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The K_*-localizations of the stunted real projective spaces

By

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0. Introduction

Given an associative ring spectrum E with unit, a CW-spectrum X is said to be *quasi E_*-equivalent* to a CW-spectrum Y if there exists a map $f: Y \to E \wedge X$ such that the composite $(\mu \wedge 1)(1 \wedge f): E \wedge Y \to E \wedge X$ is an equivalence where $\mu: E \wedge E \to E$ denotes the multiplication of E. We call such a map $f: Y \to E \wedge X$ a quasi E_*-equivalence. Let KO and KU be the real and the complex K-spectrum respectively. Since there is no difference between the KO_*- and KU_*-localizations, we denote by S_K the K_*-localization of the sphere spectrum $S = \Sigma^0$. Recall the smashing theorem [B1, Corollary 4.7] (or [R]) that the smash product $S_K \wedge X$ is actually the K_*-localization of X. This implies that two CW-spectra X and Y have the same K_*-local type if and only if X is quasi S_{K*}-equivalent to Y.

In [Y2] we studied the quasi KO_*-equivalence, and moreover in [Y3] and [Y4] we determined the quasi KO_*-types of the real projective spaces RP^n and the stunted real projective spaces $RP^n/RP^m = RP^n_{m+1}$. In this note we shall be interested in the quasi S_{K*}-equivalence in advance of the quasi KO_*-equivalence. The purpose of this note is to determine the K_*-local types of the stunted real projective spaces RP^n/RP^m along the line of [Y5], in which we have already determined the K_*-local types of the real projective spaces RP^n [Y5, Theorem 3]. Our proof will be established separately in the following three cases;

i) RP^{2s+n}/RP^{2s} $(2 \le n \le \infty)$, ii) RP^{2s+2t}/RP^{2s-1} $(t \ge 1)$ and

iii) $RP^{2s+2t+1}/RP^{2s-1}$ $(0 \le t \le \infty)$.

In the proof of [Y5, Theorem 3] we first investigated the behavior of the Adams operations ψ^k_C and ψ^k_R for the real projective spaces RP^n, and then applied a powerful tool due to Bousfield [B2, 9.8] (or see [Y5, Theorem 4]). By a quite similar argument to the old case we shall determine the K_*-local types of RP^{2s+n}_{2s+1} $(2 \le n \le \infty)$ and the Spanier-Whitehead duals DRP^{2s+2t}_{2s} $(t \ge 1)$ (Theorem 2.7 and Proposition 2.8). Since two finite spectra X and Y have the same K_*-local type if and only if their duals DX and DY have the same K_*-local type [Y5, Lemma 4.7], it is easy to determine the K_*-local types of RP^{2s+2t}_{2s} $(t \ge 1)$

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(Theorem 2.9). In order to observe the rest case we shall construct maps $g_{st}\colon \Sigma^{2s} \to Y_{st}$ modelled on the bottom cell inclusions $i\colon \Sigma^{2s} \to \Sigma^1 RP_{2s-1}^{2s+2t+1}$, where Y_{st} is a certain elementary spectrum with a few cells appearing in Theorem 2.7 admitting the same K_*-local type as $\Sigma^1 RP_{2s-1}^{2s+2t+1}$. By proving that each cofiber $C(g_{st})$ has the same K_*-local type as $\Sigma^1 RP_{2s}^{2s+2t+1}$ we shall determine the K_*-local types of $RP_{2s}^{2s+2t+1}$ $(1 \le t \le \infty)$ (Theorem 3.8).

1. Some elementary spectra with a few cells

1.1. The Moore spectrum SZ/n of type Z/n $(n \ge 2)$ is constructed by the cofiber sequence $\Sigma^0 \xrightarrow{n} \Sigma^0 \xrightarrow{i} SZ/n \xrightarrow{j} \Sigma^1$. Let M_{2m}, M'_{2m}, P_{2m} and P'_{2m} denote the cofibers of the maps $i\eta\colon \Sigma^1 \to SZ/2m$, $\eta j\colon SZ/2m \to \Sigma^0$, $\tilde{\eta}_{2m}\colon \Sigma^2 \to SZ/2m$ and $\bar{\eta}_{2m}\colon \Sigma^1 SZ/2m \to \Sigma^0$ respectively [Y2, I.4.1]. Here $\eta\colon \Sigma^1 \to \Sigma^0$ is the stable Hopf map of order 2, and $\bar{\eta}_{2m}$ and $\tilde{\eta}_{2m}$ are an extension and a coextension of η satisfying $\bar{\eta}_{2m} i = \eta$ and $j\tilde{\eta}_{2m} = \eta$. Hereafter the subscript "2" in the symbols $\bar{\eta}_2$ and $\tilde{\eta}_2$ are dropped as $\bar{\eta}$ and $\tilde{\eta}$. Notice that P'_{4m} and P_{4m} are respectively quasi KO_*-equivalent to $\Sigma^2 M_{2m}$ and $\Sigma^{-1} M'_{2m}$, and $P'_2 = C(\bar{\eta})$ and $P_2 = C(\tilde{\eta})$ are respectively quasi KO_*-equivalent to Σ^4 and Σ^{-1} (see [Y2, Corollary I. 5.4] and [Y5, (1.2)]). More precisely, it follows from [Y5, Theorem 1.2 i)] that $\Sigma^{-3} C(\tilde{\eta})$ has the same K_*-local type as $C(\bar{\eta})$.

Denote by V_{2m}, V'_{2m}, U_{2m} and U'_{2m} the cofibers of the maps $i\bar{\eta}\colon \Sigma^1 SZ/2 \to SZ/m$, $\tilde{\eta} j\colon \Sigma^1 SZ/m \to SZ/2$, $\tilde{\eta}_{4m/2}\colon \Sigma^2 SZ/2 \to SZ/4m$ and $\bar{\eta}_{4m/2}\colon \Sigma^2 SZ/4m \to SZ/2$ respectively where $\tilde{\eta}_{4m/2}$ is a coextension of $\bar{\eta}$ with $j\tilde{\eta}_{4m/2} = \bar{\eta}$ and $\bar{\eta}_{4m/2}$ is an extension of $\tilde{\eta}$ with $\bar{\eta}_{4m/2} i = \tilde{\eta}$. Then they are exhibited by the following cofiber sequences

$$(1.1)\qquad \begin{array}{ll} \Sigma^0 \xrightarrow{m\bar{i}} C(\bar{\eta}) \xrightarrow{i_V} V_{2m} \xrightarrow{j_V} \Sigma^1, & \Sigma^2 \xrightarrow{i'_V} V'_{2m} \xrightarrow{j'_V} C(\tilde{\eta}) \xrightarrow{m\tilde{j}} \Sigma^3, \\ C(\bar{\eta}) \xrightarrow{m\bar{\lambda}} \Sigma^0 \xrightarrow{i_U} U_{2m} \xrightarrow{j_U} \Sigma^1 C(\bar{\eta}), & \Sigma^3 \xrightarrow{m\tilde{\lambda}} C(\tilde{\eta}) \xrightarrow{i'_U} U'_{2m} \xrightarrow{j'_U} \Sigma^4. \end{array}$$

Here $\bar{i}\colon \Sigma^0 \to C(\bar{\eta})$ and $\tilde{j}\colon C(\tilde{\eta}) \to \Sigma^3$ denote the bottom cell inclusion and the top cell projection, and $\bar{\lambda}\colon C(\bar{\eta}) \to \Sigma^0$ and $\tilde{\lambda}\colon \Sigma^3 \to C(\tilde{\eta})$ satisfy the equalities $\bar{\lambda}\bar{i} = 4$ and $\tilde{j}\tilde{\lambda} = 4$. By virtue of [Y5, Theorem 1.2 ii) with (1.3) and (1.4)] we observe that $\Sigma^{-2} V'_{2m} \wedge C(\bar{\eta})$, $U_{2m} \wedge C(\bar{\eta})$ and $\Sigma^{-3} U'_{2m}$ have the same K_*-local type as V_{2m}.

Denote by MP_{2m} the cofiber of the map $i\eta \vee \tilde{\eta}_{2m}\colon \Sigma^1 \vee \Sigma^2 \to SZ/2m$. By use of [Y2, Lemma II.1.1] we have a cofiber sequence

$$(1.2)\qquad \Sigma^2 \xrightarrow{i_M \tilde{\eta}_{2m}} M_{2m} \xrightarrow{i_{MP}} MP_{2m} \xrightarrow{j_{MP}} \Sigma^3$$

where $i_M\colon SZ/2m \to M_{2m}$ denotes the canonical inclusion. Note that $\Sigma^4 MP_{2m}$ is quasi KO_*-equivalent to MP_{2m} [Y4, Corollary 2.7]. In [Y2, Propositions I.4.1, I.4.2, II.1.2 and II.1.3 and Corollary I.4.6] the KU- and KO-homologies of some elementary spectra with a few cells are computed. In particular, for $X = M_{2m}$, M'_{2m}, V_{2m} and MP_{2m} we have

(1.3) i) The KU-homologies KU_iX ($i=0, 1$) are tabled as follows:

$i =$	0	1
$KU_iM_{2m} \cong$	$Z\oplus Z/2m$	0
$KU_iV_{2m} \cong$	$Z/2m$	0

$i =$	0	1
$KU_iM'_{2m} \cong$	Z	$Z/2m$
$KU_iMP_{2m} \cong$	$Z\oplus Z/m$	Z

ii) The KO-homologies KO_iX ($0\le i\le 7$) are tabled as follows:

$i =$	0	1	2	3	4	5	6	7
$KO_iM_{2m} \cong$	$Z/2m$	0	$Z\oplus Z/2$	$Z/2$	$Z/4m$	0	Z	0
$KO_iM'_{2m} \cong$	Z	$Z/4m$	$Z/2$	$Z/2$	Z	$Z/2m$	0	0
$KO_iV_{2m} \cong$	Z/m	0	$Z/2$	$Z/2$	$Z/4m$	$Z/2$	$Z/2$	0
$KO_iMP_{2m} \cong$	$Z/2m$	0	Z	Z	$Z/2m$	0	Z	Z

Consider the two composite maps

(1.4)
$$i_\infty = j_{2,\infty}i\colon \Sigma^0 \longrightarrow SZ/2 \longrightarrow SZ/2^\infty, \quad \tilde\eta_\infty = j_{2,\infty}\tilde\eta_2\colon \Sigma^2 \longrightarrow SZ/2 \longrightarrow SZ/2^\infty$$

where the map $j_{2,\infty}\colon SZ/2\to SZ/2^\infty$ is the obvious map associated with the inclusion $Z/2\subset Z/2^\infty$. Evidently $i_{\infty *}(1)=1/2\in KU_0SZ/2^\infty\cong Z/2^\infty$ and $\tilde\eta_{\infty *}(1) = 1/2\in KU_2SZ/2^\infty \cong Z/2^\infty$. Since $[SZ/2^\infty, \Sigma^2KO]=0$, it is immediately shown that

(1.5) the cofiber $C(i_\infty)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^1\vee SZ/2^\infty$.

On the other hand, the KU- and KO-homologies of the cofiber $C(\tilde\eta_\infty)$ are easily computed as follows:

(1.6) i) $KU_0C(\tilde\eta_\infty)\cong Z/2^\infty$ and $KU_1C(\tilde\eta_\infty)\cong Z$.

ii) $KO_iC(\tilde\eta_\infty)\cong Z/2^\infty, 0, 0, Z$ according as $i\equiv 0, 1, 2, 3 \bmod 4$.

1.2. Let X and Y be CW-spectra which admit the same quasi KO_*-type. Let $f\colon \Sigma^0\to X$ and $g\colon \Sigma^0\to Y$ be maps related by the equality $(\iota_U\wedge 1)f = (\varepsilon_U\wedge 1)hg$ for a suitable quasi KO_*-equivalence $h\colon Y\to KO\wedge X$ where $\iota_U\colon \Sigma^0\to KU$ denotes the unit of KU and $\varepsilon_U\colon KO\to KU$ the complexification. Thus $f_*(1)\in KU_0X$ and $g_*(1)\in KU_0Y$ coincide when KU_*X and KU_*Y are identified via the quasi KO_*-equivalence h. If $\varepsilon_{U*}\colon KO_0X\to KU_0X$ is a monomorphism, then there holds the equality $(\iota_O\wedge 1)f = hg$ where $\iota_O\colon \Sigma^0\to KO$ denotes the unit of KO. In this case it is easily seen that

(1.7) the cofiber $C(f)$ is quasi KO_*-equivalent to $C(g)$.

