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Filter-regular sequences and multiplicity of blow-up rings of ideals of the principal class

By

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1. Introduction

Let R be a graded algebra generated by finitely many elements of degree 1 over a field k and I a homogeneous ideal of R. Recently J. Herzog, B. Ulrich and this author [HTU] computed the multiplicity of the associated graded ring $\mathrm{gr}_I(R)$, the Rees algebra $R[It]$, and the extended Rees algebra $R[It, t^{-1}]$ in terms of the degrees of the generators of I when I generated by a d-sequence of R. We had to require that the degrees of the elements of the d-sequence are non-decreasing, and we were able to give an explicit representation of the associated graded rings of these blow-up rings with respect to some refinement of the adic filtration of their maximal graded ideal, from which the multiplicity formulas followed.

In this paper we will compute the multiplicity of $\mathrm{gr}_I(R)$, $R[It]$, and $R[It, t^{-1}]$ when I is a homogeneous ideal of the principal class, that means I is generated exactly by $\mathrm{ht}(I)$ homogeneous elements, where $\mathrm{ht}(I)$ is the height of I. Our main tool will be an extended version of the notion of filter-regular sequences. This notion originated from the theory of generalized Cohen-Macaulay rings [CST], and it has proven to be useful in many contexts [Br], [SV], [T2]. One can easily show that if the field k is infinite, an assumption which does not cause any problem in computing the multiplicity, then every homogeneous ideal of R can be generated by a homogeneous filter-regular sequence. For the definition and some basic properties of filter-regular sequences we refer to Section 1 of this paper. Unless otherwise specified we will denote by e the multiplicity of a given local ring with respect to the maximal ideal or of a given graded ring with respect to its (uniquely determined) maximal graded ideal.

For the associated graded rings we will use the associative formula for multiplicities to derive in Section 2 the following formula

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$$e(\mathrm{gr}_J(S))=\sum e(S/\mathfrak{p})e(JS_{\mathfrak{p}};\ S_{\mathfrak{p}})$$

which holds for any ideal J of the principal class of an arbitrary local ring S, where $\mathfrak{p}$ runs all associated prime ideals of J with $\dim R/\mathfrak{p}=\dim R/J$ and $e(JS_{\mathfrak{p}};\ S_{\mathfrak{p}})$ denotes the multiplicity of the local ring $S_{\mathfrak{p}}$ with respect to the parameter ideal $JS_{\mathfrak{p}}$. This result is similar to a result of M. Nagata [Na, (24.7)] which he also called the associative formula. In particular, it implies that $e(\mathrm{gr}_J(S))\leq e(S/J)$ and that $e(\mathrm{gr}_J(S))=e(S/J)$ if and only if all the local rings $S_{\mathfrak{p}}$ are Cohen-Macaulay rings.

Using the notion of filter-regular sequences and results of M. Auslander and D. Buchsbaum on the multiplicity of parameter ideals [AB] we can then deduce the following multiplicity formula for the associated graded ring of an arbitrary homogeneous ideal $I\subset R$ of the principal class:

$$e(\mathrm{gr}_I(R))=a_1\cdots a_n e(R)\,,$$

where $a_1, \cdots, a_n$ are the degrees of the elements of a homogeneous minimal basis of I. This formula had been proven before for Buchsbaum and Cohen-Macaulay graded algebras [HTU], and we are in fact inspired of the independence of this formula upon the order of $a_1, \cdots, a_n$.

To deal with $R[It]$ and $R[It, t^{-1}]$ we have to impose, as for the case of d-sequences [HTU], the condition that the degrees of the elements of a filter-regular sequence generating I are non-decreasing. Then we can roughly estimate the associated graded rings of the Rees algebras with respect to some refinement of the adic filtration of their maximal graded ideal. When I is an ideal of the principal class, this estimation is good enough for the computation of the multiplicity of $R[It]$ and $R[It, t^{-1}]$.

Our main results concerning the Rees and extended Rees algebras can be formulated as follows: Let I be a homogeneous ideal of the principal class of R. Assume that I is generated by a homogeneous filter-regular sequence $x_1, \cdots, x_n$ with respect to I which satisfies the condition $a_1\leq\cdots\leq a_n$, where $a_i:=\deg x_i$. Then

$$e(R[It])=(1+\sum_{i=1}^{n-1}a_1\cdots a_i)e(R)\,,$$

$$e(R[It, t^{-1}])=(1+\sum_{i=l}^{n-1}a_1\cdots a_i)e(R)\,,$$

where l is the largest integer for which $a_l=1$ ($l=0$ and $a_1\cdots a_l=1$ if $a_i>1$ for all $i=1, \cdots, n$).

Unfortunately we do not know whether the additional condition on the generation of I can be dropped. However, this condition is automatically satisfied if R belongs to the large class of generalized Cohen-Macaulay rings, e.g. if R is a Buchsbaum ring [HTU, Example 3.1 and Example 3.2], or if I is generated by homogeneous elements of the same degree. The proofs for the multiplicity formulas of $R[It]$ and $R[It, t^{-1}]$ will be found in Section 3 and Section 4, respectively. In Section 3 we will also show that the multiplicity of the symmetric algebra of an ideal of the principal class (not necessary homogeneous) is always equal to the multiplicity of the Rees algebra, an easy but less known fact.

It is worth to mention that the above formulas for the multiplicity of $R[It]$ and $R[It, t^{-1}]$ bear some resemblance to those found by J. K. Verma and D. Katz [V1], [V2], [KV] in terms of the mixed multiplicities of the maximal graded ideal $\mathfrak{m}$ of R and I. Mixed multiplicities was first imtroduced by B. Teissier and J. J. Risler [Te] for two $\mathfrak{m}$-primary ideals and in this case they can be interpreted as the multiplicity of general elements or, due to D. Rees [R], of joint reductions. These interpretations suggest a probably close connection between mixed multiplicities and filter-regular sequences.

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1. Filter-regular seguences

Let S be an arbitrary noetherian commutative ring with unity and J an ideal of S. A sequence $z_1, \cdots, z_n$ of elements of S is called *filter-regular* with respect to J if $z_i \notin \mathfrak{p}$ for all associated prime ideals $\mathfrak{p} \not\supseteq J$ of $(z_1, \cdots, z_{i-1})$, $i=1, \cdots, n$. Originally, the notion of filter-regular sequences is introduced with respect to the maximal ideal of a local ring [CST] or with respect to the ideal generated by the elements of positive degree of a graded ring [T2].

It is easily seen that if $z_1, \cdots, z_n$ is a filter regular sequence with respect to J, then $z_i, \cdots, z_n$ form a filter regular sequence in the factor ring $\bar{S} := S/(z_1, \cdots, z_{i-1})$ with respect to the ideal $J\bar{S}$, $i=1, \cdots, n$. Moreover, $z_1, \cdots, z_n$ is also a filter-regular sequence with respect to any ideal contained in J.

Remark. The property of being a filter-regular sequence is not permutable. For instance, put $S=k[x, y, z] := k[X, Y, Z]/(XY, XZ)$, $J=(x, y, z)$. Then $x+y, z$ is a filter-regular sequence with respect to J, but $z, x+y$ not.

The following characterization of filter-regular sequences shows that this notion is a generalization of weak-sequences and hence of d-sequences. For more details on these sequences we refer to [Hu], [SV], and [T1].

For $i=1, \cdots, n$ we set

$$J_i = \cup_{m=1}^{\infty} (z_1, \cdots, z_{i-1}) : J^m .$$

Note that J_i is equal to the intersection of all primary components of $(z_1, \cdots, z_{i-1})$ whose associated prime ideals do not contain J.

