J. Math. Kyoto Univ. (JMKYAZ)
33-4 (1993) 1057–1070

On an effective determination of a Shintani's decomposition of the cone $\mathbf{R}_+^n$

By

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1. Introduction

1.1. Problem. Let K be a totally real algebraic number field of degree $n \geq 2$, o_K the ring of integers of K, o_K^+ the set of all totally positive elements of o_K, E_K^+ the totally positive unit group of K and φ_i $(i = 1, 2, \dots, n)$ the distinct embeddings of K into $\mathbf{R}$. We embed K into $\mathbf{R}^n$ (considered as column vectors) by identifying each element α of K with ${}^t(\varphi_1(\alpha), \varphi_2(\alpha), \dots, \varphi_n(\alpha))$. By this embedding, E_K^+ acts on the cone $\mathbf{R}_+^n$ consisting of all vectors with positive entries. A set of vectors $v_1, v_2, \dots, v_r \in \mathbf{R}^n$ generates an open polyhedral cone $C(v_1, v_2, \dots, v_r)$:

$$C(v_1, v_2, \dots, v_r) = \mathbf{R}_+ v_1 + \mathbf{R}_+ v_2 + \cdots + \mathbf{R}_+ v_r\,. \tag{1}$$

If v_1, v_2, $\dots$, v_r are linearly independent, $C(v_1, v_2, \dots, v_r)$ is called an open simplicial cone of rank r. Shintani [3] proved a theorem which states that there is a fundamental domain for the action of E_K^+ on $\mathbf{R}_+^n$, which is a finite union of open simplicial cones of various ranks generated by elements v_{jk}'s of o_K^+:

$$D = \bigcup_{j \in J} C(v_{j1}, v_{j2}, \dots, v_{jr(j)}) \qquad \text{(disjoint.)} \tag{2}$$

Here, the term fundamental domain is used in the strict sense:

$$\mathbf{R}_+^n = \bigcup_{e \in E_K^+} eD \qquad \text{(disjoint.)} \tag{3}$$

His proof itself gives a theoreticaly effective method to find such a fundamental domain. However, it is almost impossible to execute his method unless the degree n is small. Therefore, fundamental domains are actually known only for the case of $n = 2$ [3] and the case of $n = 3$ [4]. The purpose of this paper is to present an efficient method to find a fundamental domain of the form (2), applicable to the case of number fields of higher degrees.

1.2. Method and result. Our method can been seen as a sort of a generalization of the positive continued fraction expansion of Hirzebruch (see [3]), which works only in real quadratic fields. In this context we regard the Hirzebruch's continued fraction expansion as computation of a sequence of adjacent integers on the boundary of the convex hull of o_K^+ where K is a real quadratic field. In the general case, we firstly study the structure of the boundary of the convex hull of the set o_K^+ in Theorem 8. Here we replace the adjacency between integers by adjacency between hyperpolyhedras on the boundary; two integers are adjacent if and only if the two edges, which are to the "right" of those integers, are adjacent in the quadratic case. Then, our algorithm to construct a fundamental domain for the action of E_K^+ on $\mathbf{R}_+^n$ is given in Theorem 11. Unlike the original method of Shintani, it works without knowing a set of generators of E_K^+ beforhand. It rather gives us a set of generators of E_K^+ as a by-product. We remark that our algorithm viewed as an algorithm for determining the unit group can be thought of as a "positive" version of the generalized Lagrange algorithm of Buchmann [1]. The similarity becomes clearer if we modify the Buchmann's algorithm in the following way, although the modified algorithm is slower than the original graph theoretic version: attach an abstract simlex to each minmal subset, consider elements of a minimal set as vertices of the simplex, glue them to form a complex, compute a non-associated maximal connected family of simplices on that complex and then one has enough information of the unit group.

2. Preparation

We will give some notations and terminologies which will be used in this paper together with those introduced in § 1 and we also recall some basic facts. We note that some terminologies differ from standard ones for the sake of simlicity of the argument on the topic dealt with here.

2.1. Unit theorem. We use Dirichlet's unit theorem in the following form:

Proposition 1. *The totally positive unit group E_K^+ of K has the following properties.*

(i) *Let k be an integer such that $1 \leq k \leq n$. Then, there is a unit $\varepsilon_k \in E_K^+$ such that*

$$\begin{cases} 0 < \varphi_i(\varepsilon_k) < 1 & \text{for } i \neq k\,, \\ 1 < \varphi_k(\varepsilon_k)\,. \end{cases} \tag{4}$$

(ii) *The group E_K^+ acts on $\mathbf{R}_+^n/\mathbf{R}_+$ and has a fundamental domain whose topological closure is compact in the natural topology.*

2.2. Convex sets. A subset B of $\mathbf{R}^n$ is called convex if the segment connecting arbitrary two points of B is contained in B. The convex hull B of a set S is the smallest convex set containing S, i.e., the set of all points of the form $t_1p_1 + t_2p_2 + \cdots + t_rp_r$ with a positive integer r, positive real numbers t_1, t_2, $\ldots$, t_r such that $t_1 + t_2 + \cdots + t_r = 1$ and points p_1, p_2, $\ldots$, p_r of S. A hyperplane π is called a support hyperplane of a convex set B provided

(i) The hyperplane π contains a boundary point of B and

(ii) π divides $\mathbf{R}^n$ into the disjoint union of a closed half-space which completely contains B and an open half-space which is disjoint with B.