Consider the cofiber sequence $\Sigma^0 \xrightarrow{mi} SZ/2m \xrightarrow{i_C} C(mi) \xrightarrow{j_C} \Sigma^1$. The cofiber $C(mi)$ is evidently decomposed into the wedge sum $\Sigma^1\vee SZ/m$. Since the composite $i_Ci\tilde\eta\colon \Sigma^1SZ/2\to C(mi)$ is expressed as $(0, i\tilde\eta)\colon \Sigma^1SZ/2\to\Sigma^1\vee SZ/m$, we obtain two cofiber sequences

(1.8) $\Sigma^0 \xrightarrow{mk_V} V_{4m} \longrightarrow \Sigma^1 \vee V_{2m} \longrightarrow \Sigma^1$ and $\Sigma^0 \xrightarrow{2mk_M} M_{4m} \longrightarrow \Sigma^1 \vee M_{2m} \longrightarrow \Sigma^1$

where $k_V: \Sigma^0 \to SZ/2m \to V_{4m}$ and $k_M: \Sigma^0 \to SZ/4m \to M_{4m}$ denote the bottom cell inclusions. Choose a map $\tilde{\eta}_{V,4m}: \Sigma^2 \to V_{4m}$ with $j_V \tilde{\eta}_{V,4m} = \eta$. Then [Y5, Lemma 3.6] asserts that

(1.9) the cofiber $C(\tilde{\eta}_{V,4m})$ is quasi KO_*-equivalent to $\Sigma^4 P_{4m}$.

By the aid of (1.7) we show

Lemma 1.1. *Let Y be a CW-spectrum which is quasi KO_*-equivalent to the following spectrum X: 1) $\Sigma^0 \vee SZ/4m$, 2) $\Sigma^4 \vee V_{4m}$, 3) M_{4m}, 4) $\Sigma^{-2} \vee \Sigma^{-2}SZ/4m$, 5) $\Sigma^2 \vee \Sigma^{-2}V_{4m}$ or 6) $\Sigma^{-2}M_{4m}$. If a map $g: \Sigma^0 \to Y$ satisfies that $g_*(1) = (0, 2m) \in KU_0 Y \cong Z \oplus Z/4m$, then its cofiber $C(g)$ is quasi KO_*-equivalent to the following spectrum W: 1) $\Sigma^0 \vee \Sigma^1 \vee SZ/2m$, 2) $\Sigma^4 \vee \Sigma^1 \vee V_{2m}$, 3) $\Sigma^1 \vee M_{2m}$, 4) $\Sigma^{-2} \vee \Sigma^{-2}P_{4m}$, 5) $\Sigma^2 \vee \Sigma^2 P_{4m}$ or 6) $\Sigma^{-2}MP_{4m}$ corresponding to each of the above cases 1)–6).*

Proof. In each case of 1)–6) we consider the map $f: \Sigma^0 \to X$ given as follows: 1) $(0, 2mi): \Sigma^0 \to \Sigma^0 \vee SZ/4m$, 2) $(0, mk_V): \Sigma^0 \to \Sigma^4 \vee V_{4m}$, 3) $2mk_M: \Sigma^0 \to M_{4m}$, 4) $(0, \tilde{\eta}_{4m}): \Sigma^0 \to \Sigma^{-2} \vee \Sigma^{-2}SZ/4m$, 5) $(0, \tilde{\eta}_{V,4m}): \Sigma^0 \to \Sigma^2 \vee \Sigma^{-2}V_{4m}$, 6) $i_M \tilde{\eta}_{4m}: \Sigma^0 \to \Sigma^{-2}M_{4m}$. By means of (1.2), (1.8) and (1.9) we observe that each cofiber $C(f)$ is itself the spectrum W stated in the lemma except the case 5), and it is quasi KO_*-equivalent to $W = \Sigma^2 \vee \Sigma^2 P_{4m}$ in the rest case 5). Then it is easily seen that $KU_0 C(f) \cong Z \oplus Z/2m$ and $KU_1 C(f) \cong Z$, and hence $f_*(1) = (0, 2m) \in KU_0 X \cong Z \oplus Z/4m$. Since $KO_7 X = 0$ in the cases 1), 2), 3), 5) and 6), (1.7) implies our result immediately except the case 4).

In the case 4) we shall next show that (1.7) remains still valid although $\varepsilon_{U*}: KO_2\Sigma^0 \vee SZ/4m \to KU_2\Sigma^0 \vee SZ/4m$ is never a monomorphism. The map $f = (0, \tilde{\eta}_{4m}): \Sigma^0 \to \Sigma^{-2} \vee \Sigma^{-2}SZ/4m$ satisfies that $f_*(1) = (0, 1, 0) \in KO_2\Sigma^0 \vee SZ/4m \cong KO_2\Sigma^0 \oplus KO_1\Sigma^0 \oplus KO_2\Sigma^0 \cong Z/2 \oplus Z/2 \oplus Z/2$. Identify $KO_* Y$ and $KU_* Y$ with $KO_*\Sigma^{-2} \vee \Sigma^{-2}SZ/4m$ and $KU_*\Sigma^{-2} \vee \Sigma^{-2}SZ/4m$ respectively via a quasi KO_*-equivalence $h: Y \to KO \wedge (\Sigma^{-2} \vee \Sigma^{-2}SZ/4m)$. Then it is easily seen that $g_*(1) = (a, 1, b) \in KO_0 Y \cong Z/2 \oplus Z/2 \oplus Z/2$ for some a and b because $g_*(1) = (0, 2m) \in KU_0 Y \cong Z \oplus Z/4m$ by our assumption. Here both a and b may be taken to be 0 by replacing the quasi KO_*-equivalence h without the change of the complexification $(\varepsilon_U \wedge 1)h: Y \to KU \wedge (\Sigma^{-2} \vee \Sigma^{-2}SZ/4m)$. Thus $f_*(1)$ and $g_*(1)$ have the same expression in $KO_0 Y \cong KO_2\Sigma^0 \vee SZ/4m \cong Z/2 \oplus Z/2 \oplus Z/2$ as desired.

Similarly to Lemma 1.1 we obtain

Lemma 1.2. *Let Y be a CW-spectrum which is quasi KO_*-equivalent to the following spectrum: 1) $SZ/2^\infty$ or 2) $\Sigma^{-2}SZ/2^\infty$. If a map $g: \Sigma^0 \to Y$ satisfies that $g_*(1) = 1/2 \in KU_0 Y \cong Z/2^\infty$, then its cofiber $C(g)$ is quasi KO_*-equivalent to the following spectrum: 1) $\Sigma^1 \vee SZ/2^\infty$ or 2) $\Sigma^{-2}C(\tilde{\eta}_\infty)$ corresponding to each of the above cases 1) and 2).*

Proof. Set $f = i_\infty: \Sigma^0 \to SZ/2^\infty$ in the first case and $f = \tilde{\eta}_\infty: \Sigma^0 \to \Sigma^{-2} SZ/2^\infty$ in the second case. Then we can apply (1.7) to show our result since $KO_7 SZ/2^\infty = 0 = KO_1 SZ/2^\infty$.

1.3. Let $f: \Sigma^{2t-1} X \to Y$ be a map of order 2 and $\bar{f}: \Sigma^{2t-1} X \wedge SZ/2 \to Y$ and $\tilde{f}: \Sigma^{2t} X \to Y \wedge SZ/2$ be its extension and coextension with $\bar{f}(1 \wedge i) = f$ and $(1 \wedge j)\tilde{f} = f$. Then there exist maps

$$\varphi: \Sigma^{-2t-1} C(\bar{f}) \longrightarrow X \quad \text{and} \quad \psi: Y \longrightarrow C(\tilde{f})$$

of order 2 whose cofibers $C(\varphi)$ and $C(\psi)$ coincide with $\Sigma^{-2t} C(f)$ and $\Sigma^1 C(f)$ respectively. The bottom cell inclusion $i: \Sigma^0 \to SZ/2$ has an extension $\bar{i}_{2g}: C(2g) \to X \wedge SZ/2$ whose cofiber is $\Sigma^1 C(g)$ for any map $g: W \to X$. Similarly the top cell projection $j: SZ/2 \to \Sigma^1$ has a coextension $\tilde{j}_{2g'}: Y \wedge SZ/2 \to C(2g')$ whose cofiber is $C(g')$ for any map $g': Y \to W$. Consider the composite maps

$$i_g \varphi: \Sigma^{-2t-1} C(\bar{f}) \longrightarrow C(g), \quad \psi j_{g'}: \Sigma^{-1} C(g') \longrightarrow C(\tilde{f}),$$
$$\bar{f}\bar{i}_{2g}: \Sigma^{2t-1} C(2g) \longrightarrow Y, \quad \tilde{j}_{2g'}\tilde{f}: \Sigma^{2t} X \longrightarrow C(2g')$$

where $i_g: X \to C(g)$ and $j_{g'}: \Sigma^{-1} C(g') \to Y$ denote the canonical inclusion and the canonical projection respectively. By use of Verdier's lemma we can easily show the following equalities among the cofibers of the above maps.

Lemma 1.3. *$C(i_g \varphi) = \Sigma^{-2t} C(\bar{f}\bar{i}_{2g})$ and $C(\psi j_{g'}) = C(\tilde{j}_{2g'}\tilde{f})$.*

Choose maps $\bar{h}: \Sigma^3 SZ/2 \to C(\bar{\eta})$, $\bar{k}: \Sigma^5 SZ/2 \to C(\bar{\eta})$, $\tilde{h}: \Sigma^1 C(\bar{\eta}) \to SZ/2$ and $\tilde{k}: \Sigma^3 C(\bar{\eta}) \to SZ/2$ such that $\bar{j}\bar{h} = \tilde{\eta}j$, $\bar{j}\bar{k} = \tilde{\eta}\bar{\eta}$, $\tilde{h}\bar{i} = i\bar{\eta}$ and $\tilde{k}\bar{i} = \tilde{\eta}\bar{\eta}$ where $\bar{j}: C(\bar{\eta}) \to \Sigma^2 SZ/2$ and $\bar{i}: SZ/2 \to C(\bar{\eta})$ denote the canonical projection and the canonical inclusion respectively. The maps $\bar{h}$ and $\tilde{h}$ have order 2 and the maps $\bar{k}$ and $\tilde{k}$ have order 4 (use [AT, §4]). Using a fixed Adams' K_*-equivalence $A_2: \Sigma^8 SZ/2 \to SZ/2$ [Ad2] we can obtain seven kinds of maps f_t $(t \geq 1)$ [Y5, (1.13)]:

$$\begin{aligned}
&\alpha_{4r} = jA_2^r i: \Sigma^{8r-1} \longrightarrow \Sigma^0, \\
&\mu_{4r+1} = \tilde{\eta} A_2^r i: \Sigma^{8r+1} \longrightarrow \Sigma^0, \quad \mu'_{4r+1} = jA_2^r \bar{\eta}: \Sigma^{8r+1} \longrightarrow \Sigma^0, \\
&a_{4r+2} = \bar{h} A_2^r i: \Sigma^{8r+3} \longrightarrow C(\bar{\eta}), \quad a'_{4r+2} = jA_2^r \tilde{h}: \Sigma^{8r} C(\bar{\eta}) \longrightarrow \Sigma^0, \\
&m_{4r+3} = \bar{k} A_2^r i: \Sigma^{8r+5} \longrightarrow C(\bar{\eta}), \quad m'_{4r+3} = jA_2^r \tilde{k}: \Sigma^{8r+2} C(\bar{\eta}) \longrightarrow \Sigma^0.
\end{aligned} \tag{1.10}$$

Denote by $\bar{f}_t: \Sigma^{2t-1} SZ/2 \to W$ the map obtained by omitting the "i" from the composite components of the map $f_t: \Sigma^{2t-1} \to W$ for $f_t = \alpha_{4r}, \mu_{4r+1}, a_{4r+2}$ or m_{4r+3}, and similarly by $\tilde{f}_t: \Sigma^{2t} W \to SZ/2$ the map obtained by omitting the "j" from the composite components of the map $f'_t: \Sigma^{2t-1} W \to \Sigma^0$ for $f'_t = \alpha_{4r}, \mu'_{4r+1}, a'_{4r+2}$ or m'_{4r+3} (see [Y5, (2.3) and (3.2)]). Then there exist eight kinds of maps

$$f_{-t}: \Sigma^{-2t-1} C(\bar{f}_t) \longrightarrow \Sigma^0 \quad \text{and} \quad f'_{-t}: \Sigma^0 \longrightarrow C(\tilde{f}_t) \tag{1.11}$$

as given in [Y5, (2.5) and (3.4)]. Among the cofibers of these maps there hold the equalities as $C(f_{-t}) = \Sigma^{-2t} C(f_t)$ and $C(f'_{-t}) = \Sigma^1 C(f'_t)$.