Lemma 1.1. *$z_1, \cdots, z_n$ is a filter-regular sequence with respect to J if and only if*

$$(z_1, \cdots, z_{i-1}) : z_i \subseteq J_i\,, \quad i=1, \cdots, n\,.$$

Proof. ($\Rightarrow$) By definition, z_i is not contained in any associated prime ideals of J_i. Hence $(z_1, \cdots, z_{i-1}) : z_i \subseteq J_i : z_i = J_i$, $i=1, \cdots, n$.

($\Leftarrow$) Let $\mathfrak{p} \not\supseteq J$ be an associated prime ideal of $(z_1, \cdots, z_{i-1})$. Then

$$((z_1, \cdots, z_{i-1}) : z_i) S_\mathfrak{p} \subseteq J_i S_\mathfrak{p} = (z_1, \cdots, z_{i-1}) S_\mathfrak{p}.$$

This occurs only if z_i is a unit of $S_\mathfrak{p}$ because $\mathfrak{p} S_\mathfrak{p}$ is also an associated prime ideal of $(z_1, \cdots, z_{i-1}) S_\mathfrak{p}$. Hence $z_i \notin \mathfrak{p}$, as required.

Corollary 1.2. *Let $z_1, \cdots, z_n$ be a filter-regular sequence with respect to J. If $z_1, \cdots, z_n \in J$, then*

$$J_i = \cup_{m=1}^{\infty} (z_1, \cdots, z_{i-1}) : z_i^m\,, \quad i=1, \cdots, n\,.$$

Proof. By definition, $z_1, \cdots, z_{i-1}, z_i^m$ is also a filter-regular sequence with respect to J for all $m \geq 1$. Thus, by Lemma 1.1, $\cup_{m=1}^{\infty} (z_1, \cdots, z_{i-1}) : z_i^m \subseteq J_i$. On the other hand, since $z_i \in J$, we also have $J_i \subseteq \cup_{m=1}^{\infty} (z_1, \cdots, z_{i-1}) : z_i^m$.

From the above characterization of filter-regular sequences we obtain the following result which will play a crucial role in the computation of the multiplicity of Rees algebras.

Lemma 1.3. *Let $z_1, \cdots, z_n \in J$ be a filter-regular sequence with respect to J. For any form f in the polynomial ring $S[T_1, \cdots, T_n]$ which vanishes at $z_1, \cdots, z_n$, the coefficients of all terms of f with the highest degree in T_n are contained in J_n.*

Proof. Put $y_i = z_i / z_n$ for all $i=1, \cdots, n-1$. Since $z_n \in J$, we have

$$J_i S[z_n^{-1}] = (z_1, \cdots, z_{i-1}) S[z_n^{-1}]\,.$$

By Lemma 1.1, this implies $((z_1, \cdots, z_{i-1}) : z_i) S[z_n^{-1}] = (z_1, \cdots, z_{i-1}) S[z_n^{-1}]$. Hence $z_1, \cdots, z_{n-1}$ and therefore $y_1, \cdots, y_{n-1}$ form a regular sequence in $S[z_n^{-1}]$. Write f in the form $f = f_0 + f_1 T_n + \cdots + f_r T_n^r$, where $f_i \in S[T_1, \cdots, T_{n-1}]$ with deg $f_i = \deg f_r + r - i$, $f_r \neq 0$. Since $f(z_1, \cdots, z_n) = 0$, we have

$$f_0(y_1, \cdots, y_{n-1}) + f_1(y_1, \cdots, y_{n-1}) + \cdots + f_r(y_1, \cdots, y_{n-1}) = 0\,.$$

From this it follows that $f_r(y_1, \cdots, y_{n-1}) \in (y_1, \cdots, y_{n-1})^{\deg f_r + 1}$. Therefore, all coefficients of f_r are contained in $(y_1, \cdots, y_{n-1}) S[z_n^{-1}]$. But

$$(y_1, \cdots, y_{n-1}) S[z_n^{-1}] \cap S = \cup_{m=1}^{\infty} (z_1, \cdots, z_{n-1}) : z_n^m = J_n$$

by Corollary 1.2. Hence all coefficients of f_r are contained in J_n.

Remark. Lemma 1.3 is an extension of the following statement used in [HTU, Proof of Lemma 1.2]. Let $z_1, \cdots, z_n \in J$ be a d-sequence of S and f a form of the polynomial ring $S[T_1, \cdots, T_n]$ which vanishes at $z_1, \cdots, z_n$. Then the coefficients of all terms of f with the highest degree in T_n are contained in $(z_1, \cdots, z_{n-1}): z_n$. In fact, we have $(z_1, \cdots, z_{n-1}): z_n = \cup_{m=1}^{\infty}(z_1, \cdots, z_{n-1}): z_n^m$ [Hu].

Now we want to consider homogeneous filter-regular sequences. Let R be a graded ring generated by finitely many forms of degree 1 over a local ring whose residue field is infinite. Let I be an arbitrary homogeneous ideal of R. We will show that I can be generated by a homogeneous filter-regular sequence with respect to I.

Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be the degrees of the elements of an arbitrary homogeneous minimal basis of I arranged in non-decreasing order. It is well-known that the sequence $a_1, \cdots, a_n$ does not depend upon the choice of the minimal basis, and we call it the *degree sequence* of I.

Lemma 1.4. *Let I be an arbitrary homogeneous ideal of R with degree sequence $a_1 \leq a_2 \leq \cdots \leq a_n$. There exists a homogeneous minimal basis $x_1, \cdots, x_n$ of I such that $x_1, \cdots, x_n$ is a filter-regular sequence with respect to I and deg $x_i = a_{n-i+1}$, $i=1, \cdots, n$. Moreover, if $a_1 = \cdots = a_n$, then the initial forms of $x_1, \cdots, x_n$ in the associated graded ring $gr_I(R)$ also form a filter-regular sequence with respect to the ideal generated by the elements of positive degree of $gr_I(R)$.*

Proof. Set $a = a_n$. Let I_a denote the ideal generated by the homogeneous elements of degree a in I. It is obvious that I_a and I have the same radical. Since any homogeneous minimal basis of I has at least an element of degree a, I_a is not contained in $\mathfrak{m}I$, where $\mathfrak{m}$ denotes the maximal graded ideal of R. Since $R/\mathfrak{m}$ is an infinite field, there exists an element $x_1 \in I_a$ with deg $x_1 = a$ such that $x_1 \notin \mathfrak{p}$ for all associated prime ideals $\mathfrak{p} \not\supseteq I$ of R and $x_1 \notin \mathfrak{m}I$. Clearly, the ideal $\bar{I} := I/(x_1)$ of $\bar{R} := R/(x_1)$ has the degree sequence $a_1 \leq \cdots \leq a_{n-1}$. By induction on n we may assume that $\bar{I}$ is generated by a homogeneous filter-regular sequence $\bar{x}_2, \cdots, \bar{x}_n$ with respect to $\bar{I}$ and deg $\bar{x}_i = a_{n-i+1}$. Let $x_2, \cdots, x_n$ be homogeneous elements of R whose images in $\bar{R}$ are $\bar{x}_2, \cdots, \bar{x}_n$. It is easy to check that $x_1, \cdots, x_n$ is a filter-regular sequence with respect to I which generates I and deg$x_i = a_{n-i+1}$, $i=1, \cdots, n$.

Let $z_1, \cdots, z_n$ denote the initial forms of $x_1, \cdots, x_n$ in $G := gr_I(R)$ and G_+ the ideal generated by the elements of positive degree of G. If $a_1 = \cdots = a_n = a$, then $I = I_a$. For any associated prime ideal $P \not\supseteq G_+$ of $(z_1, \cdots, z_{i-1})$, the ideal generated by all elements of I whose initial forms in G belong to P do not contain I. Therefore, we can choose x_i so that z_i avoids all prime ideals P, $i=1, \cdots, n$. In this case, $z_1, \cdots, z_n$ is also a filter-regular sequence with respect to G_+.

