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On the Dirichlet problem for the nonlinear equation of the vibrating string. I

By

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0. Introduction

In this paper we shall investigate the solvability of the following Dirichlet problem for the nonlinear equation of the vibrating string

$$(1)\qquad \begin{aligned} &u_{xx} - u_{yy} + f(x, y, u) = 0, \qquad (x, y) \in \Omega \\ &u|_{\partial\Omega} = 0 \end{aligned}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain, convex relative to the characteristics $x \pm y = const.$ The function $f(x, y, u)$ is assumed to be continuous in (x, y), continuous and monotone in u.

The boundary value problems for the equation of the vibrating string have been studied by many authors. Plenty of works devoted to this theme can be divided into two main parts.

The first one deals with the Dirichlet problem for the linear equation of the vibrating string

$$(2)\qquad \begin{aligned} &u_{xx} - u_{yy} = g(x, y), \qquad && (x, y) \in \Omega \\ &u(x, y) = \psi(x, y), && (x, y) \in \partial\Omega \end{aligned}$$

J. Hadamard [1] noted that the Dirichlet problem (2) is a non-well-posed problem. D. G. Bourgin, R. Duffin [2] and D. W. Fox, C. Pucci [3] gave a complete discussion of problem (2) in the case where $\partial\Omega$ is a rectangle with sides parallel to the coordinate axes. The existence and uniqueness of continuous solutions of (2) were completely investigated by F. John [4]. The measurable solutions were considered by R. A. Aleksandryan [5]. The solvability of (2) in the Sobolev spaces $W_2^k(\Omega)$, $k \in \mathbf{Z}$ was investigated by M. V. Fokin [6] in the case of analytic boundary.

Another part of works deals with T-periodic solutions of the nonlinear equation of the vibrating string

$$(3)\qquad \begin{aligned} &u_{tt} - u_{xx} + f(x, t, u) = 0, \qquad && o < x < \pi, \ t \in \mathbf{R}, \\ &u(0, t) = u(\pi, t) = 0, && t \in \mathbf{R} \\ &u(x, t + T) = u(x, t), && o < x < \pi, \ t \in \mathbf{R} \end{aligned}$$

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where $f(x, t, u)$ is a given T-periodic function of t. In the case T/π is rational a lot of results concerning existence, uniqueness and regularity of weak solutions of (3) were obtained. We mention here the works of O. Vejvoda [7], L. Cesari [8], J. Hale [9], P. Rabinowitz [10], H. Lovicarova [11], H. Brezis [12].

In the case when T/π is irrational problem (3) is much more complicated. There are only a few works dealing with this case. We mention here the work of P. I. Plotnikov and M. N. Urgerman [13].

In the present paper for some class of domains we shall prove the existence and uniqueness of weak solutions of problem (1) if the function $f(x, y, u)$ satisfies some conditions.

1. Main notations

We rewrite the nonlinear equation of the vibrating string in the characteristic form

$$u_{xy} + f(x, y, u) = 0$$

We shall look for the solutions in $L_2(\Omega)$ of the following operator equation

$$\mathbf{A}u + f(x, y, u) = 0 \tag{4}$$

where the operator $\mathbf{A}$ is the closure in $\mathbf{L}_2(\Omega)$ of a symmetric operator $\mathbf{A}_0 u = u_{xy}$, $D(\mathbf{A}_0) = C^\infty(\bar{\Omega}) \cap \overset{0}{W}{}_2^1(\Omega)$.

The domain Ω is assumed to be bounded and convex relative to the lines $x = const$, $y = const$. We shall assume also that the boundary $\Gamma = \partial\Omega$ is infinitely smooth and the curvature of Γ at those points where the tangent is parallel to one of the coordinate axes is positive. We shall call such points "the vertices" of Γ [4].

Following [4], we define diffeomorphisms T^+, T^- of the boundary Γ: T^+ assigns to a point of the boundary another boundary point with the same coordinate y, while T^- assigns to a point of the boundary another boundary point with the same coordinate x (any vertex of Γ is a fixed point of either T^+ or T^-). We set $F = T^- \circ T^+$ (see Figure 1).

The diffeomorphism F belongs to the class C^∞ and preserves the orientation of the boundary.

Let $\Gamma = \{(x(s), y(s)) \mid 0 \le s < l\}$ be a natural parametrization of Γ, s be parameter of arc's length, l be total length of Γ. For each point $P \in \Gamma$ we assign its coordinate $S(P) \in [0, l)$. Then the diffeomorphism F can be "lifted" [14] to a map $f\colon \mathbf{R} \to \mathbf{R}$. It means that there exists increasing function $f\colon \mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ such that $0 \le f(0) < l$ and

$$f(s + l) = f(s) + l, \quad s \in \mathbf{R}; \quad S(FP) = f(S(P)) \pmod{l}, \quad P \in \Gamma.$$

The function $f(s)$ is called the "lift" of F[14]. Since $F \in C^\infty$ then $f \in \mathbf{C}^\infty(\mathbf{R})$.

If we set $f_1(s) = f(s)$, $f_k(s) = f(f_{k-1}(s))$, $k = 2, 3, \ldots$, then $f_k(s)$ is the lift of

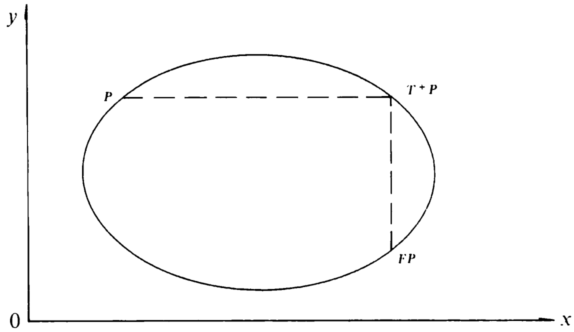

Figure 1.

F^k. It is known [14] that independently of the choice of $s \in \mathbf{R}$ there exists the following limit

$$\lim_{n \to \infty} \frac{f_n(s)}{nl} \stackrel{def}{=} \alpha(F) \in [0, 1]$$

The number $\alpha(F)$ is called the "rotation number" of F [14]. The following cases are possible:

(A) $\alpha(F) = \dfrac{m}{n}$ is a rational number, and $F^n = I$, where I is the identity mapping of Γ onto itself;

(B) $\alpha(F) = \dfrac{m}{n}$ is a rational number, $F^n \neq I$, F^n has a fixed point on Γ;

(C) $\alpha(F)$ is an irrational number, and F^k has no fixed points on Γ for any $k \in \mathbf{N}$.

In the present paper we shall consider only domains for which the condition (A) holds. We point out that the condition of rationality of T/π in problem (3) actually means that the rotation number $\alpha(F)$ of the corresponding diffeomorphism F is rational and the condition (A) holds. So problem (3) is well studied exactly in the case (A).

2. The null space of the operator A

Following [15], we shall describe here the null space of the operator **A**. For this we need to introduce some notations.

Denote $P_0, P_1, P_2, P_3 \in \Gamma$ the vertices of the boundary Γ, moreover assume point $P_0(P_2)$ has the maximal (minimal) coordinate y on Γ, $P_3(P_1)$ has the maximal (minimal) coordinate x on Γ (see Figure 2).

Then

$$T^+P_0 = P_0, \quad T^+P_2 = P_2, \quad T^-P_1 = P_1, \quad T^-P_3 = P_3$$

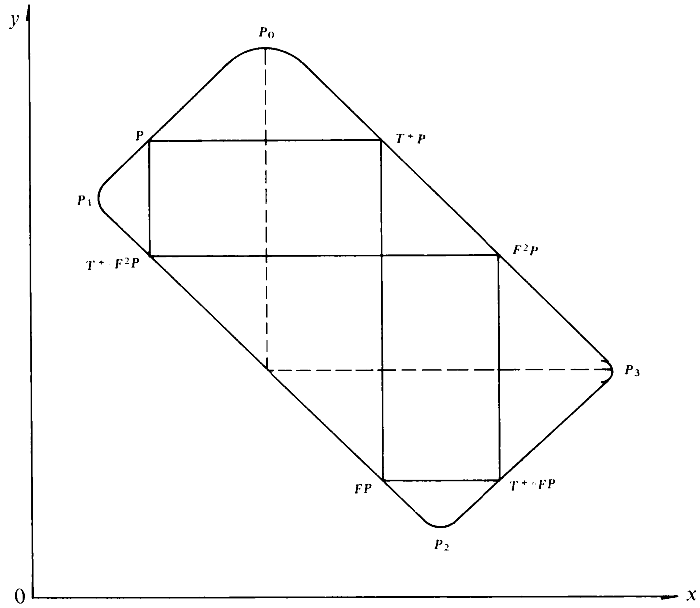

Figure 2.

We assume in present paper that condition (A) holds. So there exists number $n \in \mathbf{N}$ such, that $F^n = I$, $F^m P \neq P$ for any $P \in \Gamma$, $m = 1, 2, \ldots, n-1$ [14].

Following [6], for any $P \in \Gamma$ we shall mean by a "cycle" the following set (see Figure 2)

$$O(P) = \{P, T^+P, FP, T^+ \circ FP, F^2P, \ldots, F^{n-1}P, T^+ \circ F^{n-1}P\}$$

Then for any $P \in \Gamma$ it follows that the set $O(P)$ is invariant relative to $T^{\pm}$, F, i.e. $T^+(O(P)) = T^-(O(P)) = F(O(P)) = O(P)$.

Consider the point P_0. As far as $T^+P_0 = P_0$ then the cycle $O(P_0)$ consists of n different points. Hence it should be $O(P_0) \cap \{P_1, P_2, P_3\} \neq \emptyset$. So it is easy to check that:

(1) if n is even then $F^{\frac{n}{2}} P_0 = P_2$ and hence $P_2 \in O(P_0)$, $O(P_2) = O(P_0)$;

(2) if n is odd then $F^{\frac{n+1}{2}} P_0 = P_1$ or $F^{\frac{n+1}{2}} P_0 = P_3$.

Following [15], we define so-called generating set of the diffeomorphism F. If n is even then we denote by P^* the point from finite set $O(P_1) \cap (P_0, P_1]_\Gamma$ such that $(P_0, P^*)_\Gamma \cap O(P_1) = \emptyset$ (for any $P, Q \in \Gamma$ we denote by $(P, Q)_\Gamma$ the open arc of Γ from P to Q according to the positive orientation of Γ; $(P, Q]_\Gamma = (P, Q)_\Gamma \cup \{Q\}$). If n is odd then we denote by P^* the point from finite set $O(P_2) \cap (P_0, P_1]_\Gamma$ such that $(P_0, P^*)_\Gamma \cap O(P_2) = \emptyset$. By a "generating set" (G.S.) of the diffeomorphism F we shall mean the arc $\tilde{M}_0 = [P_0, P^*]_\Gamma$.

From the results obained in [15] it follows that the following statements hold.

Lemma 1. (1) *For any* $P, Q \in \tilde{M}_0$, $P \neq Q$ *we have* $O(P) \cap O(Q) = \emptyset$;

(2) $\bigcup_{P \in \tilde{M}_0} O(P) = \Gamma$, *i.e. for any* $Q \in \Gamma$ *there exists* $P \in \tilde{M}_0$ *that* $Q \in O(P)$.

Denote the null space of the operator $\mathbf{A}$ by $N(\mathbf{A})$. It is well known (for example, [16]) that any $u \in N(\mathbf{A})$ can be written in the form

(5) $$u(x, y) = G(x) + H(y), \qquad (x, y) \in \Omega$$

where

(6) $$G(x) + H(y) = 0, \qquad \text{a.e. } (x, y) \in \Gamma$$

Besides $G(x) \in \mathbf{L}_{2,\rho_1}(a, b)$, $H(y) \in \mathbf{L}_{2,\rho_2}(c, d)$; $a = x(P_1)$, $b = x(P_3)$, $c = y(P_2)$, $d = y(P_0)$, $\rho_1 = \rho_1(x) = \sqrt{x-a} \cdot \sqrt{b-x}$, $\rho_2 = \rho_2(y) = \sqrt{y-c} \cdot \sqrt{d-y}$, where for any $P \in \Gamma$ by $x(P)$, $y(P)$ we denote the coordinates of the point P, i.e. $P = (x(P), y(P))$.