Choose a coextension $\bar{h}_{2/2}: \Sigma^4 SZ/2 \to SZ/2 \wedge C(\bar{\eta})$ of $\bar{h}$ with $(j \wedge 1)\bar{h}_{2/2} = \bar{h}$ and an extension $\tilde{h}_{2/2}: \Sigma^1 SZ/2 \wedge C(\bar{\eta}) \to SZ/2$ of $\tilde{h}$ with $\tilde{h}_{2/2}(i \wedge 1) = \tilde{h}$. Setting $\tilde{a}'_{4r+2} = jA_2^r \tilde{h}_{2/2}: \Sigma^{8r} SZ/2 \wedge C(\bar{\eta}) \to \Sigma^0$ and $\tilde{a}_{4r+2} = \bar{h}_{2/2} A_2^r i: \Sigma^{8r+4} \to SZ/2 \wedge C(\bar{\eta})$, we obtain the following maps similar to (1.11):

$$(1.12) \quad b_{-4r-2}: \Sigma^{-8r-5} C(\tilde{a}'_{4r+2}) \longrightarrow \Sigma^{-3} C(\bar{\eta}) \quad \text{and} \quad b'_{-4r-2}: C(\bar{\eta}) \longrightarrow C(\tilde{a}_{4r+2})$$

such that $C(b_{-4r-2}) = \Sigma^{-8r-4} C(a'_{4r+2})$ and $C(b'_{-4r-2}) = \Sigma^1 C(a_{4r+2})$ (see [Y 5, (2.5) and (3.4)]).

Since $[\Sigma^3 SZ/2, \Sigma^0] \cong [\Sigma^5 SZ/2, \Sigma^0] \cong [SZ/2, \Sigma^1 C(\bar{\eta})] \cong Z/2$ and $[\Sigma^1 SZ/2, C(\bar{\eta})] = 0$, the maps $j: SZ/2 \to \Sigma^1$, $\bar{\eta}: \Sigma^1 SZ/2 \to \Sigma^0$, $\bar{h}: \Sigma^3 SZ/2 \to C(\bar{\eta})$ and $\bar{k}: \Sigma^5 SZ/2 \to C(\bar{\eta})$ give rise to the following two kinds of coextensions:

$$(1.13) \quad \begin{array}{llll} j_{2,2m}: SZ/2 \longrightarrow SZ/2m, & j_{V,4m/2}: SZ/2 \longrightarrow V_{4m}, \\ \bar{\eta}_{4m/2}: \Sigma^2 SZ/2 \longrightarrow SZ/4m, & \bar{\eta}_{V,2m/2}: \Sigma^2 SZ/2 \longrightarrow V_{2m}, \\ \bar{h}_{2m/2}: \Sigma^4 SZ/2 \longrightarrow SZ/2m \wedge C(\bar{\eta}), & \bar{h}_{U,4m/2}: \Sigma^4 SZ/2 \longrightarrow U_{4m}, \\ \bar{k}_{4m/2}: \Sigma^6 SZ/2 \longrightarrow SZ/4m \wedge C(\bar{\eta}), & \bar{k}_{U,4m/2}: \Sigma^6 SZ/2 \longrightarrow U_{4m} \end{array}$$

such that $jj_{2,2m} = j$, $j_V j_{V,4m/2} = j$, $j\bar{\eta}_{4m/2} = \bar{\eta}$, $j_V \bar{\eta}_{V,2m/2} = \bar{\eta}$, $(j \wedge 1)\bar{h}_{2m/2} = \bar{h}$, $j_U \bar{h}_{U,4m/2} = \bar{h}$, $(j \wedge 1)\bar{k}_{4m/2} = \bar{k}$ and $j_U \bar{k}_{U,4m/2} = \bar{k}$. Here $j_{2,2m}$ is the obvious map associated with the inclusion $Z/2 \subset Z/2m$.

Compose the above eight maps after the map $\tilde{\alpha}_{4r} = A_2^r i$, and also the first two maps after the map $\tilde{\mu}'_{4r+1} = A_2^r \bar{\eta}$, $\tilde{a}'_{4r+2} = A_2^r \tilde{h}$ or $\tilde{m}'_{4r+3} = A_2^r \tilde{k}$. Then we obtain the following several coextensions given into the concrete forms:

$$(1.14) \quad \begin{array}{llll} \tilde{\alpha}_{4r,l}: \Sigma^{8r} \longrightarrow SZ/2^l, & \tilde{\alpha}_{4r,V,l}: \Sigma^{8r} \longrightarrow V_{2^l}, \\ \tilde{\mu}_{4r+1,l}: \Sigma^{8r+2} \longrightarrow SZ/2^l, & \tilde{\mu}_{4r+1,V,l}: \Sigma^{8r+2} \longrightarrow V_{2^l}, \\ \tilde{a}_{4r+2,l}: \Sigma^{8r+4} \longrightarrow SZ/2^l \wedge C(\bar{\eta}), & \tilde{a}_{4r+2,U,l}: \Sigma^{8r+4} \longrightarrow U_{2^l}, \\ \tilde{m}_{4r+3,l}: \Sigma^{8r+6} \longrightarrow SZ/2^l \wedge C(\bar{\eta}), & \tilde{m}_{4r+3,U,l}: \Sigma^{8r+6} \longrightarrow U_{2^l}, \\ \tilde{\mu}'_{4r+1,l}: \Sigma^{8r+2} \longrightarrow SZ/2^l, & \tilde{\mu}'_{4r+1,V,l}: \Sigma^{8r+2} \longrightarrow V_{2^l}, \\ \tilde{a}'_{4r+2,l}: \Sigma^{8r+1} C(\bar{\eta}) \longrightarrow SZ/2^l, & \tilde{a}'_{4r+2,V,l}: \Sigma^{8r+1} C(\bar{\eta}) \longrightarrow V_{2^l}, \\ \tilde{m}'_{4r+3,l}: \Sigma^{8r+3} C(\bar{\eta}) \longrightarrow SZ/2^l, & \tilde{m}'_{4r+3,V,l}: \Sigma^{8r+3} C(\bar{\eta}) \longrightarrow V_{2^l} \end{array}$$

whenever $l \geq 2$. All the maps $\tilde{\varphi}_{t,l}: \Sigma^{2t} X \to W_{2^l}$ given in (1.14) satisfy the following condition:

$$(1.15) \quad \tilde{\varphi}_{t,l*}(1) = 2^{l-1} \in KU_{2t} W_{2^l} \cong Z/2^l.$$

For the Moore spectrum $SZ/2^l$ of type $Z/2^l$ the bottom cell inclusion $i: \Sigma^0 \to SZ/2^l$ and the top cell projection $j: SZ/2^l \to \Sigma^1$ are sometimes written as i_l and j_l with the subscript "l". Similarly the maps i_W, i'_W, j_W and j'_W ($W = U$ or V) appearing in (1.1) are written as $i_{W,l}$, $i'_{W,l}$, $j_{W,l}$ and $j'_{W,l}$ with the subscript "l" when $2m = 2^l$. Applying Lemma 1.3 to the maps given in (1.11), (1.12) and (1.14), we now obtain

Lemma 1.4. i) $C(f'_{-t}j_{l-1}) = C(\tilde{f}'_{t,l})$ *and* $C(f'_{-t}j_{V,l-1}) = C(\tilde{f}'_{t,V,l})$ *for* $l \geq 2$, *where* $f'_t = \alpha'_{4r}, \mu'_{4r+1}, a'_{4r+2}$ or m'_{4r+3} *with* $\alpha'_{4r} = \alpha_{4r}$.

ii) $C(b'_{-4r-2}(j_{l-1} \wedge 1)) = C(\tilde{a}_{4r+2,l})$ *and* $C(b'_{-4r-2}j_{U,l-1}) = C(\tilde{a}_{4r+2,U,l})$ for $l \geq 2$.

By virtue of [Y5, Lemma 3.6 ii)] we can show

(1.16) i) $C(\tilde{\mu}_{4r+1,l})$ and $C(\tilde{\mu}_{4r+1,V,l})$ have the same K_*-local types as $C(\tilde{\mu}'_{4r+1,l})$ and $C(\tilde{\mu}'_{4r+1,V,l})$ respectively.

ii) $C(\tilde{m}_{4r+3,l})$ and $C(\tilde{m}_{4r+3,U,l})$ have the same K_*-local types as $C(\tilde{m}'_{4r+3,V,l})$ and $C(\tilde{m}'_{4r+3,l})$ respectively.

Similarly to (1.13) the maps $j\colon SZ/2 \to \Sigma^1$, $\bar{\eta}\colon \Sigma^1 SZ/2 \to \Sigma^0$, $\bar{h}\colon \Sigma^3 SZ/2 \to C(\bar{\eta})$ and $\bar{k}\colon \Sigma^5 SZ/2 \to C(\bar{\eta})$ give rise to the following maps:

(1.17)
$$\begin{aligned} j_{2,\infty}&\colon SZ/2 \longrightarrow SZ/2^\infty, & \bar{\eta}_{2,\infty}&\colon \Sigma^2 SZ/2 \longrightarrow SZ/2^\infty,\\ \bar{h}_{2,\infty}&\colon \Sigma^4 SZ/2 \longrightarrow SZ/2^\infty \wedge C(\bar{\eta}), & \bar{k}_{2,\infty}&\colon \Sigma^6 SZ/2 \longrightarrow SZ/2^\infty \wedge C(\bar{\eta}). \end{aligned}$$

Composing the above four maps after the map $\tilde{\alpha}_{4r}$, and also the obvious map $j_{2,\infty}$ after the map $\tilde{\mu}'_{4r+1}, \tilde{a}'_{4r+2}$ or $\tilde{m}'_{4r+3}$, we obtain seven kinds of maps as follows:

(1.18)
$$\begin{aligned} &\tilde{\alpha}_{4r,\infty}\colon \Sigma^{8r} \longrightarrow SZ/2^\infty,\\ &\tilde{\mu}_{4r+1,\infty}\colon \Sigma^{8r+2} \longrightarrow SZ/2^\infty, & &\tilde{\mu}'_{4r+1,\infty}\colon \Sigma^{8r+2} \longrightarrow SZ/2^\infty,\\ &\tilde{a}_{4r+2,\infty}\colon \Sigma^{8r+4} \longrightarrow SZ/2^\infty \wedge C(\bar{\eta}), & &\tilde{a}'_{4r+2,\infty}\colon \Sigma^{8r+1} C(\bar{\eta}) \longrightarrow SZ/2^\infty,\\ &\tilde{m}_{4r+3,\infty}\colon \Sigma^{8r+6} \longrightarrow SZ/2^\infty \wedge C(\bar{\eta}), & &\tilde{m}'_{4r+3,\infty}\colon \Sigma^{8r+3} C(\bar{\eta}) \longrightarrow SZ/2^\infty. \end{aligned}$$

All the maps $\tilde{\varphi}_{t,\infty}\colon \Sigma^{2t} X \to W_\infty$ given in (1.18) satisfy the following condition:

(1.19)
$$\tilde{\varphi}_{t,\infty *}(1) = 1/2 \in KU_{2t} W_\infty \cong Z/2^\infty.$$

2. The K_*-localizations of RP^{2s+n}_{2s+1} and RP^{2s+2t}_{2s}

2.1. Let X_n ($n \geq 1$) denote the suspension spectrum $\Sigma^{-n} SP^2 S^n$ whose n-th term is the symmetric square $SP^2 S^n$ of the n-sphere as in [Y3, §2] or [Y5, §4], and X_∞ denote the union of X_n. In other words, X_∞ is the spectrum whose n-th term is $SP^2 S^n$ for each $n \geq 1$. For every $n \geq 1$ the Spanier-Whitehead dual DX_n is denoted by X_{-n} for convenience sake. From [U, Theorem 3.3] (or [Y3, Proposition 2.6 i)]) we recall the KU-homologies of X_n ($n \neq 0$) that $KU_0 X_n \cong Z$, $Z \oplus Z$ or $Z[1/2]$ according as $n = 2t-1$, $2t$ or ∞ and $KU_1 X_n = 0$. For each $k \neq 0$ the complex Adams operation ψ^k_C behaves in $KU_0 X_n$ ($n \neq 0$) as follows (see [Y5, Lemma 4.1 i) and Corollary 4.2 i)]):

(2.1) $\psi^k_C = A_{k,t}$ or 1 according as $n = 2t$ or otherwise.