The following fact is well known.

Proposition 2. *Let p be a boundary point of a convex set B in $\mathbf{R}^n$. Then, there exists a support hyperplane π of B contaning the point p.*

2.3. Faces. We denote by $\bar{B}$ the topological closure of a set B in $\mathbf{R}^n$. Let B be a convex set. Then, the intersection F of the set $\bar{B}$ and a support hyperplane of B is called a face of B.

Definition 3. Let B be a convex set in $\mathbf{R}^n$. Then a face P of B is called a hyperface of B if the dimension of P is $n-1$. and a face E of B is called a hyperedge of B if the dimension of E is $n-2$.

Definition 4. Two distinct hyperfaces P and P' of a convex set of the dimension n are called adjacent at a hyperedge E if P and P' contain E.

Clearly, any hyperface can have at most one hyperface adjacent at a single hyperedge.

Definition 5. A family $\mathscr{F}$ of hyperfaces of a convex set is called connected if an arbitrary pair of distinct hyperfaces P, $P' \in \mathscr{F}$ has a sequence $P = P_1$, P_2, $\dots$, $P_k = P'$ of hyperfaces in $\mathscr{F}$ such that P_i and P_{i+1} are adjacent for $i = 1, 2, \dots, k-1$.

Let $p_1, p_2, \dots, p_r$ be points in $\mathbf{R}^n$. Then we denote by $P(p_1, p_2, \dots, p_r)$ the convex hull of $p_1, p_2, \dots, p_r$. If the set $P(p_1, p_2, \dots, p_r)$ has the dimension $n-1$, it is called a convex hyperpolyhedra. In this case, we denote by $\pi(p_1, p_2, \dots, p_r)$ the unique hypeplane containing points $p_1, p_2, \dots, p_r$. We finish this section by defining the notion of open faces.

Definition 6. Let $P = P(p_1, p_2, \dots, p_r)$ be a face of a convex set B. Then, the set $P^o = P^o(p_1, p_2, \dots, p_r)$ is defined by

(i) $P^o = P$ if P has the dimension 0 or

(ii) $P^o = \{\sum_{i=1}^r t_i p_i | t_1, t_2, \dots, t_r \in \mathbf{R}_+, \sum_{i=1}^r t_i = 1\}$ otherwise.

The set P^o is called an open hyperface spanned by $p_1, p_2, \dots, p_r$.

3. Theorems

3.1. The family of hyperfaces. Let A be a finite set of points in $\mathbf{R}_+^n$, S_A the union $o_K^+ \cup A$, B_A the convex hull of S, Δ_A the boundary set of B_A and $\mathscr{D}_A$ the family of all hyperfaces of B_A. The set A is called an auxiliary set and its elements are called auxiliary points for the reason which will be stated before Theorem 15. We omit the subscript A when the set A is empty. For brevity, elements of S_A are called S_A-points.

The following lemma is fundamental:

Lemma 7. *Let π be a support hyperplane of B_A. Then, the hyperplane π has an equation of the form*

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 1 \tag{5}$$

where $c_1, c_2, \dots, c_n > 0$. Moreover, B_A is contained in the closed half-space

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \geq 1\,. \tag{6}$$

This further implies that there are at most finitely many S_A-points on any segment on Δ_A.

Proof. Let N be the map defined by

$$N\colon {}^t(x_1, x_2, \dots, x_n) \in \mathbf{R}^n_+ \mapsto x_1x_2\dots x_n \in \mathbf{R}_+ \tag{7}$$

and set $a = \min(N(A) \cup o_K^+) = \min(N(A) \cup \{1\}) > 0$. Then, the set S_A is contained in the convex set

$$x_1x_2\dots x_n \geq a\,. \tag{8}$$

This implies that all boundary points of B_A and, in particular, a point $p = (p_1, p_2, \dots, p_n)$ on π are contained in $\mathbf{R}^n_+$. Write the equation of π:

$$c_1x_1 + c_2x_2 + \cdots c_nx_n = c \tag{9}$$

and assume that the set B_A is contained in the half-space

$$c_1x_1 + c_2x_2 + \cdots c_nx_n \geq c\,. \tag{10}$$

On the other hand, one has a unit ε_k for $1 \leq k \leq n$ as stated in Proposition 1. Applying inequality (10) to sufficiently higher power of ε_k, one sees that c_k's are non-negative and that $c \leq 0$ if one of c_k's is zero. However, applying equation (9) to the point $p \in \mathbf{R}^n_+$ mentioned after (8), one has that

$$c_1p_1 + c_2p_2 + \cdots + c_np_n = c\,. \tag{11}$$

One sees that c is positive because some of c_k's must be non-zero. Therefore, all of c_k's and c are positive. One gets the desired form of equation by dividing coefficients of equation (9) by c. The last assertion is proved by observing that the support hyperplane π of B_A at the middle point of a segment σ has the form (5) and that π contains σ.

Using this lemma, we show that B_A has a similar structure to bounded convex bodies.