According to (6) each $u \in N(\mathbf{A})$ can be uniquely determined by the values of the function $G(x)$, $x \in (a, b)$. Besides, as far as for any $P \in \Gamma$ we have $x(P) = x(T^- P)$, $y(P) = y(T^+ P)$, then from (6) it follows that for almost every $P \in \Gamma$

(7) $$G(x(P)) = G(x(Q)), \qquad H(y(P)) = H(y(Q))$$

for any $Q \in O(P)$. So by (7) and Lemma 1 we have that the functions $G(x)$, $H(y)$, $u(x, y)$ are uniquely determined by the values of $G(x)$ on $\tilde{M}_0$ (as function from $\mathbf{L}_2$).

We choose the natural parametrization of Γ: $\Gamma = \{(x(s), y(s)) \mid 0 \leq s < l\}$ such that $P_0 = (x(0), y(0))$, $P_j = (x(s_j), y(s_j))$, $j = 1, 2, 3$; $0 < s_1 < s_2 < s_3 < l$.

Define the functions

(8) $$g(s) = G(x(s)), \quad h(s) = H(y(s)), \quad 0 \leq s < l$$

As it has been mentioned earlier, the diffeomorphisms T^+, T^- can be lifted to maps $f^+, f^-: \mathbf{R} \to \mathbf{R}$, i.e.

$$f^{\pm}(s + l) = f^{\pm}(s) - l, \quad f^+(0) = 0, \quad f^-(s_1) = s_1, \quad S(T^{\pm} P) = f^{\pm}(S(P)) (\text{mod } l)$$

for any $s \in \mathbf{R}$, $P \in \Gamma$.

Following [15], we define functions $f_k: \mathbf{R} \xrightarrow{\text{on}} \mathbf{R}$, $k = 0, 1, 2, \ldots$

$$f_0(s) \equiv s, \quad f_1(s) = f^+(s), \quad f_2(s) = f^-(f^+(s))$$
$$f_{2k+1}(s) = f^+(f_{2k}(s)), \quad f_{2k+2}(s) = f_2(f_{2k}(s))$$

It is easy to see that $f_{2k}(s)$ is the lift of F^k. As far as $\Gamma \in \mathbf{C}^\infty$ and the curvature of Γ at the vertices $P_0, \ldots, P_3$ is positive then $f^+, f^-, f_k \in \mathbf{C}^\infty(\mathbf{R})$, $f^{\pm}, f_{2k+1}$ are strictly decreasing functions, f_{2k} is strictly increasing function for any $k \in \mathbf{N}$ [15].

We define also functions $\hat{f}_k(s)$: $[0, 1) \xrightarrow{\text{on}} [0, l)$

$$\hat{f}_k(s) = f_k(s) \quad (\text{mod } l), \qquad s \in [0, l)$$

It is easy to see that for any $k \in \mathbf{N}$

$$\hat{f}_k \in \mathbf{C}^\infty([0, l) \setminus E)$$

where the finite set

$$E = \{S(P) | P \in O(P_0)\} \subset [0, l)$$

consists of n numbers.

The formulas (6), (7) can be rewritten in the form

$$g(s) = -h(s), \qquad \text{a.e. } s \in [0, l), \tag{9}$$

$$g(s) = g(\hat{f}_k(s)), \qquad k \in \mathbf{N}, \quad \text{a.e. } s \in [0, l) \tag{10}$$

We denote $M_0 = (0, s^*)$ where $P^* = (x(s^*), y(s^*))$. Then from (9), (10) and Lemma 1 it follows that the values of $g(s)$, $h(s)$ are determined uniquely for a.e. $s \in [0. l)$ by the values $g(s)$ on M_0.

Denote

$$M_k = \{\hat{f}_k(s) | s \in M_0\}, \qquad k = 1, \dots, 2n - 1$$

Then Lemma 1 can be reformulated as follows.

Lemma 1′. (1) $M_k \cap M_m = \emptyset$, $m \neq k$, $m, k \in \{0, \dots, 2n-1\}$;
(2) $\bigcup_{k=0}^{2n-1} M_k = [0, l) \setminus E$.

So $\hat{f}_k \in \mathbf{C}^\infty(M_j)$ for any $k \in \mathbf{N}$, $j \in \{0, 1, \dots, 2n-1\}$. Moreover it is easy to see that for any $k \in \mathbf{N}$, $j \in \{0, 1, \dots, 2n-1\}$ there exists $m = m(k, j) \in \mathbf{Z}$ that $\hat{f}_k(s) \equiv f_k(s) - m(k, j) \cdot l$, $s \in M_j$. Hence $\dfrac{d\hat{f}_k}{ds}(s) \in \mathbf{C}^\infty(\bar{M}_j)$.

Thus we have obtained that any $u \in N(A)$ is uniquely determined by the values of the corresponding function $g(s)$ on M_0. Moreover any function $g(s) \in \mathbf{L}_2(M_0)$ generates with the help of formulas (5), (8), (9), (10) a function $u \in N(\mathbf{A})$. Indeed, if $g(s)$ is some function from $\mathbf{L}_2(M_0)$ then using (10) we define $g(s)$ for all $s \in [0, l) \setminus E$. As far as $\dfrac{d\hat{f}_k(s)}{ds} \in \mathbf{C}^\infty(\bar{M}_j)$, $k \in \mathbf{N}$, $j \in \{0, \dots, 2n-1\}$ then $g(s) \in \mathbf{L}_2(0, l)$. Then according to (9) we have $h(s) = -g(s) \in \mathbf{L}_2(0, l)$. It is easy to see that the maps $x(s): [s_1, s_3] \overset{\text{on}}{\rightarrow} [a, b]$, $y(s): [0, s_2] \overset{\text{on}}{\rightarrow} [c, d]$ are one-to-one mappings. So there exist functions $\tilde{s}(x): [a, b] \overset{\text{on}}{\rightarrow} [s_1, s_3]$, $\hat{s}(y): [c, d] \overset{\text{on}}{\rightarrow} [0, s_2]$ that

$$x(\tilde{s}(x)) \equiv x, \quad x \in [a, b]; \quad y(\hat{s}(y)) \equiv y, \quad y \in [c, d]$$

Because of formula (9) we obtain that functions

$$G(x) = g(\tilde{s}(x)), \quad x \in [a, b]; \quad H(y) = h(\hat{s}(y)), \quad y \in [c, d] \tag{11}$$

satisfy (6).

As far as $\Gamma \in \mathbf{C}^\infty$ and the curvature of Γ at the vertices $P_0, \ldots, P_3$ is positive then there exist constants $c_1, c_2 > 0$ such that

$$\frac{c_1}{\rho_1(x)} \leq \frac{d\tilde{s}(x)}{dx} \leq \frac{c_2}{\rho_1(x)}, \qquad x \in (a, b)$$

$$\frac{c_1}{\rho_2(y)} \leq \frac{d\hat{s}(y)}{dy} \leq \frac{c_2}{\rho_2(y)}, \qquad y \in (c, d)$$

Hence $G(x) \in \mathbf{L}_{2,\frac{1}{\rho_1}}(a, b) \subset \mathbf{L}_{2,\rho_1}(a, b)$, $H(y) \in \mathbf{L}_{2,\frac{1}{\rho_2}}(c, d) \subset \mathbf{L}_{2,\rho_2}(c, d)$. So the function

$$u(x, y) = G(x) + H(y)$$

belongs to $N(\mathbf{A})$. Thus the following theorem holds.

Theorem 1. (1) *For any $u \in N(\mathbf{A})$ there exists a function $g(s) \in \mathbf{L}_2(M_0)$ such that equalities* (5), (8), (9), (10) *hold.*

(2) *Any function $g(s) \in \mathbf{L}_2(M_0)$ generates some function $u \in N(\mathbf{A})$ with the help of formulas* (5), (9), (10), (11).

3. The normal solvability of equation $\mathbf{A}u = f$

M. V. Fokin in his work [6] shows that if the condition (A) holds then the operator $\mathbf{A}$ is a selfadjoint operator in $\mathbf{L}_2(\Omega)$ and equation $\mathbf{A}u = f$ is normally solvable, i.e. the range $R(\mathbf{A})$ is closed in $\mathbf{L}_2(\Omega)$. Although this statement was formulated under assumption of analyticity of Γ it remains true if $\Gamma \in \mathbf{C}^\infty$ (the proof in the case $\Gamma \in \mathbf{C}^\infty$ is exactly the same as in the case of analyticity of Γ). We shall give here the sketch of the proof because we shall use it further.

Theorem 2 (M. V. Fokin). *If $\Gamma \in \mathbf{C}^\infty$ and the condition (A) holds then the range $R(\mathbf{A})$ of the operator $\mathbf{A}$ is closed in $\mathbf{L}_2(\Omega)$.*

Sketch of the proof. We shall construct an operator $\mathbf{B}$: $R(\mathbf{A}) \to \mathbf{L}_2(\Omega)$ such that for some constant $C > 0$

$$\mathbf{A} \circ \mathbf{B}u = u, \quad u \in R(\mathbf{A}) \tag{12}$$

$$\|\mathbf{B}u\|_{\mathring{\mathbf{W}}_2^1(\Omega)} \leq C \cdot \|u\|_{\mathbf{L}_2(\Omega)}, \quad u \in R(\mathbf{A}) \tag{13}$$

Clearly it is sufficient to construct such operator $\mathbf{B}$ on $R(\mathbf{A}) \cap \mathbf{C}^\infty(\bar{\Omega})$.

Let $u \in R(\mathbf{A}) \cap \mathbf{C}^\infty(\bar{\Omega})$. Then

$$(u, \phi)_{\mathbf{L}_2(\Omega)} = \int_\Omega u \cdot \bar{\phi}\, d\Omega = 0, \qquad \phi \in N(\mathbf{A}^*)$$

It means $u \in N(\mathbf{A}^*)^{\perp}$ in $\mathbf{L}_2(\Omega)$. We consider the function

$$v(x, y) = \int_0^x \int_0^y u(\xi, \eta) \, d\xi d\eta \tag{14}$$

(we set $u(x, y) = 0$ if $(x, y) \notin \Omega \cup \Gamma$). Then $v \in \mathbf{C}^\infty(\Omega \cup \Gamma)$ and $v_{xy}(x, y) = u(x, y)$, $(x, y) \in \Omega$. We shall construct such functions $G(x) \in \mathbf{C}^\infty(a, b) \cap \overset{o}{\mathbf{W}}{}_2^1(\Omega)$, $H(y) \in \mathbf{C}^\infty(c, d) \cap \overset{o}{\mathbf{W}}{}_2^1(\Omega)$ that

$$v(x, y) + G(x) + H(y) = 0, \qquad (x, y) \in \Gamma \tag{15}$$

Denote

$$v(s) = v(x(s), y(s)), \qquad s \in [0, l) \tag{16}$$

We extend $v(s)$ to a l-periodic function defined on $\mathbf{R}$ by the formula

$$v(s) = v\,(s + l), \qquad s \in \mathbf{R}$$

Then $v(s) \in \mathbf{C}^\infty(\mathbf{R})$.

