (1.20)) and define

$$(5.81) \quad \tilde{\Omega}_\gamma^\pm(\xi, \eta) := s - \lim_{t \rightarrow \pm\infty} U(\xi, \eta; 0, t) \check{\psi}_\gamma^\infty(\xi, \eta; *, t) \quad \text{in } L^2(\mathbf{R}^N)$$



for  $\gamma = \alpha, \beta$ . Then, in the same way as Lemma 5.9, we have

$$(5.82) \quad (2\pi)^{(N-2)/2} \int_{\mathbf{R}} p^\infty(-\omega; t\omega + \eta) dt \\ = e^{-i \int_{-\infty}^{\infty} v_{12}(v_0 t \omega - \eta) dt} (\tilde{\Omega}_\beta^-(-v_0 \omega, \eta), \tilde{\Omega}_\alpha^+(-v_0 \omega, \eta)) - \delta_{\beta\alpha},$$

for a.e.  $\eta \in \Pi_\omega$ . Thus, by (1.24), (5.80), we have only to prove the following lemma to finish the proof of Theorem 1.3.

**Lemma 5.13.** For  $\gamma = \alpha, \beta$  and  $\xi \in \mathbf{R}^N \setminus \{0\}$ ,  $\eta \in \Pi_\xi$ , we have

$$(5.83) \quad \tilde{\Omega}_\gamma^\pm(-\xi, \eta) = J\Omega_\gamma^\mp(\xi, \eta).$$

*Proof.* Recall that  $U(\xi, \eta; t, s)$  is the propagator of  $h_{\xi, \eta}(t)$  (see (1.18)).  $Q(t, s) := JU(-\xi, \eta; -t, -s)J$ , ( $s, t \in \mathbf{R}$ ) obviously satisfies (U-i) and (U-ii) of Sect. 1. Moreover we have

$$i\partial_t Q(t, s) = Jh_{-\xi, \eta}(-t)U(-\xi, \eta; -t, -s)J \\ = h_{\xi, \eta}(t)Q(t, s),$$

where we have used  $h_{\xi, \eta}(t) = h_{-\xi, \eta}(-t)$  in the last step. Thus we see that  $Q(t, s) = U(\xi, \eta; t, s)$  for all  $s, t \in \mathbf{R}$  by the uniqueness of propagator. Since  $J\tilde{\psi}_\gamma^\infty(-\xi, \eta; x, t) = \psi_\gamma^\infty(\xi, \eta; x, -t)$  for  $\gamma = \alpha, \beta$ , it follows that

$$\tilde{\Omega}_\gamma^\pm(-\xi, \eta) = s - \lim_{t \rightarrow \pm\infty} U(-\xi, \eta; 0, t)\tilde{\psi}_\gamma^\infty(-\xi, \eta; *, t) \\ = s - \lim_{t \rightarrow \pm\infty} JU(\xi, \eta; 0, -t)J\tilde{\psi}_\gamma^\infty(-\xi, \eta; *, t) \\ = J\Omega_\gamma^\mp(\xi, \eta)$$

## §6. Proof of Lemma 5.4

We shall prove Lemma 5.4, under assumption (V), by applying the abstract theorem obtained in Sect. 3. We fix  $s \in \mathbf{R}$  and an integer  $k \geq 0$  with  $0 \leq k \leq \varepsilon_0 - 2$ ,  $k + 1/2 < s$ , and a compact interval  $J = [e_1, e_2]$  in  $\mathbf{R}$  throughout this section. We may assume  $a = a_1 = \{1, (2, 3)\}$ , since the other case can be treated similarly.

Let

$$(6.1) \quad A_0 := (1/2i)(x \cdot \mathcal{V}_x + \mathcal{V}_x \cdot x + y \cdot \mathcal{V}_y + \mathcal{V}_y \cdot y)$$

be the generator of dilations on  $\mathbf{R}^{2N}$ , which is self-adjoint in  $\mathcal{H} = L^2(\mathbf{R}^{2N})$  with  $\mathcal{S} = \mathcal{S}(\mathbf{R}^{2N})$  as a core. For a triplet  $\theta = (\omega, \mu, v)$  with  $\omega \in S^{N-1}$ ,  $\mu \gg 1$  and  $|v - v_0| \ll 1$  for fixed  $v_0 > 0$ , we define a operator

$$(6.2) \quad A_\theta := n_a^{-1} A_0 + v\omega \cdot y = n_a^{-1} e^{-in_a v \omega \cdot y} A_0 e^{in_a v \omega \cdot y},$$

and for  $\omega \in S^{N-1}$  we define

$$(6.3) \quad A_\omega^\infty := v_0 \omega \cdot y.$$

We also write

$$(6.4) \quad H_\theta = H^\mu(\omega, v) = H^\mu - iv\omega \cdot \nabla_y \quad (\text{see (5.5)})$$

for a triplet  $\theta = (\omega, \mu, v)$ .  $A_\theta$ ,  $A_\omega^\infty$  and  $H_\theta$  are self-adjoint operators in  $\mathcal{H}$  with  $\mathcal{S}$  as a core. A direct calculation yields

$$(6.5) \quad i[H_\theta, A_\theta] = \frac{2}{n_a} H_\theta - \frac{i}{n_a} [A_0, V^\mu] - \frac{2}{n_a} V^\mu + v^2,$$

$$(6.6) \quad i[A_0, V^\mu] = V_{23}^{(1)}(x) + V_{13}^{(1)} \left( \frac{m_a}{m} x - y \right) + V_{12}^{(1)} \left( -\frac{m_a}{M_2} x - y \right)$$

on  $\mathcal{S}$ , where  $V_{jk}^{(\ell)}(x) = (x \cdot \nabla_x)^\ell V_{jk}(x)$  (see (5.12)). Thus the R.H.S. of (6.5) can be extended to a bounded operator from  $H^2(\mathbf{R}^{2N})$  to  $\mathcal{H}$ , and the commutator  $i[H_\theta, A_\theta]$  defines a self-adjoint operator  $iH_\theta^{(1)}$  in  $\mathcal{H}$ . If  $\ell \leq \varepsilon_0$ , the  $\ell$ -th commutator

$$(6.7) \quad i^\ell [A_0, [\dots, [A_0, V^\mu] \dots]] = V_{23}^{(\ell)}(x) + V_{13}^{(\ell)} \left( \frac{m_a}{m} x - y \right) + V_{12}^{(\ell)} \left( -\frac{m_a}{M_2} x - y \right)$$

is bounded by (V). Therefore, by using (6.5) we see that the  $l$ -th commutator  $i^\ell [\dots [H_\theta, A_\theta], \dots, A_\theta]$  on  $\mathcal{S}$  can be uniquely extended to a self-adjoint operator  $i^\ell H_\theta^{(\ell)}$  in  $\mathcal{H}$  for  $\ell \leq \varepsilon_0$ .

Let  $\phi$  be a  $C^\infty$ -function on  $\mathbf{R}$  such that  $0 \leq \phi \leq 1$ ,  $\phi \equiv 1$  on  $J$  and  $\text{supp } \phi \subset [e_1 - 1, e_2 + 1]$  ( $\text{supp} = \text{support}$ ).

**Lemma 6.1.** *Let  $d$  be a positive integer with  $d \leq k + 1$ .*

(i) *There exist  $M > 0$  and  $\delta_0 > 0$  such that assumption 3.1 ( $\mathbf{H} = \mathcal{H}$ ,  $H = H_\theta$ ,  $A = A_\theta$ ,  $I = J$ ) is satisfied for all triplets  $\theta = (\omega, \mu, v)$  with  $\omega \in S^{N-1}$ ,  $\mu = (M_1, M_2)$ ,  $M_1, M_2 > M$ ,  $|v - v_0| < \delta_0$ , where we can take  $C_0 = (1/2)v_0^2$  in (H-v). Furthermore,  $\|i^\ell H_\theta^{(\ell)} R_\theta(i)\|$  ( $\ell = 1, \dots, d + 1$ ) is uniformly bounded for  $\theta = (\omega, \mu, v)$ , where we write*

$$(6.8) \quad R_\theta(z) = (H_\theta - z)^{-1}.$$

(ii) *For  $\omega \in S^{N-1}$  assumption 3.1 ( $\mathbf{H} = \mathcal{H}$ ,  $H = H(\omega)$ ,  $A = A_\omega^\infty$ ,  $I = J$ ) is satisfied, where we can take  $C_0 = v_0^2$  in (H-v). Furthermore,  $i[H(\omega), A_\omega^\infty] = v_0^2$  and  $i^\ell [\dots [H(\omega), A_\omega^\infty], \dots, A_\omega^\infty] = 0$  if  $2 \leq \ell$ .*