Remark. In general there does not exist a homogeneous minimal basis

$x_1, \cdots, x_n$ for I such that $x_1, \cdots, x_n$ form a filter-regular sequence with respect to I and $\deg x_1 \leq \cdots \leq \deg x_n$. For instance, put $S = k[x, y, z] := k[X, Y, Z]/(XY, XZ)$, $I = (x^2 + y^2, z)$. Then any homogeneous element of degree 1 in I must be divisible by z which belongs to the associated prime ideal $(y, z) \not\supseteq I$ of R.

Lemma 1.5. *Let $x_1, \cdots, x_n$ be an arbitrary homogeneous filter-regular sequence with respect to I. Put $J_i := \cup_{m=1}^{\infty}(x_1, \cdots, x_{i-1}) : I^m$. There exists a homogeneous element $x \in I$ such that $J_i : x = J_i$ and $xJ_i \subseteq (x_1, \cdots, x_{i-1})$ for all $i = 1, \cdots, n$.*

Proof. First, since every associated prime ideal $\mathfrak{p}$ of J_i do not contain I, we can find a homogeneous element $x \in I$ such that $x \notin \mathfrak{p}$ for all such $\mathfrak{p}$. As a consequence, $J_i : x = J_i$. Now one only needs to replace x by a sufficiently higher power of x in order to get the relations $xJ_i \subseteq (x_1, \cdots, x_{i-1})$.

We conclude this section by showing that homogeneous filter-regular sequences behave well with respect to the multiplicity.

Lemma 1.6. *Let x be a homogeneous filter-regular sequence with respect to an ideal I of R with* $\mathrm{ht}(I) \geq 2$. *Set $a := \deg x$. Then*

$$e(R/(x)) = ae(R).$$

Proof. Set $J := \cup_{m=1}^{\infty} 0 : I^m$. Then J is the intersection of all primary components of the zeroideal of R whose associated prime ideals do not contain I. Since $\mathrm{ht}(I) \geq 2$, the zeroideal of R and J share the same d-dimensional primary components, $d := \dim R$. Therefore $e(R) = e(R/J)$. On the other hand, the definition of filter-regular sequences says that x is not contained in all associated prime ideals of J. As a consequence, $\dim R/(J, x) = d - 1 = \dim R/(x)$ and we obtain $e(R/(J, x)) = ae(R/J) = ae(R)$. Let $\mathfrak{p}$ be an arbitrary associated prime ideal of (x) with $\dim R/\mathfrak{p} = d - 1$. Then $I \not\subseteq \mathfrak{p}$ because $\dim R/I \leq d - 2$. By the definition of J, $JR_{\mathfrak{p}} = 0$ and therefore $xR_{\mathfrak{p}} = (J, x)R_{\mathfrak{p}}$. This implies that (x) and (J, x) share the same $(d-1)$-dimensional primary component associated with $\mathfrak{p}$. Hence using the associative formula for multiplicities we obtain $e(R/(x)) = e(R/(J, x)) = ae(R)$.

2. Multiplicity of the associated graded ring

First we will derive a general multiplicity formula for the associated graded ring of an arbitrary ideal of the principal class of a local ring.

For any local ring S we will denote by $e(S)$ the multiplicity of S with respect to its maximal ideal and by $e(J; S)$ the multiplicity of S with respect to an ideal J with $l(S/J) < \infty$.

Lemma 2.1. *Let S be a local ring and J an arbitrary ideal of S with*

$l(S/J)<\infty$. *Then*

$$e(\mathrm{gr}_J(S))=e(J;\ S)\,.$$

Proof. Set $G:=\mathrm{gr}_J(S)$ and let G_+ denote the ideal generated by all forms of positive degree of G. Then G_+ is a reduction of the maximal graded ideal of G. By [NR] this implies that $e(G)$ is equal to the multiplicity of G with respect to the ideal G_+. But the latter is exactly $e(J;\ S)$.

Therorem 2.2. *Let S be a local ring and J an arbitrary ideal of the principal class of S. Then*

$$e(gr_J(S))=\sum e(S/\mathfrak{p})e(JS_\mathfrak{p};\ S_\mathfrak{p})\,,$$

where $\mathfrak{p}$ runs all associated prime ideals of J with dim $S/\mathfrak{p}=$dim S/J.

Proof. Set $d:=\dim S$ and $n:=\mathrm{ht}(J)$. Then $\dim S/J=d-n$. Consider the representation:

$$\mathrm{gr}_J(S)\simeq A/(J,\ Q)\,,$$

where $A:=S[T_1,\cdots,\ T_n]$ and Q is the ideal of all forms of A vanishing at a fixed minimal basis $z_1,\cdots,\ z_n$ of J. By the associative formula for multiplicities we know that

$$e(\mathrm{gr}_J(S))=\sum e(A/P)l(A_P/(J,\ Q)A_P)\,,$$

where P runs all d-dimensional associated prime ideals of $(J,\ Q)$. Note that $\dim A/JA=\dim S/J+n=d$. Then P is also an associated prime ideal of JA. Hence P must be of the form $\mathfrak{p}A$ and

$$e(A/P)=e(S/\mathfrak{p})$$

for some $(d-n)$-dimensional associated prime ideal $\mathfrak{p}$ of J. It remains to show that

$$l(A_P/(J,\ Q)A_P)=e(JS_\mathfrak{p};\ S_\mathfrak{p})$$

and that every ideal of the form $\mathfrak{p}A$ with $\dim S/\mathfrak{p}=d-n$ is an associated prime ideal of $(J,\ Q)$. Note that $z_1,\cdots,\ z_n$ is a system of parameters of $S_\mathfrak{p}$. Then $z_1,\cdots,\ z_n$ are analytically independent in $S_\mathfrak{p}$ [ZS]. This means that the coefficients of any form of Q are contained in $\mathfrak{p}$. Therefore $Q\subseteq\mathfrak{p}A$ and $\mathfrak{p}A$ must be an associated prime ideal of $(J,\ Q)$. Now we look at the local ring $S_\mathfrak{p}$. Then

$$\mathrm{gr}_{JS_P}(S_\mathfrak{p})\simeq A_\mathfrak{p}/(J,\ Q)A_\mathfrak{p}\,,$$

where $A_\mathfrak{p}=S_\mathfrak{p}[T_1,\cdots,\ T_n]$. Since $JS_\mathfrak{p}$ is $\mathfrak{p}S_\mathfrak{p}$-primary, $\mathfrak{p}A_\mathfrak{p}$ is the only d-dimensional associated prime ideal of $(J,\ Q)A_\mathfrak{p}$. Thus, for $P=\mathfrak{p}A$, we have

$$e(\mathrm{gr}_{JS_P}(S_\mathfrak{p}))=e(A_\mathfrak{p}/PA_\mathfrak{p})l(A_P/(J, Q)A_P)=l(A_P/(J, Q)A_P)$$

because $e(A_\mathfrak{p}/PA_\mathfrak{p})=e(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})=1$. On the other hand, $e(\mathrm{gr}_{JS_P}(S_\mathfrak{p}))=e(JS_\mathfrak{p}; S_\mathfrak{p})$ by Lemma 2.1. Hence $l(A_P/(J, Q)A_P)=e(JS_\mathfrak{p}; S_\mathfrak{p})$, as required.

Remark. Theorem 2.2 can be deduced from the following result of M. Nagata [Na, (24.7)] which he also called the associative formula:

Let S be a local ring and $Q=(x_1, \cdots, x_d)$ a parameter ideal of S, $d=\dim S$. Put $J=(x_1, \cdots, x_n)$, $n\leq d$ fixed. Then

$$e(Q; S)=\sum e(S/\mathfrak{p})e(JS_\mathfrak{p}; S_\mathfrak{p}),$$

where $\mathfrak{p}$ runs all associated prime ideals $\mathfrak{p}$ of J with $\dim S/\mathfrak{p}=d-n$.