It is easy to see that to construct the functions $G(x)$, $H(y) \in \mathbf{C}^\infty$ satisfying (15) it is sufficient to find l-periodic functions $g(s)$, $h(s) \in \mathbf{C}^\infty$ satisfying

$$g(s) = g(f^-(s)) \tag{17}$$

$$h(s) = h(f^+(s)) \tag{18}$$

$$v(s) + g(s) + h(s) = 0 \tag{19}$$

for all $s \in \mathbf{R}$. Using the induction we obtain from (17), (18), (19) for any $k \in \mathbf{N}$, $s \in \mathbf{R}$

$$g(f_{2k}(s)) = g(f_{2k-1}(s)) = g(s) - \sum_{j=0}^{2k-1} (v(f_{2j+1}(s)) - v(f_{2j}(s))) \tag{20}$$

As far as $f_{2n}(s) \equiv s \pmod l$ then a necessary condition for the solvability of (17)–(19) is

$$\sum_{j=0}^{2n-1} (v(f_{2j+1}(s)) - v(f_{2j}(s)) = 0, \qquad s \in \mathbf{R} \tag{21}$$

It is easy to show that equality (21) is a necessary and sufficient condition for $u \in (N(\mathbf{A}^*))^{\perp}$. The equality (21) means exactly that u is orthogonal to all piecewise constant function from $N(\mathbf{A}^*)$ we shall discuss later. So as far as $u \in R(\mathbf{A})$ then the function $v(x, y)$ defined by (14) satisfies (21).

From (20) and Lemma 1′ it follows that any l-periodic function $g(s)$ satisfying (17)–(19) can be uniquely determined by the values of $g(s)$ for $s \in M_0$. So we denote by $g_0(s)$ l-periodic function such that $g_0(s) \equiv 0$, $s \in M_0$ and (20) holds. We set

$$h_0(s) = -\,v(s) - g_0(s), \quad s \in \mathbf{R}$$

It is easy to prove that $g_0, h_0 \in \mathbf{C}^\infty(\mathbf{R})$ and g_0, h_0 satisfy (17)–(19). We denote by $G_0(x)$, $H_0(y)$ the functions defined by the $g_0(s)$, $h_0(s)$ according to the formulas (11). It is easy to verify $G_0(x) \in \mathbf{C}^\infty(a, b) \cap \mathbf{W}_2^1(\Omega)$, $H_0(y) \in \mathbf{C}^\infty(c, d) \cap \mathbf{W}_2^1(\Omega)$ and (15) holds. Then using (11), (14), (19), (20) we can find a constant $C > 0$ which depends only on Ω and such that

$$\|v(x, y)\|_{\overset{o}{\mathbf{W}}{}_2^1(\Omega)} + \|G_0(x)\|_{\overset{o}{\mathbf{W}}{}_2^1(\Omega)} + \|H_0(y)\|_{\overset{o}{\mathbf{W}}{}_2^1(\Omega)} \leq C \cdot \|u\|_{\mathbf{L}_2(\Omega)}$$

Thus the function

$$w_0(x, y) = v(x, y) + G_0(x) + H_0(y)$$

belongs to $\mathbf{C}^\infty(\Omega) \cap \overset{o}{\mathbf{W}}{}_2^1(\Omega)$ and

$$\|w_0(x, y)\|_{\overset{o}{\mathbf{W}}{}_2^1(\Omega)} \leq C \cdot \|u\|_{\mathbf{L}_2(\Omega)}$$

Besides

$$(w_0(x, y))_{xy} = u(x, y), \qquad (x, y) \in \Omega$$

It is not difficult to show that $w_0 \in D(\mathbf{A})$ and $\mathbf{A}w_0 = w_{0_{xy}} = u$.

We set

$$\mathbf{B}_0 u = w_0$$

Then $\mathbf{B}_0$ is linear operator from $\mathbf{C}^\infty(\Omega \cup \Gamma) \cap R(\mathbf{A})$ to $\overset{o}{\mathbf{W}}{}_2^1(\Omega)$ and (12), (13) hold for all $u \in \mathbf{C}^\infty(\Omega \cup \Gamma) \cap R(\mathbf{A})$. Let $\mathbf{B} = \overline{\mathbf{B}_0}$-closure in $\mathbf{L}_2(\Omega)$. Since the operator $\mathbf{A}$ is closed in $\mathbf{L}_2(\Omega)$ then $\mathbf{B}: R(\mathbf{A}) \to D(\mathbf{A})$ and (12), (13) hold. From (12), (13) it follows that the range $R(\mathbf{A})$ is closed in $\mathbf{L}_2(\Omega)$. Theorem 2 is proved.

It is not so difficult also to prove that $\mathbf{A}$ is a selfadjoint operator. Since $\mathbf{A}$ is symmetric then to prove this it is sufficient to show that $N(\mathbf{A}) = N(\mathbf{A}^*)$. Thus $\mathbf{A} = \mathbf{A}^*$, $\dim N(\mathbf{A}) = \infty$, $R(\mathbf{A})$ is closed. So $\mathbf{L}_2(\Omega)$ has an orthogonal decomposition

$$\mathbf{L}_2(\Omega) = R(\mathbf{A}) \oplus N(\mathbf{A})$$

We denote by P_1, P_2 the operators of orthogonal projection from $\mathbf{L}_2(\Omega)$ onto $R(\mathbf{A})$, $N(\mathbf{A})$ respectively. Consider the operator

$$\mathbf{A}_R = \mathbf{A}|_{R(\mathbf{A})}: R(\mathbf{A}) \longrightarrow R(\mathbf{A})$$

Then the inverse operator $\mathbf{A}_R^{-1}: R(\mathbf{A}) \to R(\mathbf{A}) \cap D(\mathbf{A})$ can be represented in the form

$$\mathbf{A}_R^{-1} = P_1 \circ \mathbf{B}$$

So because of (13) and embedding theorems it follows that $\mathbf{A}_R^{-1}$ is compact operator. Hence the operator $\mathbf{A}$ has the following Property I [17].

Property I ([17], p. 231). *Let H be a real Hilbert space. Let $\mathbf{A}: D(\mathbf{A}) \subset H \to H$ be a closed linear operator with dense domain and closed range. Assume that*

$$N(\mathbf{A}) = N(\mathbf{A}^*) \quad (\text{or equivalently } R(\mathbf{A}) = N(\mathbf{A})^{\perp})$$

$\mathbf{A}$ *is therefore a one-one map of $D(\mathbf{A}) \cap R(\mathbf{A})$ onto $R(\mathbf{A})$. Assume furthermore that the inverse*

$$\mathbf{A}^{-1}: R(A) \longrightarrow R(\mathbf{A}) \text{ is compact.}$$

Operator $\mathbf{A}$ satisfying all these conditions will be said to have Property I.

4. The solvability of problem (4)

We denote by $(-\alpha_1)$, α_2 the first negative and first positive eigenvalue of $\mathbf{A}$ respectively. Then

$$\frac{1}{\alpha_2} \cdot \|\mathbf{A}u\|^2_{\mathbf{L}_2(\Omega)} \geq (\mathbf{A}u, u)_{\mathbf{L}_2(\Omega)} \geq \frac{1}{-\alpha_1} \cdot \|\mathbf{A}u\|^2_{\mathbf{L}_2(\Omega)}, \qquad u \in D(\mathbf{A}) \tag{22}$$

It is easy to verify that $\mathbf{A}$ has positive and negative eigenvalues. Indeed, denote $\xi = x + y$, $\eta = x - y$ and chose a rectangle $\Pi = \{p < \xi < q, r < \eta < e\} \subset \Omega$. Let $f(\xi), g(\eta) \in \mathbf{C}^\infty(\mathbf{R})$, supp $f \subset (p, q)$, supp $g \subset (r, e)$. Then

$$u(x, y) = f(x + y) \cdot g(x - y) \in \overset{0}{\mathbf{C}}{}^\infty(\Omega),$$

$$\mathbf{A}u = u_{xy} = f''(x + y) \cdot g(x - y) - f(x + y) \cdot g''(x - y),$$

$$(\mathbf{A}u, u)_{\mathbf{L}_2(\Omega)} = \int_\Omega (f''(x + y)g(x - y) - f(x + y)g''(x - y))f(x + y)g(x - y)\, dx dy$$

$$= \frac{1}{2}\int_\Pi (f''(\xi) \cdot f(\xi) \cdot g^2(\eta) - f^2(\xi) \cdot g''(\eta) \cdot g(\eta))\; d\xi\, d\eta$$

$$= \frac{1}{2}\int_p^q f^2(\xi)\, d\xi \cdot \int_r^e (g'(\eta))^2\, d\eta - \frac{1}{2}\int_p^q (f'(\xi))^2\, d\xi \cdot \int_r^e g^2(\eta)\, d\eta$$

Hence there exist $u, v \in \overset{0}{\mathbf{C}}{}^\infty(\Omega)$ such that $(\mathbf{A}u, u)_{\mathbf{L}_2} < 0 < (\mathbf{A}v, v)_{\mathbf{L}_2}$. So $(-\alpha_1) < 0 < \alpha_2$.

Assume $\mathbf{K}: H \to H$ is a (nonlinear) operator satisfying the following condition [17]:

$$\begin{cases} \text{For some positive constant } \gamma < \alpha_1 \\ (\mathbf{K}u - \mathbf{K}w, u) \geq \dfrac{1}{\gamma}\|\mathbf{K}u\|^2 - C(w), \quad \forall u, w \in H \end{cases} \tag{23}$$

were $C(w)$ depends only on w.

Theorem 3 ([17]). *Suppose* $\mathbf{A}$ *has Property I. Let* $\mathbf{K}$ *be a monotone demicontinuous* (*i.e. continuous from strong* H *into weak* H) *operator satisfying* (23). *Then*

$$R(\mathbf{A}+\mathbf{K}) \simeq R(\mathbf{A}) + conv\, R(\mathbf{K})$$

where "conv" denotes the convex hull and $C \simeq D$ *means the sets* C *and* D *have the same interiors and the same closures.*

Corollary 1 ([17], p. 233). *Under the assumptions of Theorem* 3, *if* $N(\mathbf{A}) \subset R(\mathbf{K})$ (*which is in the case when* $\|\mathbf{K}u\| \to \infty$ *as* $\|u\| \to \infty$), *then* $\mathbf{A}+\mathbf{K}$ *is onto.*

Corollary 2 ([17], p. 235). *Assume* $\mathbf{A}$ *has Property I. Suppose* $\mathbf{K}$ *is onto, and satisfies*

$$(24) \qquad (\mathbf{K}u - \mathbf{K}w, u - w) \geq \frac{1}{\gamma} \cdot \|\mathbf{K}u - \mathbf{K}w\|^2, \qquad \forall u, w \in H$$

with $\gamma < \alpha_1$.

Then for any $f \in H$ *there exists a solution of*

$$(25) \qquad \mathbf{A}u + \mathbf{K}u = f$$

and the solution is unique mod $N(\mathbf{A})$. *If furthermore* $\mathbf{K}$ *is one-one the solution is unique.*

Consider problem (4)

$$(4) \qquad \mathbf{A}u + f(x, y, u) = 0$$

We assume that $f \in \mathbf{C}^0(\Omega \times \mathbf{R})$, furthermore either f or $-f$ is nondecreasing as a function of u and satisfies for a.e. $(x, y) \in \Omega$, any $u \in \mathbf{R}$

$$(26) \qquad \eta \cdot |u| - h_1(x, y) \leq |f(x, y, u)| \leq \gamma \cdot |u| + h_2(x, y)$$

where $\eta > 0$, $h_1, h_2 \in \mathbf{L}_2(\Omega)$, and $\gamma < \alpha_1$ or $\gamma < \alpha_2$ according as f or $-f$ is nondecreasing.

Theorem 4. *Under these conditions the problem* (4) *possesses a solution in* $\mathbf{L}_2(\Omega)$.

Proof. Obviously it is sufficient to consider the case f is nondecreasing. We apply Corollary 1. For this it is necessary to prove that $\mathbf{K}u = f(x, y, u)$ satisfies (23), $N(\mathbf{A}) \subset R(\mathbf{K})$ and $\mathbf{K}: \mathbf{L}_2(\Omega) \to \mathbf{L}_2(\Omega)$ is demicontinuous operator. Assumption (23) for $\mathbf{K}u = f(x, y, u)$ follows from the following Proposition.