*Proof.* (i) (H-i) is obvious because  $\mathcal{S}$  is a common core for  $H_\theta$  and  $A_\theta$ . (H-ii) follows easily from

$$(6.9) \quad e^{itA_\theta} = e^{-in_a v \omega \cdot y} e^{itA_0/n_a} e^{in_a v \omega \cdot y}.$$

We can verify (H-iii), (H-iv) by using the arguments before this lemma and the fact that  $\mathcal{S}$  is a common core for  $H_\theta$  and  $A_\theta$ . Since  $n_a \rightarrow \infty$  as  $\mu \rightarrow \infty$ , we

obtain by (6.5)

$$(6.10) \quad \phi(H_\theta) i H_\theta^{(1)} \phi(H_\theta) \cong \left( \frac{2}{n_a} (e_1 - 1) - \frac{1}{n_a} \|[A_0, V^\mu]\| - \frac{2}{n_a} \|V^\mu\| + v^2 \right) \phi(H_\theta)^2 \\ \cong (1/2) v_0^2 \phi(H_\theta)^2$$

for all  $\theta = (\omega, \mu, v)$  with  $\omega \in S^{N-1}$ ,  $\mu \gg 1$  and  $|v - v_0| \ll 1$ . This implies (H-v) with  $C_0 = (1/2)v_0^2$ . The uniform boundedness of

$$(6.11) \quad \|i^\ell H_\theta^{(\ell)} R_\theta(i)\| = \|((2/n_a)^\ell H_\theta + \text{bounded operators}) R_\theta(i)\|$$

( $\ell = 1, \dots, d+1$ ) follows from (6.5), (6.7) and (V).

(ii)  $\mathcal{S}$  is a common core for  $H(\omega)$  and  $A_\omega^\infty$ , and (H-i) is satisfied. Noting that

$$D(H(\omega)) = \{f \in L^2(\mathbf{R}^{2N}); (-2m)^{-1} \Delta_x - i v_0 \omega \cdot \nabla_y\} f \in L^2(\mathbf{R}^{2N}),$$

we can easily see that (H-ii) holds. (H-iii)  $\sim$  (H-v) follow from  $i[H(\omega), A_\omega^\infty] = v_0^2$  on  $\mathcal{S}$ . This completes the proof.

We set  $\Xi := \{\theta = (\omega, \mu, v); M_1, M_2 > M, |v - v_0| < \delta_0, \omega \in S^{N-1}, \text{ where } \mu = (M_1, M_2)\}$ . Here  $M$  and  $\delta_0$  are as in Lemma 6.1.

We denote any of  $-i\partial_{x_j}$  or  $-i\partial_{y_j}$  ( $j = 1, \dots, N$ ) by  $D$ . For any  $f \in H^1(\mathbf{R}^{2N})$  and  $z \in \mathbf{C} \setminus \mathbf{R}$ ,

$$(6.12) \quad e^{-itD} R(\omega; z) f = (e^{-itD} H(\omega) e^{itD} - z)^{-1} e^{-itD} f.$$

Thus  $e^{-itD} R(\omega; z) f$  is strongly differentiable in  $t \in \mathbf{R}$  and

$$(6.13) \quad \left. \frac{d}{dt} e^{-itD} R(\omega; z) \right|_{t=0} = -i \{R(\omega; z) D - R(\omega; z) (DV^\infty) R(\omega; z)\} f,$$

which implies  $R(\omega; z)$  leaves  $H^1(\mathbf{R}^{2N})$  invariant and

$$(6.14) \quad DR(\omega; z) = R(\omega; z) D - R(\omega; z) (DV^\infty) R(\omega; z) \quad \text{on } H^1(\mathbf{R}^{2N})$$

for each  $z \in \mathbf{C} \setminus \mathbf{R}$ . By using (6.14) and (V), we see that

$$(6.15) \quad \sup_{\omega \in S^{N-1}, z \in K} \|R(\omega; z)\|_{\mathbf{B}(H^\ell)} < \infty$$

for any compact set  $K$  in  $\mathbf{C} \setminus \mathbf{R}$  and  $\ell \geq 0$ , where  $H^\ell = H^\ell(\mathbf{R}^{2N})$ . In the same way as above for any compact set  $K$  in  $\mathbf{C} \setminus \mathbf{R}$  and  $\ell \geq 0$  we have

$$(6.16) \quad \sup_{\theta \in \Xi, z \in K} \|R_\theta(z)\|_{\mathbf{B}(H^\ell)} < \infty.$$

Furthermore  $R_\theta(z)$  ( $z \in \mathbf{C} \setminus \mathbf{R}$ ) leaves  $\mathcal{S}$  invariant (e.g. Proposition 1.3 of [P]), and so by (6.1) we have

$$(6.17) \quad i[R_\theta(i), A_\theta] = -i R_\theta(i) [H_\theta, A_\theta] R_\theta(i) \\ = -\frac{2}{n_a} R_\theta(i) + R_\theta(i) \left\{ \frac{i}{n_a} [A_0, V^\mu] + \frac{2}{n_a} (V^\mu - i) - v^2 \right\} R_\theta(i)$$

on  $\mathcal{S}$ .

**Lemma 6.2.** *Let  $\ell$  be a nonnegative integer. Then*

$$\sup_{\theta \in \Xi} n_a^{-1} \|R_\theta(i)\|_{\mathbf{B}(H^\ell, H^{\ell+1})} < \infty .$$

*Proof.* By (6.16), (V) and the resolvent equation

$$R_\theta(i) = (H_\theta - V^\mu - i)^{-1} - (H_\theta - V^\mu - i)^{-1} V^\mu R_\theta(i) ,$$

it suffices to prove

$$(6.18) \quad \sup_{\theta \in \Xi} n_a^{-1} \|(H_\theta - V^\mu - i)^{-1}\|_{\mathbf{B}(H^\ell, H^{\ell+1})} < \infty .$$

Thus by (6.4) and the Fourier transform it suffices to show that

$$(6.19) \quad \sup_{\substack{(\omega, \mu, \nu) \in \Xi \\ \xi, \eta \in \mathbf{R}^N}} \frac{1}{n_a} \frac{|\xi| + |\eta| + 1}{|(2m_a)^{-1}|\xi|^2 + (2n_a)^{-1}|\eta|^2 + v\omega \cdot \eta + 1} < \infty .$$

Taking account of the inequality  $2ab \leq a^2 + b^2$  for real  $a, b$ , we have

$$|\xi| + |\eta| + 1 \leq |(1/2m_a)|\xi|^2 + (1/2n_a)|\eta|^2 + v\omega \cdot \eta + n_a(v^2 + 1) + 1 + (m_a/2) .$$

Therefore (6.19) follows.

*Proof of (i), (ii) of Lemma 5.4.*

(I) First we give the proof for  $R(\omega; z)$ .

When  $k = 0$ , we fix  $s > 1/2$  and  $d = 1$ . By Lemma 6.1(ii) and Theorem 3.3, we have

$$(6.20) \quad \sup_{z \in J_\pm, \omega \in S^{N-1}} \|\langle \omega \cdot y \rangle^{-s} R(\omega; z) \langle \omega \cdot y \rangle^{-s}\| < \infty$$

and the norm limits

$$(6.21) \quad \lim_{\varepsilon \downarrow 0} \langle \omega \cdot y \rangle^{-s} R(\omega; \lambda \pm i\varepsilon) \langle \omega \cdot y \rangle^{-s}$$

exist in  $\mathcal{H}$  uniformly for  $\omega \in S^{N-1}$  and  $\lambda \in J$ . From this together with  $\langle \omega \cdot y \rangle^s \langle x; y \rangle^{-s} \leq 1$ ,  $\langle x; y \rangle := (1 + |x|^2 + |y|^2)^{1/2}$ , the desired results follow.