Theorem 8. *The set B_A, Δ_A and $\mathcal{D}_A$ have the following properties:*

- (i) *the set B_A is a closed set in $\mathbf{R}^n$ and each face of B_A is of the form $P(p_1, p_2, \dots, p_r)$ with S_A-points $p_1, p_2, \dots, p_r$;*
- (ii) *there are at most finitely many faces of B_A intersecting a given bounded set;*
- (iii) *the set Δ_A is a union of all hyperfaces of B_A;*
- (iv) *the set Δ_A forms a system of representatives for $\mathbf{R}^n_+/\mathbf{R}_+$;*
- (v) *there are exactly two hyperfaces containing a given hyperedge;*
- (vi) *the family $\mathcal{D}_A$ is connected and*
- (vii) *the set Δ_A is uniquely decomposed into a union of all open faces of B_A.*

Proof. Let p be a boundary point of B_A. Then, there is a support hyperplane π contaning p of the form (5) by Proposition 2 and Lemma 7. We denote by π_t the hyperplane defined by

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 1 + t \tag{12}$$

Clearly, one can find a small positive real number t such that the set

$$1 < c_1x_1 + c_2x_2 + \cdots + c_nx_n < 1 + t \tag{13}$$

does not contain S_A-points because the hyperplane π_t together with all of coordinate hyperplanes enclose a bounded set. Let $p_1, p_2, \ldots, p_r$ be all of S_A-points on π and P the convex set $P = P(p_1, p_2, \ldots, p_r)$. Assertion (i) follows if one shows $p \in P$. Take an arbitrary point p' in B_A which is close to p. Then, p' can be written as

$$p' = t_1p_1 + t_2p_2 + \cdots + t_rp_r + t_{r+1}p_{r+1} + \cdots + t_{r'}p_{r'} \tag{14}$$

where $p_{r+1}, p_{r+2}, \ldots, p_{r'}$ are S_A-points outside π and $t_1, t_2, \ldots, t_{r'}$ are non-negative real numbers such that $t_1 + t_2 + \cdots + t_{r'} = 1$. From this, one sees that there are positive real numbers s, s', a point q on P and a point q' in the opposite side of the hyperplane π_t to the origin such that $p' = sq + sq'$. This further implies that p' is on the segment connecting a point on P and a point on $\pi_t \cap \mathbf{R}^n_+$. Hence, the point p' is in $M = P(p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_n)$ where q_i is the intersection of π_t and the i-th axis. The point p must lie in M since p' can be chosen arbitrary close to p and M is closed. This together with $\pi \cap M = P$, implies that p is in fact on P, proving assertion (i).

Let M be a bounded set. We assume, without loss of generality, that M is of the form

$$M = \{{}^t(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n_+ | x_1 + x_2 + \cdots + x_n < m\} \tag{15}$$

where m is a positive integer. Let P be a hyperface of B_A intersecting M. Then, by assertion (i), P can be written as $P = P(p_1, p_2, \ldots, p_r)$ with S_A-points p_1, p_2, $\ldots$, p_r. The assumption on the shape of M guarantees that at least one of p_i's, say $p = p_1$, is contained in M. There are at most finitely many such p's since M is bounded. Observe that the segments $P(p, p_i)$ are on Δ_A and one sees that assertion (ii) follows if one shows that there are at most finitely many S_A-points q's such that segments $P(p, q)$'s are on Δ_A. Suppose contrary, that there are infinitely many such q's. Draw a small sphere Σ at p and look at the set of intersections of $P(p, q)$'s and that sphere. Then, there is an accumulating point p_∞ on Σ of those intersection points since Lemma 7 guarantees that there are at most finitely many S_A-points on a segment in Δ_A. By Proposition 2, there is a support hyperplane π of B_A containing p_∞ of the form (5). Let $c' > 1$ be a constant such that the hyperplabe π' defined by

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = c' \tag{16}$$

does not intersect Σ. When q is in the opposite side of π' to the origin, the intersection of $P(p, q)$ and Σ is in the set $P(p, q'_1, q'_2, \ldots, q'_n)$ where q'_i is the intersection of π' and the i-th axis. On the other hand there are finitely many S_A-points in the region

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n \leq c' . \tag{17}$$

Thus, p_∞ must contact $P(p, q'_1, q'_2, \dots, q'_n)$. This together with the definition of π' implies that p and q must coniside. But q must differ p by the radious of Σ. This contradiction proves assertion (ii). Assertion (iii) follows from assertion (i), (ii) and that Δ_A has the dimesion $n-1$. Assertion (iv) follows from Lemma 7. Assertion (v) follows from assertion (i) and (ii). Assertion (iv) guarantees that a given pair of points on Δ_A can be connected by a pass γ on Δ_A which does not intersect any face of dimension lower than $n-2$ except at the given initial and terminal points. Applying this statement to the barycenters of given hyperfaces P and P', one sees that there is a sequence $P_1 = P$, P_2, ..., $P_k = P'$ such that each successive pair P_i and P_{i+1} is adjacent by taking as P_i's the hyperfaces through which the pass γ goes in order. Assertion (vi) is proved. Assertion (vii) follows from (1).