Here $A_{k,t} = \begin{pmatrix} 1/k^t & 0 \\ 1 - k^t/2k^t & 1 \end{pmatrix}$, which operates on $(Z \oplus Z) \otimes Z[1/k]$ as left action.

For each n $(1 \leq n \leq \infty)$ the real projective n-space RP^n is related to the above spectrum X_{n+1} by a cofiber sequence $RP^n \to \Sigma^0 \to X_{n+1} \to \Sigma^1 RP^n$ [JTTW]. Therefore the stunted real projective space RP^n/RP^m $(0 \leq m < n \leq \infty)$ is exhibited by the following cofiber sequence

$$RP^n/RP^m \longrightarrow X_{m+1} \longrightarrow X_{n+1} \longrightarrow \Sigma^1 RP^n/RP^m. \tag{2.2}$$

For simplicity RP^n/RP^m is often abbreviated to be RP^n_{m+1} as usual. we first investigate the behavior of the complex Adams operation ψ^k_C on $KU_*RP^n_{m+1}$ and $KU^*RP^n_{m+1}$ (cf. [Ad 1]).

Lemma 2.1. i) *The KU-homologies* $KU_*RP^n_{m+1}$ $(0 \leq m < n \leq \infty)$ *and their Adams operations* ψ^k_C *for each* $k \neq 0$ *are tabled as follows*:

X	$= RP^{2s+2t+1}_{2s+1}$	RP^{2s+2t}_{2s+1}	RP^{∞}_{2s+1}	$RP^{2s+2t+1}_{2s}$	RP^{2s+2t}_{2s}	RP^{∞}_{2s}
KU_0X	$\cong$ 0	0	0	Z	Z	Z
ψ^k_C	=			$1/k^s$	$1/k^s$	$1/k^s$
$KU_{-1}X$	$\cong Z \oplus Z/2^t$	$Z/2^t$	$Z/2^{\infty}$	$Z \oplus Z/2^t$	$Z/2^t$	$Z/2^{\infty}$
ψ^k_C	$= A_{k,s+t+1}$	1	1	$A_{k,s+t+1}$	1	1

ii) *The KU-cohomologies* $KU^*RP^n_{m+1}$ $(0 \leq m < n \leq \infty)$ *and their Adams operations* ψ^k_C *for each* $k \neq 0$ *are tabled as follows*:

X	$= RP^{2s+2t+1}_{2s+1}$	RP^{2s+2t}_{2s+1}	RP^{∞}_{2s+1}	$RP^{2s+2t+1}_{2s}$	RP^{2s+2t}_{2s}	RP^{∞}_{2s}
KU^0X	$\cong Z/2^t$	$Z/2^t$	$\hat{Z}_2$	$Z \oplus Z/2^t$	$Z \oplus Z/2^t$	$Z \oplus \hat{Z}_2$
ψ^k_C	= 1	1	1	$A_{k,-s}$	$A_{k,-s}$	$A_{k,-s}$
$KU^{-1}X$	$\cong Z$	0	0	Z	0	0
ψ^k_C	$= k^{s+t+1}$			k^{s+t+1}		

where $\hat{Z}_2$ *denotes the 2-completion of the integers.*

Proof. i) The $s=0$ case has been proved in [Y5, Lemma 4.1 ii)]. Recall that $KU_0RP^{2s+n}_{2s+1}=0$ and the sequence $0 \to KU_{-1}RP^{2s} \to KU_{-1}RP^{2s+n} \to KU_{-1}RP^{2s+n}_{2s+1} \to 0$ is exact for each n. Since the Adams operation ψ^k_C on $KU_{-1}RP^{2s+n} \otimes Z[1/2]$ behaves as $\psi^k_C = A_{k,s+t+1}$ or 1 according as $n=2t+1$ or otherwise, the $X = RP^{2s+n}_{2s+1}$ case follows immediately. On the other hand, the cofiber sequence $\Sigma^{2s} \to RP^{2s+n}_{2s} \to RP^{2s+n}_{2s+1} \to \Sigma^{2s+1}$ induces two isomorphisms $KU_{-1}RP^{2s+n}_{2s} \xrightarrow{\cong} KU_{-1}RP^{2s+n}_{2s+1}$ and $KU_0\Sigma^{2s} \xrightarrow{\cong} KU_0RP^{2s+n}_{2s}$ for each n. Hence the $X = RP^{2s+n}_{2s}$ case is immediate, too.

ii) The $s=0$ case has been proved in [Y5, Corollary 4.2 ii)]. Note that there exist isomorphisms $KU^{-1}RP^{2s+n}_{2s+1} \xrightarrow{\cong} KU^{-1}RP^{2s+n}$ and $KU^{-1}RP^{2s+n}_{2s+1} \cong KU^{-1}RP^{2s+n}_{2s}$ for each n. On the other hand, the cofiber sequence (2.2) induces an exact sequence $0 \to KU^{-1}RP^{2s+n}_{2s+\varepsilon} \to KU^0X_{2s+n+1} \to KU^0X_{2s+\varepsilon+1} \to KU^0RP^{2s+n}_{2s+\varepsilon} \to 0$ for each n where $\varepsilon = 0$ or 1. Our result is now immediate from [Y5, Corollary 4.2].

2.2. In [Y5] we dealt with CW-spectra X satisfying the following property:

(I_{2m}) $KU_0X \cong Z/2m$ on which $\psi_C^k = 1$ and $KU_1X = 0$;

(I_{2^∞}) $KU_0X \cong Z/2^\infty$ on which $\psi_C^k = 1$ and $KU_1X = 0$; or

$(II_{2m})_t$ $KU_0X \cong Z \oplus Z/2m$ on which $\psi_C^k = A_{k,t}$ and $KU_1X = 0$

where $A_{k,t} = \begin{pmatrix} 1/k^t & 0 \\ 1 - k^t/2k^t & 1 \end{pmatrix}$, which operates on $(Z \oplus Z/2m) \otimes Z[1/k]$ as left action. As an immediate result of Lemma 2.1 we notice that

(2.3) $\Sigma^1 RP_{2s+1}^{2s+2^t}$, $\Sigma^1 RP_{2s+1}^{\infty}$, $\Sigma^1 RP_{2s+1}^{2s+2^t+1}$ and $DRP_{2s}^{2s+2^t}$ satisfy the property (I_{2^t}), (I_{2^∞}), $(II_{2^t})_{s+t+1}$ and $(II_{2^t})_{-s}$ respectively.

In order to determine the quasi KO_*-types of RP_{2s+1}^{2s+n} $(1 \le n \le \infty)$ and DRP_{2s}^{2s+2t} $(t \ge 0)$ we need the following calculations (see [FY] or [Y4, Lemma 3.4]).

Lemma 2.2. i) $KO_{4m}RP_{4m+1}^{4m+n} = 0 = KO_{4m}RP_{4m-1}^{4m+n}$ *if* $n \equiv 1, 2, 3, 4, 5 \bmod 8$, *and hence if* $n = \infty$.

ii) $KO_{4m+4}RP_{4m+1}^{4m+n} = 0 = KO_{4m+4}RP_{4m-1}^{4m+n}$ *if* $n \equiv 0, 1, 5, 6, 7 \bmod 8$, *and hence if* $n = \infty$.

iii) $KO_{4m+6}RP_{4m+1}^{4m+n} = 0 = KO_{4m+6}RP_{4m-1}^{4m+n}$ *for all* n.

iv) $KO^{4m-3}RP_{4m}^{4m+2t} = 0 = KO^{4m-3}RP_{4m-2}^{4m+2t}$ *if* $t \equiv 1, 2 \bmod 4$.

v) $KO^{4m-7}RP_{4m}^{4m+2t} = 0 = KO^{4m-7}RP_{4m-2}^{4m+2t}$ *if* $t \equiv 0, 3 \bmod 4$.

vi) $KO^{4m-5}RP_{4m}^{4m+2t} = 0 = KO^{4m-5}RP_{4m-2}^{4m+2t}$ *for all* t.

Proof. The first three parts have been shown in [Y4, Lemma 3.4]. The latter three parts are similarly shown by a dual argument.

Proposition 2.3 (cf. [Y4, Theorem 2 i) and iii)]). i) $\Sigma^{-4m+1}RP_{4m+1}^{4m+n}$ *is quasi* KO_**-equivalent to* $SZ/2^{4r}$, $M_{2^{4r}}$, $V_{2^{4r+1}}$, $\Sigma^4 \vee V_{2^{4r+1}}$, $V_{2^{4r+2}}$, $M_{2^{4r+2}}$, $SZ/2^{4r+3}$, $\Sigma^0 \vee SZ/2^{4r+3}$ *according as* $n = 8r, 8r+1, \ldots, 8r+7$. *In addition*, $\Sigma^{-4m+1}RP_{4m+1}^{\infty}$ *is quasi* KO_**-equivalent to* $SZ/2^\infty$.

ii) $\Sigma^{-4m+1}RP_{4m-1}^{4m+n-2}$ *is quasi* KO_**-equivalent to* $SZ/2^{4r}$, $\Sigma^0 \vee SZ/2^{4r}$, $SZ/2^{4r+1}$, $M_{2^{4r+1}}$, $V_{2^{4r+2}}$, $\Sigma^4 \vee V_{2^{4r+2}}$, $V_{2^{4r+3}}$, $M_{2^{4r+3}}$ according as $n = 8r$, $8r+1, \ldots, 8r+7$. *In addition*, $\Sigma^{-4m+1}RP_{4m-1}^{\infty}$ *is quasi* KO_**-equivalent to* $SZ/2^\infty$.

iii) $\Sigma^{4m}DRP_{4m}^{4m+2t}$ *is quasi* KO_**-equivalent to* $\Sigma^0 \vee SZ/2^{4r}$, $\Sigma^0 \vee \Sigma^4 V_{2^{4r+1}}$, $\Sigma^0 \vee \Sigma^4 V_{2^{4r+2}}$, $\Sigma^0 \vee SZ/2^{4r+3}$ *according as* $t = 4r, 4r+1, 4r+2, 4r+3$.

iv) $\Sigma^{4m}DRP_{4m-2}^{4m+2t-2}$ *is quasi* KO_**-equivalent to* $M_{2^{4r}}$, $M_{2^{4r+1}}$, $\Sigma^4 M_{2^{4r+2}}$, $\Sigma^4 M_{2^{4r+3}}$ *according as* $t = 4r, 4r+1, 4r+2, 4r+3$.

Proof. Use Lemmas 2.1 and 2.2, and then apply [Y3, Theorem 2.5] when n or t is finite and [B2, Theorem 3.3] when n is infinite.

Proposition 2.4 (cf. [Y4, Theorem 2 ii) and iv)]. i) $\Sigma^{-4m+1}RP_{4m}^{4m+n}$ *is quasi* KO_**-equivalent to* $\Sigma^1 \vee SZ/2^{4r}$, $\Sigma^1 \vee M_{2^{4r}}$, $\Sigma^1 \vee V_{2^{4r+1}}$, $\Sigma^1 \vee \Sigma^4 \vee V_{2^{4r+1}}$, $\Sigma^1 \vee V_{2^{4r+2}}$, $\Sigma^1 \vee M_{2^{4r+2}}$, $\Sigma^1 \vee SZ/2^{4r+3}$, $\Sigma^1 \vee \Sigma^0 \vee SZ/2^{4r+3}$ *according as* $n = 8r$, $8r+1, \ldots, 8r+7$. *In addition*, $\Sigma^{-4m+1}RP_{4m}^{\infty}$ *is quasi* KO_**-equivalent to* $\Sigma^1 \vee SZ/2^\infty$.

ii) $\Sigma^{-4m+1}RP^{4m+n-2}_{4m-2}$ *is quasi* KO_**-equivalent to* $P_{2^{4r+1}}$, $\Sigma^0 \vee P_{2^{4r+1}}$, $P_{2^{4r+2}}$, $\Sigma^4 MP_{2^{4r+2}}$, $\Sigma^4 P_{2^{4r+3}}$, $\Sigma^4 \vee \Sigma^4 P_{2^{4r+3}}$, $\Sigma^4 P_{2^{4r+4}}$, $\Sigma^4 MP_{2^{4r+4}}$ *according as* $n = 8r$, $8r+1, \dots, 8r+7$. *In addition,* $\Sigma^{-4m+5}RP^{\infty}_{4m-2}$ *is quasi* KO_**-equivalent to* $C(\tilde{\eta}_\infty)$.