When  $k \geq 1$ , we have only to prove

$$(6.22) \quad \sup_{z \in J_\pm, \omega \in S^{N-1}} \|\langle x; y \rangle^{-s} D^\gamma R(\omega; z) \langle D \rangle^{-k} \langle x; y \rangle^{-s}\| < \infty ,$$

$$(6.23) \quad \lim_{\varepsilon, \varepsilon' \downarrow 0} \|\langle x; y \rangle^{-s} D^\gamma [R(\omega; \lambda \pm i\varepsilon) - R(\omega; \lambda \pm i\varepsilon')] \langle D \rangle^{-k} \langle x; y \rangle^{-s}\| = 0$$

$$\langle D \rangle := (-\Delta_x - \Delta_y + 1)^{1/2}$$

uniformly for  $\lambda \in J$ ,  $\omega \in S^{N-1}$  for  $s > k + 1/2$  and  $|\gamma| \leq k$ . By using (6.14) repeatedly and by taking account of  $\langle x; y \rangle^s D^\gamma \langle D \rangle^{-k} \langle x; y \rangle^{-s} \in \mathbf{B}(\mathcal{H})$  for  $|\gamma| \leq k$ , in order to prove (6.22) and (6.23), it turns out to be sufficient to show

$$(6.24) \quad \sup_{z \in J_\pm, \omega \in S^{N-1}} \|\langle \omega \cdot y \rangle^{-s} R(\omega; z) V_{\gamma_1}^\infty R(\omega; z) \dots V_{\gamma_\ell}^\infty R(\omega; z) \langle \omega \cdot y \rangle^{-s}\| < \infty ,$$

$$(6.25) \quad \lim_{\varepsilon, \varepsilon' \downarrow 0} \|\langle \omega \cdot y \rangle^{-s} \{R(\omega; \lambda \pm i\varepsilon) V_{\gamma_1}^\infty R(\omega; \lambda \pm i\varepsilon) \dots V_{\gamma_\ell}^\infty R(\omega; \lambda \pm i\varepsilon) \\ - R(\omega; \lambda \pm i\varepsilon') V_{\gamma_1}^\infty R(\omega; \lambda \pm i\varepsilon') \dots V_{\gamma_\ell}^\infty R(\omega; \lambda \pm i\varepsilon')\} \langle \omega \cdot y \rangle^{-s}\| = 0 ,$$

uniformly for  $\lambda \in J$ ,  $\omega \in S^{N-1}$ , where  $V_{\gamma_j}^\infty = D^{\gamma_j} V^\infty$  and  $1 \leq \ell \leq k$ ,  $|\gamma_j| \leq k$  ( $j = 1, \dots, \ell$ ). Since  $[V_{\gamma_j}^\infty, A_\omega^\infty] = 0$  for  $j = 1, \dots, \ell$ , taking account of Lemma 6.1, we can apply Theorem 3.3 with  $\mathbf{H} = \mathcal{H}$ ,  $H = H(\omega)$ ,  $A = A_\omega^\infty$ ,  $I = J$ ,  $d = \ell + 1$  and  $W_j = V_{\gamma_j}^\infty$  for  $j = 1, \dots, \ell$  to conclude (6.24) and (6.25). This completes the proof of (i), (ii) for  $R(\omega; z)$ .

(II) We next prove (i), (ii) for  $R_\theta(z) = R^\mu(\omega, v; z)$ .

When  $k = 0$ , we fix  $s$  with  $1/2 < s < 1$  and shall show that

$$(6.26) \quad \sup_{z \in J_\pm, \theta \in \Xi} \|\langle x; y \rangle^{-s} R_\theta(z) \langle x; y \rangle^{-s}\| < \infty,$$

$$(6.27) \quad \lim_{\varepsilon, \varepsilon' \downarrow 0} \sup_{\lambda \in J, \theta \in \Xi} \|\langle x; y \rangle^{-s} [R_\theta(\lambda \pm i\varepsilon) - R_\theta(\lambda \pm i\varepsilon')] \langle x; y \rangle^{-s}\| = 0.$$

By the resolvent equation we get

$$(6.28) \quad R_\theta(z) = R_\theta(i) + (z - i)R_\theta(i)^2 + (z - i)^2 R_\theta(i)R_\theta(z)R_\theta(i)$$

for  $z \in \mathbb{C} \setminus \mathbb{R}$ . Thus, to obtain (6.26), (6.27) it suffices to show that

$$(6.29) \quad \sup_{z \in J_\pm, \theta \in \Xi} \|\langle x; y \rangle^{-s} R_\theta(i)R_\theta(z)R_\theta(i) \langle x; y \rangle^{-s}\| < \infty,$$

$$(6.30) \quad \lim_{\varepsilon, \varepsilon' \downarrow 0} \sup_{\lambda \in J, \theta \in \Xi} \|\langle x; y \rangle^{-s} R_\theta(i) [R_\theta(\lambda \pm i\varepsilon) - R_\theta(\lambda \pm i\varepsilon')] R_\theta(i) \langle x; y \rangle^{-s}\| = 0.$$

By Lemma 6.1(i) and Theorem 3.3 with  $d = 1$ , we have

$$(6.31) \quad \sup_{z \in J_\pm, \theta \in \Xi} \|\langle A_\theta \rangle^{-s} R_\theta(z) \langle A_\theta \rangle^{-s}\| < \infty,$$

$$(6.32) \quad \lim_{\varepsilon, \varepsilon' \downarrow 0} \sup_{\lambda \in J, \theta \in \Xi} \|\langle A_\theta \rangle^{-s} [R_\theta(\lambda \pm i\varepsilon) - R_\theta(\lambda \pm i\varepsilon')] \langle A_\theta \rangle^{-s}\| = 0.$$

We have  $A_\theta R_\theta(i) = R_\theta(i)A_\theta + [A_\theta, R_\theta(i)]$  on  $\mathcal{S}$ .  $[A_\theta, R_\theta(i)]$  is uniformly bounded for  $\theta \in \Xi$  by (6.6) and (6.17). Since

$$(6.33) \quad A_\theta = -i(n_a)^{-1} \nabla_x \cdot x - i(n_a)^{-1} \nabla_y \cdot y + v\omega \cdot y + (N/in_a),$$

we have  $\sup_{\theta \in \Xi} \|R_\theta(i)A_\theta \langle x; y \rangle^{-1}\| < \infty$  by Lemma 6.2, and so we get

$$\sup_{\theta \in \Xi} \|A_\theta R_\theta(i) \langle x; y \rangle^{-1}\| < \infty.$$

By using interpolation this yields

$$(6.34) \quad \sup_{\theta \in \Xi} \|\langle A_\theta \rangle^s R_\theta(i) \langle x; y \rangle^{-s}\| < \infty$$

for  $0 \leq s \leq 1$ . Thus, (6.29) and (6.30) follows from (6.34) together with (6.31) and (6.32), and so (6.26) and (6.27) are obtained.

When  $k \geq 1$ , we fix a real  $s$  with  $k + 1/2 < s < k + 1$ . In the same way as (I), it suffices to prove

$$(6.35) \quad \sup_{z \in J_{\pm}, \theta \in \Xi} \|\langle x; y \rangle^{-s} R_{\theta}(z) V_{\gamma_1}^{\mu} R_{\theta}(z) \dots V_{\gamma_{\ell}}^{\mu} R_{\theta}(z) \langle x; y \rangle^{-s}\| < \infty,$$

$$(6.36) \quad \lim_{\varepsilon, \varepsilon' \downarrow 0} \sup_{\lambda \in J, \theta \in \Xi} \|\langle x; y \rangle^{-s} \{R_{\theta}(\lambda \pm i\varepsilon) V_{\gamma_1}^{\mu} R_{\theta}(\lambda \pm i\varepsilon) \dots V_{\gamma_{\ell}}^{\mu} R_{\theta}(\lambda \pm i\varepsilon) \\ - R_{\theta}(\lambda \pm i\varepsilon') V_{\gamma_1}^{\mu} R_{\theta}(\lambda \pm i\varepsilon') \dots V_{\gamma_{\ell}}^{\mu} R_{\theta}(\lambda \pm i\varepsilon')\} \langle x; y \rangle^{-s}\| = 0,$$

where  $V_{\gamma}^{\mu} = D^{\gamma} V^{\mu}$  and  $1 \leq \ell \leq k$ ,  $|\gamma| \leq k$ .