To see this one has to apply Nagata's result to the local ring T of $\mathrm{gr}_J(S)$ at its maximal graded ideal and a minimal reduction of the maximal ideal of T which contains the initial forms of the elements of J in $\mathrm{gr}_J(S)$. Other details are left to the readers.

The above multiplicity formula provides a close relationship between $e(\mathrm{gr}_J(S))$ and $e(S/J)$ because

$$e(S/J)=\sum e(S/\mathfrak{p})l(S_\mathfrak{p}/JS_\mathfrak{p}),$$

where $\mathfrak{p}$ runs all associated prime ideals of J with $\dim S/\mathfrak{p}=\dim S/J$.

Corollary 2.3. *Let S be a local ring and J an ideal of the principal class of S. Then*

$$e(gr_J(S))\leq e(S/J)$$

and $e(\mathrm{gr}_J(S))=e(S/J)$ if and only if $S_\mathfrak{p}$ is a Cohen-Macaulay ring for all associated prime ideals $\mathfrak{p}$ of J with $\dim S/\mathfrak{p}=\dim S/J$.

Proof. Since $JS_\mathfrak{p}$ is a parameter ideal of $S_\mathfrak{p}$, we have

$$e(JS_\mathfrak{p}; S_\mathfrak{p})\leq l(S_\mathfrak{p}/JS_\mathfrak{p}),$$

and $e(JS_\mathfrak{p}; S_\mathfrak{p})=l(S_\mathfrak{p}/JS_\mathfrak{p})$ if and only if $S_\mathfrak{p}$ is a Cohen-Macaulay ring.

In the following we will use the notion of filter-regular sequences in the computation of the multiplicity of $\mathrm{gr}_J(S)$. Note that without restriction J may be assumed to be minimally generated by a filter-regular sequence with respect to J.

Lemma 2.4. *Let J be an ideal of the principal class of a local ring S. Let $z_1, \cdots, z_n$ be a filter-regular sequence with respect to J which minimally generates J. Then*

$$e(\mathrm{gr}_J(S))=e(S/(J_n, z_n)),$$

where $J_n := \cup_{m=1}^{\infty}(z_1, \cdots, z_{n-1}) : J^m$.

Proof. Set $d := \dim S$. If $n=d$, J is a parameter ideal of S and we have to show that

$$e(\mathrm{gr}_J(S)) = l(S/(J_n, z_n)) .$$

By Lemma 2.1, $e(\mathrm{gr}_J(S)) = e(J; S)$. Let $\mathfrak{m}$ denote the maximal ideal of S. By the definition of filter-regular sequence, $z_i \notin \mathfrak{p}$ for all associated prime ideals $\mathfrak{p} \neq \mathfrak{m}$ of $(z_1, \cdots, z_{i-1})$. Therefore, by a double application of [AB, Corollary 4.8] we have

$$e(J; S) = l(S/JS) - l((z_1, \cdots, z_{n-1}) : z_n/(z_1, \cdots, z_{n-1})) = e(z_n\bar{S}; \bar{S}) ,$$

where $\bar{S} := S/(z_1, \cdots, z_{n-1})$. Note that

$$z_n{}^r\bar{S}/z_n{}^{r+1}\bar{S} \simeq S/((z_1, \cdots, z_{n-1}) : z_n{}^r, z_n) = S/(J_n, z_n)$$

for r large. Then

$$e(z_n\bar{S}; \bar{S}) = l(S/(J_n, z_n)) .$$

Hence $e(\mathrm{gr}_J(S)) = l(S/(J_n, z_n))$, as required. If $n < d$, we use the formula

$$e(\mathrm{gr}_J(S)) = \sum e(S/\mathfrak{p}) e(JS_\mathfrak{p}; S_\mathfrak{p})$$

of Theorem 2.2, where $\mathfrak{p}$ runs all $(d-n)$-dimensional associated prime ideals of J. Since $JS_\mathfrak{p}$ is a parameter ideal of $S_\mathfrak{p}$,

$$e(JS_\mathfrak{p}; S_\mathfrak{p}) = l(S_\mathfrak{p}/(J_n, z_n)S_\mathfrak{p}) .$$

It is easily seen that J_n and $(z_1, \cdots, z_{n-1})$ have the same radical. So $\sqrt{(J_n, z_n)} = \sqrt{J}$. Hence (J_n, z_n) and J share the same $(d-n)$-dimensional associated prime ideals, and we obtain

$$e(\mathrm{gr}_J(S)) = \sum e(S/\mathfrak{p}) l(S_\mathfrak{p}/(J_n, z_n)S_\mathfrak{p}) = e(S/(J_n, z_n))$$

by the associative formula for multiplicities.

Finally we will use the above result to compute the multiplicity of a homogeneous ideal of the principal class.

Theorem 2.5. *Let R be a graded algebra generated by finitely many forms of degree 1 over a field. Let I be a homogeneous ideal of the principal class of R and $a_1, \cdots, a_n$ the degrees of the elements of a homogeneous minimal basis of I. Then*

$$e(\mathrm{gr}_I(R)) = a_1 \cdots a_n e(R) .$$

Proof. By Lemma 1.4 we may assume that I is generated by a filter-regular sequence $x_1, \cdots, x_n$ and $a_i = \deg x_i$, $i=1, \cdots, n$. By Lemma 2.4,

$$e(\mathrm{gr}_I(R)) = e(R/(J_n, x_n)),$$

where J_n is the ideal $\cup_{m=1}^{\infty}(x_1, \cdots, x_{n-1}): I^m$. Note that J_n is the intersection of all primary components of $(x_1, \cdots, x_{n-1})$ whose associated prime ideals do not contain I. Then x_n is a non-zerodivisor of J_n, whence

$$e(R/(J_n, x_n)) = a_n e(R/J_n).$$

Since the minimal prime ideals of $(x_1, \cdots, x_{n-1})$ do not contain I, $(x_1, \cdots, x_{n-1})$ and J_n share the same isolated primary components. By the associative formula for multiplicities, this implies

$$e(R/J_n) = e(R/(x_1, \cdots, x_{n-1})) = a_1 \cdots a_{n-1} e(R),$$

where the latter equality follows from Lemma 1.6. Summing up we will obtain

$$e(\mathrm{gr}_I(R)) = a_1 \cdots a_n e(R).$$

3. Multiplicity of the Rees algebra

Let $R = \oplus_{i=0}^{\infty} R_i$ be a graded ring generated by finitely many forms of degree 1 over a local ring R_0. Let I be a graded ideal in R and $x_1, \cdots, x_n$ a homogeneous minimal basis of I. Let A denotes the polynomial ring $R[T_1, \cdots, T_n]$ in n variables over R. If we map $x_i t$ to T_i, $i=1, \cdots, n$, we obtain a representation for the Rees algebra:

$$R[It] \simeq A/Q,$$

where Q is the ideal generated by all forms vanishing at $x_1, \cdots, x_n$.

Let M denote the ideal of A generated by $T_1, \cdots, T_n$ and the elements of degree 1 of R. As in [HTU], to compute the multiplicity of $R[It]$ we use the following refinement of the M-adic filtration of A. First we note that $A = \oplus A_h$ is a $\mathbf{N}^{n+1}$-graded ring with $A_h := R_{\alpha_0} T_1^{\alpha_1} \cdots T_n^{\alpha_n}$, $h = (\alpha_0, \cdots, \alpha_n) \in \mathbf{N}^{n+1}$. Define the following degree-lexicographic order on $\mathbf{N}^{n+1}$:

$$(\alpha_0, \cdots, \alpha_n) < (\beta_0, \cdots, \beta_n)$$

if the first non-zero component from the leftside of

$$\left(\sum_{i=0}^{n} \alpha_i - \sum_{i=0}^{n} \beta_i, \ \alpha_0 - \beta_0, \cdots, \alpha_n - \beta_n\right)$$

is negative. Set $\mathcal{F}_h A := \oplus_{h' \geq h} A_{h'}$. Then $\mathcal{F} := (\mathcal{F}_h A)_{h \in \mathbf{N}^{n+1}}$ is a filtration of A which is finer than the M-adic filtration. Indeed, we have $M^i = \mathcal{F}_{h_i} A$ with $h_i := (0, \cdots, 0, i)$ for all $i \geq 0$.