On the other hand, we have the obvious action of E_K^+ on the family of subsets of $\mathbf{R}_+^n$ defined by $e \in E_K^+ \colon G \subset \mathbf{R}_+^n \mapsto eG = \{ep | p \in G\}$. Two subsets G and G' of $\mathbf{R}_+^n$ are called associated if there is a unit e in E_K^+ such that $G = eG'$, and a family $\mathscr{F}$ of subsets of $\mathbf{R}^n$ is called non-associated if no pair of distinct sets from $\mathscr{F}$ are associated. When the auxiliary set is empty, the group E_K^+ acts on $B = B_\phi$, on $\Delta = \Delta_\phi$ and on the family of all support hyperplanes of B. Thus, the group E_K^+ also acts on the family $\mathscr{D} = \mathscr{D}_\phi$. This action has the following properties:

Theorem 9. *Assume that the auxiliary set A is empty. Then, one firstly has that $\mathscr{D}$ decomposes into finitely many E_K^+-orbits. Secondly, let $\mathscr{M}$ be a maximal connected non-associated family of hyperfaces of B. Then, $\mathscr{M}$ is finite and is a system of representatives for the action of E_K^+. Lastly, let $\mathscr{E}$ be the family of all hyperedges contained in exactly one hyperface in $\mathscr{M}$. Then, there is exactly one unit $\varepsilon(E) \neq 1$ in E_K^+ such that $\varepsilon(E)E \in \mathscr{E}$. The group E_K^+ is generated by those units $\{\varepsilon(E) | E \in \mathscr{E}\}$.*

Proof. By assertion (iv) of Theorem 8, Δ is thought of as $\mathbf{R}_+^n/\mathbf{R}_+$. Proposition 1 says that there is a compact closure D_0 of a fundamental domain of the action of E_K^+ on the set Δ. Let $\mathscr{M}_0$ be the family of all hyperfaces which intersect D_0. Then, the family $\mathscr{M}_0$ is finite by assertion (ii) of Theorem 8. This implies that any non-associated family of hyperfaces is finite, i.e., Δ decomposes into finitely many E_K^+-orbits. Thus, the first assertion is proved. Now, it is clear that there is a maximal connected non-associated family $\mathscr{M}$. The family $\mathscr{M}$ is finite by the first assertion. To see that $\mathscr{M}$ is a system of representatives, we pick up an arbitrary hyperface P from $\mathscr{D}$ and a hyperface P' from $\mathscr{M}$. Assertion (vi) of Theorem 8 guarantees the existance of a sequence $P_1 = P$, P_2, ..., $P_k = P'$ in which successive hyperfaces are adjacent. We will shorten this sequence by substituting P_1 by an associated hyperface while $k > 1$ as follows. Firstly, we substitute P' by P_{k-1} if P_{k-1} belongs to $\mathscr{M}$. If this is not the case, there must be the associated hyperface P'' of P_{k-1} in $\mathscr{M}$ from the maximality of

$\mathcal{M}$. Let ε be the unit which transfers P_{k-1} to P''. We substitute P by εP, P' by P'' and the sequence $P_1, P_2, \ldots, P_k$ by the sequence $\varepsilon P_1, \varepsilon P_2, \ldots, \varepsilon P_{k-1}$. We obviously get a shorter sequence in either case. Repeating these procedures, we will finally reach to a sequence of length 1, i.e., the associated hyperface in $\mathcal{M}$ of the original P. This proves that the family $\mathcal{M}$ is a system of representatives. This argument also proves the last assertion except the uniqueness of $\varepsilon(E)$. Let E be a hyperedge in $\mathcal{E}$ and assume that a unit $\varepsilon \in E_K^+$ carries E to a hyperface εE which also belongs to $\mathcal{E}$. Further, let P be the hyperface in $\mathcal{M}$ containing E and P' the hyperface in $\mathcal{M}$ containing εE. Then, $\varepsilon^{-1}P'$ must be the adjacent hyperface to P at E. Thus, the hyperface P' is determined as the unique representative in $\mathcal{M}$ of the orbit of the adjacent hyperface of P at E. Therefore, the condition $\varepsilon E \in \mathcal{E}$ uniquely determines the unit $\varepsilon(E) = \varepsilon$.

In fact, Theorem 9 is the key to the construction of a fundamental domain for the action of E_K^+ on $\mathbf{R}_+^n$ as follows:

Theorem 10. *Let $\mathcal{M}$ be a maximal connected non-associated family of hyperfaces of B, D_1 the union of all hyperfaces in $\mathcal{M}$ and $\mathcal{M}^o$ the family of all open faces contained in D_1. Then, one can find a subfamily $\mathcal{M}_0^o$ of $\mathcal{M}^o$ such that the union*

$$D_0 = \bigcup_{F \in \mathcal{M}_0^o} F \qquad (disjoint) \tag{18}$$

is a fundamental domain for the action of E_K^+ on Δ. Moreover, one has the fundamental domain D for the action of E_K^+ on $\mathbf{R}_+^n$ defined by

$$D = \mathbf{R}_+ D_0 \tag{19}$$

$$= \bigcup_{F \in \mathcal{M}_0^o} \mathbf{R}_+ F \qquad (disjoint) \tag{20}$$

where, $\mathbf{R}_+F$'s are open polyhedral cones. The fundamental domain D has a decomposition into an disjoint union of open simplicial cones.

Proof. The theorem follows from Theorems 8 and 9.

3.2. Effectivity. For speaking of the effectivity, we assume that a generator α of K over $\mathbf{Q}$ is given by its minimal equation, that a basis $\beta_1 = 1, \beta_2, \ldots, \beta_n$ of the integral ring of integers is given by a set of polynomials in α and that the following operations are effective:

(i) the basic operations of real numbers to the desired precision (i.e. addition, subtraction, multiplication and division),

(ii) the exact basic operations of algebraic integers (i.e. addition, subtraction, multiplication, test for divisibility and division in the divisible case.)