Proof. According to [Y2, Corollary I.1.6], X is quasi KO_*-equivalent to Y if and only if the Spanier-Whitehead dual DY is quasi KO_*-equivalent to DX. Hence Proposition 2.3 iii) and iv) imply immediately our result when n is even. We next use the cofiber sequences $\Sigma^{2s-1} \xrightarrow{f_{s,t}} RP^{2s+2t+1}_{2s-1} \to RP^{2s+2t+1}_{2s} \to \Sigma^{2s}$ and $\Sigma^{2s-1} \xrightarrow{f_{s,\infty}} RP^{\infty}_{2s-1} \to RP^{\infty}_{2s} \to \Sigma^{2s}$. From Lemma 2.1 i) it follows that $f_{s,t*}(1) = (0, 2^t) \in KU_{2s-1}RP^{2s+2t+1}_{2s-1} \cong Z \oplus Z/2^{t+1}$ and $f_{s,\infty *}(1) = 1/2 \in KU_{2s-1}RP^{\infty}_{2s-1} \cong Z/2^\infty$. Applying Lemmas 1.1 and 1.2 with the aid of Proposition 2.3 i) and ii) we can easily obtain our result when n is odd or infinite.

2.3. Recall the behavior of the real Adams operation ψ^k_R on $KO_iX_n \otimes Z[1/k]$ $(0 \le i \le 7)$ for each $k \neq 0$ (see [Y5, (4.3)]):

(2.4) i) When n is odd or infinite, $\psi^k_R = k^2$ or 1 according as $i = 4$ or otherwise;
ii) When $n = 4s+2$, $\psi^k_R = 1$, $1/k^{2s}$, k^2 or $1/k^{2s-2}$ according as $i = 0, 2, 4$ or 6;
iii) When $n = 4s \neq 0$, $\psi^k_R = A_{k,2s}$, $k^2 A_{k,2s}$ or 1 according as $i = 0, 4$ or otherwise.

We here investigate the behavior of the real Adams operation ψ^k_R for RP^{2s+n}_{2s+1} $(1 \le n \le \infty)$ and DRP^{2s+2t}_{2s} $(t \ge 0)$, which is useful to determine their K_*-local types.

Proposition 2.5. *When* $X = \Sigma^{-4m+1}RP^{4m+n}_{4m+1}$, $\Sigma^{-4m+1}RP^{4m+n}_{4m-1}$, $\Sigma^{4m}DRP^{4m+2t}_{4m}$ *or* $\Sigma^{4m}DRP^{4m+2t}_{4m-2}$, *the Adams operation* ψ^k_R *acts on* $KO_iX \otimes Z[1/k]$ $(0 \le i \le 7)$ *for each* $k \neq 0$ *as follows*:

i) *The* $X = \Sigma^{-4m+1}RP^{4m+n}_{4m\pm 1}$ *cases*: 1) *When n is even or infinite,* $1/k^{2m}\psi^k_R = k^2$ *or* 1 *according as* $i = 4$ *or otherwise*; 2) *When* $n = 4s+1$, $1/k^{2m}\psi^k_R = 1/k^{2m+2s}$, k^2, $1/k^{2m+2s-2}$ *or* 1 *according as* $i = 2, 4, 6$ *or otherwise*; 3) *When* $n = 4s+3$, $1/k^{2m}\psi^k_R = A_{k,2m+2s+2}$, $k^2 A_{k,2m+2s+2}$ *or* 1 *according as* $i = 0, 4$ *or otherwise.*

ii) *The* $X = \Sigma^{4m}DRP^{4m+2t}_{4m}$ *case*: $k^{2m}\psi^k_R = A_{k,-2m}$, $k^2 A_{k,-2m}$ *or* 1 *according as* $i = 0, 4$ *or otherwise.*

iii) *The* $X = \Sigma^{4m}DRP^{4m+2t}_{4m-2}$ *case*: $k^{2m}\psi^k_R = k^{2m}$, k^2, k^{2m+2} *or* 1 *according as* $i = 2, 4, 6$ *or otherwise.*

Proof. Use the cofiber sequence $RP^n_{m+1} \to X_{m+1} \to X_{n+1} \to \Sigma^1 RP^n_{m+1}$ of (2.2) and its dual sequence $\Sigma^{-1}DRP^n_{m+1} \to X_{-n-1} \to X_{-m-1} \to DRP^n_{m+1}$. By a quite similar argument to [Y5, Lemma 4.4] with the aid of (2.4) our result is easily shown.

To determine the K_*-local types of RP^{2s+n}_{2s} $(0 \le n \le \infty)$ we shall not need to investigate the behavior of their real Adams operations ψ^k_R. Neverthless we dare to give the following result, whose proof is almost the same as in Proposition 2.5 (or [Y5, Lemma 4.4]).

Proposition 2.6. *When* $X = \Sigma^{-4m+1}RP^{4m+n}_{4m}$ *or* $\Sigma^{-4m+1}RP^{4m+n}_{4m-2}$ *the Adams* *operation* ψ_R^k *acts on* $KO_iX \otimes Z[1/k]$ $(0 \le i \le 7)$ *for each* $k \ne 0$ *as follows*:

i) *The* $X = \Sigma^{-4m+1} RP_{4m}^{4m+n}$ *case*: 1) *When n is even or infinite*, $1/k^{2m}\psi_R^k = 1/k^{2m}, k^2, 1/k^{2m-2}$ *or* 1 *according as* $i = 1, 4, 5$ *or otherwise*; 2) *When* $n = 4s+1$, $1/k^{2m}\psi_R^k = 1/k^{2m}, 1/k^{2m+2s}, k^2, 1/k^{2m-2}, 1/k^{2m+2s-2}$ *or* 1 *according as* $i = 1, 2, 4, 5, 6$ *or otherwise*; 3) *When* $n = 4s+3$, $1/k^{2m}\psi_R^k = A_{k,2m+2s+1}, 1/k^{2m}, k^2 A_{k,2m+2s+1}, 1/k^{2m-2}$ *or* 1 *according as* $i = 0, 1, 4, 5$ *or otherwise.*

ii) *The* $X = \Sigma^{-4m+1} RP_{4m-2}^{4m+n}$ *case*: 1) *When n is even or infinite*, $1/k^{2m}\psi_R^k = 1/k^{2m-2}, k^2, 1/k^{2m-4}$ *or* 1 *according as* $i = 3, 4, 7$ *or otherwise*; 2) *When* $n = 4s+1$, $1/k^{2m}\psi_R^k = 1, 1/k^{2m+2s}, 1/k^{2m-2}, k^2, 1/k^{2m+2s-2}, 1/k^{2m-4}$ *according as* $i = 0, 2, 3, 4, 6$ *or* 7; 3) *When* $n = 4s+3$, $1/k^{2m}\psi_R^k = A_{k,2m+2s+2}, 1/k^{2m-2}, k^2 A_{k,2m+2s+2}, 1/k^{2m-4}$ *or* 1 *according as* $i = 0, 3, 4, 7$ *or otherwise.*

We now determine the K_*-local types of RP_{2s+1}^{2s+n} as the first part of our main result (cf. [DM, Theorem 4.2]).

Theorem 2.7. *The stunted real projective space* $\Sigma^1 RP_{2s+1}^{2s+n}$ $(2 \le n \le \infty)$ *has the same* K_**-local type as the elementary spectrum tabled below*:

$s \setminus n$	$8r$	$8r+1$	$8r+2$	$8r+3$
$4m-1$	$SZ/2^{4r}$	$C(i_{4r}\alpha_{4m+4r})$	$SZ/2^{4r+1}$	$C(i_{4r+1}\mu_{4m+4r+1})$
$4m$	$SZ/2^{4r}$	$C(i_{4r}\mu_{4m+4r+1})$	$V_{2^{4r+1}}$	$C(i_{V,4r+1}a_{4m+4r+2})$
$4m+1$	$SZ/2^{4r} \wedge C(\bar{\eta})$	$C((i_{4r} \wedge 1)a_{4m+4r+2})$	$SZ/2^{4r+1} \wedge C(\bar{\eta})$	$C((i_{4r+1} \wedge 1)m_{4m+4r+3})$
$4m+2$	$SZ/2^{4r} \wedge C(\bar{\eta})$	$C((i_{4r} \wedge 1)m_{4m+4r+3})$	$U_{2^{4r+1}}$	$C(i_{U,4r+1}\alpha_{4m+4r+4})$

$s \setminus n$	$8r+4$	$8r+5$	$8r+6$	$8r+7$
$4m-1$	$V_{2^{4r+2}}$	$C(i_{V,4r+2}a_{4m+4r+2})$	$V_{2^{4r+3}}$	$C(i_{V,4r+3}m_{4m+4r+3})$
$4m$	$V_{2^{4r+2}}$	$C(i_{V,4r+2}m_{4m+4r+3})$	$SZ/2^{4r+3}$	$C(i_{4r+3}\alpha_{4m+4r+4})$
$4m+1$	$U_{2^{4r+2}}$	$C(i_{U,4r+2}\alpha_{4m+4r+4})$	$U_{2^{4r+3}}$	$C(i_{U,4r+3}\mu_{4m+4r+5})$
$4m+2$	$U_{2^{4r+2}}$	$C(i_{U,4r+2}\mu_{4m+4r+5})$	$SZ/2^{4r+3} \wedge C(\bar{\eta})$	$C((i_{4r+3} \wedge 1)a_{4m+4r+6})$

$n \setminus s$	$4m-1$	$4m$	$4m+1$	$4m+2$
∞	$SZ/2^\infty$	$SZ/2^\infty$	$SZ/2^\infty \wedge C(\bar{\eta})$	$SZ/2^\infty \wedge C(\bar{\eta})$

Proof. Put (2.3) and Propositions 2.3 and 2.5 together and then apply [Y5, Theorems 1.2 and 2.6 with (2.8)] as in the RP^n case [Y5, Theorem 4.6 ii)].

Applying [Y5, Theorem 2.6 with (2.8)] we can similarly obtain

Proposition 2.8. *The Spanier-Whitehead dual* $DRP_{2s}^{2s+2^t}$ $(t \ge 1)$ *has the same* K_**-local type as the cofiber of the map tabled below*:

$s \backslash t$	$4r$	$4r+1$	$4r+2$	$4r+3$
$4m$	$i_{4r}\alpha_{-4m}$	$i_{U,4r+1}\alpha_{-4m}$	$i_{U,4r+2}\alpha_{-4m}$	$i_{4r+3}\alpha_{-4m}$
$4m+1$	$i_{U,4r}\mu_{-4m-1}$	$i_{U,4r+1}\mu_{-4m-1}$	$i_{4r+2}\mu_{-4m-1}$	$i_{4r+3}\mu_{-4m-1}$
$4m+2$	$(i_{4r}\wedge 1)b_{-4m-2}$	$i'_{U,4r+1}b_{-4m-2}$	$i'_{U,4r+2}b_{-4m-2}$	$(i_{4r+3}\wedge 1)b_{-4m-2}$
$4m+3$	$i_{4r}m_{-4m-3}$	$i_{4r+1}m_{-4m-3}$	$i_{U,4r+2}m_{-4m-3}$	$i_{U,r+3}m_{-4m-3}$

According to [Y5, Lemma 4.7], two finite spectra X and Y have the same K_*-local type if and only if their Spanier-Whitehead duals DX and DY have the same K_*-local type. As a dual of Proposition 2.8 we can show immediately the second part of our main result by using Lemma 1.4 and (1.16) with the aid of [Y5, (2.7) and (3.7)].