Using (6.28) repeatedly, we get

$$(6.37) \quad R_{\theta}(z) = \left\{ \sum_{\ell_1, \ell_2; \text{finite}} (z-i)^{\ell_1} R_{\theta}(i)^{\ell_2} \right\} + (z-i)^{2(k+1)} R_{\theta}(i)^{k+1} R_{\theta}(z) R_{\theta}(i)^{k+1}$$

for  $z \in \mathbb{C} \setminus \mathbb{R}$ . Thus, by substituting (6.37) in (6.35) and (6.36), we finally see that the proof of (6.35), (6.36) can be reduced to that of the following

$$(6.38) \quad \sup_{z \in J_{\pm}, \theta \in \Xi} \|\langle A_{\theta} \rangle^{-s} R_{\theta}(z) U_2 R_{\theta}(z) \dots U_m R_{\theta}(z) \langle A_{\theta} \rangle^{-s}\| < \infty,$$

$$(6.39) \quad \lim_{\varepsilon, \varepsilon' \downarrow 0} \sup_{\lambda \in J, \theta \in \Xi} \|\langle A_{\theta} \rangle^{-s} \{R_{\theta}(\lambda \pm i\varepsilon) U_2 R_{\theta}(\lambda \pm i\varepsilon) \dots U_m R_{\theta}(\lambda \pm i\varepsilon) \\ - R_{\theta}(\lambda \pm i\varepsilon') U_2 R_{\theta}(\lambda \pm i\varepsilon') \dots U_m R_{\theta}(\lambda \pm i\varepsilon')\} \langle A_{\theta} \rangle^{-s}\| = 0,$$

$$(6.40) \quad \sup_{\theta \in \Xi} \{\|\langle x; y \rangle^{-s} U_1 \langle A_{\theta} \rangle^s\| + \|\langle A_{\theta} \rangle^s U_{m+1} \langle x; y \rangle^{-s}\|\} < \infty$$

for  $2 \leq m \leq k+1$ , where each  $U_j$  is the form

$$(6.41) \quad U_j = R_{\theta}(i) Q_1 R_{\theta}(i) \dots Q_h R_{\theta}(i) \quad (h \geq k)$$

with  $Q_n = 1$  or  $V_{\gamma}^{\mu}$  ( $|\gamma| \leq k$ ).

We first prove (6.38) and (6.39) by applying Theorem 3.3. It follows from (6.17) and  $k \leq \varepsilon_0 - 2$  that  $q$ -th commutators ( $0 \leq q \leq k+2$ )  $[\dots [U_j, A_{\theta}], \dots, A_{\theta}]$  on  $\mathcal{S}$  can be extended to bounded operators  $U_j^{(q)}$  on  $\mathcal{H}$ , and their operator norms are uniformly bounded for  $\theta \in \Xi$ . Thus each  $U_j$  ( $j = 1, \dots, m+1$ ) satisfies Assumption 3.2 with  $A = A_{\theta}$  and  $d \leq k+1$ , and so (6.38) and (6.39) follow from Lemma 6.1 (i) and Theorem 3.3 with  $d \leq k+1$ . Next we shall prove (6.40). We have

$$(6.42) \quad U_1 A_{\theta}^{k+1} = A_{\theta}^{k+1} U_1 + [U_1, A_{\theta}^{k+1}] \quad \text{on } \mathcal{S}.$$

$A_{\theta}^j$  ( $j = 0, \dots, k+1$ ) has the form

$$(6.43) \quad A_{\theta}^j = \sum_{|\gamma_1 + \gamma_2|, |\gamma_3| \leq j} C_{\gamma_1 \gamma_2 \gamma_3} x^{\gamma_1} y^{\gamma_2} \left( \frac{1}{n_a} D \right)^{\gamma_3},$$

where  $C_{\gamma_1 \gamma_2 \gamma_3}$  are constants uniformly bounded for  $\theta \in \Xi$ . Since  $U_1$  contains at least  $(k+1)R_{\theta}(i)$ , we obtain

$$(6.44) \quad \sup_{\theta \in \Xi} \left\| \left( \frac{1}{n_a} D \right)^{\gamma} U_1 \right\| < \infty \quad \text{for } |\gamma| \leq k+1$$

by Lemma 6.2 and (V), and so we have

$$(6.45) \quad \sup_{\theta \in \Xi} \|\langle x; y \rangle^{-k-1} A_\theta^{k+1} U_1\| < \infty.$$

The commutator  $[U_1, A_\theta^{k+1}]$  has the form

$$[U_1, A_\theta^{k+1}] = \sum_{j=0}^k C_j A_\theta^j U_1^{(k+1-j)},$$

where  $C_j$  are constants independent of  $\theta \in \Xi$ . We note that (6.17), (V) and Lemma 6.2 yield

$$(6.46) \quad \sup_{\theta \in \Xi} \left\| \left( \frac{1}{n_a} D \right)^\gamma U_1^{(k+1-j)} \right\| < \infty$$

for  $|\gamma| \leq k+1$  and for  $j = 0, \dots, k$ , and so we obtain

$$(6.47) \quad \sup_{\theta \in \Xi} \|\langle x; y \rangle^{-k-1} [U_1, A_\theta^{k+1}]\| < \infty.$$

It follows from (6.45), (6.47) that

$$(6.48) \quad \sup_{\theta \in \Xi} \|\langle x; y \rangle^{-k-1} U_1 \langle A_\theta \rangle^{k+1}\| < \infty.$$

Similarly, we have

$$(6.49) \quad \sup_{\theta \in \Xi} \|\langle A_\theta \rangle^{k+1} U_{m+1} \langle x; y \rangle^{-k-1}\| < \infty.$$

Therefore (6.40) follows by interpolation. This completes the proof of (i), (ii) of Lemma 5.4.

It remains to prove (iii), (iv) of Lemma 5.4.

**Lemma 6.3.** *Let  $\ell$  be a nonnegative integer and  $f \in H^\ell$ . Then for each  $\delta > 0$ ,*

$$(6.50) \quad s - \lim_{\lambda' \rightarrow \lambda, \mu \rightarrow \infty, v \rightarrow v_0} R^\mu(\omega, v; \lambda' + i\delta) f = R(\omega; \lambda + i\delta) f \quad \text{in } H^\ell$$

uniformly for  $\omega \in S^{N-1}$ .

*Proof.* By (6.15) and (6.16) we may assume  $f \in \mathcal{S}$ . We have

$$(6.51) \quad [R^\mu(\omega, v; \lambda' + i\delta) - R(\omega; \lambda + i\delta)] f = -R^\mu(\omega, v; \lambda' + i\delta) \\ \times \left[ \left( \frac{1}{2m} - \frac{1}{2m_a} \right) A_x - \frac{1}{2n_a} A_y - i(v - v_0) \omega \cdot \nabla_y - \lambda' + \lambda \right] R(\omega; \lambda + i\delta) f \\ - R^\mu(\omega, v; \lambda' + i\delta) [V^\mu - V^\infty] R(\omega; \lambda + i\delta) f.$$

Taking account of (6.15), (6.16), we see that the first term in the R.H.S. goes to zero in  $H^\ell$  uniformly for  $\omega \in S^{N-1}$  as  $\mu \rightarrow \infty$ ,  $v \rightarrow v_0$ ,  $\lambda' \rightarrow \lambda$ . It is obvious that

$$(6.52) \quad \lim_{\mu \rightarrow \infty} \|[V^\mu - V^\infty] R(\omega; \lambda + i\delta) f\|_{\ell, 0} = 0$$

for each  $\omega \in S^{N-1}$ . Since  $R(\omega; \lambda + i\delta)f$  is  $H^l$ -valued strongly continuous function of  $\omega \in S^{N-1}$ , we can see that (6.52) hold uniformly for  $\omega$  by the finite covering argument (see (5.37)). Thus, by (6.15), the second term in the R.H.S. of (6.51) goes to zero in  $H^l$  uniformly for  $\omega \in S^{N-1}$  as  $\mu \rightarrow \infty$ ,  $v \rightarrow v_0$ ,  $\lambda' \rightarrow \lambda$ . This proves the lemma.

*Proof of (iii) of Lemma 5.4.* Fix  $f \in H_s^k$  for  $s > k + 1/2$  and a sufficiently small  $\tau > 0$ . By (ii) of Lemma 5.4 already shown, we can take a  $\delta > 0$  such that

$$(6.53) \quad \sup_{\substack{\lambda, \lambda' \in J \\ \theta = (\omega, \mu, v) \in \Xi}} \{ \| [R_\theta(\lambda' + i0) - R_\theta(\lambda + i\delta)]f \|_{H_s^k} \\ + \| [R(\omega; \lambda + i\delta) - R(\omega; \lambda + i0)]f \|_{H_s^k} \} < \tau.$$

This together with Lemma 6.3 gives the desired result.