For every polynomial $f \in A$ we denote by f^* the *initial term* of f with respect to the above order, i.e. $f^* = f_{h'}$ if $f = \sum_{h \in \mathbf{N}^{n+1}} f_h$ and $h' := \min\{h | f_h \neq 0\}$. Let Q^* denote the ideal of A generated by all elements f^*, $f \in Q$. Set

$$J_i := \cup_{m=1}^{\infty} (x_1, \cdots, x_{i-1}) : I^m ,$$

$$P := (J_1 T_1, \cdots, J_n T_n) .$$

Lemma 3.1. *Assume that $x_1, \cdots, x_n$ is a filter-regular sequence with respect to I and* $\deg x_1 \leq \cdots \leq \deg x_n$. *Then $Q^* \subseteq P$.*

Proof. Let f be an arbitrary element of Q and r the degree of f in T_n. If $r = 0$, $f \in R[T_1, \cdots, T_{n-1}]$ and we get $f^* \in P$ by induction on n. If $r > 0$, $f = g + hT_n^r$, where $g \in A$ has degree in T_n less than r and $h \in R[T_1, \cdots, T_{n-1}]$. By Lemma 1.3, the coefficient of all terms of h belongs to J_n. Hence we may assume that the degree of f^* in T_n is less than r, i.e. $f^* = g^*$. By Lemma 1.5, there exists a homogeneous element $x \in I$ such that $J_i : x = J_i$ for all $i = 1, \cdots, n$ and $xJ_n \subseteq (x_1, \cdots, x_{n-1})$. Then $xh = x_1 g_1 + \cdots + x_{n-1} g_{n-1}$ for some polynomials $g_1, \cdots, g_{n-1} \in R[T_1, \cdots, T_{n-1}]$. Now consider the polynomial

$$e = xf - [(x_1 T_n - x_n T_1) g_1 + \cdots + (x_{n-1} T_n - x_n T_{n-1}) g_{n-1}] T_n^{r-1}$$

$$= xg + x_n (T_1 g_1 + \cdots + T_{n-1} g_{n-1}) T_n^{r-1} .$$

It is obvious that e is obtained from xf by replacing the part xhT_n^r by terms whose orders are higher than that of xf^* ($\deg x_n \geq \deg x_i$ for all $i = 1, \cdots, n-1$). Hence $e^* = xf^*$. Since $e \in Q$ and the degree of e in T_n is less than r, by induction we may assume that $e^* \in P$. So $f^* \in P : x = P$, as required.

Remark. (1) When I is generated by a d-sequence, $J_i = (x_1, \cdots, x_{i-1}) : x_i$ for all $i = 1, \cdots, n$ [Hu]. Since every element xT_i with $x \in (x_1, \cdots, x_{i-1}) : x_i$ is the initial term of a linear form of $R[T_1, \cdots, T_i]$ vanishing at $x_1, \cdots, x_i$, we can conclude that $Q^* = P$. This equality is basic for the study of $R[It]$ in [HTU] and has been proven by a slightly different method.
(2) Lemma 3.1 will be also applied to the associated graded ring $\mathrm{gr}_I(R)$ in the proof of Lemma 4.1. That is the reason why till now we work over a graded algebra over a local ring.

From now on let R be a graded algebra generated by finitely many forms of degree 1 over a field. Then M is the maximal graded ideal of A. Since $\mathcal{F}$ is a finer filtration of the M-adic filtration, we have

$$e(R[It]) = e(A/Q^*) .$$

To compute $e(A/Q^*)$ we shall need the following lemma.

Lemma 3.2. *Let $Q_1 \supseteq Q_2$ be two homogeneous ideals of A. Suppose that Q_1 is contained in all primary components V of Q_2 with* $\dim A/V = \dim A/Q_2$.

Then

$$e(A/Q_1)=e(A/Q_2)\,.$$

Proof. From the assumption we deduce that $\dim A/Q_1=\dim A/Q_2$ and that Q_1, Q_2 share the same primary components V with $\dim A/V=\dim A/Q_2$. Therefore, applying the associative formula for multiplicities we obtain

$$e(A/Q_1)=\sum e(A/V)=e(A/Q_2)\,.$$

Theorem 3.3. *Let I be a homogeneous ideal of the principal class which is minimally generated by a filter-regular sequence $x_1, \cdots, x_n$. Put $a_i=\deg x_i$ and suppose that $a_1\leq\cdots\leq a_n$. Then*

$$e(R[It])=(1+\sum_{i=1}^{n-1}a_1\cdots a_i)e(R)\,.$$

Proof. Set $I_i:=(x_1, \cdots, x_{i-1})R$, $i=1, \cdots, n$, and

$$L:=(I_2T_2, \cdots, I_nT_n)\,.$$

Then L is the ideal generated by the initial form of the relations $x_iT_j-x_jT_i$. Hence $L\subseteq Q^*$. We will see that $e(A/L)$ has the same multiplicity formula as above. It is easy to check that the ideal L has the following decomposition

$$L=\bigcap_{i=1}^{n}(I_i, T_{i+1}, \cdots, T_n)\,.$$

Since $\dim R/I_i=\dim R-i+1$ for all $i=1, \cdots, n$, every component of the above decomposition has dimension $d+1$, while the sum of every couple of them has a smaller dimension. Thus, using the associative formula for multiplicities we obtain

$$e(A/L)=\sum_{i=1}^{n}e(A/(I_i, T_{i+1}, \cdots, T_n))=\sum_{i=1}^{n}e(R/I_i)\,.$$

By Lemma 1.6, $e(R/I_i)=a_1\cdots a_{i-1}e(R)$ for all $i=1, \cdots, n$. Hence

$$e(A/L)=(1+\sum_{i=1}^{n-1}a_1\cdots a_i)e(R)\,.$$

It is also clear that every $(d+1)$-dimensional primary component of L must be of the form $(\mathfrak{q}, T_{i+1}, \cdots, T_n)$ for some primary component $\mathfrak{q}$ of I_i with $\dim R/\mathfrak{q}=\dim R-i+1$, $i=1, \cdots, n$. Let $\mathfrak{p}$ denote the associated prime ideal of $\mathfrak{q}$. Then $I\nsubseteq\mathfrak{p}$ because $\dim R/I=\dim R-n<\dim R/\mathfrak{p}$. Therefore $J_iR_\mathfrak{p}=(\cup_{m=1}^{\infty}I_i:I^m)R_\mathfrak{p}=I_iR_\mathfrak{p}$. From this we deduce that $\mathfrak{q}\supseteq J_i$. On the other hand, the ideal $P=(J_1T_1, \cdots, J_nT_n)$ has the following decomposition

$$P=\cap_{i=1}^{n}(J_i, T_{i+1}, \cdots, T_n)\cap(T_1, \cdots, T_n)\,.$$

So P is contained in all $(d+1)$-dimensional primary components of L. But $Q^* \subseteq P$ by Lemma 3.1. Therefore, by Lamma 3.2, we obtain

$$e(A/Q^*)=e(A/L)\,.$$

Since $e(R[It])=e(A/Q^*)$, the conclusion is now immediate.

In the following we will show that the multiplicities of the Rees algebra and the symmetric algebra of an ideal of the principal class (not necessarily homogeneous) are equal. For an ideal J in a ring S we will denote by $Sym(J)$ the symmetric algebra of the ideal J.