Now, we present the following algorithm for finding a fundamental domain and later fill in the details of that algorithm.

Theorem 11. *The following algorithm effectively leads to a fundamental domain D for the action of E_K^+ on $\mathbf{R}_+^n$:*

- *Firstly, find out one particular hyperface of B by the method to be described in Proposition* 15.
- *Secondly, using the method to be described in Proposition* 14, *successively search for adjacent hyperfaces which are adjacent to one of the hyperfaces so far found and associated to none of the hyperfaces so far found. We surely reaches to a maximal connected non-associated family $\mathscr{M}$ of hyperfaces in finite steps.*
- *Thirdly, divide the hyperfaces (found in the second step) into open faces, then select maximal non-associated family of faces from those.*
- *Lastly, we divide the faces (found in the third step) into open simplices to find a family $\mathscr{S}$ of simplices on Δ spanned by o_K^+-points. Now, the union $D = \bigcup_{\sigma \in \mathscr{S}} \mathbf{R}_+\sigma$ is the desired form of a fundamental domain.*

Furthermore, let $\mathscr{E}$ be the family of all hyperedges which are contained in exactly one hyperface of $\mathscr{M}$. Then, there exists exactly one unit $\varepsilon(E) \neq 1$ of K such that $\varepsilon(E)E$ belongs to $\mathscr{E}$, and the totally positive unit group E_K^+ is generated by $\{\varepsilon(E) | E \in \mathscr{E}\}$.

Proof. The effectivity of the first and the second steps will be proved by Propositions 14, 15 and the fact that non-associatedness of hyperfaces can be checked by computing quotients of their vertices. The third and fourth steps are no doubt effective. Therefore, Theorems 9 and 10 guarantee that the repetition in the second step terminates and that the set D is a fundamental domain for the action of E_K^+ on $\mathbf{R}_+^n$. We note that the faces found in the third step are not always simplices so that the last step is neccessary. The second assertion is a part of Theorem 9.

Lemma 12. *Let $P = P(p_1, p_2, \ldots, p_r)$ be a hyperface of B_A and assume that vectors $p_1, p_2, \ldots, p_n$ are linearly independent. Then, points $p_1, p_2, \ldots, p_{n-1}$ are contained in a hyperedge of B_A if and only if all determinants $|p_1, p_2, \ldots, p_{n-1}, p_i|$ for $i = n, n+1, \ldots, r$ are simultaneously non-negative or non-positive. Moreover, if $p_1, p_2, \ldots, p_{n-1}$ are contained in a hyperedge E, then one has that $E = P(p_{\ell(1)}, p_{\ell(2)}, \ldots, p_{\ell(r')})$ where $\ell(i)$'s are all of indices such that the determinants $|p_1, p_2, \ldots, p_{n-1}, p_{\ell(i)}| = 0$.*

Proof. We note that r must be greater than or equal to n for P to be a hyperface. Then the proof of this lemma is obvious.

Remark. The above mentioned algorithm seems very slow since the number of integers on a hyperface can be very large as it is observed in the example of § 3.3. In fact, we can traverse faces of co-dimension one from one of them for any finite convex body, by a similar method to Lemma 13. The latter method seems faster but unfortunately the author is unaware of the time complexity of the algorithm described in Lemma 13. Therefore, the author decided to avoid detailed description of the latter method which requires a little more complicated data structure of recursive nature. Here we just mention that the combinatoric complexity of the former is the most time consuming in the above mentioned example.

Let $p_1, p_2, \ldots, p_n$ be linearly independent vectors in $\mathbf{R}_+^n$. We denote by $\pi_t = \pi_t(p_1; p_2, \ldots, p_n)$ the hyperplane containing $tp_1, p_2, p_3, \ldots, p_n$. When π_t does not pass through the origin, we denote by $\mathbf{c}_t = \mathbf{c}_t(p_1; p_2, \ldots, p_n)$ the normal vector of π_t such that $\pi_t = \{p \in \mathbf{R}^n | (\mathbf{c}_t, p) = 1\}$. When $\mathbf{c}_1$ consists of positive entries, we denote by $t_\infty = t_\infty(p_1; p_2, \ldots, p_n)$ the real number $\inf\{t \in \mathbf{R}_+ | \mathbf{c}_t \in \mathbf{R}_+^n\}$. We also denote by $k(p_1; p_2, \ldots, p_n)$ the index $\min\{i | i\text{-th entry of } \mathbf{c}_{t_\infty} \text{ is zero}\}$. Well-definedness of these numbers are easily verified as follows. Let q_i be the intersection of π_t and the i-th axis for $i = 1, 2, \ldots, n$. Then, q_i's approach to the origin as t goes to 0. But π_t contains a point p_2. Thus, the coordinate of one of q_i's, say, q_k, becomes negative at a small t, for otherwise $p_2 \in \mathbf{R}_+^n$ must be arbitrary close to the origin. This implies that the coordinate of q_k becomes 0 or q_k diverges at certain t before q_k has a negative coordinate since coordinate of q_k is written as a linear fractional function of t. However, the q_k can't be the origin except at $t = 0$. Therefore, q_k diverges at certain t such that $0 < t < 1$. It is clear, that $\mathbf{c}_k$ becomes zero at this point. Well-definedness of t_∞ and $k(p_1; p_2, \ldots, p_n)$ follows since there are finitely many indices k.