*Proof of (iv) of Lemma 5.4.* By (i) of Lemma 5.4 already shown, we may assume  $f \in \mathcal{L}$ . For any  $\xi \in S^{N-1}$ , we have

$$(6.54) \quad [R(\xi; \lambda \pm i0) - R(\omega; \lambda \pm i0)]f = [R(\xi; \lambda \pm i0) - R(\xi; \lambda \pm i\varepsilon)]f \\ + [R(\omega; \lambda \pm i\varepsilon) - R(\omega; \lambda \pm i0)]f \\ + [R(\xi; \lambda \pm i\varepsilon) - R(\omega; \lambda \pm i\varepsilon)]f.$$

By using the resolvent equation,

$$(6.55) \quad R(\xi; z) - R(\omega; z) = R(\omega; z)iv_0(\xi - \omega) \cdot \nabla_y R(\omega; z)$$

for  $z \in \mathbf{C} \setminus \mathbf{R}$ , we can easily show that the last term goes to zero as  $\omega \rightarrow \xi$  in  $H_s^k(\mathbf{R}^{2N})$  for each  $\varepsilon > 0$ . By Lemma 5.4 (ii), the others go to zero uniformly for  $\xi$ ,  $\omega \in S^{N-1}$  as  $\varepsilon \rightarrow 0$ . This completes the proof.

## §7. Proof of Lemma 5.11 and Lemma 5.12

**Lemma 7.1.** *Let  $k > 1/2$ ,  $s > (N - 1)/2$ . Then for any  $\varepsilon > 0$  and any  $0 < \delta \leq 1/2$  there exist positive constants  $R_0 = R_0(\varepsilon, \delta, s, k, N)$ ,  $C = C(s, k, N)$  such that*

$$(7.1) \quad \left| R^{N-1} \int_{S^{N-1}} F(R(\omega' - \omega))h(\omega')d\omega' - h(\omega) \int_{\Pi_\omega} F(\eta)d\eta \right| \\ \leq \varepsilon \|h\|_{L^\infty(S^{N-1})} \|F\|_{k,s} + C \sup_{|\omega - \omega'| < \delta} |h(\omega') - h(\omega)| \|F\|_{k,s}$$

for all  $R \geq R_0$ ,  $h \in C(S^{N-1})$ ,  $F \in H_s^k(\mathbf{R}^N)$  and  $\omega \in S^{N-1}$ .

*Proof of Lemma 5.11.* Immediate from Lemma 7.1.

Lemma 7.1 will be proved after the series of lemmas.



**Lemma 7.2.** *Let  $k > 1/2$ ,  $s > (N - 1)/2$ . Then for any  $0 < \delta < 1$  there exists a positive constant  $C = C(\delta, s, k, N)$  such that*

$$(7.2) \quad R^{N-1} \int_{|\omega - \omega'| > \delta} |F(R(\omega' - \omega))| d\omega' \leq C \cdot R^{-s + ((N-1)/2)} \|F\|_{k,s}$$

for all  $R \geq 1$ ,  $F \in \mathcal{S}(\mathbf{R}^N)$  and  $\omega \in S^{N-1}$ .

*Proof.* For each  $\omega \in S^{N-1}$  there exists a  $\chi_\omega \in C^\infty(\mathbf{R}^N)$  such that

$$\begin{aligned} 0 \leq \chi_\omega \leq 1, \quad \text{supp } \chi_\omega &\subset \{\xi \in \mathbf{R}^N; |\hat{\xi} - \omega| > \delta/2, |\xi| > 1/2\}, \\ \chi_\omega = 1 \quad \text{on } \{\xi \in \mathbf{R}^N; |\hat{\xi} - \omega| \geq \delta, |\xi| \geq 1\}, \quad (\hat{\xi} := \xi/|\xi|), \\ \sup_{\omega \in S^{N-1}, \xi \in \mathbf{R}^N} |D_\xi^\gamma \chi_\omega(\xi)| &< 0 \quad \text{for each } \gamma. \end{aligned}$$

Let  $dS_R(\xi)$  be the Lebesgue measure on  $S_R := \{\xi \in \mathbf{R}^N; |\xi| = R\}$ . Then

$$\begin{aligned} (7.3) \quad R^{N-1} \int_{|\omega - \omega'| > \delta} |F(R(\omega' - \omega))| d\omega' &= \int_{|\hat{\xi} - \omega| > \delta} |F(\xi - R\omega)| dS_R(\xi) \\ &\leq \|\chi_\omega(\xi) F(\xi - R\omega)\|_{L^1(S_R)} \\ &\leq C_1 \cdot R^{-s + ((N-1)/2)} \cdot \|\langle \xi \rangle^s \chi_\omega(\xi) F(\xi - R\omega)\|_{L^2(S_R)} \\ &\leq C_2 \cdot R^{-s + ((N-1)/2)} \cdot \|\langle \xi \rangle^s \chi_\omega(\xi) F(\xi - R\omega)\|_{H^k(\mathbf{R}^N)} \\ &\leq C_2 \cdot R^{-s + ((N-1)/2)} \|\langle D_\xi \rangle^k \langle \xi \rangle^s \langle \xi - R\omega \rangle^{-s} \chi_\omega \langle D_\xi \rangle^{-k}\|_{\mathbf{B}(L^2)} \\ &\quad \times \|\langle \xi - R\omega \rangle^s F(\xi - R\omega)\|_{H^k}, \end{aligned}$$

where in the second step we have used the Schwarz inequality, and in the last step but one we have used the fact that

$$(7.4) \quad \|f(\xi)\|_{L^2(S_R)} \leq C \|f\|_{H^k(\mathbf{R}^N)}$$

for all  $R \geq 1$  and  $f \in H^k(\mathbf{R}^N)$ , where  $C$  is independent of  $R$  (cf. Proposition (2.1) of [G-M]). Note that

$$\|\langle \xi - R\omega \rangle^s F(\xi - R\omega)\|_{H^k} = \|\langle \xi \rangle^s F(\xi)\|_{H^k} \leq C \|F\|_{H_s^k}.$$

Since  $\omega \cdot \xi \leq (1 - (\delta^2/8))|\xi|$  for  $\xi \in \text{supp } \chi_\omega$ , we have

$$|\xi - R\omega|^2 \geq (1 - (\delta^2/8))(|\xi| - R)^2 + (\delta^2/8)(|\xi|^2 + R^2) \geq (\delta^2/8)\langle \xi \rangle^2$$

for  $\xi \in \text{supp } \chi_\omega$ ,  $\omega \in S^{N-1}$  and  $R \geq 1$ , and so we obtain for any multi-index  $\gamma$

$$(7.5) \quad \sup_{R \geq 1, \omega \in S^{N-1}, \xi \in \mathbf{R}^N} |D_\xi^\gamma \langle \xi \rangle^s \langle \xi - R\omega \rangle^{-s} \chi_\omega(\xi)| < \infty.$$

This means that  $\|\langle D_\xi \rangle^k \langle \xi \rangle^s \langle \xi - R\omega \rangle^{-s} \chi_\omega \langle D_\xi \rangle^{-k}\|_{\mathbf{B}(L^2(\mathbf{R}^N))}$  is uniformly bounded for  $R \geq 1$  and  $\omega \in S^{N-1}$ . Thus we have obtained (7.2).

**Lemma 7.3.** *Let  $0 \leq t \leq 1$ ,  $k - (1/2) > t$ ,  $s - ((N - 1)/2) > t$ . Then there exists a positive constant  $C = C(t, s, k, N)$  such that*

$$(7.6) \quad \int_{|\eta| < \delta R} |F(\eta, \sqrt{R^2 - \eta^2} - R) - F(\eta, 0)| d\eta \leq C \cdot \delta^t \|F\|_{k,s}$$

for all  $R \geq 1$ ,  $0 < \delta \leq 1/2$  and  $F \in \mathcal{S}(\mathbf{R}^N)$ , where  $\eta \in \mathbf{R}^{N-1}$ .