Proposition 1 ([17], p. 317). *Let* Ω *be a measure space. Let* $\alpha > 0$. *Assume* $f(x, u): \Omega \times \mathbf{R} \to \mathbf{R}$ *is measurable in* x *and continuous nondecreasing in* u. *Suppose*

$$|f(x, u)| \le \theta \cdot |u| + h(x) \quad a.e.\ x \in \Omega, \ \forall u \in \mathbf{R}$$

with $\theta < \alpha$ *and* $h \in \mathbf{L}_2(\Omega)$. *Set* $(\mathbf{K}u)(x) = f(x, u(x))$. *Then for some* $\gamma < \alpha$

$$(\mathbf{K}u - \mathbf{K}w, u - v)_{\mathbf{L}_2} \ge \frac{1}{\gamma} \cdot \|\mathbf{K}u\|^2_{\mathbf{L}_2} - C(v, w) \quad \forall u, v, w \in \mathbf{L}_2(\Omega)$$

Let $v \in \mathbf{L}_2(\Omega)$. Set

$$u(x, y) = \max\{u \in \mathbf{R} \mid f(x, y, u) = v(x, y)\}, \quad (x, y) \in \Omega$$

Then from [19, Theorem 12.4] it follows that $u(x, y)$ is measurable. Besides from (26) and $f(x, y, u(x, y)) = v(x, y)$ we obtain

$$|u(x, y)| \le \frac{1}{\eta} |v(x, y)| + \frac{1}{\eta} h_1(x, y), \quad (x, y) \in \Omega$$

Hence $u \in \mathbf{L}_2(\Omega)$. Since v is arbitrary then $R(\mathbf{K}) = \mathbf{L}_2(\Omega)$. So $N(\mathbf{A}) \subset R(\mathbf{K})$. Demicontinuity of the operator $\mathbf{K}$ follows from the following Lemma 2 which will be proved in Appendix.

Lemma 2. *Let* Ω *be a bounded domain,* $f(x, y, u) \in \mathbf{C}^0(\Omega \times \mathbf{R})$ *and for a.e.* $(x, y) \in \Omega$, *any* $u \in \mathbf{R}$

$$|f(x, y, u)| \le C \cdot |u| + h(x, y)$$

for some $h \in \mathbf{L}_2(\Omega)$ *and some constant* $C > 0$. *Then the operator* $\mathbf{K}$ *is a continuous operator from* $\mathbf{L}_2(\Omega)$ *into itself.*

So Theorem 4 follows from Corollary 1

Theorem 5. *In Theorem 4 if we add the condition*

$$|f(x, y, u) - f(x, y, v)| \le \gamma |u - v| \quad a.e.\ (x, y), \quad \forall u, v \in \mathbf{R} \tag{27}$$

with $\gamma < \alpha_1$ *or* $\gamma < \alpha_2$ *respectively.*

Then the solution of (4) *is unique* mod $N(\mathbf{A})$. *If, furthermore,* f *is strictly monotone in* u *for every* $(x, y) \in \Omega$, *then the solution is unique.*

Theorem 5 follows from Corollary 2.

5. General structure of domains with the property (A)

In paper [4] F. John notes that the null space $N(\mathbf{A})$ is invariant relative to the following change of variables

$$x_1 = f(x), \quad y_1 = g(y) \tag{28}$$

Following this idea we shall prove

Theorem 6. (1) *Let* Ω *be a bounded domain, convex relative to the lines* $x = const$, $y = const$; $\Gamma = \partial\Omega \in \mathbf{C}^\infty$; *the curvature of* Γ *at the vertices be* *positive. Assume* $\alpha(F_\Gamma) = \frac{m}{n}$; $(m, n) = 1$; $(F_\Gamma)^n = I_\Gamma$. *Let* $f(x) \in \mathbf{C}^\infty[a, b]$, $g(y) \in \mathbf{C}^\infty[c, d]$, $f', g' \geq \delta > 0$. *Consider*

$$\Omega_1 = \{(f(x), g(y)) | (x, y) \in \Omega\} \tag{29}$$

Then $\alpha(F_{\Gamma_1}) = \frac{m}{n}$; $(F_{\Gamma_1})^n = I_{\Gamma_1}$; Ω_1 *is convex relative to the lines* $x = \text{const}$, $y = \text{const}$; *the curvature of* $\Gamma_1 = \partial\Omega_1$ *at the vertices is positive.*

(2) *Let* Ω, Ω_1 *be bounded domains, convex relative to the lines* $x = \text{const}$, $y = \text{const}$; $\Gamma = \partial\Omega \in \mathbf{C}^\infty$, $\Gamma_1 = \partial\Omega_1 \in \mathbf{C}^\infty$; *the curvature of* Γ, Γ_1 *at the vertices be positive*; $\alpha(F_\Gamma) = \alpha(F_{\Gamma_1}) = \frac{m}{n}$, $(m, n) = 1$; $(F_\Gamma)^n = I_\Gamma$, $(F_{\Gamma_1})^n = I_{\Gamma_1}$. *Then there exist functions* $f(x)$, $g(y)$ *and number* $\delta > 0$ *such that*

$$f(x) \in \mathbf{C}^1[a, b] \cap \mathbf{C}^\infty[a, b) \quad or \quad f(x) \in \mathbf{C}^1[a, b] \cap \mathbf{C}^\infty(a, b];$$
$$g(y) \in \mathbf{C}^1[c, d] \cap \mathbf{C}^\infty(c, d];$$
$$f'(x) \geq \delta > 0, \; x \in [a, b], \quad g'(y) \geq \delta > 0, \; y \in [c, d]$$

and (29) *holds.*

(3) *Let* Ω, Ω_1 *be bounded domains, convex relative to the lines* $x = \text{const}$, $y = \text{const}$; $\alpha(F_\Gamma) = \alpha(F_{\Gamma_1}) = \frac{m}{n}$, $(m, n) = 1$; $(F_\Gamma)^n = I_\Gamma$, $(F_{\Gamma_1})^n = I_{\Gamma_1}$. *Then there exist strictly increasing functions* $f(x)$, $g(y)$ *such that* (29) *holds.*

Proof. We shall widely use here the notations introduced earlier.

1. Since the map (28) transforms any segment of the line $x = \text{const}$ ($y = \textit{const}$) into segment of line $x_1 = \text{const}$ ($y_1 = \text{const}$) then Ω_1 is convex relative to the lines $x_1 = \text{const}$, $y_1 = \text{const}$ and any cycle $O(P)$, $P \in \Gamma$ is transformed into cycle $O(P_1)$, $P_1 = (P)_1 \in \Gamma_1$ (for any $P = (x, y) \in \Omega \cup \Gamma$ we denote $(P)_1 = (f(x), g(y)) \in \Omega_1 \cup \Gamma_1$). Besides $T^\pm_{\Gamma_1}(P_1) = (T^\pm_\Gamma P)_1$, $F_{\Gamma_1}(P)_1 = (F_\Gamma P)_1$, $P \in \Gamma$. Hence $(F_{\Gamma_1})^n = I_{\Gamma_1}$, $\alpha(F_{\Gamma_1}) = \frac{m}{n} = \alpha(F_\Gamma)$. It easy to see that $(P_0)_1, \ldots, (P_3)_1$ are the vertices of Γ_1. Consider, for example, the "left" vertex $(P_1)_1 = (f(x(s_1)), g(y(s_1)))$. Since $(x(s), y(s))$ is natural parametrization of Γ then $x'(s_1) = 0$, $y'(s_1) = -1$ and the curvature of Γ at P_1 is equal to

$$x'(s_1)y''(s_1) - x''(s_1)y'(s_1) = x''(s_1) > 0$$

So the curvature of Γ_1 at $(P_1)_1$ is equal to

$$((f(x(s)))' \cdot (g(y(s)))'' - (f(x(s)))'' \cdot (g(y(s)))')|_{s=s_1}$$
$$= f'(x(s_1)) \cdot g'(y(s_1)) \cdot (x''(s_1)) > 0.$$

2. Let $M_0 = (0, s^*)$, $M_0^1 = (0, s_1^*)$ be the generating sets for $\Gamma = \{(x(s), y(s)) | 0 \leq s < l\}$, $\Gamma_1 = \{(x_1(s), y_1(s)) | 0 \leq s < l_1\}$ respectively. Let $h_0(s)$ be some function such that $h_0: [0, s^*] \overset{\text{on}}{\to} [0, s_1^*]$; $h_0 \in \mathbf{C}^\infty [o, s^*]$; $h_0'(s) > 0$, $s \in [0, s^*]$. Define a map $h: [0, l] \overset{\text{on}}{\to} [0, l_1]$ by the following formulas

$$(30) \qquad \begin{cases} h(s) = h_0(s), & s \in [0, s^*] = \overline{M_0}; \\ h(s) = \hat{f}_k^{\Gamma_1}(h(\hat{f}_{-k}^{\Gamma}(s))), & s \in \overline{M_k}, \ k = 1, 2, \ldots, 2n-1 \end{cases}$$

where $\hat{f}_{-k}^{\Gamma} = (\hat{f}_k^{\Gamma})^{-1}$ is inverse function for $\hat{f}_k^{\Gamma} = \hat{f}_k$ defined for $\Gamma = \partial\Omega$ in section 2. Using Lemma 1′ and properties of $f_k, \hat{f}_k$ we obtain that $h(s): [0, l] \overset{\text{on}}{\to} [0, l_1]$ is strictly increasing function, $h \in \mathbf{C}[0, l] \cap \mathbf{C}^\infty(\overline{M_k})$, $k = 0, 1, \ldots, 2n-1$. Besides, since $|(\hat{f}_k(s))'| \geq \delta > 0$, $s \in \overline{M_m}$, $m, k \in \{0, 1, \ldots, 2n-1\}$ [15] then $h'(s) \geq \delta_1 > 0$, $s \in \overline{M_k}$, $k = 0, \ldots, 2n-1$ for some $\delta_1 > 0$. We shall show that $h \in \mathbf{C}^\infty[0, l]$. We extend $h(s)$ to a function $h(s): \mathbf{R} \overset{\text{on}}{\to} \mathbf{R}$ by the following formula

$$(31) \qquad h(s + kl) = h(s) + kl_1, \quad k \in \mathbf{Z}, \ s \in [0, l)$$

Then $h \in \mathbf{C}(\mathbf{R}) \cap \mathbf{C}^\infty(f_k(\overline{M_m}))$, $k \in \mathbf{Z}$, $m \in \{0, \ldots, 2n-1\}$. Since $[0, l] \setminus \bigcup_{k=0}^{2n-1} M_k = \{\hat{f}_j(s_i) | j = 0, \ldots, 2n-1, \ i = 0, 1, 2, 3\}$ where $P_i = (x(s_i), y(s_i))$, $i = 0, 1, 2, 3$ are the vertices of Γ then we need to prove

$$(32) \qquad \frac{d^k h}{ds^k}(\hat{f}_j(s_i) + 0) = \frac{d^k h}{ds^k}(\hat{f}_j(s_i) - 0)$$

for $k \in \mathbf{N}$, $j \in \{0, \ldots, 2n-1\}$, $i \in \{0, \ldots, 3\}$. As far as $f_j(s) \in \mathbf{C}^\infty(\mathbf{R})$, $j \in \mathbf{N}$ then from (31) it follows that to prove (32) it is sufficient to prove

$$(32') \qquad \frac{d^k h}{ds^k}(s_i + 0) = \frac{d^k h}{ds^k}(s_i - 0), \quad i = 0, 1, 2, 3.$$

Let ε be some positive number that $(s_1 - \varepsilon) \in (0, s_1)$, $f^-(s_1 - \varepsilon) \in (s_1, s_2)$. Then

$$f^-: [s_1 - \varepsilon, s_1] \xrightarrow{\text{on}} [s_1, f^-(s_1 - \varepsilon)], \quad f^-(s_1) = s_1, \ f^-(f^-(s)) \equiv s.$$

Besides it is not difficult to prove that $f^- \in \mathbf{C}^\infty[s_1 - \varepsilon, f^-(s_1 + \varepsilon)]$, $(f^-)' \leq -\delta < 0$ and

$$(33) \qquad f^-(s) \equiv 2s_1 - s + g(s), \quad s \in \mathbf{R}$$

where $g \in \mathbf{C}^\infty$ and $g^{(k)}(s_1) = 0$, $k = 0, 1, \ldots$.