Lemma 3.4. *Let S be a local ring and J an ideal of the principal class in S. Then*

$$e(Sym(J))=e(S[Jt])\,.$$

Proof. Suppose that J is minimally generated by the elements $z_1, \cdots, z_n$. Set $B:=S[T_1, \cdots, T_n]$. Let Q_1 resp. Q_2 denote the ideal of all forms resp. linear forms of B vanishing at $z_1, \cdots, z_n$. Then

$$S[Jt]\simeq B/Q_1\,,$$

$$Sym(J)\simeq B/Q_2\,.$$

By Lemma 3.2 we only need to show that Q_1 is contained in all primary components V of Q_2 with $\dim B/V=\dim Sym(J)$. Note that $S[Jt, 1/z_i]=S[1/z_i][t]$. Then

$$Q_1B[1/z_i]=Q_2B[1/z_i]$$

for all $i=1, \cdots, n$. From this it follows that $Q_1=Q_2$ or $Q_1=Q_2\cap U$ for some ideal U of B whose associated prime ideals contain J. Since

$$\dim B/Q_1=\dim S+1\,,$$

$$\dim B/U\leq \dim B/JB=\dim S/J+n=\dim S\,,$$

we conclude that $\dim B/Q_2=\dim S+1$ and that Q_1 and Q_2 share the same primary components V with $\dim B/V=\dim S+1$.

Corollary 3.5. *Let I be a homogeneous ideal of the principal class as in Theorem 3.3. Then*

$$e(Sym(I))=(1+\sum_{i=1}^{n-1} a_1\cdots a_i)e(R)\,.$$

Now we will apply the above results to generalized Cohen-Macaulay graded algebras. Recall that R is called a *generalized Cohen-Macaulay ring* if $R_\mathfrak{p}$ is Cohen-Macaulay and $\dim R/\mathfrak{p}+\mathrm{ht}(\mathfrak{p})=\dim R$ for all prime ideals $\mathfrak{p}\neq$

$\mathfrak{m}$ of R, $\mathfrak{m}$ being the maximal graded ideal of R. For instances, this is the case if R is a Buchsbaum ring. See [CST] and [T3] for more details on the theory of generalized Cohen-Macaulay rings.

It is well-known that if R is a generalized Cohen-Macaulay graded algebra, every homogeneous system of parameters of R is filter-regular with respect to $\mathfrak{m}$. Therefore, any sequence $x_1, \cdots, x_n$ of homogeneous elements of R which minimally generates an ideal I of the principal class is filter-regular with respect to I, and we may assume that $\deg x_1 \leq \cdots \leq \deg x_n$.

Corollary 3.6. (cf. [HTU, Example 3.1 and Example 3.2]). *Let R be a generalized Cohen-Macaulay graded algebra and $I \subset R$ a homogeneous ideal of the principal class with the degree sequence $a_1 \leq \cdots \leq a_n$. Then*

$$e(R[It]) = e(Sym(I)) = (1 + \sum_{i=1}^{n-1} a_1 \cdots a_i) e(R).$$

Remark. For a local ring $(S, \mathfrak{m})$ and an arbitrary ideal $J \subset S$ with positive height, J. K. Verma [V1], [V2] has found the following multiplicity formula:

$$e(S[Jt]) = \sum_{i=0}^{d-1} e_i(\mathfrak{m}|J),$$

where $e_i(\mathfrak{m}|J)$ denotes the ith mixed multiplicity of $\mathfrak{m}$ and J and $d = \dim S$. It is known that $e_0(\mathfrak{m}|J) = e(S)$ and $e_i(\mathfrak{m}|J) = 0$ for $i \geq l$, where l is the analytic spread of J [KV]. If J is an $\mathfrak{m}$-primary ideal, $e_i(\mathfrak{m}|J)$ is the multiplicity of an ideal generated by $d-i$ elements from $\mathfrak{m}$ and i elements from J chosen sufficiently general [Te] or of a joint reduction of $d-i$ copies of $\mathfrak{m}$ and i copies of J [R]. For the situation of Theorem 3.3, this suggests that probably $e_i(\mathfrak{m}|I) = a_1 \cdots a_i e(S)$ and that one may find another proof for Theorem 3.3 by the theory of mixed multiplicities. Unfortunately, for an arbitrary ideal J, there is at present no interpretation for mixed multiplicities like those in [Te] or [R].

4. Multiplicity of the extended Rees algebra

Let R be a graded algebra generated by finitely many forms of degree 1 over a field and I a homogeneous ideal of the principal class of R. To compute the multiplicity of the extended Rees algebra $R[It, t^{-1}]$ we will follow the approach of [HTU] and Section 3.

Let $\mathcal{A} = R[T_1, \cdots, T_n, U]$ be a polynomial ring over R in $n+1$ variables and Q the ideal of $\mathcal{A}$ generated by all forms of $R[T_1, \cdots, T_n]$ vanishing at $x_1, \cdots, x_n$ and the relations $x_1 - UT_1, \cdots, x_n - UT_n$. It is well-known that the extended Rees algebra $R[It, t^{-1}]$ of the ideal $I = (x_1, \cdots, x_n)$ has the presentation

$$R[It, t^{-1}] \simeq \mathcal{A}/Q\ .$$

Lemma 4.1. *Assume that $x_1, \cdots, x_n$ is a filter-regular sequence with respect to I. Set*

$$J_n := \cup_{m=1}^{\infty}(x_1, \cdots, x_{n-1}):I^m\ .$$

For any polynomial $f \in Q$, the coefficient of all terms of f which have the highest degree in T_n and which are not divided by U is contained in J_n.

Proof. Using the substitutions $T_iU \to x_i$ whenever it is possible, we can transform every polynomial f of $\mathcal{A}$ to a polynomial g of the form $g = g_1 + Ug_2$ with $g_1 \in R[T_1, \cdots, T_n]$ and $g_2 \in R[U]$. If $f \in Q$, then $g \in Q$ too. In this case,

$$g(x_1t, \cdots, x_nt, t^{-1}) = g_1(x_1t, \cdots, x_nt) + t^{-1}g_2(t^{-1}) = 0\ ,$$

hence $g_2 = 0$ and $g = g_1$ is a polynomial of $R[T_1, \cdots, T_n]$ vanishing at $x_1, \cdots, x_n$. If we do the substitutions with preference to $T_nU \to x_n$, the terms of g with the highest degree in T_n are exactly the terms of f which have the highest degree in T_n and which are not divided by U. Now we only need to apply Lemma 1.3 to get the statement.

Let $\mathcal{M}$ denotes the maximal ideal $(\mathfrak{m}, T_1, \cdots, T_n, U)$ of $\mathcal{A}$. As in Section 3, to compute the multiplicity of $R[It, t^{-1}]$ with respect to $\mathcal{M}$ we introduce a refinement $\mathcal{F}$ of the $\mathcal{M}$-adic filtration of $\mathcal{A}$. First we note that $\mathcal{A} = \oplus \mathcal{A}_h$ is a $\mathbf{N}^{n+2}$-graded ring with $\mathcal{A}_h := R_{a_0} T_1^{a_1} \cdots T_n^{a_n} U^{a_{n+1}}$, $h = (a_0, \cdots, a_{n+1}) \in \mathbf{N}^{n+2}$. For every term $f \in \mathcal{A}_h$ we call the sum $\Sigma_{i=0}^{n+1} a_i$ the *total degree* of f. Define the following degree-lexicographic order on $\mathbf{N}^{n+2}$:

$$(a_0, \cdots, a_{n+1}) < (\beta_0, \cdots, \beta_{n+1})$$

if the first non-zero component from the left side of

$$\left(\sum_{i=0}^{n+1} a_i - \sum_{i=0}^{n+1} \beta_i,\ a_0 - \beta_0, \cdots, a_{n+1} - \beta_{n+1}\right)$$

is negative. Set $\mathcal{F}_h\mathcal{A} := \oplus_{h' \geq h} \mathcal{A}_{h'}$. Then $\mathcal{F} := (\mathcal{F}_h\mathcal{A})_{h \in \mathbf{N}^{n+2}}$ is a filtration of $\mathcal{A}$ which is finer than the $\mathcal{M}$-adic filtration. Notice that this filtration induces the filtration $\mathcal{F}$ on $R[T_1, \cdots, T_n]$ introduced in Section 3.