*Proof.* Using the Fourier transform, we have

$$(7.7) \quad F(\eta, \sqrt{R^2 - \eta^2} - R) - F(\eta, 0) = (2\pi)^{-N/2} \int_{-\infty}^{\infty} [e^{ix_N(\sqrt{R^2 - \eta^2} - R)} - 1] dx_N \\ \times \int e^{ix' \cdot \eta \hat{F}(x', x_N)} dx' \quad (x' \in \mathbf{R}^{N-1}).$$

Noting that  $|e^{ir} - 1| \leq 2|r|^t$  for all  $r \in \mathbf{R}$  and all  $0 \leq t \leq 1$ , and that

$$|x_N(\sqrt{R^2 - \eta^2} - R)| = \eta^2 \cdot (R + \sqrt{R^2 - \eta^2})^{-1} |x_N| \leq \delta |\eta| |x_N| \quad \text{for } |\eta| \leq \delta R,$$

we have

$$(7.8) \quad |e^{ix_N(\sqrt{R^2 - \eta^2} - R)} - 1| \leq 2\delta^t |\eta|^t |x_N|^t \quad \text{for } |\eta| \leq \delta R.$$

Thus, by using the Schwarz inequality, we obtain

$$(7.9) \quad |F(\eta, \sqrt{R^2 - \eta^2} - R) - F(\eta, 0)|^2 \leq C \cdot \delta^{2t} |\eta|^{2t} \int_{-\infty}^{\infty} \langle x_N \rangle^{2k} dx_N \left| \int e^{ix' \cdot \eta \hat{F}(x', x_N)} dx' \right|^2,$$

since  $k - (1/2) > t$ . (7.9) and the Schwarz inequality give

$$(7.10) \quad \int_{|\eta| < \delta R} |F(\eta, \sqrt{R^2 - \eta^2} - R) - F(\eta, 0)| d\eta \\ \leq C \cdot \left\{ \int_{|\eta| < \delta R} \langle \eta \rangle^{2(s-t)} |F(\eta, \sqrt{R^2 - \eta^2} - R) - F(\eta, 0)|^2 d\eta \right\}^{1/2} \\ \leq C \cdot \delta^t \left\{ \int_{-\infty}^{\infty} \langle x_N \rangle^{2k} dx_N \int \langle \eta \rangle^{2s} d\eta \left| \int e^{ix' \cdot \eta \hat{F}(x', x_N)} dx' \right|^2 \right\}^{1/2} \\ \leq C \cdot \delta^t \left\{ \int_{-\infty}^{\infty} \langle x_N \rangle^{2k} dx_N \int |\langle D_{x'} \rangle^s \hat{F}(x', x_N)|^2 dx' \right\}^{1/2} \\ \leq C \cdot \delta^t \|F\|_{k,s},$$

where we have used the Parseval equality in  $N - 1$  variables in the third step.

**Lemma 7.4.** *Let  $k > 1/2$ ,  $s > (N - 1)/2$ . Then for any  $\varepsilon > 0$  there exists a positive constant  $R_0 = R_0(\varepsilon, s, k) \geq 1$  such that*

$$(7.11) \quad \left| R^{N-1} \int_{S^{N-1}} F(R(\omega' - \omega)) d\omega' - \int_{\Pi_\omega} F(\eta) d\eta \right| \leq \varepsilon \|F\|_{k,s}$$

for all  $R \geq R_0$ ,  $F \in \mathcal{S}(\mathbf{R}^N)$  and  $\omega \in S^{N-1}$ .

*Proof.* Fix a sufficiently small positive constant  $\delta$ , and let  $\delta' = \delta\sqrt{1 - (\delta^2/4)}$ . We have

$$\begin{aligned}
 (7.12) \quad & \int_{S^{N-1}} F(\xi - R\omega) dS_R(\xi) - \int_{\Pi_\omega} F(\eta) d\eta = \int_{|\xi-\omega|>\delta} F(\xi - R\omega) dS_R(\xi) \\
 & + \left\{ \int_{|\xi-\omega|<\delta} F(\xi - R\omega) dS_R(\xi) - \int_{\substack{|\eta|<\delta'R \\ \eta \in \Pi_\omega}} F(\eta) \frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} d\eta \right\} \\
 & + \int_{\substack{|\eta|<\delta'R \\ \eta \in \Pi_\omega}} F(\eta) \cdot \left[ \frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} - 1 \right] d\eta - \int_{\substack{|\eta|>\delta'R \\ \eta \in \Pi_\omega}} F(\eta) d\eta \\
 & = I_1 + I_2 + I_3 + I_4 .
 \end{aligned}$$

Applying Lemma 7.2 to  $I_1$ , we have

$$(7.13) \quad |I_1| \leq C_1 R^{-s+((N-1)/2)} \|F\|_{k,s}, \quad R \geq 1,$$

where  $C_1 = C_1(\delta, s, k, N) > 0$ . Next we claim that

$$(7.14) \quad |I_2| \leq C_2 \delta'^t \|F\|_{k,s}$$

for any  $t > 0$  satisfying  $\text{Min}\{s - ((N-1)/2), k - (1/2), 1\} > t$ , where  $C_2 = C_2(t, s, k, N) > 0$ . Indeed, in the case  $\omega = (0, \dots, 0, 1)$ , noting that

$$(7.15) \quad \int_{|\xi-\omega|<\delta} F(\xi - R\omega) dS_R(\xi) = \int_{|\eta|<\delta'R} F(\eta, \sqrt{R^2 - \eta^2} - R) \frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} d\eta,$$

and that  $\frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} \leq 2$  for  $|\eta| < \delta'R$  for sufficiently small  $\delta$ , and applying Lemma

7.3, we have (7.14). In other cases we have only to change the coordinates.

Noting that for  $|\eta| < \delta'R$

$$\left| \frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} - 1 \right| \leq C \cdot \frac{|\eta|^2}{|R|^2} \leq C\delta'^2,$$

we have

$$(7.16) \quad |I_3| \leq C \cdot \delta'^2 \|\langle \eta \rangle^s F\|_{L^2(\Pi_\omega)} \leq C_3 \delta'^2 \|F\|_{k,s},$$

where  $C_3 = C_3(s, k, N)$ , and we have used the trace theorem in the last step.  $I_4$  is estimated as follows.

$$(7.17) \quad |I_4| \leq \left\{ \int_{\substack{|\eta| > \delta'R \\ \eta \in \Pi_\omega}} \langle \eta \rangle^{-2s} d\eta \right\}^{1/2} \left\{ \int_{\eta \in \Pi_\omega} \langle \eta \rangle^{2s} |F(\eta)|^2 d\eta \right\}^{1/2} \\ \leq C_4 (\delta'R)^{-s + ((N-1)/2)} \|F\|_{k,s},$$

where  $C_4 = C_4(s, k, N) > 0$ . Thus we have the desired result by taking  $\delta$  sufficiently small and then taking  $R$  sufficiently large.

*Proof of Lemma 7.1.* We may assume  $F \in \mathcal{S}(\mathbf{R}^N)$  by the approximation, since  $\mathcal{S}(\mathbf{R}^N)$  is dense in  $H_s^k(\mathbf{R}^N)$ . We have

$$(7.18) \quad R^{N-1} \int_{S^{N-1}} F(R(\omega' - \omega)) h(\omega') d\omega' = R^{N-1} \int_{S^{N-1}} F(R(\omega' - \omega)) d\omega' \cdot h(\omega) \\ + R^{N-1} \int_{|\omega - \omega'| < \delta} F(R(\omega' - \omega)) [h(\omega') - h(\omega)] d\omega' \\ + R^{N-1} \int_{|\omega - \omega'| > \delta} F(R(\omega' - \omega)) [h(\omega') - h(\omega)] d\omega' \\ = J_1 + J_2 + J_3.$$

We first show that

$$(7.19) \quad R^{N-1} \int_{|\omega - \omega'| < \delta} |F(R(\omega' - \omega))| d\omega' \leq C_1 \|F\|_{k,s},$$

where  $C_1 = C_1(s, k, N) > 0$ . By the change of coordinates we may assume  $\omega = (0, \dots, 0, 1)$ . Then,

$$(7.20) \quad R^{N-1} \int_{|\omega - \omega'| < \delta} |F(R(\omega' - \omega))| d\omega' \leq C \cdot \int_{|\eta| < \delta'R} |F(\eta, \sqrt{R^2 - \eta^2} - R)| d\eta,$$

where  $\delta' = \delta \sqrt{1 - (\delta^2/4)}$  (see (7.15)). By Lemma 7.3 we have

$$(7.21) \quad \left| \int_{|\eta| < \delta'R} |F(\eta, \sqrt{R^2 - \eta^2} - R)| d\eta - \int_{|\eta| < \delta'R} |F(\eta, 0)| d\eta \right| \\ \leq \int_{|\eta| < \delta'R} |F(\eta, \sqrt{R^2 - \eta^2} - R) - F(\eta, 0)| d\eta \\ \leq C \cdot \|F\|_{k,s},$$

and by the trace theorem we have

$$(7.22) \quad \int_{|\eta| < \delta'R} |F(\eta, 0)| d\eta \leq C \cdot \|F\|_{k,s}.$$

Thus, by (7.20) ~ (7.22), we obtain (7.19). (7.19) yields

$$(7.23) \quad |J_2| \leq C_1 \cdot \sup_{|\omega - \omega'| < \delta} |h(\omega') - h(\omega)| \cdot \|F\|_{k,s}.$$

By Lemma 7.2 we have

$$(7.24) \quad |J_3| \leq C_2 R^{-s+(N-1)/2} \|h\|_{L^\infty} \|F\|_{k,s},$$

where  $C_2 = C_2(\delta, s, k, N) > 0$ .