From (30), (31) it follows that $h(s)$ satisfies

$$h(s) = f_{\Gamma_1}^-(h(f_\Gamma^-(s))), \quad s \in \mathbf{R}$$

So

$$(34) \quad h(s) = 2h(s_1) - h(f_\Gamma^-(s)) + g_1(f_\Gamma^-(s)) = 2h(s_1) - h(2s_1 - s + g(s)) + g_1(f_\Gamma^-(s))$$

where $f_{\Gamma_1}^-(s) = 2h(s_1) - s + g_1(s)$, $g_1^{(k)}(h(s_1)) = 0$, $k = 0, 1, \ldots$. As far as $g^{(k)}(s_1) = g_1^{(k)}(h(s_1)) = 0$ then from (34) it follows that (32') holds for $i = 1$. The cases $i = 0, 2, 3$ can be considered similarly. Thus $h \in \mathbf{C}^\infty(\mathbf{R})$.

According to the definition of $\hat{f}^{\pm}(s)$ we have

$$y(s) = y(\hat{f}_\Gamma^+(s)), \quad x(s) = x(\hat{f}_\Gamma^-(s)), \quad s \in [0, l)$$

$$y_1(s) = y_1(\hat{f}_{\Gamma_1}^+(s)), \quad x_1(s) = x_1(\hat{f}_{\Gamma_1}^-(s)), \quad s \in [0, l_1)$$

So from (30) it follows that $h(s)$ satisfies the following equalities:

$$(35) \qquad y_1(h(s)) = y_1(h(\hat{f}_\Gamma^+(s))), \quad x_1(h(s)) = x_1(h(\hat{f}_\Gamma^-(s))), \quad s \in [0, l)$$

We set

$$(36) \qquad f(x) = x_1(h(\tilde{s}(x))), \quad g(y) = y_1(h(\hat{s}(y))), \quad (x, y) \in \Omega \cup \Gamma$$

From (35) it folows that the map (36) transforms Γ onto Γ_1. Since Ω, Ω_1 are convex relative to the lines $x = \text{const}$, $y = \text{const}$ then from (35) it follows that (29) holds.

Obviously $f \in \mathbf{C}^\infty(a, b)$, $g \in \mathbf{C}^\infty(c, d)$; $f'(x) > 0$, $x \in (a, b)$; $g'(y) > 0$, $y \in (c, d)$. Further

$$\begin{aligned}
\lim_{x \to a+0} f'(x) &= \lim_{x \to a+0} x_1'(h(\tilde{s}(x))) \cdot h'(\tilde{s}(x)) \cdot (\tilde{s}(x))' \\
&= \lim_{x \to a+0} \frac{x_1'(h(\tilde{s}(x)))}{x'(\tilde{s}(x))} \cdot h'(\tilde{s}(x)) = \lim_{s \to s_1+0} \frac{x_1'(h(s))}{x'(s)} \cdot h'(s) \\
&= (h'(s_1))^2 \cdot \lim_{s \to s_1+0} \frac{x_1''(h(s))}{x''(s)} = (h'(s_1))^2 \cdot \frac{x_1''(h(s_1))}{x''(s_1)}
\end{aligned}$$

Since Γ, Γ_1 have positive curvature at the vertices then $x_1''(h(s_1)) > 0$, $x''(s_1) > 0$. Hence $f'(a+0) \in (0, \infty)$. Similarly we can obtain $f'(b-0)$, $g'(c+0)$, $g'(d-0) \in (0, \infty)$. Thus $f \in \mathbf{C}^1[a, b]$, $g \in \mathbf{C}^1[c, d]$, $f', g' \geq \delta > 0$.

As it was mentioned before the following three cases are possible:

(1) n is even, $F^{\frac{n}{2}} P_0 = P_2$;

(2) n is odd, $F^{\frac{n+1}{2}} P_0 = P_1$;

(3) n is odd, $F^{\frac{n+1}{2}} P_0 = P_3$.

We shall consider only the case (1). The cases (2), (3) can be considered similarly.

Let $n = 2k$, $F^k P_0 = P_2$. Since $P^* = (x(s^*), y(s^*)) \in O(P_1) = O(P_3)$ then there exists a number $m \in \{0, \ldots, n-1\}$ that $F^m P^* = P_3$. Then $\overline{M_{2m}} = [\hat{f}_{2m}^\Gamma(0), s_3] \subseteq [s_1, s_3]$, where $P_j = (x(s_j), y(s_j))$. Let $\varepsilon > 0$ be so small that $\hat{s}(y) \in (o, s^*) = M_0$ for any $y \in (d - \varepsilon, d)$, $\tilde{s}(x) \in (\hat{f}_{2m}(0), s_3) = M_{2m}$ for any $x \in (b - \varepsilon, b)$ and

$$(37) \qquad \hat{s}(d - \varepsilon) < \hat{f}^\Gamma_{-2m}(\tilde{s}(b - \varepsilon))$$

Then

(38) $$g(y) = y_1(h_0(\hat{s}(y))), \quad y \in (d - \varepsilon, d)$$

(39) $$f(x) = x_1(\hat{f}^{\Gamma_1}_{2m}(h_0(\hat{f}^{\Gamma}_{-2m}(\tilde{s}(x))))), \quad x \in (b - \varepsilon, b)$$

Let

$$g_0(y) : [d - \varepsilon, d] \xrightarrow{\text{on}} [y_1(\tfrac{1}{3}s_1^*), d_1],$$

$$f_0(x) : [b - \varepsilon, b] \xrightarrow{\text{on}} [x_1(\hat{f}^{\Gamma_1}_{2m}(\tfrac{2}{3}s_1^*)), b_1]$$

be arbitrary functions such that

$$g_0 \in \mathbf{C}^\infty[d - \varepsilon, d], \quad f_0 \in \mathbf{C}^\infty[b - \varepsilon, b], \quad g_0', f_0' \geq \delta > 0$$

where $[a_1, b_1]$, $[c_1, d_1]$ are the projections of $\Omega_1 \cup \Gamma_1$ onto the x and y axes. We set

$$h_1(s) = \hat{s}_1(g_0(y(s))), \quad s \in [0, \hat{s}(d - \varepsilon)]$$

$$h_2(s) = \hat{f}^{\Gamma_1}_{-2m}(\tilde{s}_1(f_0(x(\hat{f}^{\Gamma}_{2m}(s))))), \quad s \in [\hat{f}^{\Gamma}_{-2m}(\tilde{s}(b - \varepsilon)), s^*]$$

Then

$$h_1 : [0, \hat{s}(d - \varepsilon)] \xrightarrow{\text{on}} [0, \tfrac{1}{3}s_1^*],$$

$$h_2 : [\hat{f}^{\Gamma}_{-2m}((\tilde{s}(b - \varepsilon)), s^*] \xrightarrow{\text{on}} [\tfrac{2}{3}s_1^*, s_1^*],$$

$$h_1 \in \mathbf{C}^\infty[0, \hat{s}(d - \varepsilon)], \quad h_2 \in \mathbf{C}^\infty[\hat{f}^{\Gamma}_{-2m}(\tilde{s}(b - \varepsilon)), s^*], \quad h_1', h_2' \geq \delta_1 > 0$$

Because of (37) there exists function $h_0 : [0, s^*] \xrightarrow{\text{on}} [0, s_1^*]$, $h_0 \in \mathbf{C}^\infty[0, s^*]$, $h_0' \geq \delta > 0$ such that

$$\begin{cases} h_0(s) = h_1(s), & s \in [0, \hat{s}(d - \varepsilon)] \\ h_0(s) = h_2(s), & s \in [\hat{f}^{\Gamma}_{-2m}(\tilde{s}(b - \varepsilon)), s^*] \end{cases}$$

So if we define $f(x)$, $g(y)$ by formulas (30), (36) then

$$f(x) = f_0(x), \qquad x \in [b - \varepsilon, b]$$

$$g(y) = g_0(y), \qquad y \in [d - \varepsilon, d]$$

Thus $f(x) \in \mathbf{C}^\infty(a, b]$, $g(y) \in \mathbf{C}^\infty(c, d]$.

3. The proof of the existence of functions $f(x)$, $g(y)$ such that (29) holds in the case (3) is exactly the same as in the case (2). Thus Theorem 6 is completely proved.

Remark. It is easy to construct domains Ω, Ω_1 satisfying the conditions of the case (2) and such that there exist no functions $f(x)$, $g(y)$ such that $f \in \mathbf{C}^\infty[a, b]$, $g \in \mathbf{C}^\infty[c, d]$ and (29) holds.

It is not so difficult to obtain a necessary and sufficient conditions for existence $f\in\mathbf{C}^\infty[a,b]$, $g\in\mathbf{C}^\infty[c,d]$ that (29) holds. We shall write here such conditions without proof in the case when n is even and $F^{\frac{n}{2}}P_0 = P_2$. Denote

$$\phi(x) = x(f_n(\tilde{s}(x))), \quad x\in[a,b], \qquad \psi(y) = y(f_n(\hat{s}(y))), \quad y\in[c,d]$$

$$\phi_1(x) = x_1(f_n^{\Gamma_1}(\tilde{s}_1(x))), \quad x\in[a_1,b_1], \quad \psi_1(y) = y_1(f_n^{\Gamma_1}(\hat{s}_1(y))), \quad y\in[c_1,d_1]$$

Then there exist $f\in\mathbf{C}^\infty[a,b]$, $g\in\mathbf{C}^\infty[c,d]$, $f', g' \geq \delta > 0$ such that (29) holds if and only if

$$\phi_1(p(\phi(x)))\in\mathbf{C}^\infty[a, a+\varepsilon)$$

$$\psi_1(q(\psi(y)))\in\mathbf{C}^\infty[c, c+\varepsilon)$$

for some ε, $p(x)$, $q(y)$ such that $\varepsilon > 0$; $p\in\mathbf{C}^\infty(b-\varepsilon, b]$, $p(b) = b_1$, $p' \geq \delta > 0$; $q\in\mathbf{C}^\infty(d-\varepsilon, d]$, $q(d) = d_1$, $q' \geq \delta > 0$.

Corollary 1. *Let Ω be bounded domain convex relative to the lines $x =$ const, $y =$ const, $\alpha(F_\Gamma) = \frac{m}{n}$, $(m,n) = 1$, $F_\Gamma^n = I_\Gamma$. Then from the statement* (3) *of Theorem* 6 *it follows that the topological structure of F_Γ coincides with the structure of homoeomorphism $F_{\Gamma_n^m}$, where $\Gamma_n^m = \partial\Pi_n^m$ is the boundary of rectangle*

$$\Pi_n^m = \left\{(x,y)\,\middle|\, 0 < x + y < \frac{m}{\sqrt{2}},\ 0 < x - y < \frac{n-m}{\sqrt{2}}\right\}$$

$\left(\textit{if } F_\Gamma^n = I_\Gamma,\ \alpha(F) = \frac{m}{n} \textit{ then always } 0 < m < n\right)$.

The boundary Γ_n^m is very convenient for studying of the topological properties of F because all functions $f^\pm(s)$, $f_k(s)$ are linear and can be written in explicit form. For example, $f^+(s) = n - s$, $f^-(s) = m + n - s$, $f_2(s) = f^-(f^+(s)) = m + s$, $\alpha(F) = \frac{m}{n}\in(0,1)$.

In other words, the rectangle Π_n^m is the simplest representative of the class of domains $E(m,n)$ such that any $\Omega\in E(m,n)$ is bounded domain convex relative to the lines $x =$ const, $y =$ const and $\alpha(F_\Gamma) = \frac{m}{n}$, $F_\Gamma^n = I_\Gamma$, $\Gamma = \partial\Omega$.