Lemma 13. *Let E be a hyperedge of B_A whose S_A-points are known, P a hyperface of B_A containing E, P' the hyperface adjacent to P at E, p_1 a point on P which is not on E, $p_2, p_3, \ldots, p_n$ lineary independent points on E, p_1' a vector in $\mathbf{R}^n$ whose entries are zero except the $k = k(p_1; p_2, \ldots, p_n)$-th entry is the inverse of the k-th entry of $\mathbf{c}_1(p_1; p_2, \ldots, p_n)$ (i.e., the intersection of the k-th axis and $\pi(p_1, p_2, \ldots, p_n)$) and $U_s = P(0, sp_1', p_2, p_3, \ldots, p_n)$ for real number $s > 1$. Then, U_s for sufficiently large s contains a point from S_A which does not belong to E. Moreover, one can effectively determine the minimum s_0 of such s and the set $V = S_A \cap \pi(s_0 p_1', p_2, p_3, \ldots, p_n)$ including a point q outside E provided that A consists of points of K. The set V spans the hyperface P'.*

Proof. Proposition 1 says that there exists a unit $\varepsilon \in E_K^+$ such that each component $\varphi_i(\varepsilon)$ is less than 1 except $\varphi_k(\varepsilon) > 1$. Taking sufficiently high power of ε, one finds an algebraic integer ε' which is very close to the k-th axis. It is clear, that ε' is contained in U_s for sufficiently large s. We proved the first assertion. The last assertion is clear. For the second assertion, we adopt the following algorithm which determines the minimum of s.

- Firstly, compute all integral points within U_s for $s = 2, 4, 8, \ldots$ until an integral point in $U_s - E$ is found. Let s_1 be the first s such that $U_s - E$ is found to contain an integral point and $q_1, q_2, \ldots, q_{r'}$ all of integral point in $U_{s_1} - E$.
- Secondly, compute determinants

$$|q_j, p_2 - q_i, p_3 - q_i, \ldots, p_n - q_i| \tag{21}$$

for $1 \leq i, j \leq r'$ to the absolute precision of $1/(4m^n)$ with $m \in \mathbf{Z}$ such that $mA \subset o_K$. Then, one finds a $q = q_\ell$ such that

$$|q_j, p_2 - q, p_3 - q, \ldots, p_n - q|/|q_\ell, p_2 - q, p_3 - q, \ldots, p_n - q| \qquad (22)$$

is greater than or equal to 1 for $1 \le j \le r'$ (comparison should be done paying respect to the precision).

- Lastly, compute the intersection p_1'' of $\pi(q, p_2, p_3, \ldots, p_n)$ and the k-th axis. Then, the number s_0 is the quotient of the k-th coordinates of p_1'' and p_1'. Moreover, the point $q \notin E$ is on the hyperplane $\pi(s_0 p_1', p_2, p_3, \ldots, p_n)$ and the set V is the union of the S_A-point on E and the set of all q_j's such that the quotient (22) equals to 1 (comparison are done paying respect to the precision).

The first assertion guarantees that this algorithm terminates. On the other hand, it is clear that this algorithm computes the desired result if computation could be exact. Thus, we only need to verify that the precision refered to in the second step is sufficient for obtaining the correct answer. Let d be the discriminant of K. Then, determinants $|q_j, p_2 - q_i, p_3 - q_i, \ldots, p_n - q_i|$ are *a priori* known to be greater than or equal to $\sqrt{d}/m^n$. Therefore, the precision of $1/4m^n$ is sufficient.

A face F of B_A is said to be effectivey determined if all of S_A-points on F is effectivey determined.

Proposition 14. *Assume that the auxiliary set A consists of algebraic numbers of K. And let $P = P(p_1, p_2, \ldots, p_r)$ be a hyperface of B_A whose S_A-points are known. Then, one can effectively list all hyperedges contained in P by computing determinants $|p_{\ell(1)}, p_{\ell(2)}, \ldots, p_{\ell(n)}|$ for all possible injections $\ell\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, r\}$, as stated in Lemma* 12.

Let E be a hyperedge contained in P and P' the hyperface adjacent to P at E. And assume that p_2, p_3, $\ldots$, p_n are independent points on E and that p_1 is on $P - E$. Then, by Lemma 13, *one can effectively determine P'.*

Proof. The proof is obvious.

To find one hyperface from which we can compute successive hyperfaces, we fake the method of Proposition 14 by adjoining auxiliary points to S to form a "known" hyperface.