Theorem 2.9. The stunted real projective space $\Sigma^1 RP_{2s}^{2s+2t}$ $(t \geq 1)$ has the same K_*-local type as the cofiber of the map tabled below:

$s \backslash t$	$4r$	$4r+1$	$4r+2$	$4r+3$
$4m$	$\tilde{\alpha}_{4m,4r+1}$	$\tilde{\alpha}_{4m,V,4r+2}$	$\tilde{\alpha}_{4m,V,4r+3}$	$\tilde{\alpha}_{4m,4r+4}$
$4m+1$	$\tilde{\mu}_{4m+1,V,4r+1}$	$\tilde{\mu}_{4m+1,V,4r+2}$	$\tilde{\mu}_{4m+1,4r+3}$	$\tilde{\mu}_{4m+1,4r+4}$
$4m+2$	$\tilde{a}_{4m+2,4r+1}$	$\tilde{a}_{4m+2,U,4r+2}$	$\tilde{a}_{4m+2,U,4r+3}$	$\tilde{a}_{4m+2,4r+4}$
$4m+3$	$\tilde{m}_{4m+3,U,4r+1}$	$\tilde{m}_{4m+3,U,4r+2}$	$\tilde{m}_{4m+3,4r+3}$	$\tilde{m}_{4m+3,4r+4}$

3. The K_*-localizations of $RP_{2s}^{2s+2t+1}$

3.1. Let p be a fixed prime and r be a positive integer such that $r \equiv \pm 3 \bmod 8$ when $p=2$ and r generates the group of units of Z/p^2 when p is odd. Denote by $\mathscr{J}_{(p)}$ the fiber of the map $\psi_R^r - 1\colon KOZ_{(p)} \to KOZ_{(p)}$ where $KOZ_{(p)} = KO \wedge SZ_{(p)}$ is the real K-spectrum with coefficients $Z_{(p)}$. Consider the map $\kappa_{(p)}\colon \mathscr{J}_{(p)} \to \Sigma^{-1}SQ$ inducing an isomorphism $\kappa_{(p)*}\colon \pi_{-1}\mathscr{J}_{(p)} \otimes Q \xrightarrow{\cong} \pi_0 SQ \otimes Q$. According to [B1, Theorem 4.3] (or [R]) the fiber of the map $\kappa_{(p)}$ is actually the $KZ_{(p)*}$-localization of the sphere spectrum S. Thus we have cofiber sequences

$$\text{(3.1)}\quad \text{i)}\quad S_{KZ_{(p)}} \xrightarrow{\iota_1} \mathscr{J}_{(p)} \xrightarrow{\kappa_{(p)}} \Sigma^{-1}SQ \xrightarrow{\pi_1} \Sigma^1 S_{KZ_{(p)}}$$

$$\text{ii)}\quad \mathscr{J}_{(p)} \xrightarrow{\iota_2} KOZ_{(p)} \xrightarrow{\psi_R^r - 1} KOZ_{(p)} \xrightarrow{\pi_2} \Sigma^1 \mathscr{J}_{(p)}$$

where $S_{KZ_{(p)}} = S_K \wedge SZ_{(p)}$ for the K_*-localization S_K of S. The unit $\iota_O\colon S \to KO$ is factorized through S_K as $\iota_O = \iota_K l_K$ for the K_*-localization map $l_K\colon S \to S_K$. Note that the composite $\iota_2\iota_1\colon S_{KZ_{(p)}} \to KOZ_{(p)}$ is just the map $\iota_K\colon S_K \to KO$ smashed with $SZ_{(p)}$.

Let J be a set of primes. The obvious map $l_{(J)}\colon S \to SZ_{(J)}$ associated with the inclusion $Z \subset Z_{(J)}$ gives rise to the $SZ_{(J)*}$-localization map $l_{(J)} \wedge 1\colon X \to SZ_{(J)} \wedge X$. For each map $f\colon Y \to X$ we denote by $f_{(J)}\colon Y \to SZ_{(J)} \wedge X$ the J-local map given by the composite $(l_{(J)} \wedge 1)f$.

Lemma 3.1. *Let J be a fixed set of primes, W and X be CW-spectra with W finite and $f: W \to S_K \wedge X$ be a map such that the composite $(\iota_K \wedge 1)f: W \to KO \wedge X$ is trivial. Assume that $[\Sigma^2 W, SQ \wedge X] = 0$ and $[\Sigma^1 W, KOZ_{(p)} \wedge X] = 0$ for each prime $p \in J$. Then the J-local map $f_{(J)}: W \to S_{KZ_{(J)}} \wedge X$ becomes trivial.*

Proof. Under our assumptions it is immediate that $(\iota_K \wedge 1)_*: [W, S_{KZ_{(p)}} \wedge X] \to [W, KOZ_{(p)} \wedge X]$ is a monomorphism for each $p \in J$. Therefore the p-local map $f_{(p)}: W \to S_{KZ_{(p)}} \wedge X$ becomes trivial for each $p \in J$. Since there exists an isomorphism $[W, S_K \wedge X] \otimes Z_{(p)} \xrightarrow{\simeq} [W, S_{KZ_{(p)}} \wedge X]$ under the assumption that W is finite, we can find a positive integer n_p prime to p such that $n_p f = 0 \in [W, S_K \wedge X]$ for every $p \in J$. Consequently we get a positive integer n prime to all $p \in J$ such that $nf = 0 \in [W, S_K \wedge X]$. This implies that the J-local map $f_{(J)}: W \to S_{KZ_{(J)}} \wedge X$ is trivial as desired.

Lemma 3.2. *Let p be a fixed prime and W, X and Y be CW-spectra. Let $f: W \to S_K \wedge X$, $g: W \to Y$ and $h': Y \to S_K \wedge X$ be maps such that f and $h'g$ coincide when they are carried into $[W, S_{KZ[1/p]} \wedge X]$ and $[W, KO \wedge X]$. Assume that $[\Sigma^2 W, SQ \wedge X] = 0 = [\Sigma^1 Y, SQ \wedge X]$ and $g^*: [\Sigma^1 Y, KOZ_{(p)} \wedge X] \to [\Sigma^1 W, KOZ_{(p)} \wedge X]$ is an epimorphism. Then there exists a map $h: Y \to S_K \wedge X$ satisfying $f = hg \in [W, S_K \wedge X]$. Further the map h is taken to be a quasi S_{K*}-equivalence whenever h' is so.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
[\Sigma^1 Y, KOZ_{(p)} \wedge X] & \xrightarrow{(\pi_2 \wedge 1)_*} & [Y, \mathscr{J}_{(p)} \wedge X] & \xleftarrow{(\iota_1 \wedge 1)_*} & [Y, S_{KZ_{(p)}} \wedge X] \\
{\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g^*} \\
[\Sigma^1 W, KOZ_{(p)} \wedge X] & \xrightarrow[(\pi_2 \wedge 1)_*]{} & [W, \mathscr{J}_{(p)} \wedge X] & \xleftarrow[(\iota_1 \wedge 1)_*]{} & [W, S_{KZ_{(p)}} \wedge X]
\end{array}$$

in which the left vertical arrow g^* and the right upper arrow $(\iota_1 \wedge 1)_*$ are epimorphisms and the right lower arrow $(\iota_1 \wedge 1)_*$ is a monomorphism. By a routine diagram chasing we can easily find a map $h'': Y \to S_{KZ_{(p)}} \wedge X$ such that $f_{(p)} = h''g \in [W, S_{KZ_{(p)}} \wedge X]$ and $(\iota_K \wedge 1)h'_{(p)} = (\iota_K \wedge 1)h'' \in [Y, KOZ_{(p)} \wedge X]$ since $(\iota_K \wedge 1)f_{(p)} = (\iota_K \wedge 1)h'_{(p)}g \in [W, KOZ_{(p)} \wedge X]$. Note that the rationalizations of h' and h'' coincide. Using [B1, Proposition 2.10] we then obtain a unique map $h: Y \to S_K \wedge X$ such that $h_{(p)} = h'' \in [Y, S_{KZ_{(p)}} \wedge X]$ and $h_{(p^c)} = h'_{(p^c)} \in [Y, S_{KZ[1/p]} \wedge X]$ where p^c denotes the complement of the single prime set $\{p\}$. Evidently this map h satisfies the desired equality $hg = f \in [W, S_K \wedge X]$ because $h''g = f_{(p)} \in [W, S_{KZ_{(p)}} \wedge X]$ and $h'_{(p^c)}g = f_{(p^c)} \in [W, S_{KZ[1/p]} \wedge X]$.

If the old map $h': Y \to S_K \wedge X$ is a quasi S_{K*}-equivalence, then it induces an isomorphism $h'_*: K_* Y \to K_* S_K \wedge X \xleftarrow{\simeq} K_* X$ where $K = KU$ or KO. This implies that $h''_*: KZ_{(p)*} Y \to KZ_{(p)*} S_{KZ_{(p)}} \wedge X \xleftarrow{\simeq} KZ_{(p)*} X$ is an isomorphism because $(\iota_K \wedge 1)h'_{(p)} = (\iota_K \wedge 1)h''$. Therefore we can observe that $h_*: K_* Y \to K_* S_K \wedge X \xleftarrow{\simeq} K_* X$ is an isomorphism since $h_{(p)} = h''$ and $h_{(p^c)} = h'_{(p^c)}$. Thus the new map $h: Y \to S_K \wedge X$ becomes a quasi S_{K*}-equivelence, too.

Putting Lemmas 3.1 and 3.2 together we obtain

Proposition 3.3. *Let W, X and Y be CW-spectra with W finite, and $f: W \to S_K \wedge X$, $g: W \to Y$ and $h': Y \to S_K \wedge X$ be maps related by the equality $(\iota_K \wedge 1)f = (\iota_K \wedge 1)h'g \in [W, KO \wedge X]$. Assume that the following three conditions are satisfied for a certain prime p:* i) $[\Sigma^2 W, SQ \wedge X] = 0 = [\Sigma^1 Y, SQ \wedge X]$, ii) $[\Sigma^1 W, KO \wedge X] \otimes Z[1/p] = 0$ *and* iii) $g^*: [\Sigma^1 Y, KOZ_{(p)} \wedge X] \to [\Sigma^1 W, KOZ_{(p)} \wedge X]$ *is an epimorphism. Then there exists a map $h: Y \to S_K \wedge X$ satisfying $f = hg \in [W, S_K \wedge X]$. Further the map h is taken to be a quasi S_{K*}-equivalence whenever h' is so.*

Proof. Take J in Lemma 3.1 as the set p^c of all primes but only the prime p and f in Lemma 3.1 as the map $f - h'g$. Then Lemma 3.1 asserts that $f_{(p^c)} = h'_{(p^c)}g \in [W, S_{KZ[1/p]} \wedge X]$. Since the assumptions in Lemma 3.2 are all satisfied, we can now apply Lemma 3.2 to get a desired map $h: Y \to S_K \wedge X$.

As an immediate result of Proposition 3.3 we can show

Corollary 3.4. *Let W, X and Y be CW-spectra with W finite, and $f: W \to X$ and $g: W \to Y$ be maps. Assume that the conditions* i), ii) *and* iii) *stated in Proposition* 3.3 *are all satisfied for a certain prime p. If there exists a quasi S_{K*}-equivalence $h': Y \to S_K \wedge X$ satisfying $(\iota_O \wedge 1)f = (\iota_K \wedge 1)h'g \in [W, KO \wedge X]$,* then the cofiber $C(f)$ is quasi *S_{K*}-equivalent to $C(g)$.*

3.2. Concerning the conditions i), ii) and iii) stated in Proposition 3.3 we have

Lemma 3.5. *Let Y be a CW-spectrum which is quasi KO_*-equivalent to the following spectrum X:* 1) $\Sigma^0 \vee SZ/4m$, 2) $\Sigma^4 \vee V_{4m}$, 3) M_{4m}, 4) $\Sigma^{-2} \vee \Sigma^{-2}SZ/4m$, 5) $\Sigma^2 \vee \Sigma^{-2}V_{4m}$, 6) $\Sigma^{-2}M_{4m}$, 7) $SZ/2^\infty$ *or* 8) $\Sigma^{-2}SZ/2^\infty$. *Let $g: \Sigma^0 \to Y$ be a map satisfying the following condition: $g_*(1) = (0, 1) \in KU_0 Y \cong Z \oplus Z/4m$ in the case* 1); *$g_*(1) = (0, 2m) \in KU_0 Y \cong Z \oplus Z/4m$ in the cases* 2)–6); *$g_*(1) = 1/2 \in KU_0 Y \cong Z/2^\infty$ in the cases* 7)–8). *Then $KO_1 Y \otimes Z[1/2] = 0 = [\Sigma^1 Y, SQ \wedge Y]$ and $g^*: [\Sigma^1 Y, KO \wedge Y] \to [\Sigma^1, KO \wedge Y]$ is an epimorphism.*

Proof. It is obvious that $KO_1 Y \otimes Z[1/2] \cong KO_1 X \otimes Z[1/2] = 0$ and $[\Sigma^1 Y, SQ \wedge Y] \cong \prod_i \mathrm{Hom}(\pi_{i-1} Y \otimes Q, \pi_i Y \otimes Q) = 0$ because $KO_{2j+1} Y \otimes Q \cong KO_{2j+1} X \otimes Q = 0$ for each j. As is observed in the proofs of Lemmas 1.1 and 1.2, we can choose a certain map $f: \Sigma^0 \to X$ such that $(\iota_O \wedge 1)f = hg$ with a suitable quasi KO_*-equivalence $h: Y \to KO \wedge X$. For any CW-spectrum W the quasi KO_*-equivalence h induces an isomorphism $h^\#: [X, KO \wedge W] \to [Y, KO \wedge W]$ defined by $h^\#(x) = (\mu \wedge 1)(1 \wedge x)h$ where $\mu: KO \wedge KO \to KO$ denotes the multiplication of KO. Therefore it is sufficient to show that the map $f: \Sigma^0 \to X$ in place of $g: \Sigma^0 \to Y$ induces an epimorphism $f^*: [\Sigma^1 X, KO \wedge X] \to [\Sigma^1, KO \wedge X]$. In the cases 2), 3) and 7) our assertion is trivial because $KO_1 X = 0$ for $X = \Sigma^4 \vee V_{4m}$, M_{4m} or $SZ/2^\infty$.