Corollary 2. *The boundary Γ_n^m of the rectangle Π_n^m is not smooth. The simplest representative of $E(m,n)$ with smooth boundary is the ellipse E_n^m* [4]

$$\partial E_n^m = \left\{\left(\sin 2\pi\left(t + \frac{1}{4} - \frac{1}{2}\frac{m}{n}\right),\ \cos 2\pi t\right)\,\middle|\, t\in\mathbf{R}\right\}$$

where t is called the canonical parameter for the diffeomorphism F [4]. *It is easy to verify that, roughly speaking, $T^+: t\mapsto -t$, $T^-: t\mapsto \frac{m}{n} - t$, $F: t\mapsto t + \frac{m}{n}$,*

$t \in \mathbf{R}$. *So* $F^n: t \mapsto t + m$, $t \in \mathbf{R}$. *Hence* $F^n = I$, $\alpha(F) = \dfrac{m}{n}$.

Corollary 3. *Let* $\Omega \in E(m, n)$, $\partial\Omega = \Gamma \in \mathbf{C}^\infty$, *the curvature of* Γ *at the vertices is positive. Let* $f(x) \in \mathbf{C}^\infty[a, b]$, $g(y) \in \mathbf{C}^\infty[c, d]$, $f', g' \geq \delta > 0$ *and*

$$\Omega_1 = \{(f(x), g(y)) | (x, y) \in \Omega\}$$

Consider the following Dirichlet problem in Ω

$$\begin{cases} u_{xy}(x, y) + h(x, y, u) = 0, & (x, y) \in \Omega \\ u(x, y) = \phi(x, y), & (x, y) \in \Gamma \end{cases} \tag{40}$$

Denote

$$u_1(x, y) = u(f^{-1}(x), g^{-1}(y)), \quad (x, y) \in \Omega_1 \cup \Gamma_1$$

$$h_1(x, y, u) = h(f^{-1}(x), g^{-1}(y), u) \cdot f'(f^{-1}(x)) \cdot g'(g^{-1}(y)), \quad (x, y) \in \Omega_1, u \in \mathbf{R}$$

$$\phi_1(x, y) = \phi(f^{-1}(x), g^{-1}(y)), \quad (x, y) \in \Gamma_1 = \partial\Omega_1$$

Consider the following Dirtichlet problem in Ω_1

$$\begin{cases} u_{1xy}(x, y) + h_1(x, y, u) = 0, & (x, y) \in \Omega_1 \\ u_1(x, y) = \phi_1(x, y), & (x, y) \in \Gamma_1 \end{cases} \tag{41}$$

Then

(1) $u \in \mathbf{C}^k(\Omega \cup \Gamma) \Leftrightarrow u_1 \in \mathbf{C}^k(\Omega_1 \cup \Gamma_1)$;
(2) $u \in N(\mathbf{A}) \Leftrightarrow u_1 \in N(\mathbf{A}_1)$;
(3) $u|_\Gamma = \phi \Leftrightarrow u_1|_{\Gamma_1} = \phi_1$;
(4) *for any* $k = 0, 1, \ldots$ *there exist constants* $C_1(k)$, $C_2(k)$ *such that*

$$C_1(k) \| u \|_{\mathbf{W}_2^k(\Omega)} \leq \| u_1 \|_{\mathbf{W}_2^k(\Omega_1)} \leq C_2(k) \| u \|_{\mathbf{W}_2^k(\Omega)}$$

(5) u *is a solution of* (40) $\Leftrightarrow u_1$ *is a solution of* (41).

6. The orthogonal projection onto $N(\mathbf{A})$

After we proved the solvability of the problem (4) it is interesting to study the regularity of the solutions. If we follow the techniques developed in [18] we have to study the regularity properties of the orthogonal projection P_N from $\mathbf{L}_2(\Omega)$ onto $N(\mathbf{A})$. We shall obtain here the explicit form of P_N and consider some properties.

Let $u \in \mathbf{L}_2(\Omega)$. Denote $u_2 = P_N u \in N(\mathbf{A})$. According to Theorem 1 there exists some function $g_2(s) \in \mathbf{L}_2(M_0)$ which generates the function u_2 with the help of formulas (5) (9) (10) (11). So it is sufficient to define $g_2(s)$ for any $s \in M_0$.

Following [15] we introduce the piecewise constant functions (P. C. functions)

$v(x, y, \gamma) \in N(\mathbf{A})$, $\gamma \in M_0$. The function $v(x, y, \gamma)$ is generated by the following function $g(s, \gamma) \in \mathbf{L}_2(M_0)$

$$g(s, \gamma) = \begin{cases} 0, & s \leq \gamma; \\ 1, & s > \gamma. \end{cases} = \Theta(s - \gamma), \quad s \in \overline{M_0} \tag{42}$$

where $\Theta(x)$ is the Heaviside's function.

We shall describe below the structure of the P. C. function $v(x, y, \gamma) \in N(\mathbf{A})$. According to Corollary 1 of Theorem 6 it is sufficient to consider the structure of $v(x, y, \gamma)$ in the case $\Omega = \Pi_n^m$.

Consider the rectangle Π_n^m, $n, m \in \mathbf{N}$, $n > m$ (see Figure 3).

Then $s_1 = \frac{m}{2}$, $s_2 = \frac{n}{2}$, $s_3 = \frac{n+m}{2}$, $l = n$. As far as $\hat{f}_k(s)$, $k \in \{0, \ldots, 2n-1\}$ are linear functions and $\hat{f}_k'(s) \equiv (-1)^k$, then all intervals M_k have a common length. So from Lemma 1′ it follows that $M_0 = \left(0, \frac{l}{2n}\right) = \left(0, \frac{1}{2}\right)$.

Let $\gamma \in \left(0, \frac{1}{2}\right)$. From (10), (42) and Lemma 1′ it follows that

$$g(s, \gamma) = \begin{cases} 1, & s \in \bigcup\limits_{k=0}^{n-1} (\gamma + k, k + 1 - \gamma); \\ 0, & s \in (0, l) \setminus \bigcup\limits_{k=0}^{n-1} (\gamma + k, k + 1 - \gamma). \end{cases} \tag{43}$$

$$= \sum_{k=0}^{n-1} (\Theta(k + 1 - \gamma - s) - \Theta(k + \gamma - s))$$

Since $\tilde{s}(x) = \sqrt{2}x + \frac{m}{2}$, $\hat{s}(y) = \frac{m}{2} - \sqrt{2}y$ then from (5), (9), (11) we obtain

$$v(x, y, \gamma) = g\left(\sqrt{2}x + \frac{m}{2}, \gamma\right) - g\left(\frac{m}{2} - \sqrt{2}y, \gamma\right), \qquad (x, y) \in \Pi_n^m \tag{44}$$

Because of (43), (44) we have for $(x, y) \in \Pi_n^m$

$$v(x, y, \gamma) = \sum_{k=0}^{n-1} \left(\Theta\left(k + 1 - \gamma - \frac{m}{2} - \sqrt{2}x\right) - \Theta\left(k + \gamma - \frac{m}{2} - \sqrt{2}x\right)\right) - \sum_{k=0}^{n-1} \left(\Theta\left(k + 1 - \gamma - \frac{m}{2} + \sqrt{2}y\right) - \Theta\left(k + \gamma - \frac{m}{2} + \sqrt{2}y\right)\right) \tag{45}$$

So the structure of $v(x, y, \gamma)$ is the following (see Figure 3): each $\gamma \in M_0 = (0, \frac{1}{2})$ generates a system of rectangles $\Pi_k(\gamma) \subset \Pi_n^m$, $k = 1, \ldots, (n-m)m$, such that

(1) for any $k \in \{1, \ldots, (n-m)m\}$ there exist $j, i \in \{1, \ldots, n-1\}$ that

$$\Pi_k(\gamma) = \left\{ (x, y) \,\middle|\, \frac{i - \gamma - \frac{m}{2}}{\sqrt{2}} < x < \frac{i + \gamma - \frac{m}{2}}{\sqrt{2}}, \frac{\frac{m}{2} - (j+1) + \gamma}{\sqrt{2}} < y < \frac{\frac{m}{2} - j - \gamma}{\sqrt{2}} \right\}$$

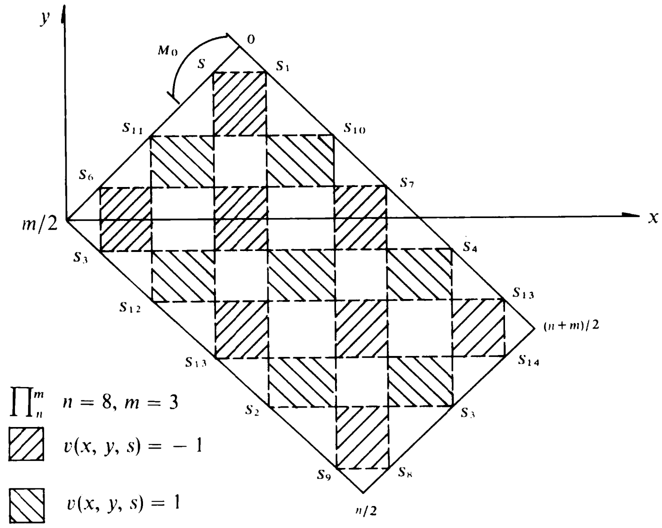

Figure 3.

if k is odd and

$$\Pi_k(\gamma) = \left\{ (x, y) \,\middle|\, \frac{i + \gamma - \frac{m}{2}}{\sqrt{2}} < x < \frac{j + 1 - \gamma - \frac{m}{2}}{\sqrt{2}},\ \frac{\frac{m}{2} - \gamma - i}{\sqrt{2}} < y < \frac{\frac{m}{2} - i + \gamma}{\sqrt{2}} \right\}$$

if k is even;

(2) for any $(x, y) \in \Pi_n^m$

$$v(x, y, \gamma) = \begin{cases} (-1)^k, & (x, y) \in \Pi_k(\gamma), \\ 0, & (x, y) \notin \bigcup_{k=1}^{(n-m)m} \Pi_k(\gamma) \end{cases} \tag{46}$$

Using Corollary 1 of Theorem 6 we obtain that if $\Omega \in E(m, n)$ for some $m, n \in \mathbf{N}$, $n > m$ then any $\gamma \in M_0$ generates a system of rectangles $\Pi_k(\gamma)$, $k = 1, \ldots, (n - m)m$ with the sides parallel to x and y axes and such that (46) holds.

We denote by $N_1(\mathbf{A})$ a subset of $N(\mathbf{A})$ consisting from all functions $u \in N(\mathbf{A})$ which have the generating functions $g(s) \in \mathbf{C}^1(\overline{M_0}) = \mathbf{C}^1[0, s^*]$. Since for any constant C functions $g(s)$, $g(s) + C$ generate the same $u \in N(\mathbf{A})$ then for any $u \in N_1(\mathbf{A})$ we denote by $g_u(s) \in \mathbf{C}^1[o, s^*]$ the generating function such that $g_u(0) = 0$.

Let $u \in N_1(\mathbf{A})$. Then for any $s \in [0, s^*]$

$$g_u(s) = \int_0^{s^*} g_u'(\eta) \cdot \Theta(s - \eta)\, d\eta = \int_0^{s^*} g_u'(\eta) \cdot g(s, \eta)\, d\eta \tag{47}$$

It is easy to verify that operator A which assigns with the help of (5), (9), (10), (11) to any $g(s) \in \mathbf{L}_2(M_0)$ the corresponding function $u \in N(\mathbf{A})$ is linear bounded operator from $\mathbf{L}_2(M_0)$ to $\mathbf{L}_2(\Omega)$. So from (47) it follows that for any $u \in N_1(\mathbf{A})$

(48) $$u(x, y) = \Lambda g_u(s) = \int_0^{s^*} g_u'(\eta) \cdot \Lambda g(s, \eta)\, d\eta = \int_0^{s^*} g_u'(\eta) \cdot v(x, y, \eta)\, d\eta$$

Let $u, w \in N_1(\mathbf{A})$. Then

(49) $$(u, w)_{\mathbf{L}_2(\Omega)} = \int_0^{s^*} \int_0^{s^*} g_u'(\eta) \cdot g_w'(\gamma) \cdot k(\eta, \gamma)\, d\eta\, d\gamma$$

where

(50) $$k(\eta, \gamma) = (v(x, y, \eta), v(x, y, \gamma))_{\mathbf{L}_2(\Omega)}, \qquad \eta, \gamma \in [0, s^*]$$

Using the structure of P. C. functions $v(x, y, \gamma)$ we can obtain explicit formula for $k(\eta, \gamma)$. First, we consider the case of domain represented in Figure 4.