Proposition 15. *One can effectively find a basis $\beta_1' = 1$, β_2', $\ldots$, β_n' of o_K consisting of totally positive integers. Set $\gamma_i = n\beta_i'/\mathrm{tr}\,\beta_i'$ for $1 \le i \le n$ and $A_i = \{\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_n\}$ for $0 \le i \le n$, (in particular, $A_n = \phi$.) Then, the set $P_1 = P(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a hyperface of B_{A_1} such that $S_A \cap P_1 = A_0$. One can inductivly and effectively define P_i's which are hyperfaces of B_{A_i}'s as follows: One can effectively determine a hyperedge $E_i \subset P_i - \{\gamma_{i+1}\}$ by Lemma* 12 *and the adjacent hyperface P_{i+1} of P_i at E_i by Lemma* 13. *In particular, P_n is an effectively determined hyperface of B.*

Proof. The first assertion is verified by recalling that each algebraic integer can be computed to the precision of 1/2. The second assertion is proved by the fact that the hyperplane $\pi: x_1 + x_2 + \cdots + x_n = n$ contacts the convex set $x_1 x_2 \ldots x_n \geq 1$, which contains all totally positive integers, at $\gamma_1 = {}^t(1, 1, \ldots, 1)$. Note that $B_{A_0} = B_{A_1}$. The third and the fourth assertions are clear.

3.3. Example. For a demonstration of Theorems 9 and 11, we give the following example. Let ζ be a primitive 11-th root of unity, θ be the sum of ζ and its complex conjugate. Then θ has the minimal polynomial

$$X^5 + X^4 - 4X^3 - 3X^2 + 3X + 1 .$$

We take the field $\mathbf{Q}(\theta)$ as K. Then, the field K is a totally real number field of degree 5 and has the discriminant 14641. One can take

$$1, \theta, \theta^2, \theta^3, \theta^4$$

as a basis for o_K. Computation shows that one can take $\mathscr{M}$ to be the family consisting of the following 9 hyperfaces:

$$P(\alpha_1, \alpha_3, \alpha_5, \alpha_8, \alpha_{10}),$$
$$P(\alpha_5, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}),$$
$$P(\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_8, \alpha_9),$$
$$P(\alpha_1, \alpha_5, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{16}),$$
$$P(\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_8, \alpha_{10}),$$
$$P(\alpha_5, \alpha_7, \alpha_{10}, \alpha_{13}, \alpha_{14}, \alpha_{18}),$$
$$P(\alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{13}, \alpha_{19}),$$
$$P(\alpha_8, \alpha_9, \alpha_{10}, \alpha_{15}, \alpha_{16}, \alpha_{17}),$$
$$P(\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{13})$$

where α_i's are given as follows:

$$\alpha_1 = -1 - 3\theta + 5\theta^2 + \theta^3 - \theta^4 ,$$
$$\alpha_2 = -1 - 4\theta + \theta^2 + 5\theta^3 + 2\theta^4 ,$$
$$\alpha_3 = -1 - 4\theta + 4\theta^2 + 3\theta^3 ,$$
$$\alpha_4 = -2\theta - \theta^2 + 3\theta^3 + 2\theta^4 ,$$
$$\alpha_5 = \theta^2 ,$$
$$\alpha_6 = \theta^2 + 2\theta^3 + \theta^4 ,$$
$$\alpha_7 = 2\theta^2 + \theta^3 ,$$
$$\alpha_8 = 1 - 2\theta - \theta^2 + 2\theta^3 + \theta^4 ,$$
$$\alpha_9 = 1 - 2\theta^2 + \theta^4 ,$$

$$\begin{aligned}
\alpha_{10} &= 1\,, \\
\alpha_{11} &= 1 + \theta - 2\theta^2 - \theta^3 + \theta^4\,, \\
\alpha_{12} &= 1 + 2\theta - 2\theta^2 - \theta^3 + \theta^4\,, \\
\alpha_{13} &= 1 + 2\theta + \theta^2\,, \\
\alpha_{14} &= 1 + 5\theta + 6\theta^2 - 2\theta^3 - 2\theta^4\,, \\
\alpha_{15} &= 10 - 5\theta - 12\theta^2 + 2\theta^3 + 3\theta^4\,, \\
\alpha_{16} &= 2 - 3\theta + \theta^3\,, \\
\alpha_{17} &= 3 - 2\theta - 3\theta^2 + \theta^3 + \theta^4\,, \\
\alpha_{18} &= 3 + 11\theta + 5\theta^2 - 4\theta^3 - 2\theta^4\,, \\
\alpha_{19} &= 4 - 4\theta^2 + \theta^4\,.
\end{aligned}$$

It turns out that there are too many lower dimensional faces to list up here. Therefore, we only list association between hyperedges. Association between hyperedges within the family $\mathscr{E}$ of Theorem 9 are as follows:

$$\begin{aligned}
P(\alpha_4, \alpha_6, \alpha_{13}, \alpha_{12}) &= \eta_7 P(\alpha_5, \alpha_9, \alpha_{10}, \alpha_{11})\,, \\
P(\alpha_2, \alpha_4, \alpha_8, \alpha_9) &= \eta_4 P(\alpha_5, \alpha_{10}, \alpha_{11}, \alpha_{12})\,, \\
P(\alpha_2, \alpha_5, \alpha_6, \alpha_7) &= \eta_3 P(\alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12})\,, \\
P(\alpha_7, \alpha_8, \alpha_{10}, \alpha_{13}) &= \eta_{10} P(\alpha_5, \alpha_9, \alpha_{11}, \alpha_{12})\,, \\
P(\alpha_5, \alpha_{10}, \alpha_{12}, \alpha_{13}) &= \eta_{11} P(\alpha_1, \alpha_8, \alpha_9, \alpha_{16})\,, \\
P(\alpha_1, \alpha_8, \alpha_{10}, \alpha_{16}) &= \eta_9 P(\alpha_5, \alpha_{10}, \alpha_{13}, \alpha_{18})\,, \\
P(\alpha_2, \alpha_7, \alpha_8, \alpha_{13}) &= \eta_4 P(\alpha_5, \alpha_{10}, \alpha_{14}, \alpha_{18})\,, \\
P(\alpha_2, \alpha_4, \alpha_6, \alpha_{12}) &= \eta_6 P(\alpha_3, \alpha_7, \alpha_8, \alpha_{10})\,, \\
P(\alpha_4, \alpha_8, \alpha_{12}, \alpha_{13}) &= \eta_{12} P(\alpha_2, \alpha_3, \alpha_5, \alpha_7)\,, \\
P(\alpha_2, \alpha_5, \alpha_6, \alpha_{12}) &= \eta_2 P(\alpha_8, \alpha_{10}, \alpha_{13}, \alpha_{19})\,, \\
P(\alpha_2, \alpha_6, \alpha_7, \alpha_{13}) &= \eta_3 P(\alpha_9, \alpha_{10}, \alpha_{12}, \alpha_{19})\,, \\
P(\alpha_5, \alpha_7, \alpha_{18}, \alpha_{14}) &= \eta_8 P(\alpha_8, \alpha_9, \alpha_{10}, \alpha_{19})\,, \\
P(\alpha_3, \alpha_5, \alpha_7, \alpha_{10}) &= \eta_1 P(\alpha_8, \alpha_{12}, \alpha_{13}, \alpha_{19})\,, \\
P(\alpha_4, \alpha_8, \alpha_9, \alpha_{12}) &= \eta_{12} P(\alpha_1, \alpha_2, \alpha_3, \alpha_5)\,, \\
P(\alpha_2, \alpha_3, \alpha_7, \alpha_8) &= \eta_3 P(\alpha_9, \alpha_{10}, \alpha_{15}, \alpha_{16})\,, \\
P(\alpha_1, \alpha_2, \alpha_5, \alpha_9) &= \eta_2 P(\alpha_{10}, \alpha_{16}, \alpha_{15}, \alpha_8)\,, \\
P(\alpha_5, \alpha_7, \alpha_{10}, \alpha_{14}) &= \eta_8 P(\alpha_8, \alpha_9, \alpha_{10}, \alpha_{15})\,,
\end{aligned}$$

$$P(\alpha_{10}, \alpha_{12}, \alpha_{13}, \alpha_{19}) = \eta_{11} P(\alpha_8, \alpha_9, \alpha_{15}, \alpha_{16}),$$

$$P(\alpha_8, \alpha_9, \alpha_{12}, \alpha_{19}) = \eta_{12} P(\alpha_1, \alpha_3, \alpha_5, \alpha_{10}),$$

$$P(\alpha_5, \alpha_7, \alpha_{13}, \alpha_{18}) = \eta_{11} P(\alpha_1, \alpha_3, \alpha_8, \alpha_{10}),$$

$$P(\alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{13}) = \eta_7 P(\alpha_1, \alpha_5, \alpha_9, \alpha_{10}, \alpha_{16}),$$

$$P(\alpha_2, \alpha_4, \alpha_5, \alpha_9, \alpha_{12}) = \eta_5 P(\alpha_7, \alpha_{10}, \alpha_{13}, \alpha_{14}, \alpha_{18}),$$

$$P(\alpha_5, \alpha_6, \alpha_7, \alpha_{12}, \alpha_{13}) = \eta_{11} P(\alpha_1, \alpha_2, \alpha_3, \alpha_8, \alpha_9),$$

where η_i's are given as follows:

$$\eta_1 = -2 - 5\theta + 11\theta^2 + \theta^3 - 3\theta^4,$$

$$\eta_2 = \theta^2,$$

$$\eta_3 = 2\theta^2 + \theta^3,$$

$$\eta_4 = 1 - 2\theta - \theta^2 + 2\theta^3 + \theta^4,$$

$$\eta_5 = 1 - 2\theta^2 + \theta^4,$$

$$\eta_6 = 1 + 2\theta - 2\theta^2 - \theta^3 + \theta^4,$$

$$\eta_7 = 1 + 2\theta + \theta^2,$$

$$\eta_8 = 1 + 5\theta + 6\theta^2 - 2\theta^3 - 2\theta^4,$$

$$\eta_9 = 2 - 3\theta + \theta^3,$$

$$\eta_{10} = 2 + \theta,$$

$$\eta_{11} = 3 + 11\theta + 5\theta^2 - 4\theta^3 - 2\theta^4,$$

$$\eta_{12} = 4 - 4\theta^2 + \theta^4.$$

One can check that units $\eta_1, \eta_2, \ldots, \eta_{12}$ are written as products of powers of

$$\eta_2 = \theta^2,$$

$$\eta_5 = (\theta - 1)^2(\theta + 1)^2,$$

$$\eta_7 = (\theta + 1)^2,$$

$$\eta_{10} = \theta + 2.$$

Moreover, one has the identity $\eta_2\eta_5\eta_{10}(\theta^2 - 3)^2 = 1$. This implies that the group E_K^+ is generated by square units. Thus, the unit group E_K of K is generated by

$$\theta,\ \theta+1,\ \theta-1,\ \theta^2-3\,.$$

This is consistent with a result in [2].

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