By taking  $R$  sufficiently large in (7.24), the desired result follows from (7.23), (7.24) and Lemma 7.4.

*Proof of Lemma 5.12.* The proof is similar to that of Lemma 7.1, and so we give a sketch of the proof. We may assume  $0 < s < 1$ . Let  $0 < \delta \ll 1$  and let  $\delta'$  be the same as in the proof of Lemma 7.4. We write

$$(7.25) \quad \begin{aligned} & \int_{S^{N-1}} |F(\xi - R\omega)|^2 dS_R(\xi) - \int_{\Pi_\omega} |F(\eta)|^2 d\eta = \int_{|\xi-\omega|>\delta} |F(\xi - R\omega)|^2 dS_R(\xi) \\ & + \left\{ \int_{|\xi-\omega|<\delta} |F(\xi - R\omega)|^2 dS_R(\xi) - \int_{\substack{|\eta|<\delta'R \\ \eta \in \Pi_\omega}} |F(\eta)|^2 \frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} d\eta \right\} \\ & + \int_{\substack{|\eta|<\delta'R \\ \eta \in \Pi_\omega}} |F(\eta)|^2 \left[ \frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} - 1 \right] d\eta - \int_{\substack{|\eta|>\delta'R \\ \eta \in \Pi_\omega}} |F(\eta)|^2 d\eta \\ & = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

In the same way as Lemma 7.2 we have

$$(7.26) \quad |I_1| \leq C_1 \langle R \rangle^{-2s} \|\langle \xi \rangle^s \chi_\omega(\xi) F(\xi - R\omega)\|_{k,0}^2 \leq C_2 \langle R \rangle^{-2s} \|F\|_{k,s}^2,$$

where  $C_2 = C_2(\delta, s, k, N)$ . We next estimate  $I_2$ , and assume  $\omega = (0, \dots, 0, 1)$ . Since  $0 < \delta < 1$ , we have by the trace theorem

$$\int_{|\eta|<\delta'R} |F(\eta, 0)|^2 \frac{1}{\sqrt{1 - \frac{|\eta|^2}{|R|^2}}} d\eta \leq C_3 \|F\|_{k,0}^2,$$

Furthermore, taking  $t = s$  in (7.9), and integrating it w.r.t.  $\eta$ , we get

$$\int_{|\eta|<\delta'R} |F(\eta, \sqrt{R^2 - \eta^2} - R) - F(\eta, 0)|^2 d\eta \leq C_4 \delta^{2s} \|F\|_{k,s}^2.$$

Therefore, by using the inequality

$$\|f\|^2 - \|g\|^2 \leq \|f - g\|^2 + 2\|f - g\| \|g\|,$$

we obtain

$$(7.27) \quad |I_2| \leq C_5 \delta^s \|F\|_{k,s}^2.$$

In a way similar to the proof of Lemma 7.4,  $I_3$  and  $I_4$  can be estimated as follows;

$$(7.28) \quad |I_3| \leq C_6 \delta'^2 \|F\|_{k,s}^2, \quad |I_4| \leq C_7 \langle \delta'R \rangle^{-2s} \|F\|_{k,s}^2.$$

Here note that  $C_j = C_j(s, k, N)$  for  $j = 3, \dots, 7$ . Thus we have

$$|I_1 + I_2 + I_3 - I_4| \leq C_8 \{C(\delta) \langle R \rangle^{-2s} + \delta^s + \langle \delta' R \rangle^{-2s}\} \|F\|_{k,s}^2,$$

where  $C_8 = C_8(s, k, N)$ , which implies the desired result.

### §8. Proof of formula (2.10)

We shall prove here formula (2.10) of Theorem 2.3. The superscript  $\mu$  will be omitted in the proof. Let  $J$  be a compact interval of  $(\lambda_{\beta\alpha}, \infty) \setminus A$  (see below (1.15) for  $\lambda_{\beta\alpha} = \lambda_{\beta\alpha}^\mu$ ,  $A = A^\mu$ ), and fix  $f_\gamma \in L_s^2(\mathbf{R}^N)$ ,  $s > 1/2$ , such that  $E_{T_\gamma}(J)f_\gamma = f_\gamma$  for  $\gamma = \alpha, \beta$ , where  $E_{T_\gamma}(\cdot)$  is the spectral measure of  $T_\gamma$  (see (1.9)). Considering in the momentum space, we can see that such  $f_\gamma$ 's form a dense set in  $E_{T_\gamma}(J)L^2(\mathbf{R}^N)$ .

We denote the resolvents of  $H$ ,  $h_c \otimes I + I \otimes (-(2n_c)^{-1}A_{y_c})$ ,  $T_\gamma$  by  $R(z)$ ,  $R_\gamma(z)$  and  $r_\gamma(z)$  ( $D(\gamma) = c$ ), respectively (see (1.7) for  $h_c$ ). The following relation are obvious:

$$(8.1) \quad R_\gamma(z)P_\gamma = P_\gamma r_\gamma(z) \quad (\text{see (1.9)}).$$

Using the intertwining relation, we have

$$(8.2) \quad \begin{aligned} (S_{\beta\alpha}f_\alpha, f_\beta) &= (W_{\alpha-}f_\alpha, W_{\beta+}f_\beta) \\ &= s - \lim_{t \rightarrow -\infty} (e^{itH}P_\alpha e^{-itT_\alpha}f_\alpha, W_{\beta+}f_\beta) \\ &= s - \lim_{t \rightarrow -\infty} (P_\alpha e^{-itT_\alpha}f_\alpha, W_{\beta+}e^{-itT_\beta}f_\beta) \\ &= \lim_{\varepsilon \downarrow 0} 2\varepsilon \int_{-\infty}^0 e^{2\varepsilon t} (P_\alpha e^{-itT_\alpha}f_\alpha, W_{\beta+}e^{-itT_\beta}f_\beta) dt. \end{aligned}$$

Let  $\chi(t) = 1$  for  $t < 0$  and  $= 0$  for  $t \geq 0$ . Then, for each  $\varepsilon > 0$ , the inverse Fourier transforms of the vector-valued functions

$$P_\alpha e^{\varepsilon t - itT_\alpha} \chi(t) f_\alpha, \quad W_{\beta+} e^{\varepsilon t - itT_\beta} \chi(t) f_\beta$$

are  $i(2\pi)^{-1/2} P_\alpha r_\alpha(\lambda - i\varepsilon) f_\alpha$ ,  $i(2\pi)^{-1/2} W_{\beta+} r_\beta(\lambda - i\varepsilon) f_\beta$ , respectively. Therefore, applying the Parseval equality to the above integral in (8.2), we get

$$(8.3) \quad \begin{aligned} (S_{\beta\alpha}f_\alpha, f_\beta) &= \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} (P_\alpha r_\alpha(\lambda - i\varepsilon) f_\alpha, W_{\beta+} r_\beta(\lambda - i\varepsilon) f_\beta) d\lambda \\ &= \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_J (P_\alpha r_\alpha(\lambda - i\varepsilon) f_\alpha, W_{\beta+} r_\beta(\lambda - i\varepsilon) f_\beta) d\lambda, \end{aligned}$$

where we have used  $E_{T_\gamma}(J)f_\gamma = f_\gamma$  for  $\gamma = \alpha, \beta$ .

Set  $u_\gamma = r_\gamma(\lambda - i\varepsilon) f_\gamma$  ( $\gamma = \alpha, \beta$ ). Similarly to the above, we obtain

$$(8.4) \quad \begin{aligned} (P_\alpha u_\alpha, W_{\beta+} u_\beta) &= \lim_{\delta \downarrow 0} \frac{\delta}{\pi} \int_{-\infty}^{\infty} (R(\zeta + i\delta) P_\alpha u_\alpha, P_\beta r_\beta(\zeta + i\delta) u_\beta) d\zeta \\ &= \lim_{\delta \downarrow 0} \frac{\delta}{\pi} \int_J (R(\zeta + i\delta) P_\alpha u_\alpha, P_\beta r_\beta(\zeta + i\delta) u_\beta) d\zeta, \end{aligned}$$

where we have used  $\int_{-\infty}^{\infty} \|R(\zeta + i\delta)P_\alpha u_\alpha\|^2 d\zeta = \frac{\pi}{\delta} \|P_\alpha u_\alpha\|^2$ , and  $E_{T_\beta}(J)f_\beta = f_\beta$  in the last step. For  $z \in \mathbf{C} \setminus \mathbf{R}$  we define  $K(z) := -I_a + I_b R(z)I_a$  ( $a = D(\alpha)$ ,  $b = (\beta)$ );  $I_a := V - V_{j_3}$  for  $a = \{i, (j, 3)\}$ . Then the following relation

$$R(\zeta + i\delta) = R_\alpha(\zeta + i\delta) + R_\beta(\zeta + i\delta)K(\zeta + i\delta)R_\alpha(\zeta + i\delta)$$

is obtained by using the resolvent equations:

$$R(z) = R_\alpha(z) - R(z)I_a R_\alpha(z), \quad R(z) = R_\beta(z) - R_\beta(z)I_b R(z).$$