Let $\eta > \gamma$. Then for any $s \in [0, s^*]$ we have

$$\Pi_1(s) = (x(x_7), x(s_1)) \times (y(s_7), y(s_1))$$
$$\Pi_2(s) = (x(s_2), x(s_6)) \times (y(s_2), y(s_6))$$
$$\Pi_3(s) = (x(s_5), x(s_3)) \times (y(s_5), y(s_3))$$

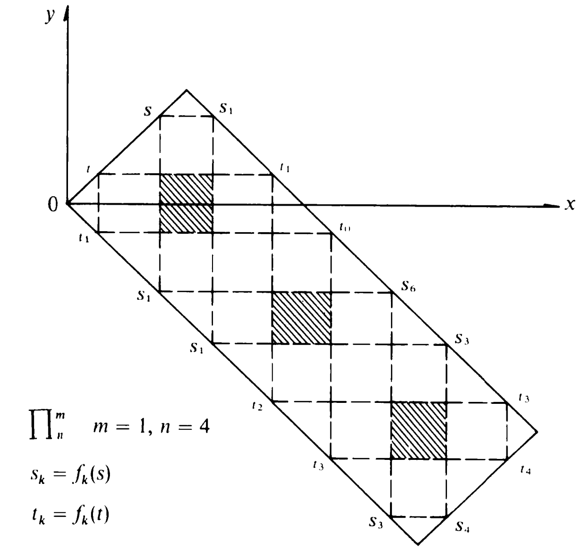

Figure 4.

where $s_k = \hat{f}_k(s)$, $k = 1, \dots, 7$. So from (46) it follows that $k(\eta, \gamma)$ is equal to the total area of shaded rectangles

(51) $$\begin{aligned} k(\eta, \gamma) &= (x(\hat{f}_1(\gamma)) - x(\hat{f}_7(\gamma))) \cdot (y(\hat{f}_1(\eta)) - y(\hat{f}_7(\eta))) \\ &+ (y(\hat{f}_6(\gamma)) - y(\hat{f}_2(\gamma))) \cdot (x(\hat{f}_6(\eta)) - x(\hat{f}_2(\eta))) \\ &+ (x(\hat{f}_3(\gamma)) - x(\hat{f}_5(\gamma))) \cdot (y(\hat{f}_3(\eta)) - y(\hat{f}_5(\eta))) \end{aligned}$$

$$= a_1(\gamma)b_1(\eta) + a_2(\gamma)b_2(\eta) + a_3(\gamma)b_3(\eta)$$

From (50) it follows that $k(\eta, \gamma) = k(\gamma, \eta)$. So for any $\eta, \gamma \in [0, s^*]$, $\eta \neq \gamma$

$$k(\eta, \gamma) = \Theta(\eta - \gamma) \cdot \sum_{k=1}^{3} a_k(\gamma) \cdot b_k(\eta) + \Theta(\gamma - \eta) \cdot \sum_{k=1}^{3} a_k(\eta) \cdot b_k(\gamma)$$

It is easy to see that for any $k = 1, 2, 3$, $a_k, b_k \in \mathbf{C}^\infty[0, s^*]$; $a_k, b_k \geq 0$; a_k is strictly increasing function, $a_k(0) = 0$; b_k is strictly decreasing function, $b_k(s^*) = 0$.

Let $\Omega \in E(m, n)$, $\partial\Omega = \Gamma \in \mathbf{C}^\infty$, the curvature of Γ at the vertices is positive. Then for any $\eta, \gamma \in [0, s^*]$, $\eta \neq \gamma$

$$k(\eta, \gamma) = \Theta(\eta - \gamma) \cdot \sum_{k=1}^{m(n-m)} a_k(\gamma) \cdot b_k(\eta) + \Theta(\gamma - \eta) \cdot \sum_{k=1}^{m(n-m)} a_k(\eta) \cdot b_k(\gamma) \tag{52}$$

where for any $k = 1, \dots, m(n - m)$

(1) $a_k, b_k \in \mathbf{C}^\infty[0, s^*]$;

(2) $a_k(\gamma), b_k(\gamma) > 0$, $\gamma \in (0, s^*)$, $a_k(0) = b_k(s^*) = 0$;

(3) $a'_k(\gamma), (-b'_k(\gamma)) \geq \delta > 0$, $\gamma \in [0, s^*]$.

For simplicity we omit here the proof of (52). It easy to note that according to Corollary 1 of Theorem 6 it is sufficient to prove (52) for the case $\Omega = \Pi_n^m$.

Let $u \in \mathbf{C}[0, s^*]$ satisfies the following integral equation:

$$\int_0^{s^*} k(\eta, \gamma) \cdot u(\gamma)\, d\gamma = h(\eta), \quad \eta \in [0, s^*] \tag{53}$$

Then from (52), (53) it follows

$$\begin{aligned} &u(\eta) \cdot \sum_{k=1}^{m(n-m)} (b'_k(\eta)a_k(\eta) - a'_k(\eta)b_k(\eta)) \\ &+ \sum_{k=1}^{m(n-m)} \left(b''_k(\eta) \int_0^\eta a_k(\gamma)u(\gamma)\, d\gamma + a''_k(\eta) \int_\eta^{s^*} b_k(\gamma)u(\gamma)\, d\gamma \right) \\ &= h''(\eta) \end{aligned} \tag{54}$$

We denote

$$q(\eta) = \sum_{k=1}^{m(n-m)} (b'_k(\eta) \cdot a_k(\eta) - a'_k(\eta) \cdot b_k(\eta)), \qquad \eta \in [0, s^*]$$

Since for any $k = 1, \dots, m(n - m)$, $a_k(\gamma)$, $b_k(\gamma) > 0$, $\gamma \in (0, s^*)$; $a_k(0) = b_k(s^*) = 0$; $a'_k(\gamma), (-b'_k(\gamma)) \geq \delta > 0$, $\gamma \in [0, s^*]$ then $(b'_k(\eta)a_k(\eta) - a'_k(\eta)b_k(\eta)) < 0$, $\eta \in [0, s^*]$. So $q(\eta) \in \mathbf{C}^\infty[0, s^*]$ and $q(\eta) \leq -\delta < 0$, $\eta \in [0, s^*]$ for some $\delta > 0$. We denote

$$k_1(\eta, \gamma) = \frac{1}{q(\eta)}\left(\Theta(\eta - \gamma) \cdot \sum_{k=1}^{m(n-m)} b''_k(\eta)a_k(\gamma) + \Theta(\gamma - \eta) \cdot \sum_{k=1}^{m(n-m)} a''_k(\eta)b_k(\gamma)\right)$$

Then equation (54) can be rewritten in the form

(55) $$u(\eta) + \int_0^{s^*} k_1(\eta, \gamma) \cdot u(\gamma)\, d\gamma = \frac{h''(\eta)}{q(\eta)}$$

So we have obtained the Fredholm integral equation of the second kind with the bounded kernel $k_1(\eta, \gamma)$.

The homogeneous problem (55) has only trivial solution in $\mathbf{L}_2(0, s^*)$. Indeed, if there exists $v(\eta) \in \mathbf{L}_2(0, s^*)$ such that

$$v(\eta) + \int_0^{s^*} k_1(\eta, \gamma) \cdot v(\gamma)\, d\gamma = 0, \quad \eta \in [0, s^*]$$

then $v \in \mathbf{C}[0, s^*]$ and for the function

$$h(\eta) = \int_0^{s^*} k(\eta, \gamma) \cdot v(\gamma)\, d\gamma$$

we have $h''(\eta) = 0$, $\eta \in [0, s^*]$. Besides as far as $k(0, \gamma) = k(s^*, \gamma) = 0$, $\gamma \in [0, s^*]$ then $h(0) = h(s^*) = 0$ Hence $h(\eta) \equiv 0$. Denote $g(s) = \int_0^s v(\eta)\, d\eta$, $s \in [0, s^*]$. Then $g \in \mathbf{C}^1[0, s^*]$, $g(0) = 0$ and g generates some function $u_g \in N_1(\mathbf{A})$. From (49) it follows

$$0 = \int_0^{s^*} h(\eta) v(\eta)\, d\eta = \int_0^{s^*} \int_0^{s^*} k(\eta, \gamma) v(\gamma) v(\eta)\, d\gamma\, d\eta = \| u_g \|^2_{\tilde{\mathbf{L}}_2(\Omega)}$$

Hence $u_g = 0$. So $g(s) = \int_0^s v(\eta)\, d\eta = 0$, $s \in [0, s^*]$ and consequently $v = 0$.

From Fredholm's alternative theorem it follows that for any $h \in \mathbf{W}_2^2(0, s^*)$ there exists a unique solution of the equation (55) in $\mathbf{L}_2(0, s^*)$.

We denote by $\mathbf{R}_1$ the resolvent of the operator

$$(\mathbf{I} + \mathbf{K}_1) v(\eta) = v(\eta) + \int_0^{s^*} k_1(\eta, \gamma) u(\gamma)\, d\gamma$$

i.e.

$$\mathbf{R}_1 \circ (\mathbf{I} + \mathbf{K}_1) = (\mathbf{I} + \mathbf{K}_1) \circ \mathbf{R}_1 = \mathbf{I}$$

Let $u \in \tilde{\mathbf{L}}_2(\Omega) = R(\mathbf{A}) \oplus N_1(\mathbf{A})$, $u = u_1 + u_2$, $u_1 \in R(\mathbf{A})$, $u_2 \in N_1(\mathbf{A})$. Consider the function

$$h(\eta) = \int_\Omega u(x, y) v(x, y, \eta)\, dx\, dy = \int_\Omega u_2(x, y) v(x, y, \eta)\, dx\, dy = \int_0^{s^*} g'_{u_2}(s) k(s, \eta)\, ds$$

where $g_{u_2}(s)$ is the generating function for $u_2 \in N_1(\mathbf{A})$. As far as $g_{u_2} \in \mathbf{C}^1[0, s^*]$ then $h \in \mathbf{C}^2[0, s^*]$ and $g'_{u_2}(s)$ satisfies (55). So

$$g'_{u_2}(s) = \mathbf{R}_1 \frac{h''(\eta)}{q(\eta)} = \mathbf{R}_1 \left(\frac{1}{q(\eta)} \cdot \frac{d^2}{d\eta^2} \left(\int_\Omega u(x, y) v(x, y, \eta)\, dx\, dy \right) \right)$$

Since $u_2 = \Lambda g_{u_2}$ (we have denoted by Λ the linear bounded operator which assigns to any $g(s) \in \mathbf{L}_2(M_0)$ the corresponding function $u \in N(\mathbf{A})$) then

$$u_2 = A\left(\int_0^s \mathbf{R}_1\left(\frac{1}{q(\eta)}\cdot\frac{d^2}{d\eta^2}\left(\int_\Omega u(x, y)v(x, y, \eta)\,dx\,dy\right)\right)d\eta\right) \stackrel{def}{=} P_{N_1}u \tag{56}$$

Thus the operator P_{N_1} defined by (56) is the operator of orthogonal projection from $\tilde{\mathbf{L}}_2(\Omega)$ onto $N_1(\mathbf{A})$.

We point out that the function

$$h_u(\eta) = \int_\Omega u(x, y)\cdot v(x, y, \eta)\,dx\,dy, \quad \eta\in[0, s^*] \tag{57}$$

belongs to $\mathbf{W}_2^1(0, s^*)$ for any $u\in\mathbf{L}_2(\Omega)$. However $h_u(\eta)\in\mathbf{W}_2^2(0, s^*)$ if and only if $u\in\tilde{\mathbf{L}}_2(\Omega)$.