For every polynomial $f \in \mathcal{A}$ we denote again by f^* the initial term of f with respect to the above order and by Q^* the ideal of $\mathcal{A}$ generated by all elements f^*, $f \in Q$. Then

$$e(R[It, t^{-1}]) = e(\mathcal{A}/Q^*)\ .$$

We shall estimate Q^* by an ideal $\mathcal{P}$ of $\mathcal{A}$ which is defined as follows. Let l be the largest integer for which $\deg x_l = 1$, $l = 0$ if $\deg x_i > 1$ for all $i = 1, \cdots, n$. If $l < n$, set

$$\mathcal{P}=(J_{l+1}, T_{l+1}U, \cdots, T_nU, J_{l+2}T_{l+2}, \cdots, J_nT_n),$$

where $J_i=\bigcup_{m=1}^{\infty}(x_1, \cdots, x_{i-1}): I^m$, $i=l+1, \cdots, n$, and if $l=n$,

$$\mathcal{P}=K\mathcal{A},$$

where $K:=\bigcup_{m=1}^{\infty}(I_nI^{m-1}+I^{m+1}): x_n^m$.

Lemma 4.2. *Assume that $x_1, \cdots, x_n$ is a filter-regular sequence with respect to I and* $\deg x_1 \leq \cdots \leq \deg x_n$. *Then $Q^* \subseteq \mathcal{P}$.*

Proof. Let f be an arbitrary element of Q. If $l=n$, we may assume that f is not divided by U and that f is quasi-homogeneous with respect to the weight w: $w(T_1)=\cdots=w(T_l)=1$, $w(U)=-1$. Then $f=f_0+f_1U+\cdots+f_sU^s$ for some homogeneous forms $f_i \in R[T_1, \cdots, T_n]$ with $\deg f_i=\deg f_0+i$, $i=1, \cdots, s$, $f_0 \neq 0$. Since all elements $x_1, \cdots, x_n$ have degree 1, we may further assume that all coefficients of f_i have degree $d-i$ for some positive integer $d \geq s$. Under these assumptions the total degree of every term of f_iU^i is equal to that of f_0 plus i. Hence we have $f^*=f_0^*$. Since the polynomial $f_0+f_1+\cdots+f_s$ vanishes at $x_1, \cdots, x_n$, $f_0(x_1, \cdots, x_n) \in I^{\deg f_0+1}$. Let $\bar{x}_1, \cdots, \bar{x}_n$ denote the initial forms of $x_1, \cdots, x_n$ in the associated graded ring $\mathrm{gr}_I(R)$ and $\bar{f}_0$ the image of f_0 in $(R/I)[T_1, \cdots, T_n]$. Then $\bar{f}_0$ is a relation of the elements $\bar{x}_1, \cdots, \bar{x}_n$. By Lemma 1.4 we may assume that these elements form a filter-regular sequence of $\mathrm{gr}_I(R)$. Then we can apply Lemma 3.1 to deduce that $(\bar{f}_0)^* \in (\bar{J}_1T_1, \cdots, \bar{J}_nT_n)$, where $\bar{J}_i=\bigcup_{m=1}^{\infty}(\bar{x}_1, \cdots, \bar{x}_{i-1}): \bar{x}_i^m$. If the coefficient of f_0^* does not belong to I, $\overline{f_0^*}=(\bar{f}_0)^*$. Since $\bar{J}_1 \subseteq \cdots \subseteq \bar{J}_n$, the coefficient of $\overline{f_0^*}$ belongs to $\bar{J}_n$ or, more precisely, to the zero-graded piece $[\bar{J}_n]_0$. It is easily seen that $[\bar{J}_n]_0 = K/I$. Hence the coefficient of f_0^* belongs to K, and we obtain $f^*=f_0^* \in \mathcal{P}$.

If $l<n$, let r be the degree of f in T_n. If $r=0$, $f \in R[T_1, \cdots, T_{n-1}, U]$ and we have $f^* \in \mathcal{P}$ by induction on n. If $r>0$, we write $f=g+hT_n^r$, where g is a polynomial of $\mathcal{A}$ whose degree in T_n is less than r and $h \in R[T_1, \cdots, T_{n-1}, U]$. By Lemma 4.1 the coefficient of all terms of h which is not divided by U belongs to the ideal J_n. Hence we may assume that the degree of f^* in T_n is less than r and $h=h_1+Uh_2$ with $h_1 \in J_nR[T_1, \cdots, T_{n-1}]$ and $h_2 \in R[T_1, \cdots, T_{n-1}, U]$. By Lemma 1.5 there exists a homogeneous element $x \in I$ such that $J_i: x = J_i$ for all $i=1, \cdots, n$ and $xJ_n \in (x_1, \cdots, x_{n-1})$. Then $xh_1=x_1g_1+\cdots+x_{n-1}g_{n-1}$ for some polynomials $g_1, \cdots, g_{n-1} \in R[T_1, \cdots, T_{n-1}]$. It follows that

$$\begin{aligned} xhT_n = & (x_1T_n-x_nT_1)g_1+\cdots+(x_{n-1}T_n-x_nT_{n-1})g_{n-1} \\ & +x_nT_1g_1+\cdots+x_nT_{n-1}g_{n-1}-(x_n-T_nU)xh_2+x_nxh_2 . \end{aligned}$$

Consider the polynomial

$$\begin{aligned} e=xf-[& (x_1T_n-x_nT_1)g_1+\cdots+(x_{n-1}T_n-x_nT_{n-1})g_{n-1} \\ & -(x_n-T_nU)xh_2]T_n^{r-1} \end{aligned}$$

$$=xg+x_n(T_1g_1+\cdots+T_{n-1}g_{n-1}+xh_2)T_n^{r-1}.$$

It is obvious that e is obtained from xf by replacing the part xhT_n^r by terms whose orders are higher than that of xf^* ($\deg x_n\geq\deg x_i$ for all $i=1,\cdots,n-1$ and $\deg x_n>1$). Therefore $e^*=xf^*$. Since $e\in Q$ and the degree of e in T_n is less than r, by induction we may assume that $e^*\in\mathcal{P}$. So we obtain $f^*\in\mathcal{P}:x=\mathcal{P}$.

Theorem 4.3. *Let I be a homogeneous ideal of the principal class which is minimally generated by a filter-regular sequence $x_1,\cdots,x_n$. Put $a_i=\deg x_i$. Suppose that $a_1\leq\cdots\leq a_n$. Then*

$$e(R[It,t^{-1}])=(1+\sum_{i=l}^{n-1}a_1\cdots a_i)e(R),$$

where l is the largest integer for which $a_l=1$ ($l=0$ and $a_1\cdots a_l=1$ if $a_i>1$ for all $i=1,\cdots,n$).