Substituting this in (8.4), we have

$$(8.5) \quad (P_\alpha u_\alpha, W_{\beta+} u_\beta) = \lim_{\delta \downarrow 0} \frac{\delta}{\pi} \int_J (R_\alpha(\zeta + i\delta)P_\alpha u_\alpha, P_\beta r_\beta(\zeta + i\delta)u_\beta) d\zeta, \\ + \lim_{\delta \downarrow 0} \frac{\delta}{\pi} \int_J (K(\zeta + i\delta)P_\alpha r_\alpha(\zeta + i\delta)u_\alpha, P_\beta r_\beta(\zeta - i\delta)r_\beta(\zeta + i\delta)u_\beta) d\zeta,$$

where we have used (8.1). Reversing the argument used for showing (8.3), we see that the first term in the R.H.S. of (8.5) is equal to

$$\lim_{t \rightarrow +\infty} (P_\alpha e^{-itT_\alpha} u_\alpha, P_\beta e^{-itT_\beta} u_\beta) = (W_{\alpha+} u_\alpha, W_{\beta+} u_\beta) \\ = \delta_{\beta\alpha} (P_\alpha r_\alpha(\lambda - i\varepsilon)f_\alpha, P_\beta r_\beta(\lambda - i\varepsilon)f_\beta).$$

In the last step we have used the fact that  $\text{Ran } W_{\alpha+} \perp \text{Ran } W_{\beta+}$  ( $\text{Ran} = \text{Range}$ ) for  $\alpha \neq \beta$ .

We next consider the second term of (8.5). By the first resolvent equation we have

$$r_\alpha(\zeta + i\delta)u_\alpha = \frac{1}{(\zeta - \lambda) + i(\varepsilon + \delta)} \{r_\alpha(\zeta + i\delta) - r_\alpha(\lambda - i\varepsilon)\}f_\alpha, \\ r_\beta(\zeta + i\delta)r_\beta(\zeta - i\delta)u_\beta = (2i\delta)^{-1} \left[ \frac{1}{(\zeta - \lambda) + i(\varepsilon + \delta)} \{r_\beta(\zeta + i\delta) - r_\beta(\lambda - i\varepsilon)\}f_\beta \right. \\ \left. + \frac{1}{(\zeta - \lambda) + i(\varepsilon - \delta)} \{r_\beta(\zeta - i\delta) - r_\beta(\lambda - i\varepsilon)\}f_\beta \right].$$

Thus we obtain

$$(P_\alpha r_\alpha(\lambda - i\varepsilon)f_\alpha, W_{\beta+} r_\beta(\lambda - i\varepsilon)f_\beta) \\ = \delta_{\beta\alpha} (P_\alpha r_\alpha(\lambda - i\varepsilon)f_\alpha, P_\beta r_\beta(\lambda - i\varepsilon)f_\beta) + \lim_{\delta \downarrow 0} \left[ \frac{i}{2\pi} \int_J \frac{1}{(\zeta - \lambda)^2 + (\varepsilon + \delta)^2} \right. \\ \times (K(\zeta + i\delta)P_\alpha \{r_\alpha(\zeta + i\delta) - r_\alpha(\lambda - i\varepsilon)\}f_\alpha, P_\beta \{r_\beta(\zeta + i\delta) - r_\beta(\lambda - i\varepsilon)\}f_\beta) d\zeta \\ - \frac{i}{2\pi} \int_J \frac{1}{\{(\zeta - \lambda) + i(\varepsilon + \delta)\} \{(\zeta - \lambda) - i(\varepsilon - \delta)\}} \\ \left. \times (K(\zeta + i\delta)P_\alpha \{r_\alpha(\zeta + i\delta) - r_\alpha(\lambda - i\varepsilon)\}f_\alpha, P_\beta \{r_\beta(\zeta - i\delta) - r_\beta(\lambda - i\varepsilon)\}f_\beta) d\zeta \right].$$

The norm limits  $P_\beta^*K(\zeta + i0)P_\alpha := \lim_{\delta \downarrow 0} P_\beta^*K(\zeta + i\delta)P_\alpha$ , and  $r_\gamma(\zeta \pm i0) := \lim_{\delta \downarrow 0} r_\gamma(\zeta \pm i\delta)$  exist in  $\mathbf{B}(L^2_{-s}(\mathbf{R}^N_{y_a}), L^2_s(\mathbf{R}^N_{y_b}))$ ,  $\mathbf{B}(L^2_s(\mathbf{R}^N_{y_c}), L^2_{-s}(\mathbf{R}^N_{y_d}))$ , respectively, where  $c = D(\gamma)$ , and  $s > 1/2$ . Indeed, the former follows from Lemmas 2.1, 2.2 and (V), the latter is well known (cf. [R-S]IV, XIII.8). Therefore we can write

$$(P_\alpha r_\alpha(\lambda - i\varepsilon)f_\alpha, W_{\beta+} r_\beta(\lambda - i\varepsilon)f_\beta) = \delta_{\beta\alpha}(P_\alpha r_\alpha(\lambda - i\varepsilon)f_\alpha, P_\beta r_\beta(\lambda - i\varepsilon)f_\beta) \\ + \frac{i}{2\pi} \int_J \frac{1}{(\zeta - \lambda)^2 + \varepsilon^2} (h_+(\lambda, \zeta, \varepsilon) - h_-(\lambda, \zeta, \varepsilon)) d\zeta,$$

$$h_\pm(\lambda, \zeta, \varepsilon) := (K(\zeta + i0)P_\alpha\{r_\alpha(\zeta + i0) - r_\alpha(\lambda - i\varepsilon)\}f_\alpha, P_\beta\{r_\beta(\zeta \pm i0) - r_\beta(\lambda - i\varepsilon)\}f_\beta).$$

By substituting in (8.3) we have

$$(8.6) \quad (S_{\beta\alpha}f_\alpha, f_\beta) = \delta_{\beta\alpha}(f_\alpha, f_\beta) \\ + \lim_{\varepsilon \downarrow 0} \frac{i}{2\pi} \int_J d\lambda \int_J \frac{\varepsilon}{\pi\{(\zeta - \lambda)^2 + \varepsilon^2\}} (h_+(\lambda, \zeta, \varepsilon) - h_-(\lambda, \zeta, \varepsilon)) d\zeta.$$

Since  $h_\pm(\lambda, \zeta, \varepsilon)$  is continuous in  $(\lambda, \zeta) \in J \times J$  for each  $\varepsilon > 0$  and

$$h_\pm(\lambda, \zeta) := \lim_{\varepsilon \downarrow 0} h_\pm(\lambda, \zeta, \varepsilon) \\ = (K(\zeta + i0)P_\alpha\{r_\alpha(\zeta + i0) - r_\alpha(\lambda - i0)\}f_\alpha, P_\beta\{r_\beta(\zeta \pm i0) - r_\beta(\lambda - i0)\}f_\beta)$$

uniformly for  $(\lambda, \zeta) \in J \times J$ , the limit in the R.H.S. of (8.6) converges to

$$\frac{i}{2\pi} \int_J (h_+(\lambda, \lambda) - h_-(\lambda, \lambda)) d\lambda = \frac{i}{2\pi} \int_J h_+(\lambda, \lambda) d\lambda,$$

because of  $h_-(\lambda, \lambda) = 0$ . Thus, by noting that

$$Z_\gamma^*(\lambda)Z_\gamma(\lambda) = \frac{1}{2\pi i} [r_\gamma(\lambda + i0) - r_\gamma(\lambda - i0)] \quad (\text{see (1.12)}),$$

for  $\gamma = \alpha, \beta$ , we obtain

$$(S_{\beta\alpha}f_\alpha, f_\beta) = \delta_{\beta\alpha}(f_\alpha, f_\beta) + 2\pi i \int_J (K(\lambda + i0)P_\alpha Z_\alpha^*(\lambda)Z_\alpha(\lambda) f_\alpha, P_\beta Z_\beta^*(\lambda)Z_\beta(\lambda) f_\beta) d\lambda.$$

This implies (2.10).

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