As far as $N_1(\mathbf{A})$ is dense in $N(\mathbf{A})$ then we obtain

$$P_N = \overline{P_{N_1}} \tag{58}$$

where $\overline{P_{N_1}}$ means closure of P_{N_1} in $\mathbf{L}_2(\Omega)$.

Let $u\in\mathbf{C}^k(\Omega\cup\Gamma)$, $k\in\mathbf{N}$. Then $h_u(\eta)\in\mathbf{C}^{k+1}[0, s^*]$. Indeed it is easy to verify that

$$h_u(\eta) = \sum_{m=0}^{2n-1}(-1)^m\cdot U_u(x(\hat{f}_m(\eta)), y(\hat{f}_m(\eta)))$$

$$U_u(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} u(\xi, \kappa)\,d\xi\,d\kappa$$

where we assume $u(\xi, \kappa) = 0$, $(\xi, \kappa)\notin\Omega$. As far as $x(s), y(s)\in\mathbf{C}^\infty[0, l]$, $\hat{f}_m(s)\in\mathbf{C}^\infty[0, s^*]$, $u\in\mathbf{C}^k(\Omega\cup\Gamma)$ then $U_u\in\mathbf{C}^{k+1}(\Omega\cup\Gamma)$, $h_u\in\mathbf{C}^{k+1}[0, s^*]$. Since $k\geq 1$ then $g'_{u_2}(s)$ (where $u_2 = P_N u = P_{N_1}u\in N_1(\mathbf{A})$) satisfies equation (55) which can be written in the form

$$g'_{u_2}(\eta) + \sum_{p=1}^{m(n-m)}\frac{b''_p(\eta)}{q(\eta)}\cdot\int_0^\eta a_p(\gamma)g'_{u_2}(\gamma)\,d\gamma + \sum_{p=1}^{m(n-m)}\frac{a''_p(\eta)}{q(\eta)}\cdot\int_\eta^{s^*} b_p(\gamma)g'_{u_2}(\gamma)\,d\gamma = \frac{h''_u(\eta)}{q(\eta)}$$

As far as $a_p, b_p, q\in\mathbf{C}^\infty[0, s^*]$, $h''_u\in\mathbf{C}^{k-1}[0, s^*]$, $q(\eta)\leq -\delta < 0$ then $g'_{u_2}\in\mathbf{C}^{k-1}[0, s^*]$. Then

$$g_{u_2}(\eta) = \int_0^\eta g'_{u_2}(s)\,ds\in\mathbf{C}^k[0, s^*]$$

Using (10) it can be shown $g_{u_2}\in\mathbf{C}^k[0, l]$. Since

$$u_2(x, y) = P_N u(x, y) = g_{u_2}(\tilde{s}(x)) - g_{u_2}(\hat{s}(y))$$

then $u_2\in\mathbf{C}^k(\Omega\cup\Gamma\setminus\{P_0,\ldots,P_3\})$ because of $\tilde{s}(x)\in\mathbf{C}^\infty(a, b)$, $\hat{s}(y)\in\mathbf{C}^\infty(c, d)$, $\tilde{s}'(x)\xrightarrow[\substack{x\to a+0\\ x\to b-0}]{} +\infty$, $\hat{s}'(y)\xrightarrow[\substack{y\to c+0\\ y\to d-0}]{} +\infty$.

Thus we have verified

Theorem 7. *If* $u \in \mathbf{C}^k(\Omega \cup \Gamma)$ *then* $u_2 = P_N u \in \mathbf{C}^k(\Omega \cup \Gamma \setminus \{P_0, \dots, P_3\})$.

Remark 1. It seems to us that it can be proved $P_N\colon \mathbf{C}^k(\Omega \cup \Gamma) \to \mathbf{C}^k(\Omega \cup \Gamma)$. The plan of the proof may be following: First, we transform Ω onto the rectangle Π_n^m by the mapping (28). Then for any $k \in \mathbf{N}$ the space $\mathbf{C}^k(\Omega \cup \Gamma)$ corresponds to a space $\mathbf{C}^k_{\rho_k}(\Pi_n^m)$ where ρ_k is corresponding weight function. Second, we obtain $P_N\colon \mathbf{C}^k_{\rho_k}(\Pi_n^m) \to \mathbf{C}^k_{\rho_k}(\Pi_n^m)$. Third, we transform Π_n^m onto Ω by the inverse mapping (28). But, of course, this question needs a special consideration.

Remark 2. It is easy to see that if $\Omega = \Pi_n^m$ then for any $k \in \mathbf{N}$, $P_N\colon \mathbf{C}^k(\Omega \cup \Gamma) \to \mathbf{C}^k(\Omega \cup \Gamma)$. So using the techniques developed in [18] we can obtain that under the conditions of Theorem 4 there exists a $\mathbf{C}^\infty$-solution of problem (4) if $f \in \mathbf{C}^\infty$ and $|f_u| \geq \varepsilon > 0$.

Remark 3. It is interesting to point out that in the domain Π_n^m with piecewise smooth boundary problem (4) is in a sense "better" than in the case $\partial\Omega \in \mathbf{C}^\infty$. It is due to the hyperbolic character of the problem (4).

Appendix

Proof of Lemma 2. Let Ω be a bounded domain, $f(x, y, u) \in \mathbf{C}^0(\Omega \times \mathbf{R})$ and for a.e. $(x, y) \in \Omega$, any $u \in \mathbf{R}$

$$|f(x, y, u)| \leq C \cdot |u| + h(x, y) \tag{59}$$

for some $h \in \mathbf{L}_2(\Omega)$ and some constant $C > 0$. Let $u_n \xrightarrow[n\to\infty]{\mathbf{L}_2(\Omega)} u$. We shall show that $f(x, y, u_n(x, y)) \xrightarrow[n\to\infty]{\mathbf{L}_2(\Omega)} f(x, y, u(x, y))$. Let $\varepsilon > 0$ be an arbitrary positive number. There exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\|u_n\|_{\mathbf{L}_2(\Omega)} \leq 2\|u\|_{\mathbf{L}_2(\Omega)}, \qquad n > N_1 \tag{60}$$

From (59) it follows that for $n > N_1$ and for any domain $G \subset \Omega$

$$\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2(G)} \leq 2\|h\|_{\mathbf{L}_2(G)} + 3C\|u\|_{\mathbf{L}_2(G)} \tag{61}$$

So from the properties of Lebesgue integral it follows that there exists $\delta > 0$ such that

$$\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2(\Omega\setminus\Omega_\delta)} < \frac{\varepsilon}{5} \tag{62}$$

where

$$\Omega_\delta = \{(x, y) \mid \operatorname{dist}((x, y), \partial\Omega) > \delta\}$$

Consider

$$\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2(\Omega_\delta)}$$

Using Chebyshev's inequality we obtain that for any $r > 0$

(63) $$\mu(\Omega_r) = \mu(\{(x, y) \in \Omega \mid u^2(x, y) \geq r^2\}) \leq \frac{1}{r^2} \cdot \|u\|^2_{\mathbf{L}_2(\Omega)}$$

So from the properties of Lebesgue integral it follows that there exists $r > 0$ such that for any $n > N_1$

$$\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2(\Omega_r)} \leq 2\|h\|_{\mathbf{L}_2(\Omega_r)} + 3C\|u\|_{\mathbf{L}_2(\Omega_r)} < \frac{\varepsilon}{5}$$

Consider for $n > N_1$

(65) $$\begin{aligned} &\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2(\Omega_\delta \setminus \Omega_r)} \\ &\leq \|f(x, y, u) - f(x, y, u_n)\|_{\mathbf{L}_2((\Omega_\delta \setminus \Omega_r) \setminus \Omega_p^n)} \\ &\quad + \|f(x, y, u) - f(x, y, u_n)\|_{\mathbf{L}_2((\Omega_\delta \setminus \Omega_r) \cap \Omega_p^n)} \end{aligned}$$

where

$$\Omega_p^n = \{(x, y) \mid u_n^2(x, y) \geq p^2\}$$

(66) $$\mu(\Omega_p^n) \leq \frac{1}{p^2} \cdot \|u_n\|^2_{\mathbf{L}_2(\Omega)} \leq \frac{4}{p^2} \cdot \|u\|^2_{\mathbf{L}_2(\Omega)}$$

From (59), (61), (66) it follows that there exists $p > 0$ such that for any $n > N_1$

(67) $$\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2((\Omega_\delta \setminus \Omega_r) \cap \Omega_p^n)} \leq \frac{\varepsilon}{5}$$

Consider for $n > N_1$

$$\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2((\Omega_\delta \setminus \Omega_r) \setminus \Omega_p^n)}$$

As far as $f(x, y, u) \in \mathbf{C}^0(\Omega \times \mathbf{R})$, $\overline{\Omega_\delta} \subset \Omega$ then $f(x, y, u)$ is uniformly continuous for $(x, y, u) \in (\overline{\Omega_\delta} \times [-d, +d])$ where $d = \max(r, p)$. So there exists continuous function $\rho(\kappa)$, $\rho(\kappa) \xrightarrow[\kappa \to 0]{} 0$ such that

(68)
$$|f(x, y, u(x, y)) - f(x, y, u_n(x, y))| \leq \rho(u(x, y) - u_n(x, y)), \quad (x, y) \in (\Omega_\delta \setminus \Omega_r) \setminus \Omega_p^n$$

Since $\rho(\kappa) \xrightarrow[\kappa \to 0]{} 0$ then there exists $\kappa_1 > 0$ such that for any $|\kappa| < \kappa_1$

(69) $$\rho(\kappa) < \frac{\varepsilon}{5} \cdot (\mu(\Omega))^{-\frac{1}{2}}$$

From (61), (68), (69) it follows for any $n > N_1$

$$\|f(x, y, u(x, y)) - f(x, y, u_n(x, y))\|_{\mathbf{L}_2((\Omega_\delta \setminus \Omega_r) \setminus \Omega_p^n)}$$

$$\leq \left(\int_{((\Omega_\delta \backslash \Omega_r) \backslash \Omega_p^n) \cap \{|u(x,y) - u_n(x,y)| \geq \kappa_1\}} (3Cu + 2h)^2 \, dx dy \right)^{\frac{1}{2}}$$

$$+ \left(\int_{((\Omega_\delta \backslash \Omega_r) \backslash \Omega_p^n) \cap \{|u(x,y) - u_n(x,y)| < \kappa_1\}} \rho^2(u(x, y) - u_n(x, y)) \, dx dy \right)^{\frac{1}{2}} \tag{70}$$

$$\leq \| 3Cu + 2h \|_{\mathbf{L}_2(\{|u(x,y) - u_n(x,y)| \geq \kappa_1\})} + \frac{\varepsilon}{5}$$

As far as

$$\mu(\{(x, y) | |u(x, y) - u_n(x, y)| \geq \kappa_1\}) \leq \frac{1}{\kappa_1^2} \cdot \| u - u_n \|^2_{\mathbf{L}_2(\Omega)} \xrightarrow[n \to \infty]{} 0$$

then there exists $N_2 > N_1$ such that for any $n > N_2$

$$\| 3Cu + 2h \|_{\mathbf{L}_2(\{(x,y) \in \Omega | |u(x,y) - u_n(x,y)| \geq \kappa_1\})} < \frac{\varepsilon}{5} \tag{71}$$

So from (62), (64), (67), (70), (71) we obtain for any $n > N_2$

$$\| f(x, y, u(x, y)) - f(x, y, u_n(x, y)) \|_{\mathbf{L}_2(\Omega)} < \varepsilon$$

As far as $\varepsilon > 0$ is arbitrary it means that $f(x, y, u_n(x, y)) \xrightarrow[n \to \infty]{\mathbf{L}_2(\Omega)} f(x, y, u(x, y))$. Thus the operator $\mathbf{K}u = f(x, y, u)$ is continuous operator from $\mathbf{L}_2(\Omega)$ into itself.

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