In the non-trivial cases we recall that the map $f\colon \Sigma^0 \to KO \wedge X$ is chosen in the proofs of Lemmas 1.1 and 1.2 as follows: 1) $(0, i)\colon \Sigma^0 \to \Sigma^0 \vee SZ/4m$; 4) $(0, \tilde{\eta}_{4m})\colon \Sigma^0 \to \Sigma^{-2} \vee \Sigma^{-2}SZ/4m$; 5) $(0, \tilde{\eta}_{V,4m})\colon \Sigma^0 \to \Sigma^2 \vee \Sigma^{-2}V_{4m}$; 6) $i_M\tilde{\eta}_{4m}\colon \Sigma^0 \to \Sigma^{-2}M_{4m}$; 8) $j_{2,\infty}\tilde{\eta}\colon \Sigma^0 \to \Sigma^{-2}SZ/2^\infty$. As is easily checked, the induced homomorphisms $i^*\colon [\Sigma^1 SZ/4m,\ KO \wedge (\Sigma^0 \vee SZ/4m)] \to [\Sigma^1,\ KO \wedge (\Sigma^0 \vee SZ/4m)]$, $\tilde{\eta}^*_{4m}\colon [\Sigma^1 SZ/4m,\ KO \wedge SZ/4m] \to [\Sigma^3,\ KO \wedge SZ/4m]$, $\tilde{\eta}^*_{V,4m}\colon [\Sigma^1 V_{4m}, KO \wedge V_{4m}] \to [\Sigma^3, KO \wedge V_{4m}]$, $i^*_M\colon [\Sigma^1 M_{4m}, KO \wedge M_{4m}] \to [\Sigma^1 SZ/4m, KO \wedge M_{4m}]$ and $\tilde{\eta}^*_{4m}\colon [\Sigma^1 SZ/4m,\ KO \wedge M_{4m}] \to [\Sigma^3,\ KO \wedge M_{4m}]$ are all epimorphisms. Further $j^*_{2,\infty}\colon [\Sigma^1 SZ/2^\infty,\ KO \wedge SZ/2^\infty] \to [\Sigma^1 SZ/2,\ KO \wedge SZ/2^\infty]$ and $\tilde{\eta}^*\colon [\Sigma^1 SZ/2,\ KO \wedge SZ/2^\infty] \to [\Sigma^3,\ KO \wedge SZ/2^\infty]$ are isomorphisms, because there exists an isomorphism $[W,\ KO \wedge SZ/2^\infty] \cong \mathrm{Hom}(KO_4 W,\ Z/2^\infty)$ for any CW-spectrum W (use [Y1, (3.1)] or [An]). Consequently we can verify that $f^*\colon [\Sigma^1 X, KO \wedge X] \to [\Sigma^1,\ KO \wedge X]$ is also an epimorphism in the non-trivial cases 1), 4), 5), 6) and 8).

Fix non-negative integers m and r, and then for simplicity set the elementary spectra appearing in Theorem 2.7 as follows:

$$(3.2)\qquad \begin{aligned}
Y_{01} &= C(i_{4r+1}\mu_{4m+4r+1}) & Y_{21} &= C((i_{4r+1} \wedge 1)m_{4m+4r+3}) \\
Y_{02} &= C(i_{V,4r+2}a_{4m+4r+2}) & Y_{22} &= C(i_{U,4r+2}\alpha_{4m+4r+4}) \\
Y_{03} &= C(i_{V,4r+3}m_{4m+4r+3}) & Y_{23} &= C(i_{U,4r+3}\mu_{4m+4r+5}) \\
Y_{04} &= C(i_{4r+4}\alpha_{4m+4r+4}) & Y_{24} &= C((i_{4r+4} \wedge 1)a_{4m+4r+6}) \\
Y_{11} &= C(i_{V,4r+1}a_{4m+4r+2}) & Y_{31} &= C(i_{U,4r+1}\alpha_{4m+4r+4}) \\
Y_{12} &= C(i_{V,4r+2}m_{4m+4r+3}) & Y_{32} &= C(i_{U,4r+2}\mu_{4m+4r+5}) \\
Y_{13} &= C(i_{4r+3}\alpha_{4m+4r+4}) & Y_{33} &= C((i_{4r+3} \wedge 1)a_{4m+4r+6}) \\
Y_{14} &= C(i_{4r+4}\mu_{4m+4r+5}) & Y_{34} &= C((i_{4r+4} \wedge 1)m_{4m+4r+7}).
\end{aligned}$$

The elementary spectrum Y_{0j} is quasi KO_*-equivalent to $M_{2^{4r+1}}$, $\Sigma^4 \vee V_{2^{4r+2}}$, $M_{2^{4r+3}}$ or $\Sigma^0 \vee SZ/2^{4r+4}$ according as $j = 1, 2, 3$ or 4, and Y_{1j} is quasi KO_*-equivalent to $\Sigma^4 \vee V_{2^{4r+1}}$, $M_{2^{4r+2}}$, $\Sigma^0 \vee SZ/2^{4r+3}$ or $M_{2^{4r+4}}$ according as $j = 1, 2, 3$ or 4. On the other hand, Y_{2j} and Y_{3j} are respectively quasi KO_*-equivalent to $\Sigma^4 Y_{0j}$ and $\Sigma^4 Y_{1j}$ for each j $(1 \le j \le 4)$.

For each pair (i, j), $0 \le i \le 3$ and $1 \le j \le 4$, we consider the following coextensions $\tilde{\varphi}_{4m+i,4r+j}\colon \Sigma^{8m+2i} \to W_{2^{4r+j}}$ given in (1.14):

$$(3.3)\qquad \begin{aligned}
&\tilde{\alpha}_{4m,4r+1}\colon \Sigma^{8m} \longrightarrow SZ/2^{4r+1} & &\tilde{a}_{4m+2,4r+1}\colon \Sigma^{8m+4} \longrightarrow SZ/2^{4r+1} \wedge C(\bar{\eta}) \\
&\tilde{\alpha}_{4m,V,4r+2}\colon \Sigma^{8m} \longrightarrow V_{2^{4r+2}} & &\tilde{a}_{4m+2,U,4r+2}\colon \Sigma^{8m+4} \longrightarrow U_{2^{4r+2}} \\
&\tilde{\alpha}_{4m,V,4r+3}\colon \Sigma^{8m} \longrightarrow V_{2^{4r+3}} & &\tilde{a}_{4m+2,U,4r+3}\colon \Sigma^{8m+4} \longrightarrow U_{2^{4r+3}} \\
&\tilde{\alpha}_{4m,4r+4}\colon \Sigma^{8m} \longrightarrow SZ/2^{4r+4} & &\tilde{a}_{4m+2,4r+4}\colon \Sigma^{8m+4} \longrightarrow SZ/2^{4r+4} \wedge C(\bar{\eta}) \\
&\tilde{\mu}_{4m+1,V,4r+1}\colon \Sigma^{8m+2} \longrightarrow V_{2^{4r+1}} & &\tilde{m}_{4m+3,U,4r+1}\colon \Sigma^{8m+6} \longrightarrow U_{2^{4r+1}} \\
&\tilde{\mu}_{4m+1,V,4r+2}\colon \Sigma^{8m+2} \longrightarrow V_{2^{4r+2}} & &\tilde{m}_{4m+3,U,4r+2}\colon \Sigma^{8m+6} \longrightarrow U_{2^{4r+2}}
\end{aligned}$$

$$\tilde{\mu}_{4m+1,4r+3}\colon \Sigma^{8m+2} \longrightarrow SZ/2^{4r+3} \quad \tilde{m}_{4m+3,4r+3}\colon \Sigma^{8m+6} \longrightarrow SZ/2^{4r+3} \wedge C(\bar{\eta})$$
$$\tilde{\mu}_{4m+1,4r+4}\colon \Sigma^{8m+2} \longrightarrow SZ/2^{4r+4} \quad \tilde{m}_{4m+3,4r+4}\colon \Sigma^{8m+6} \longrightarrow SZ/2^{4r+4} \wedge C(\bar{\eta}).$$

By composing the canonical inclusion $i_{ij}\colon W_{2^{4r+j}} \to Y_{ij}$ after the above map $\tilde{\varphi}_{4m+i,4r+j}\colon \Sigma^{8m+2i} \to W_{2^{4r+j}}$, we introduce the following map

$$(3.4) \qquad g_{ij} = i_{ij}\tilde{\varphi}_{4m+i,4r+j}\colon \Sigma^{8m+2i} \longrightarrow W_{2^{4r+j}} \longrightarrow Y_{ij}.$$

From (1.15) it follows that all the maps $g_{ij}\colon \Sigma^{8m+2i} \to Y_{ij}$ satisfy the following condition:

$$(3.5) \qquad g_{ij*}(1) = (0,\ 2^{4r+j-1}) \in KU_{8m+2i} Y_{ij} \cong Z \oplus Z/2^{4r+j}.$$

Set $Y_{0\infty} = Y_{1\infty} = SZ/2^{\infty}$ and $Y_{2\infty} = Y_{3\infty} = SZ/2^{\infty} \wedge C(\bar{\eta})$, and consider the following maps $g_{i\infty}\colon \Sigma^{8m+2i} \to Y_{i\infty}$ given in (1.18):

$$(3.6) \qquad g_{0\infty} = \tilde{\alpha}_{4m,\infty},\quad g_{1\infty} = \tilde{\mu}_{4m+1,\infty},\quad g_{2\infty} = \tilde{a}_{4m+2,\infty},\quad g_{3\infty} = \tilde{m}_{4m+3,\infty}.$$

Then Lemma 3.5 with (3.5) and (1.19) implies

Lemma 3.6. i) $KO_1 Y_{ij} \otimes Z[1/2] = 0 = [\Sigma^1 Y_{ij},\ SQ \wedge Y_{ij}]$, *and*
ii) *the maps* $g_{ij}\colon \Sigma^{8m+2i} \to Y_{ij}$ *given in* (3.4) *and* (3.6) *induce epimorphisms* $g_{ij}^*\colon [\Sigma^1 Y_{ij},\ KO \wedge Y_{ij}] \to [\Sigma^{8m+2i+1},\ KO \wedge Y_{ij}]$ *if* (i, j) *is neither* (0, 4) *nor* (2, 4).

3.3. We next discuss the maps $g_{04} = i_{04}\tilde{\alpha}_{4m,4r+4}\colon \Sigma^{8m} \to SZ/2^{4r+4} \to Y_{04}$ and $g_{24} = i_{24}\tilde{a}_{4m+2,4r+4}\colon \Sigma^{8m+4} \to SZ/2^{4r+4} \wedge C(\bar{\eta}) \to Y_{24}$. Recall that $\tilde{\alpha}_{4m,4r+4} = j_{2,2q}A_2^m i\colon \Sigma^{8m} \to \Sigma^{8m}SZ/2 \to SZ/2 \to SZ/2q$ and $\tilde{a}_{4m+2,4r+4} = \bar{h}_{2q/2}A_2^m i\colon \Sigma^{8m+4} \to \Sigma^{8m+4}SZ/2 \to \Sigma^4 SZ/2 \to SZ/2q \wedge C(\bar{\eta})$ with $q = 2^{4r+3}$ where $j_{2,2q}$ is the obvious map and $\bar{h}_{2q/2}$ is the extension of $\bar{h}$ obtained in (1.13). Using the cofiber sequences (3.1) it is easily computed (cf. [B1, Corollary 4.5] or [R, Theorem 8.5]) that

$$(3.7) \qquad \pi_0 S_K \cong \pi_0 KO \oplus \pi_1 KO \cong Z \oplus Z/2,$$
$$\pi_0 S_K \wedge SZ/2 \cong \pi_{8m} S_K \wedge SZ/2 \cong KO_{8m} SZ/2 \oplus KO_{8m+1} SZ/2 \cong Z/2 \oplus Z/2,$$
$$\pi_{8m} S_K \wedge SZ/2q \cong Z/2^{v+1} \oplus Z/2 \subset KO_{8m} SZ/2q \oplus KO_{8m+1} SZ/2q \cong Z/2q \oplus Z/2 \quad \text{and}$$
$$\pi_{8m+4} S_K \wedge SZ/2q \wedge C(\bar{\eta}) \cong Z/8 \oplus Z/2 \subset KO_{8m+4} SZ/2q \wedge C(\bar{\eta}) \oplus KO_{8m+5} SZ/2q \wedge C(\bar{\eta}) \cong Z/2q \oplus Z/2$$