Proof. If $l=n$, let b be an integer such that $K=(I_nI^{b-1}+I^{b+1}):x_n^b$. It is easy to check that

$$(KT_n^b,I)\subseteq Q^*\subseteq K\mathcal{A},$$

where the latter inclusion follows from Lemma 4.2. Since $(KT_n^b,I)=K\mathcal{A}\cap(T_n,I)$ and $\dim\mathcal{A}/K\mathcal{A}=d+1>d=\dim\mathcal{A}/(T_n^b,I)$, we can apply Lemma 3.2 and obtain

$$e(R[It,t^{-1}])=e(\mathcal{A}/Q^*)=e(\mathcal{A}/K\mathcal{A})=e(R/K).$$

With the notation of the proof of Lemma 4.2 we have

$$R/K=\mathrm{gr}_I(R)/(\bar{J}_n,\bar{x}_1,\cdots,\bar{x}_n)=\mathrm{gr}_I(R)/(\bar{J}_n,\bar{x}_n),$$

where $\bar{x}_1,\cdots,\bar{x}_n$ may be assumed to be a filter-regular sequence with respect to the ideal $(\bar{x}_1,\cdots,\bar{x}_n)$ of $\mathrm{gr}_I(R)$. By Lemma 2.4 and Theorem 2.5,

$$e(\mathrm{gr}_I(R)/(\bar{J}_n,\bar{x}_n))=e(\mathrm{gr}_I(R))=e(R).$$

Summing up we will obtain $e(R[It,t^{-1}])=e(R)$, as required.

If $l<n$, set

$$\mathcal{L}:=(I_{l+1},T_{l+1}U,\cdots,T_nU,I_{l+2}T_{l+2},\cdots,I_nT_n),$$

where $I_j=(x_1,\cdots,x_{j-1})$, $j=l+1,\cdots,n+1$. It is clear that $\mathcal{L}$ is the ideal generated by the initial terms of the relations $x_iT_j-x_jT_i$ and x_i-T_iU, whence $\mathcal{L}\subseteq Q^*$. It is easy to check that $\mathcal{L}$ has the following decomposition

$$\mathcal{L}=(I_{l+1},T_{l+1},\cdots,T_n)\cap\bigcap_{i=l+1}^{n}(I_i,T_{i+1},\cdots,T_n,U).$$

Every component of this decomposition has dimension $d+1$, but the sum of every couple of them has a smaller dimension. Therefore, using the associative formula for multiplicities we obtain

$$e(\mathcal{A}/\mathcal{L})=e(\mathcal{A}/(I_{l+1}, T_{l+1}, \cdots, T_n))+\sum_{i=l+1}^{n} e(\mathcal{A}/(I_i, T_{i+1}, \cdots, T_n, U))$$
$$=e(R/I_{l+1})+\sum_{i=l+1}^{n} e(R/I_i)\,.$$

But $e(R/I_i)=a_1\cdots a_{i-1}e(R)$ for $i=1, \cdots, n$ by Lemma 1.6 ($e(R/I_1)=e(R)$). Hence

$$e(\mathcal{A}/\mathcal{L})=(1+\sum_{i=l}^{n-1} a_1\cdots a_i)e(R)\,.$$

Moreover, every $(d+1)$-dimensional primary component of $\mathcal{L}$ must be either of the form $(\mathfrak{q}, T_{l+1}, \cdots, T_n)$ or of the form $(\mathfrak{q}, T_{i+1}, \cdots, T_n, U)$ for some primary component $\mathfrak{q}$ of I_{l+1} with $\dim R/\mathfrak{q}=\dim R-l$ or of I_i with $\dim R/\mathfrak{q}=\dim R-i+1$, $i=l+1, \cdots, n$, respectively. We have seen in the proof of Theorem 3.3 that such a primary ideal $\mathfrak{q}$ must contain J_{l+1} resp. J_i. On the other hand, the ideal $\mathcal{P}=(J_{l+1}, T_{l+1}U, \cdots, T_nU, J_{l+2}T_{l+2}, \cdots, J_nT_n)$ has the following decomposition

$$\mathcal{P}=(J_{l+1}, T_{l+1}, \cdots, T_n)\cap\bigcap_{i=l+1}^{n}(J_i, T_{i+1}, \cdots, T_n, U)\,.$$

So it is clear that $\mathcal{P}$ is contained in all $(d+1)$-dimensional primary components of $\mathcal{L}$. But $Q^*\subseteq\mathcal{P}$ by Lemma 4.2. Therefore, by Lemma 3.2, we obtain

$$e(\mathcal{A}/Q^*)=e(\mathcal{A}/\mathcal{L})\,.$$

Since $e(R[It, t^{-1}])=e(\mathcal{A}/Q^*)$, the conclusion is now immediate.

Like in the case of Rees algebras we immediately obtain the following consequence of Theorem 4.3.

Corollary 4.4. (cf. [HTU, Example 3.1 and Example 3.2]). *Let R be a generalized Cohen-Macaulay graded algebra and $I\subset R$ a homogeneous ideal of the principal class with the degree sequence $a_1\leq\cdots\leq a_n$. Then*

$$e(R[It, t^{-1}])=(1+\sum_{i=l}^{n-1} a_1\cdots a_i)e(R)\,,$$

where l is the largest integer for which $a_l=1$, $l=0$ if $a_i>1$ for all $i=1, \cdots, n$.

Remark. For a local ring $(S, \mathfrak{m})$ and an arbitrary ideal $J\subseteq\mathfrak{m}^2$ with positive height, D. Katz and J. K. Verma [KV, Proof of (3.7)] have found the following multiplicity formula for the extended Rees algebra in terms of mixed multiplicities:

$$e(S[Jt, t^{-1}])=e(R)+\sum_{i=0}^{d-1}e_i(\mathfrak{m}|J).$$

Compared with Theorem 4.3, this suggests again a close connection between mixed multiplicities and filter-regular sequences as we have mentioned at the end of Section 3.

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References

[AB] A. Auslander and D. Buchsbaum, Codimension and multiplicity, Ann. Math., **68** (1958), 625-657.
[Br] M. Brodmann, Local cohomology of certain Rees- and form rings I, J. Algebra, **81** (1983), 29-57.
[CST] N. T. Cuong, P. Schenzel, and N. V. Trung, Verallgemeinerte Cohen-Macaulay-Moduln, Math. Nachr., **85** (1978), 57-73.
[HTU] J. Herzog, N. V. Trung, and B. Ulrich, On the multiplicity of blow-up rings of ideals generated by d-sequences, J. Pure Appl. Algebra, **80** (1992), 273-297.
[Hu] C. Huneke, The theory of d-sequences and powers of ideals, Adv. Math., **46** (1982), 249-279.
[KV] D. Katz and J. K. Verma, Extended Rees algebras and mixed multiplicities, Math. Z., **202** (1989), 111-128.
[Na] M. Nagata, Local rings, Interscience, 1960.
[NR] D. G. Northcott and D. Rees, Reduction of ideals in local rings, Math. Proc. Cambridge Phil. Soc., **50** (1954), 145-158.
[R] D. Rees, Generalisations of reductions and mixed multiplicities, J. Lnondon Math. Soc., **29** (1984), 397-414.
[SV] J. Stückrad and W. Vogel, Buchsbaum rings and applications, Springer, 1986.
[Te] B. Teissier, Cycles évanescents, sectiones planes, et conditions de Whitney, in: Singularitiés à Cargèse, Asterisque, **7**-8 (1973), 285-362.
[T1] N. V. Trung, Absolutely superficial sequences, Math. Proc. Cambridge Phil. Soc., **93** (1983), 35-47.
[T2] N. V. Trung, Reduction exponent and degree bound for the defining equations of graded algebras, Proc. Amer. Math. Soc., **102** (1987), 229-236.
[T3] N. V. Trung, Toward a theory of generalized Cohen-Macaulay moduls, Nagoya Math. J., **2** (1986), 145-167.
[V1] J. K. Verma, Rees algebras and mixed multiplicities, Proc. Amer. Math. Soc., **104** (1988), 1036-1044.
[V2] J. K. Verma, Multi-graded Rees algebras and mixed multiplicities, J. Pure Appl. Algebra, **77** (1992), 219-228.
[ZS] O. Zariski and P. Samuel, Commutative algebra, Van Nostrand, 1960.