where $v = \mathrm{Min}\,\{4r + 3,\ \nu_2(8m)\}$ with $\nu_2(8m)$ the exponent of 2 in the prime power decomposition of $8m$. Further we can compute that

$$(3.8) \qquad \pi_{8m} S_K \wedge Y_{03} \cong Z/2^{u+1} \subset KO_{8m} Y_{03} \cong Z/q,$$
$$\pi_{8m+4} S_K \wedge Y_{23} \cong Z/8 \subset KO_{8m+4} Y_{23} \cong Z/q,$$
$$\pi_{8m} S_K \wedge Y_{04} \cong Z/2^{v+1} \oplus Z/2 \oplus Z/2 \subset KO_{8m} Y_{04} \oplus KO_{8m+1} Y_{04} \cong Z \oplus Z/2q \oplus Z/2 \oplus Z/2 \quad \text{and}$$

$$\pi_{8m+4}S_K \wedge Y_{24} \cong Z/8 \oplus Z/2 \oplus Z/2 \subset KO_{8m+4}Y_{24} \oplus KO_{8m+5}Y_{24}$$
$$\cong Z \oplus Z/2q \oplus Z/2 \oplus Z/2$$

where $u = \text{Min}\{4r+2,\ v_2(8m)\}$ and $v = \text{Min}\{4r+3,\ v_2(8m)\}$, because $\psi_R^k = 1$ on $KO_0 Y_{03} \cong Z/q$, $\psi_R^k = k^2$ on $KO_4 Y_{23} \cong Z/q$, $\psi_R^k = A_{k,4m+4r+4}$ on $KO_0 Y_{04} \cong Z \oplus Z/2q$ and $\psi_R^k = k^2 A_{k,4m+4r+6}$ on $KO_4 Y_{24} \cong Z \oplus Z/2q$ for any k prime to 2 (see [Y5, (2.1) and Lemma 2.2 i)]).

Lemma 3.7. *The maps $g_{04}: \Sigma^{8m} \to Y_{04}$ and $g_{24}: \Sigma^{8m+4} \to Y_{24}$ satisfy that $g_{04*}(1, 0) = (2^v, 0, 0) \in \pi_{8m}S_K \wedge Y_{04} \cong Z/2^{v+1} \oplus Z/2 \oplus Z/2$ and $g_{24*}(1, 0) = (4, 0, 0) \in \pi_{8m+4}S_K \wedge Y_{24} \cong Z/8 \oplus Z/2 \oplus Z/2$ where $(1, 0) \in \pi_0 S_K \cong Z \oplus Z/2$ stands for the element represented by the localization map $l_K: S \to S_K$.*

Proof. A routine computation shows that the cofiber $C(\bar{h}_{2q/2})$ is quasi KO_*-equivalent to $\Sigma^4 SZ/q$ since $C(\bar{\eta})$ and $C(\bar{h})$ are quasi KO_*-equivalent to Σ^4. As is easily seen, the induced homomorphisms $j_{2,2q*}: \pi_{8m}S_K \wedge SZ/2 \to \pi_{8m}S_K \wedge SZ/2q$ and $\bar{h}_{2q/2*}: \pi_{8m}S_K \wedge SZ/2 \to \pi_{8m+4}S_K \wedge SZ/2q \wedge C(\bar{\eta})$ are respectively expressed as $\begin{pmatrix} 2^v & 0 \\ 0 & 0 \end{pmatrix}: Z/2 \oplus Z/2 \to Z/2^{v+1} \oplus Z/2$ and $\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}: Z/2 \oplus Z/2 \to Z/8 \oplus Z/2$. Using these expressions we verify immediately that the induced homomorphisms $g_{04*}: \pi_0 S_K \to \pi_{8m}S_K \wedge Y_{04}$ and $g_{24*}: \pi_0 S_K \to \pi_{8m+4}S_K \wedge Y_{24}$ are expressed as $\begin{pmatrix} 2^v & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}: Z \oplus Z/2 \to Z/2^{v+1} \oplus Z/2 \oplus Z/2$ and $\begin{pmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}: Z \oplus Z/2 \to Z/8 \oplus Z/2 \oplus Z/2$ respectively.

By virtue of Corollary 3.4 and Lemmas 3.6 and 3.7 we finally determine the K_*-local types of $RP_{2s}^{2s+2t+1}$ as the last part of our main result.

Theorem 3.8. *The stunted real projective space $\Sigma^1 RP_{2s}^{2s+2t+1}$ $(0 \le t \le \infty)$ has the same K_*-local type as the cofiber of the map tabled below:*

$s \diagdown t$	$4r$	$4r+1$
$4m$	$i_{4r+1}\mu_{4m+4r+1} \vee \tilde{\alpha}_{4m,4r+1}$	$i_{V,4r+2}a_{4m+4r+2} \vee \tilde{\alpha}_{4m,4r+2}$
$4m+1$	$i_{V,4r+1}a_{4m+4r+2} \vee \tilde{\mu}_{4m+1,V,4r+1}$	$i_{V,4r+2}m_{4m+4r+3} \vee \tilde{\mu}_{4m+1,V,4r+2}$
$4m+2$	$(i_{4r+1} \wedge 1)m_{4m+4r+3} \vee \tilde{a}_{4m+2,4r+1}$	$i_{U,4r+2}\alpha_{4m+4r+4} \vee \tilde{a}_{4m+2,U,4r+2}$
$4m+3$	$i_{U,4r+1}\alpha_{4m+4r+4} \vee \tilde{m}_{4m+3,U,4r+1}$	$i_{U,4r+2}\mu_{4m+4r+5} \vee \tilde{m}_{4m+3,U,4r+2}$

$s \diagdown t$	$4r+2$	$4r+3$
$4m$	$i_{V,4r+3}m_{4m+4r+3} \vee \tilde{\alpha}_{4m,V,4r+3}$	$i_{4r+4}\alpha_{4m+4r+4} \vee \tilde{\alpha}_{4m,4r+4}$
$4m+1$	$i_{4r+3}\alpha_{4m+4r+4} \vee \tilde{\mu}_{4m+1,4r+3}$	$i_{4r+4}\mu_{4m+4r+5} \vee \tilde{\mu}_{4m+1,4r+4}$
$4m+2$	$i_{U,4r+3}\mu_{4m+4r+5} \vee \tilde{a}_{4m+2,U,4r+3}$	$(i_{4r+4} \wedge 1)a_{4m+4r+6} \vee \tilde{a}_{4m+2,4r+4}$
$4m+3$	$(i_{4r+3} \wedge 1)a_{4m+4r+6} \vee \tilde{m}_{4m+3,4r+3}$	$(i_{4r+4} \wedge 1)m_{4m+4r+7} \vee \tilde{m}_{4m+3,4r+4}$

$t \backslash s$	$4m$	$4m+1$	$4m+2$	$4m+3$
∞	$\tilde{\alpha}_{4m,\infty}$	$\tilde{\mu}_{4m+1,\infty}$	$\tilde{a}_{4m+2,\infty}$	$\tilde{m}_{4m+3,\infty}$

Proof. The $t=0$ case is obvious because $RP_{2s}^{2s+1} = \Sigma^{2s} \vee \Sigma^{2s+1}$. So we may assume that $t \geq 1$. When $(s, t+1) = (4m+i, 4r+j)$ or $(4m+i, \infty)$ we shall show that $\Sigma^1 RP_{2s}^{2s+2t+1}$ has the same K_*-local type as the cofiber $C(g_{ij})$ of the map $g_{ij}: \Sigma^{8m+2i} \to Y_{ij}$ given in (3.4) or (3.6), because the cofiber $C(g_{ij})$ coincides with the cofiber of the map tabled in the theorem (use [Y2, Lemma II.1.1]). We first take the maps f and g in Corollary 3.4 as the canonical inclusion $f_{s,t}: \Sigma^{2s-1} \to RP_{2s-1}^{2s+2t+1}$ and the above map $g_{ij}: \Sigma^{8m+2i} \to Y_{ij}$ respectively where $(s, t+1) = (4m+i, 4r+j)$ or $(4m+i, \infty)$. According to Theorem 2.7 $\Sigma^1 RP_{2s-1}^{2s+2t+1}$ has the same K_*-local type as the spectrum Y_{ij}. Note that $\pi_{2s+1} RP_{2s-1}^{2s+2t+1} \otimes Q = 0$ whenever $t \geq 1$. Then Lemma 3.6 shows that all of the conditions i), ii) and iii) stated in Proposition 3.3 are satisfied for the prime 2 unless $(s, t) = (2n, 4r+3)$. Therefore we can apply Corollary 3.4 to observe that $\Sigma^1 RP_{2s}^{2s+2t+1}$ and $C(g_{ij})$ have the same K_*-local type unless $(s, t) = (2n, 4r+3)$.

We shall next show that our assertion is valid even in the case when $(s, t) = (2n, 4r+3)$. Consider the commutative diagram

$$\begin{array}{ccccc}
\pi_0 S_K & \xrightarrow{f_{5*}} & \pi_{4n-1} S_K \wedge RP_{4n-1}^{4n+8r+5} & \xrightarrow{i_{K*}} & KO_{4n-1} RP_{4n-1}^{4n+8r+5} \\
\| & & \downarrow & & \downarrow \\
\pi_0 S_K & \xrightarrow[f_{7*}]{} & \pi_{4n-1} S_K \wedge RP_{4n-1}^{4n+8r+7} & \xrightarrow[i_{K*}]{} & KO_{4n-1} RP_{4n-1}^{4n+8r+7}
\end{array}$$

where $f_k: \Sigma^{4n-1} \to RP_{4n-1}^{4n+8r+k}$ $(k=5, 7)$ denotes the canonical inclusion. Recall that $\Sigma^1 RP_{4n-1}^{4n+8r+k}$ $(k=5$ and $7)$ are respectively quasi S_{K*}-equivalent to Y_{03} and Y_{04} when n is even, and they are quasi S_{K*}-equivalent to Y_{23} and Y_{24} when n is odd. From (3.7) and (3.8) it follows that $\pi_0 S_K \cong Z \oplus Z/2$, $\pi_{4n-1} S_K \wedge RP_{4n-1}^{4n+8r+5} \cong Z/2^{u+1}$ and $\pi_{4n-1} S_K \wedge RP_{4n-1}^{4n+8r+7} \cong Z/2^{v+1} \oplus Z/2 \oplus Z/2$ where $u = \mathrm{Min}\{4r+2, \nu_2(4n)\}$ and $v = \mathrm{Min}\{4r+3, \nu_2(4n)\}$. Since $f_{5*}(1) = 2^{4r+2} \in KO_{4n-1} RP_{4n-1}^{4n+8r+5} \cong Z/2^{4r+3}$, it is easily seen that $f_{5*}(1, 0) = 2^u \in \pi_{4n-1} S_K \wedge RP_{4n-1}^{4n+8r+5} \cong Z/2^{u+1}$. This implies immediately that $f_{7*}(1, 0) = (2^v, 0, 0) \in \pi_{4n-1} S_K \wedge RP_{4n-1}^{4n+8r+7} \cong Z/2^{v+1} \oplus Z/2 \oplus Z/2$. On the other hand, Lemma 3.7 asserts that the map $g_{i4}: \Sigma^{8m+2i} \to Y_{i4}$ $(i = 0, 2)$ satisfies the equality $g_{i4*}(1, 0) = (2^v, 0, 0) \in \pi_{8m+2i} S_K \wedge Y_{i4} \cong Z/2^{v+1} \oplus Z/2 \oplus Z/2$ where $v = \mathrm{Min}\{4r+3, \nu_2(8m)\}$ or 2 according as $i=0$ or 2. Theorefore the map $(i_K \wedge 1) f_7: \Sigma^{4n} \to S_K \wedge \Sigma^1 RP_{4n-1}^{4n+8r+7}$ coincides with the map $(i_K \wedge 1) g_{i4}: \Sigma^{8m+2i} \to S_K \wedge Y_{i4}$ for $i=0$ or 2 when $S_K \wedge \Sigma^1 RP_{4n-1}^{4n+8r+7}$ is identified with $S_K \wedge Y_{i4}$ $(i = 2n-4m)$ via a suitable quasi S_{K*}-equivalence. Hence we can easily observe that $\Sigma^1 RP_{4n-1}^{4n+8r+7}$ has the same K_*-local type as the cofiber $C(g_{04})$ or $C(g_{24})$ according as $n = 2m$ or $2m+1$.

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